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Information Frames, Implication Systems and Modalities *

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Abstract

We investigate the logical systems which result from introducing the modalities $\Box$ and $\Diamond$ into the family of substructural implication logics (including relevant, linear and intuitionistic implication). Our results lead to the formulation of a uniform labelled refutation system for these logics.

1 Introduction

The classical theory of modalities is based on the notion of *possible world*. Possible worlds can be seen as *maximally consistent* sets (or infinite conjunctions) of propositions. However, the idea of something being *possible relative to* or *accessible from* something else is by no means restricted to such a context. For instance, we may sensibly speak of a theory T_2 being "possible" relative to another theory T_1, meaning that T_2 is obtained from T_1 by modifying some peripheral assumption which is not part of the "hard-core" of the theory. In this context, neither T_1 nor T_2 need to be maximally consistent sets of propositions: indeed, they are certainly not maximal and, perhaps, not even consistent.

It is not difficult to envisage other contexts in which we may want to consider an "accessibility" relation between objects which are not "worlds" in the classical sense, and use modal operators accordingly. We may, for example, sensibly speak of states, pieces of information, or resources being "accessible" or "reachable" from each other. In general, modal notions may turn out useful when we are faced with a complex system **S** that can be represented as a directed graph whose nodes may or may not "verify" propositions, but lack the consistency and completeness

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properties of classical "worlds". Indeed, the case in which the "verification process" is *classical*, i.e. satisfies the classical truth-conditions for the logical operators, is just a very special case. In many applications of logic to Computer Science and AI, we deal with systems which are better modelled by some weaker logic (e.g. intuitionistic, relevant or linear; see, for instance, [Gar89] for an application of relevance logic to modular reasoning systems, and [Ale94] for an overview of computational applications of Linear Logic). The introduction of modalities into such logics adds a new dimension which can explicitly and naturally account for the "accessibility" relations involved in the processes that are being modelled.

Our program is that of developing a suitable proof theory for a family of logical systems which arise from grafting modalities — interpreted via accessibility relations — onto logics weaker than classical logic. This will lead us to consider frames whose objects are not necessarily *worlds* (i.e. maximally consistent theories) as in the classical tradition, but more generic "points" whose interpretation is left open and depends on the underlying propositional logic. Classical Modal Logic can be construed as a limiting case which arises when such points are assumed to be maximally consistent pieces of information.

Previous work on this topic has been concentrating on intuitionistic modal logics ([Ser77, BD84, PS86, AP93, Sym93]). Here we start to develop a more general approach which fits naturally into the framework for substructural logics presented in [DG94]. The latter consists of a labelled generalization of the classical refutation system **KE** [DM94], where the labels can be interpreted as "pieces of information" that may or may not "verify" a given formula. The methods introduced in this paper can be useful for a variety of applications where modalities are needed in the context of non-classical information processing. In particular they have interesting connections with the models of belief developed in AI, for which see [Hin62, Kon86]. In this paper we shall restrict our attention to the fragment containing the operators $\Box$, $\Diamond$ and $\rightarrow$. The interplay between modalities and the other operators will be discussed in a subsequent paper. Owing to space constraints, proper comparisons with related work will also have to be postponed to another occasion.

2 The classical KE system

The system **KE**, like the tableau method and resolution, is a refutation system for classical logic. Unlike resolution, however, **KE** is not restricted to clausal form and, unlike the tableau method, it includes a cut rule which *cannot*, in general, be eliminated. This classical cut rule is called PB (from Principle of Bivalence) and has the following forms, depending on whether we deal with signed or unsigned formulae:

$$\frac{}{TA \mid FA} \qquad\qquad \frac{}{A \mid \neg A}$$

The formula A introduced by an application of this rule is called *PB-formula* or *cut formula*[1]. Once such a cut rule has been allowed, the branching elimination rules typical of the tableau method become unnecessarily strong and can be replaced by weaker *non-branching* rules (with two premises).

The rules of the **KE** system (for unsigned formulae) are illustrated in Table 1. The two-premiss elimination rules correspond to familiar principles of inference: *modus ponens, modus tollens, disjunctive syllogism* and its dual. The one-premiss elimination rules are the same as the tableau rules. A **KE**-refutation of a set of formulae Γ is, as usual, a closed tree of formulae constructed according to the rules of **KE** starting from formulae in Γ.

