LDS - Natural Deduction for Substructural Logics

Krysia Broda,* Marcelo Finger,† and Alessandra Russo*
email: {kb,ar3}@doc.ic.ac.uk, mfinger@im.usp.br‡

March 8, 1996

Recent work on labelled tableau methods [DG94] has developed a uniform approach to substructural logics. In this work, inspired by [Gab94], formulas are replaced by *declarative units* of the form $A : x$, where A is a formula and x is a label. A simple interpretation of the label x is the resource needed to prove A. The inference rules operate on the label part of a declarative unit as well as on the formula part. Modus Ponens, for example, allows the deduction of $B : a \circ b$ from $A \rightarrow B : a$ and $A : b$. That is, B can be deduced from the resources a combined with b.

The labels are drawn from a labelling algebra $\mathcal{L}_A$, which is a complete lattice formed from a set of atomic labels and the monoid operation $\circ$, with the identity element 1. The operation $\circ$ is continuous with respect to $\bigsqcup$ and hence is order-preserving in the lattice. An atomic label a associated with a formula A represents the minimum resource to prove A. The differences between logics are captured by different additional properties imposed on the elements of $\mathcal{L}_A$. For example, linear logic requires only that $\circ$ is commutative, whereas relevance logic also requires contraction: $x \circ x \sqsubseteq x$. Intuitionistic logic needs monotonicity, $x \sqsubseteq x \circ y$, too. Within this approach, known as LDS (Labelled Deductive Systems), to show that a formula A is a theorem, we prove $A : 1$, where 1 represents the neutral resource.

However, proof using LDS is not restricted to tableau methods. Natural deduction can be used as well. In this paper, we show a uniform natural deduction proof system for the fragment $\{\rightarrow, \otimes\}$, of substructural logics. The rules are equally applicable to any member of the family and the process of detecting which logic a given formula is a theorem of is reduced to resolving the set of required constraints on the labels generated by the proof. (See [BDR96] for a method of relating KE-tableaux and natural deduction proofs.)

In Table 1 the natural deduction rules for the connectives $\rightarrow$ and $\otimes$ are shown. The box in the $\rightarrow\mathcal{I}$ serves as a new context indicator, whereas in $\otimes\mathcal{I}$ the two boxes are present only to separate the two sub-proofs. The lemma rule can be seen as a way of incorporating the cut-rule into the natural deduction system. Imposed constraints (IC) arising from the $\otimes\mathcal{E}$ rule together with the given properties of $\circ$ are used to prove the required constraints (RC) of the proof, which arise from the tick and $\otimes\mathcal{I}$ rules.

We will give here an example of the uniform property of the above set of natural deduction rules. We show that Linear Logic (LL) and Relevance Logic (RL) can be accommodated within the system in a way that the decision whether a formula A is a theorem of LL or of RL can be postponed until the end of a proof of the corresponding labelling formula $A : 1$. The difference between the two logics is in structural properties of their consequence relations [Dôs93, DG94], LL imposes the permutativity property and RL requires in addition the contractive property. In proof theoretical terms, these properties imply that in the case of LL assumptions must be used exactly once whereas in RL they can be used at least once. These properties are captured by additional

*Department of Computing, Imperial College, 180 Queen's Gate, London SW7 2BZ
†Departamento de Ciência da Computação, Instituto de Matemática e Estatística, Universidade de São Paulo
‡This work was supported by EPSRC grant GR/J14745

conditions imposed on the labelling algebra (i.e. $x \circ y \sqsubseteq y \circ x$ and $x \circ x \sqsubseteq x$ respectively). The natural deduction style proof system makes use of a *resource abduction.* Instead of trying to prove the formula $A : 1$ corresponding to a (LL or RL) theorem A, we prove $A : \gamma$, for a variable label γ. The final instantiation of γ will allow us to decide in which logic (if either) A is a theorem.

Figure 1 shows an attempted proof of the formula $A \rightarrow ((A \otimes A \rightarrow B) \rightarrow B)$, theorem only of RL. In order to complete the proof the RSs $a \sqsubseteq \gamma_1$, $a \sqsubseteq \gamma_2$, $\gamma_1 \circ \gamma_2 \sqsubseteq \alpha$ and $b \circ \alpha \sqsubseteq \gamma \circ a \circ b$ must be satisfied. This is done by a modified unification algorithm based on AC-unification. (See [BF95].) For LL, to succeed a constraint of the form $x \sqsubseteq y$, it suffices to check $x = y$ modulo commutativity. For RL , we note that the contraction property allows x to contain more occurrences of any atomic label than occur in y. To accommodate RL within the same algorithm we add a unique variable δ, called a slack-variable, to the right-hand-side (RHS) of any RC which is bound to the unwanted *extra* occurrences from the left-hand-side (LHS). For the example in Figure 1, the constraints to be solved become: $a \sqsubseteq \gamma_1 \circ \delta_1$, $a \sqsubseteq \gamma_2 \circ \delta_2$, $\gamma_1 \circ \gamma_2 \sqsubseteq \alpha \circ \delta_3$ and $b \circ \alpha \sqsubseteq \gamma \circ a \circ b \circ \delta_4$. The first two can be satisfied by binding $\gamma_1 = \gamma_2 = a$ and $\delta_1 = \delta_2 = 1$. The third requires $a \circ a \sqsubseteq \alpha \circ \delta_3$, and α can be either a or $a \circ a$, but not 1 as δ_3 can only take up additional occurrences of a, not all of them. The last constraint can be satisfied if $\alpha = a$, $\delta_4 = a$ and $\gamma = 1$. It follows that $A \rightarrow ((A \otimes A \rightarrow B) \rightarrow B)$ is a theorem in RL, but not in LL, where the only value for γ is a and not 1 as required for theorems. By *resource abduction*, $\gamma = a$ tells us that if we add "$A \rightarrow$" to the beginning of the formula, we get an LL-theorem, i.e. $A \rightarrow (A \rightarrow ((A \otimes A \rightarrow B) \rightarrow B))$.