A crucial property of **KE** is the *analytic cut property*: the applications of the cut rule can be restricted to subformulae of the formulae occurring above in the branch without loss of completeness. This property allows for systematic and efficient refutation procedures. Indeed, results in [DM94] imply that any refutation procedure which can be formulated in terms of the tableau rules can be efficiently (linearly) simulated by means of the **KE** rules, but there are efficient and systematic **KE**-procedures which cannot be polynomially simulated by means of the tableau rules. One of these procedures is *the canonical procedure*, consisting in giving priority to the linear elimination rules over the cut rule, so that the cut rule is applied only when no elimination rule is further applicable, and the choice of the cut formulae is restricted to pairs A,$\neg A$ such that A is an immediate subformula of a formula of type β (in the Smullyan notation) occurring above in the branch.

In [DG94] the system **KE** is extended into a labelled deductive system (in the sense of [Gab94]) which provides a unifying proof framework for (the "multiplicative" fragment of) classical and intuitionistic substructural logics.

3 Information Frames

In this section we briefly review the treatment of substructural implication systems in [DG94] to which we refer the reader for further details.

Definition 1 An *information frame* or *quantale frame* is a structure $(Q, \circ, 1, \sqsubseteq)$ such that:

1. Q is a non-empty set of elements called *pieces of information* or *information tokens*;

2. $\sqsubseteq$ is a partial ordering which makes Q into a complete lattice; $x \sqsubseteq y$ can be interpreted as "y contains at least the same information as x";

3. $\circ$ is a binary operation on Q which is

[1] To see that PB plays the same role as the cut rule in the classical sequent calculus, think of it as a rule that allows one to construct a closed **KE**-tree for Γ, given a closed **KE**-tree for Γ, A and a closed **KE**-tree for $\Gamma, \neg A$; for a discussion of this point see [DM94].

Disjunction Rules

$$\frac{A \vee B \quad \neg A}{B}\,\mathrm{E}\vee 1 \qquad \frac{A \vee B \quad \neg B}{A}\,\mathrm{E}\vee 2 \qquad \frac{\neg(A \vee B)}{\neg A \quad \neg B}\,\mathrm{E}\neg\vee$$

Conjunction Rules

$$\frac{\neg(A \wedge B) \quad A}{\neg B}\,\mathrm{E}\neg\wedge 1 \qquad \frac{\neg(A \wedge B) \quad B}{\neg A}\,\mathrm{E}\neg\wedge 2 \qquad \frac{A \wedge B}{A \quad B}\,\mathrm{E}\wedge$$

Implication Rules

$$\frac{A \to B \quad A}{B}\,\mathrm{E}\to 1 \qquad \frac{A \to B \quad \neg B}{\neg A}\,\mathrm{E}\to 2 \qquad \frac{\neg(A \to B)}{A \quad \neg B}\,\mathrm{E}\neg\to$$

Negation Rule

$$\frac{\neg\neg A}{A}\,\mathrm{E}\neg\neg$$

Principle of Bivalence

$$\frac{}{A \mid \neg A}\,\mathrm{PB}$$

Table 1: **KE**-rules for unsigned formulae.

(a) associative: $x \circ (y \circ z) = (x \circ y) \circ z$;

(b) distributive over $\bigsqcup$: for every family $\{z_i\} \subseteq Q$, $\bigsqcup\{z_i \circ x\} = \bigsqcup\{z_i\} \circ x$ and $\bigsqcup\{x \circ z_i\} = x \circ \bigsqcup\{z_i\}$;

4. $1 \in Q$ and for every $x \in Q$, $x \circ 1 = 1 \circ x = x$;

Observe that the properties of $\circ$ imply that this operation is order-preserving, i.e.

$$x_1 \sqsubseteq x_2 \Longrightarrow x_1 \circ y \sqsubseteq x_2 \circ y \text{ and } y \circ x_1 \sqsubseteq y \circ x_2.$$

We can define classes of quantale frames which satisfy additional conditions on the ordering $\sqsubseteq$:

Definition 2 We say that a quantale frame is:

Commutativity	$x \circ y \sqsubseteq y \circ x$	$\dfrac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$
Contraction	$x \circ x \sqsubseteq x$	$\dfrac{\Gamma, A, A, \Delta \vdash C}{\Gamma, A, \Delta \vdash C}$
Expansion	$x \sqsubseteq x \circ x$	$\dfrac{\Gamma, A, \Delta \vdash C}{\Gamma, A, A, \Delta \vdash C}$
Monotonicity	$x \sqsubseteq x \circ y$	$\dfrac{\Gamma, \Delta \vdash C}{\Gamma, A, \Delta \vdash C}$

Table 2: Correspondence between classes of frames and structural rules.

commutative	if	$x \circ y \sqsubseteq y \circ x$
contractive	if	$x \circ x \sqsubseteq x$
expansive	if	$x \sqsubseteq x \circ x$
monotonic	if	$x \sqsubseteq x \circ y$

In [DG94] different substructural logics are seen to correspond to different classes of quantale frames in the expected way, each condition on $\sqsubseteq$ being associated with a structural property of consequence relations. The correspondence is summarised in Table 2. In this context, by a *consequence relation* we intend a binary relation $\vdash$ between *sequences* of formulae and formulae satisfying the following two conditions:

(Identity) $$A \vdash A$$

(Surgical Cut) $$\Gamma \vdash A \text{ and } \Delta, A, \Lambda \vdash B \Longrightarrow \Delta, \Gamma, \Lambda \vdash B$$

In particular, by a *substructural implication logic* we mean a consequence relation characterised by the following universal condition on the operator $\rightarrow$:

$\mathrm{C}_{\rightarrow}$ $$\Gamma, A \vdash B \Longleftrightarrow \Gamma \vdash A \rightarrow B$$

and any subset of the structural rules in the right column of Table 2.

Definition 3 Given a quantale frame $\mathcal{F}$, a *valuation over* $\mathcal{F}$ is a function $V : \mathbf{F} \times Q \mapsto \{T, F\}$, where $\mathbf{F}$ is the set of formulae of the language and Q is the set of information tokens in the frame $\mathcal{F}$, satisfying the following conditions:

1. *(Heredity)* For all formulae A, if $V(A, x) = T$ and $x \sqsubseteq y$, then $V(A, y) = T$.
2. *(Continuity)* For each given A and every non-empty $S \subseteq Q$

- $V(A, \bigsqcup S) = \bigsqcup\{V(A,x)|x \in S\}$ and
- $V(A, \bigsqcap S) = \bigsqcap\{V(A,x)|x \in S\}$.

As usual, when $V(A,x) = T$ we shall also say "A is true at x" and sometimes we shall write "$x\|\!\!-A$". Similarly, when $V(A,x) = F$ we shall also say "A is false at x" and sometimes write "$x\|\!\!\not-A$".

Definition 4 An *implication structure* is a pair $(\mathcal{F}, V)$ where $\mathcal{F}$ is a quantale frame and V is a valuation over $\mathcal{F}$ satisfying the following condition:

$$V(A \rightarrow B, x) = T \Longleftrightarrow \forall y, V(A,y) = F \text{ or } V(B, x \circ y) = T. \quad (1)$$

The valuation clause for the operator $\rightarrow$ is clearly a generalization of Urquhart's semantics of relevant implication [Urq72]. The only difference is that the underlying algebraic structure is a quantale rather than a semilattice. Different combinations of the properties of $\circ$ listed in Table 2 identify different classes of frames which, in turn, characterize different substructural implication logics. (For instance, if $\circ$ satisfies only the commutativity condition, the resulting logic is the implication fragment of Girard's Linear Logic [Gir87, Avr88]; if it satisfies both commutativity and contraction, then the system of relevant implication [AB75, Dun86] is obtained; finally, if $\circ$ satisfies all the structural conditions on $\circ$, the resulting logic is intuitionistic implication.) Let **S** be one of such classes. An implication formula is *valid for* **S** if it is true at the identity point 1 of all implication structures whose underlying frames belong to the class **S**. We also say that a finite sequence $A_1, \ldots, A_n$ of formulae *implies* a formula A with respect to **S** if the implication formula $A_1 \rightarrow (A_2 \rightarrow \cdots \rightarrow (A_n \rightarrow A) \cdots)$ is valid for **S**.

4 Implicational LKE

An inferential characterization of substructural implication logics can be obtained by turning the "semantics" described above into the rules of a labelled deductive system (in the sense of [Gab94]). In [DG94], D'Agostino and Gabbay presented a labelled refutation system consisting of a generalization of the classical system KE investigated in [DM94]. The rules of this labelled refutation system, that we call LKE, are tree- expansion rules which are immediately justified by (and are indeed equivalent to) our previous definitions. The rules for the implication fragment are listed[2] in Table 3.