In general, a proof generates a collection of RCs and ICs. Each IC always has the form $a \circ b \sqsubseteq z$, where a and b are atomic and are used in the algorithm by substituting occurrences of $a \circ b$ by z in any LHS of an RC. As a consequence of the natural deduction rules, for any particular RC, variables occuring on its LHS are disjoint from any on its RHS and any non slack-variable occurring on the left of a RC must also occur on the right of at least one other RC, Hence the constraints can be topologically sorted into a sequence. The first element of the sequence is the constraint that arises from the innermost leftmost subproof, which has a ground LHS. Following this order, the variable instantiations are propagated, grounding the LHSs in turn until the last constraint – which involves the goal variable – is reached. A similar technique, namely adding slack-variables onto the LHS of a RC will allow monotonicity to be incorporated and hence Intuitionistic Logic to be captured.

What is the price paid for such generality? It is known that the multiplicative fragment of LL $(\otimes, \rightarrow, \neg)$ is NP-complete. Our algorithm is EXP-time for the multiplicative fragments of LL and RL, which appears to be a reasonable complexity, given the theoretical lower bounds.

References

[BDR96] Krysia Broda, Marcello D'Agostino, and Alessandra Russo. Transformation methods in LDS. In *Logic, Language and Reasoning. An Essay in Honor of Dav Gabbay.* Kluwer Academic Publishers, To appear 1996.

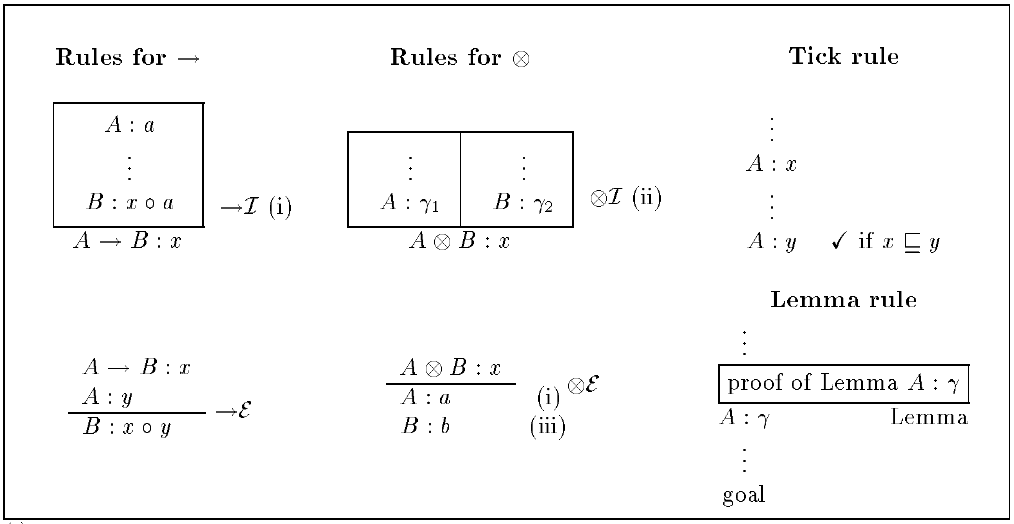
[BF95] Krysia Broda and Marcelo Finger. Ke-tableaux for a fragment of linear logic. Technical report, 4th Workshop on Theorem Proving with Analytic Tableaux and Related Methods, Ed. Peter Baumgartner, University of Koblenz, 1995.

[DG94] Marcello D'Agostino and Dov M. Gabbay. A generalization of analytic deduction via labelled deductive systems.Part I: Basic substructural logics. *Journal of Automated Reasoning*, 13:243–281, 1994.

[Dôs93] Kosta Dôsen. A historical introduction to substructural logics. In P. Schroeder Heister and Kosta Dôsen, editors, *Substructural Logics*, pages 1–31. Oxford University Press, 1993.

[Gab94] Dov M. Gabbay. Labelled Deductive Systems, Volume 1 - Foundations. Technical Report MPI-I-94-223, Max-Planck-Institut Für Informatik, 1994. Preliminary partial draft of a book intended for Oxford University Press.

Table 1: ND Rules for $\{\rightarrow, \otimes\}$ Fragment of Substructural Logics



(i) a is a new atomic label
(ii) $\gamma_1 \circ \gamma_2 \sqsubseteq x$
(iii) $a \circ b \sqsubseteq x$

1	$A : a$			
2	$A \otimes A \rightarrow B : b$			
3	$A : \gamma_1$	$\surd$ if $a \sqsubseteq \gamma_1$	$A : \gamma_2$	$\surd$ if $a \sqsubseteq \gamma_2$
4	$A \otimes A : \alpha$			$\otimes\mathcal{I}$ if $\gamma_1 \circ \gamma_2 \sqsubseteq \alpha$
5	$A \otimes A : \alpha$			lemma
6	$B : b \circ \alpha$			$\rightarrow\mathcal{E}$
7	$B : \gamma \circ a \circ b$			$\surd$ if $b \circ \alpha \sqsubseteq \gamma \circ a \circ b$
8	$(A \otimes A \rightarrow B) \rightarrow B : \gamma \circ a$			$\rightarrow\mathcal{I}$
9	$A \rightarrow ((A \otimes A \rightarrow B) \rightarrow B) : \gamma$			$\rightarrow\mathcal{I}$

Figure 1: An Example of Natural DeductionProof