In these rules the declarative units are not just signed formulae as in the classical KE system (or in the system of analytic tableaux) but *labelled signed formulae*[3].

[2] In fact, the closure rule which is necessary to guarantee completeness is, for some implication logics, more general than the one given here which is just a special case. However, this simple special case is sufficient to cover most interesting implication systems (the reader is referred to [DG94] for the more general closure rule).

[3] A similar approach is used by Fitting in his "prefixed" tableaux for classical modal logics [Fit83].

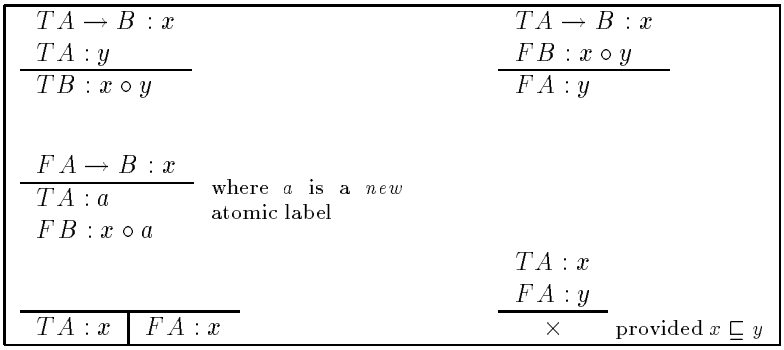

Table 3: The LKE rules for substructural implication.

The points of the quantale frames are turned into "labels", while signs play the usual role, so that $TA : x$ is interpreted as "A is true at point x" and $FA : x$ is interpreted as "A is false at point x". Different classes of frames correspond to different *labelling algebras*, i.e. different sets of rules that can be used in manipulating the labelling terms to verify whether or not the condition for the application of the closure rule is satisfied. A proof of the validity of a formula A for a class of frames **S** consists in a refutation of the assumption that A is false at the identity element 1 of some quantale frame belonging to **S**. Such a refutation is represented by a closed LKE tree starting with the labelled signed formula $FA : 1$, where the conditions of the labelling algebra corresponding to **S** may be used in order to close a branch. The whole family of substructural implication logics is, therefore, characterised by the same tree-expansion rules, and different members of the family are identified by the different "labelling algebras" that can be employed to check branch-closure. This is what we call the "separation-by-closure" property of the LKE system. An example of an LKE proof is given in Table 4. This example does not make any use of the branching rule of "generalized bivalence". Indeed, this rule introduces a good deal of "non-determinism" into the system, in that it allows for the use of (a) arbitrary formulae and (b) arbitrary labels in each rule application. However, this "non-determinism" can be tamed to some extent. As for (a), it can be shown that the applications of PB can be restricted to *analytic* ones, i.e. involving only subformulae of formulae previously occurring in the branch (and even further to *canonical* applications as in the canonical procedure for the classical system outlined in Section 2). As for (b), it can be shown that similar restrictions hold for the labels and the best strategy is to apply the PB rule with a *variable* as label. In this way the closure of a branch depends on the solution of an inequation in the given algebra of the labels and the closure of the whole tree on the simultaneous solution of a system of inequations. This is a well defined algebraic problem which can be addressed via unification-like techniques. For a detailed and systematic presenta-

$F(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B) : 1$
$TA \rightarrow (A \rightarrow B) : a$
$FA \rightarrow B : a$
$TA : b$
$FB : a \circ b$
$TA \rightarrow B : a \circ b$
$TB : a \circ b \circ b$

Table 4:
The contraction axiom is valid for the class of contractive frames (i.e. those satisfying the condition $x \sqsubseteq x \circ x$, for all x). For, $b \circ b \sqsubseteq b$ and, since $\circ$ is order-preserving, $a \circ b \circ b \sqsubseteq a \circ b$. Therefore, this one-branch tree is closed. The axiom is not valid in Linear Logic which is characterised by a class of non-contractive frames.

tion of the LKE system (including the other "multiplicative" operators different from $\rightarrow$), and more examples of LKE proofs, the reader is referred to [DG94].

5 The modal operators

We introduce in our quantale frames a binary relation R between their points, called the *accessibility relation*. We then extend valuations to the modal operator $\Box$ as follows:

$$V(\Box A, x) = T \text{ iff } V(A, y) = T \text{ for all } y, \text{ such that } xRy \tag{2}$$

Similarly, we can have the following clause for $\Diamond$:

$$V(\Diamond A, x) = T \text{ iff } V(A, y) = T \text{ for some } y, \text{ such that } xRy \tag{3}$$

In order to preserve the "hereditary" property of valuations (see point 1 of Definition 3), R must also satisfy the following conditions (see [BD84] and [PS86]):

$$x \sqsubseteq y \text{ and } xRz \Longrightarrow (\exists z')(yRz' \text{ and } z \sqsubseteq z') \tag{4}$$

and

$$x \sqsubseteq y \text{ and } yRz \Longrightarrow (\exists z')(xRz' \text{ and } z' \sqsubseteq z). \tag{5}$$

Definition 5 A *modal quantale frame* is a pair $(\mathcal{F}, R)$ where $\mathcal{F}$ is a quantale frame and R is a binary accessibility relation defined on the domain of $\mathcal{F}$ and satisfying conditions (4) and (5).

If R is defined as above, any atomic valuation V over a quantale frame can be extended to a valuation over a modal quantale frame by means of the valuation clauses for $\Box$ and $\Diamond$, preserving the "heredity" property of valuations.

Definition 6 A *modal implication structure* is a triple $(\mathcal{F}, R, V)$, where $(\mathcal{F}, R)$ is a modal quantale frame and V is a valuation satisfying (1), (2) and (3).

Let M be a modal implication structure $(\mathcal{F}, R, V)$ and let $\equiv$ be the relation that holds between two points x and y that verify exactly the same formulae, i.e. $\forall A, x\|\!\!-A$ iff $y\|\!\!-A$. We can of course consider the quotient structure $\mathcal{F}/\equiv$ and define a valuation V' on it as follows:

$$V'(A, [x]) = T \text{ iff } V(A, x) = T$$

where $[x]$ denotes the equivalence class of x under $\equiv$. Similary, we can define $[x]R'[y]$ iff xRy. Clearly, the structure $(\mathcal{F}/\equiv, R', V')$ is a modal implication structure, and we can restrict our attention, without loss of generality, to modal implication structures with the additional property that any two points verifying exactly the same formulae are identical.

6 The labelling algebra for the modal operators

We now want to incorporate our interpretation of the modal operators into the proof-theoretical framework of Section 4. For this purpose we shall first introduce some convenient notation that will allow us to formulate simple elimination rules for $\Box$ and $\Diamond$. These rules fit perfectly into the LKE system for substructural implication and, most importantly, preserve the separation-by-closure property: different modal logics — which arise from combining different properties of the accessibility relation with different substructural implication systems — are characterised *only* by the conditions under which a branch can be closed.

Let us denote by $S(x)$ *the sphere* of x, i.e. the set of all pieces of information accessible from x. We shall also use $\bigsqcup$ and $\bigsqcap$ to denote the usual lattice join and meet. In this paper we shall restrict our attention to frames such that for all x, $S(x) \neq \emptyset$, called *serial* frames (non-serial frames will be discussed in a subsequent paper). We now define the two unary operators ! and ? as follows:

$$!x =_{\text{def}} \bigsqcap S(x) \tag{6}$$

and

$$?x =_{\text{def}} \bigsqcup S(x) \tag{7}$$

Notice that the restriction to serial frames is equivalent to the assumption that, for all points x,

$$!x \sqsubseteq ?x \tag{8}$$

We assume that the operators ! and ? are both *order-preserving*:

$$x \sqsubseteq y \Longrightarrow !x \sqsubseteq !y \text{ and } ?x \sqsubseteq ?y \tag{9}$$

This assumption is equivalent to the conjunction of conditions (4) and (5) of Section 5.

Given our restriction to serial frames, the valuation clauses for $\Box$ and $\Diamond$ can be reformulated more concisely as below:

$$V(\Box A, x) = T \quad \text{iff} \quad V(A, !x) = T \tag{10}$$

and

$$V(\Diamond A, x) = T \quad \text{iff} \quad V(A, ?x) = T \tag{11}$$

Let us now show the equivalence, in serial frames, of the valuation clauses (10) and (11) with (2) and (3) respectively. First, we assume that the clause in (2) holds and show that (10) holds too.

For the only-if direction, let us assume $V(\Box A, x) = T$; then by (2), $V(A, y) = T$ for all $y \in S(x)$. Now, by the definition of valuation, A remains true at the greatest lower bound of $S(x)$, that is at $!x$. For the if-direction, let us assume $V(\Box A, x) = F$. It follows that there exists a point $y \in S(x)$ such that A is false at y. Since, $!x \sqsubseteq y$, A must be false at $!x$.

Now we show that (10) implies (2). For the only-if direction, suppose $V(\Box A, x) = T$. Then, by (10), $V(A, !x) = T$. So, A is true at every point y such that $!x \sqsubseteq y$ and, therefore, at every point in $S(x)$. For the if-direction, assume that A is true at every point in $S(x)$. Suppose, that $\Box A$ is false at x. Then, by (10), $V(A, !x) = F$. So the assumption that A is true at every point in $S(x)$ is contradicted. Hence, $\Box A$ must be true at x. A similar argument shows the equivalence of (3) and (11).

We can now exploit our new notation to express complex statements about the accessibility relation R concisely, as simple inequalities of the form $\alpha \sqsubseteq \beta$, where α and β are expressions built up from atomic terms by means of the operators $\circ$, ! and ?. For this purpose, we shall make a crucial use of the following assumption:

Assumption 1 *If $S(x) \neq \emptyset$, then both $?x$ and $!x$ are accessible from x.*

So, when $S(x)$ is not empty, the points $!x$ and $?x$ have a special status: $!x$ is a point in $S(x)$ that verifies all and only the formulae which are true at *all* points in $S(x)$, while $?x$ is a point in $S(x)$ that verifies all and only the formulae which are true at *at least one* point in $S(x)$.

Let us now consider some of the most familiar properties of the accessibility relation R.

Seriality We have said that a frame is *serial* if for every point x, $S(x) \neq \emptyset$, that is, for every point x there exists a y such that xRy. As mentioned above, in our approach, seriality corresponds to the assumption that $!x \sqsubseteq ?x$ for all x. Since we are restricting our attention to serial frames, this condition will always be satisfied as far as this paper is concerned.

Reflexivity A frame is said to be *reflexive* if xRx for all x. In our notation this property can be expressed by the following condition:

(Reflexivity) $\quad (\forall x)(!x \sqsubseteq x)$

The equivalence between the two formulations is immediately seen as follows. First, assume that $!x \sqsubseteq x$. Then every formula true at $!x$ is true also at x. Suppose xRx does *not* hold for some x. This means that there exists a point a such that $\neg aRa$. Now, consider a valuation V and a formula A such that $V(A,x) = T$ for all $x \in S(a)$, i.e. $V(\Box A, a) = T$, but $V(A,a) = F$ (which is possible, since $a \notin S(a)$). For this V, we have that $V(A,!a) = T$ and, since $!a \sqsubseteq a$, $V(A,a) = T$. This is a contradiction. Hence, if $!x \sqsubseteq x$ for all x, then xRx for all x. Conversely, if xRx for all x, every formula that is true at all points accessible from x is true also at x, i.e. for all x, $!x \sqsubseteq x$.

Transitivity A frame is said to be *transitive* if the following condition holds:

$$(\forall x)(\forall y)(\forall z)(xRy \text{ and } yRz \Longrightarrow xRz).$$

In our notation, transitivity is expressed— more concisely — by the following condition:

(Transitivity) $\quad (\forall x)(!x \sqsubseteq !!x)$

Assume that $!x \sqsubseteq !!x$ for all x. Now, suppose that the frame is not transitive. Then, there exists a point a, a valuation V and a formula A such that $V(\Box A, a) = T$ but $V(\Box\Box A, a) = F$. This implies by (10) that $V(A,!a) = T$ and $V(A,!!a) = F$. But, since $!a \sqsubseteq !!a$, this is impossible. Hence, if $!x \sqsubseteq !!x$ for all x, the frame is transitive.

Conversely, suppose the frame is transitive. We show that $!x \sqsubseteq !!x$. Let A be any formula which is true at $!x$. Then A is true at all points in $S(x)$. Now, since $S(x)$ is not empty, $!x$ is accessible from x. Since the frame is transitive, if a point y is accessible from $!x$, it is also accessible from x. So, if A is true at all the points accessible from x it is also true at all the points accessible from $!x$, that is $!x \sqsubseteq !!x$.

Symmetry A frame is said to be *symmetric* if the following condition holds:

$$(\forall x)(\forall y)(xRy \Longrightarrow yRx).$$

In our notation, symmetry is expressed by the following condition:

(Symmetry) $\quad (\forall x)(x \sqsubseteq ?!x)$

We leave it to the reader to verify the equivalence between the two formulations.

Euclidean Property A frame is said to be *Euclidean* if the following condition holds true:

$$(\forall x)(\forall y)(\forall z)(xRy \text{ and } xRz \Longrightarrow yRz).$$

The condition corresponding to this property is the one stated below:

(Euclideanism) $\quad (\forall x)(?x \sqsubseteq ?!x).$

Again, the reader can verify that the two formulations are equivalent.

The following table summarises all the conditions on the algebra of the labels we have been considering so far:

Structural Conditions		Modal Conditions	
commutativity	$x \circ y \sqsubseteq y \circ x$	seriality	$!x \sqsubseteq ?x$
contraction	$x \circ x \sqsubseteq x$	reflexivity	$!x \sqsubseteq x$
expansion	$x \sqsubseteq x \circ x$	transitivity	$!x \sqsubseteq !!x$
monotonicity	$x \sqsubseteq x \circ y$	symmetry	$x \sqsubseteq ?!x$
		Euclideanism	$?x \sqsubseteq ?!x$

We shall now state a crucial lemma. Let σ_1 and σ_2 be (possibly empty) strings of ! and ?. The *dual* of σ_i, denoted by σ_i' is the string obtained by interchanging ! and ? in σ_i. Moreover, let $\tau \subseteq \mathbf{M}$, where $\mathbf{M}$ is the set of modal conditions listed above, and let $\mathbf{S}_\tau$ be the set of frames satisfying all the conditions in τ. We have the following *duality principle*:

Duality Lemma *Assume that $\sigma_1 x \sqsubseteq \sigma_2 x$ holds for all x in all members of a given class $\mathbf{S}_\tau$ of frames, i.e. there is a chain of inequalities $\phi_0 \sqsubseteq \cdots \sqsubseteq \phi_n$ such that (i) $\phi_0 = \sigma_1 x$, (ii) $\phi_n = \sigma_2 x$, and (iii) each inequality $\phi_i \sqsubseteq \phi_{i+1}$, with $i = 0, \ldots, n-1$, is one of of the primitive inequalities in τ. Then $\sigma_2' x \sqsubseteq \sigma_1' x$ also holds for all x in all members of $\mathbf{S}_\tau$.*

The proof is an easy induction on the length of the chain $\phi_0 \sqsubseteq \cdots \sqsubseteq \phi_n$.

Base: $n = 1$. Then the inequality $\sigma_1 x \sqsubseteq \sigma_2 x$ is an element of τ. The reader can easily verify that the lemma holds for all primitive inequalities in $\mathbf{M}$.

Step: $n = k+1$. Let ρ be the prefix of ϕ_k, i.e. $\phi_k = \rho x$. By inductive hypothesis, $\rho' x \sqsubseteq \sigma_1' x$. Moreover, since $\rho x \sqsubseteq \sigma_2 x$, again by inductive hypothesis we have that $\sigma_2' x \sqsubseteq \rho' x$. Hence, $\sigma_2' x \sqsubseteq \sigma_1' x$.

7 Modal LKE rules

The valuation clauses (10) and (11) immediately imply simple LKE-style rules for the modal operators.

Elimination rules for $\Box$

$$\frac{T\Box A : x}{TA : !x}\ ET\Box \qquad\qquad \frac{F\Box A : x}{FA : !x}\ EF\Box$$

Elimination rules for $\Diamond$

$$\frac{T\Diamond A : x}{TA : ?x}\ ET\Diamond \qquad\qquad \frac{F\Diamond A : x}{FA : ?x}\ EF\Diamond$$

The rules have a clear intuitive interpretation: if $\Box A$ is true at a point x, then it is true at all points accessible from it and, therefore, it is true at the special point $!x$ which verifies all and only the formulae which are true at all points in $S(x)$. Conversely, if $\Box A$ is false at x, then it must be false at $!x$.

Similarly, if $\Diamond A$ is true at a point x, then it is true at some point accessible from it and, therefore, at the special point $?x$ which verifies all and only the formulae which are true at at least one point in $S(x)$. Conversely, if $\Diamond A$ is false at x, then it must be false at $?x$.

8 Examples

Example 1 *The formula*

$$(A \rightarrow (A \rightarrow B)) \rightarrow (\Box A \rightarrow B)$$

is valid in all contractive and reflexive frames.

$$\begin{array}{c} F(A \rightarrow (A \rightarrow B)) \rightarrow (\Box A \rightarrow B) : 1 \\ | \\ TA \rightarrow (A \rightarrow B) : a \\ | \\ F\Box A \rightarrow B : a \\ | \\ T\Box A : b \\ | \\ FB : a \circ b \\ | \\ TA : !b \\ | \\ TA \rightarrow B : a \circ !b \\ | \\ TB : a \circ !b \circ !b \end{array}$$

In all contractive frames $!b \circ !b \sqsubseteq !b$, so $a \circ !b \circ !b \sqsubseteq a \circ !b$. Moreover, in all reflexive frames $!b \sqsubseteq b$. So, in all contractive and reflexive frames $a \circ !b \circ !b \sqsubseteq a \circ b$.

Example 2 *The formula*

$$\Box A \rightarrow (B \rightarrow \Box\Box A)$$

is valid in all frames which are both monotonic and transitive.

$$F\Box A \to (B \to \Box\Box A) : 1$$
$$|$$
$$T\Box A : a$$
$$|$$
$$FB \to \Box\Box A : a$$
$$|$$
$$TB : b$$
$$|$$
$$F\Box\Box A : a \circ b$$
$$|$$
$$TA :!a$$
$$|$$
$$F\Box A :!(a \circ b)$$
$$|$$
$$FA :!!(a \circ b)$$

This one-branch tree is closed for all monotonic-transitive frames. For, $a \sqsubseteq a \circ b$ by monotonicity; therefore, $!a \sqsubseteq !(a \circ b)$ by (9); moreover, by transitivity, $!(a \circ b) \sqsubseteq !!(a \circ b)$; so, $!a \sqsubseteq !!(a \circ b)$ and the branch is closed.

Example 3 *The formula*

$$A \to (A \to \Box\Diamond A)$$

is valid in all frames which are expansive and symmetric.

$$FA \to (A \to \Box\Diamond A) : 1$$
$$|$$
$$TA : a$$
$$|$$
$$FA \to \Box\Diamond A : a$$
$$|$$
$$F\Box\Diamond A : a \circ a$$
$$|$$
$$F\Diamond A :!(a \circ a)$$
$$|$$
$$FA :?!(a \circ a)$$

This one-branch tree is closed in all expansive-symmetric frames. For, $a \sqsubseteq a \circ a$ by expansion; $(a \circ a) \sqsubseteq ?!(a \circ a)$ by symmetry. These two inequalities imply that $a \sqsubseteq ?!(a \circ a)$.

9 Completeness

Let $\mathbf{C}$ be any set of constraints taken from the following list:

commutativity	$x \circ y \sqsubseteq y \circ x$
contraction	$x \circ x \sqsubseteq x$
expansion	$x \sqsubseteq x \circ x$
monotonicity	$x \sqsubseteq x \circ y$
seriality	$!x \sqsubseteq ?x$
reflexivity	$!x \sqsubseteq x$
transitivity	$!x \sqsubseteq !!x$
symmetry	$x \sqsubseteq ?!x$
Euclideanism	$?x \sqsubseteq ?!x$

By $\mathbf{C}$-*frame* we mean a modal quantale frame such that $\sqsubseteq$ satisfies all the constraints in $\mathbf{C}$. We call $\mathbf{C}$-*model* any modal implication structure whose underlying frame is a $\mathbf{C}$-frame. Finally, let LKE + $\mathbf{C}$ be the system obtained from LKE by adding the constraints in $\mathbf{C}$ to the algebra of the labels. Let us denote by $\vdash_{\mathrm{LKE+C}}$ the deducibility relation defined as follows:

$$A_1, \ldots, A_n \vdash_{\mathrm{LKE+C}} B \iff \begin{array}{l}\text{there is a closed LKE + C tree starting}\\ \text{from the labelled signed formulae}\\ TA_1 : a_1, \ldots, TA_n : a_n, FB : a_1 \circ \cdots \circ a_n\end{array}$$

where a_i are distinct atomic labels. Then we can prove the following theorem:

Theorem $\emptyset \vdash_{\mathrm{LKE+C}} A$ *iff* A *is true at the identity point 1 of any* $\mathbf{C}$*-model.*

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