GENERALISING PROPOSITIONAL MODAL LOGIC USING LABELLED DEDUCTIVE SYSTEMS

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Abstract

A family of labelled deductive systems called *Propositional Modal Labelled Deductive* (PMLD) systems is described. These logics combine the standard syntax of propositional modal logic with a simple subset of first–order predicate logic, called a *labelling algebra*, to allow syntactic reference to a Kripke–like structure of possible worlds. PMLD systems are a generalisation of normal propositional modal logic in that they facilitate reasoning about what is true at different points in a (possibly singleton) structure of actual worlds, called a *configuration*. A model–theoretic semantics (based on first–order logic) is provided and its equivalence to Kripke semantics for normal propositional modal logics is shown whenever the initial configuration is a single point. A sound and complete natural deduction style proof system is also described. Unlike traditional proof systems for modal logics, this system is uniform in that every deduction rule is applicable to (the generalisation of) each normal modal logic extension of K obtained by adding combinations of the axiom schemas $T, 4, 5, D$ and B.

1. Introduction

This paper builds upon the methodology in [6] to develop generalisations of normal propositional modal logics, called *Propositional Modal Labelled Deductive* (PMLD) systems. The idea is to define a logical framework in which explicit reference can be made to particular possible worlds and to the relationships between possible worlds, while retaining the conventional syntax of modal logic. This is partly to provide a logic, based on modal reasoning, able to fulfil the increasing need in some application areas for using not only sets, but *structures* of information (in the Discussion section an example application in distributed database systems is given). The combined approach described in this paper shows how explicit references to structures of arbitrary modal theories can be made, whilst retaining a concise proof–theoretical presentation of modal logic. This is achieved by combining a modal logic with a *labelling algebra*. The labelling algebra is defined as a binary first–order theory which axiomatises the Kripke semantic notion of an accessibility relation. The modal language is given in a traditional way as a modal extension of a propositional language.

The two languages (modal language and labelling language) are combined via the notion of *declarative unit*. The declarative unit $\lambda : \alpha$ expresses that the modal formula α is true at the label (i.e. possible world) λ. This combined approach retains the advantages of both implicit (e.g. [3]) and explicit (e.g. [11]) traditional formalisations for modal logic. As in the implicit approach, statements such as "necessary α" can be captured succinctly using the modal operator $\Box$. This statement is simply written as the single declarative unit $W_a : \Box\alpha$ (where W_a is the labelling algebra representation of the actual world). Like the explicit approach, the language is rich enough to allow explicit syntactic reference to the accessibility relation between possible worlds.

PMLD systems are in fact generalisations of normal propositional modal logics in that they facilitate reasoning about what is true at different points in a (possibly singleton) structure of actual worlds. Information like "A holds at a possible world W_1 accessible from W_0 and $\neg A$ holds at a possible world W_2 not accessible from W_0", which could not be formalised at all in the implicit approach, can be directly represented as information about the relation between the possible worlds, together with modal formulae holding locally at W_1 and W_2. Loosely speaking, the PMLD theory would in this case be the set $\{R(W_0, W_1), \neg R(W_0, W_2), W_1 : A, W_2 : \neg A\}$. Of course, the same kind of generalisation could be achieved using explicit (first–order translation) methods, although less succinctly.

Furthermore, the explicit syntactic reference to a Kripke–like accessibility relation allows a natural deduction system to be developed for PMLD systems which is uniform, in that every deduction rule can be applied to each propositional modal logic. The difference between one modal logic and another is captured entirely by the labelling algebra[1].

[1]The term "algebra" has been chosen here to reflect the terminology adopted in [6].

The paper is organized as follows. In Section 2 the language and syntax of PMLD systems is defined together with the notion of a *configuration* – a PMLD system's equivalent to a modal theory. In Section 3, a natural deduction style proof system for PMLD systems is given in which inference rules are applied to configurations. In Section 4, a model–theoretic semantics, based on a translation method into classical logic, is described together with a notion of semantic entailment. Its equivalence to Kripke semantics for normal propositional modal logics is also shown whenever the initial configuration is a single point. Soundness and completeness results of the proof system described in Section 3 are stated with respect to this semantics. The paper ends with a general discussion in Section 5. [13] is a longer version of this paper which includes detailed definitions, proofs of theorems and further examples.

Some remarks may be helpful regarding notation. Throughout the paper constant and predicate symbols begin with an upper-case letter, whereas variables and function symbols begin with a lower-case letter. Greek-letter meta-variables are used to refer in general to terms and expressions in the system. Larger entities such as structures, sets, theories and languages are symbolised in calligraphic font, $\mathcal{A}, \mathcal{B}, \mathcal{C}$, etc.. The power set of a given set $\mathcal{A}$ is denoted by $PW(\mathcal{A})$. A wide family of PMLD systems, corresponding to each of the normal modal logic extensions of K obtained using combinations of the axiom schemas $T, 4, 5, D$ and B, (e.g. $K, T, K4, K45, KD, KD4, KB$, etc.), will be considered. These systems will be referred to as the K–PMLD system, the T–PMLD system, etc. respectively. The symbol S will generally refer to an arbitrary PMLD system within this family.

2. Language and Syntax

In this section PMLD systems are described formally. Basic definitions of the PMLD language and syntax are given together with the notion of a *configuration* – a PMLD system's equivalent to a modal theory.

2.1. LANGUAGES

A PMLD language is defined as an ordered pair $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$, where $\mathcal{L}_L$ is a *labelling language* and $\mathcal{L}_M$ is a *propositional modal language*. As in the implicit approach, $\mathcal{L}_M$ is composed of a countable set of propositional letters, $\{p, q, r, \ldots\}$, the logical connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$, and the two modal operators $\Box$ and $\Diamond$. The labelling language $\mathcal{L}_L$ is a binary fragment of a first–order language composed of a countable set of constant symbols $\{W_0, W_1, W_2, \ldots\}$, a countable set of variables, $\{x, y, z, \ldots\}$, a unary function symbol *succ*, a binary predicate R, the set of logical connectives, $\{\neg, \wedge, \vee, \rightarrow, \equiv\}$, and the quantifiers $\forall$ and $\exists$. The first–order language $Func(\mathcal{L}_L, \mathcal{L}_M)$ is an extension of $\mathcal{L}_L$ defined as follows.

Definition 2.1 Let $\mathcal{L}_M$ be a modal propositional language and $\{\alpha_1,\alpha_2,\ldots\}$ be the set of all wffs of $\mathcal{L}_M$. The first–order language $Func(\mathcal{L}_L, \mathcal{L}_M)$ is defined as the language $\mathcal{L}_L$ extended with the sets of unary function symbols $\{f_{\alpha_1}, f_{\alpha_2}, \ldots\}$ and $\{box_{\alpha_1}, box_{\alpha_2}, \ldots\}$.

The binary predicate R represents the accessibility relation between possible worlds, whereas the ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$, also called *labels*, refer to the possible worlds. Intuitively, for each possible world (label) λ and modal formula α, the term $f_\alpha(\lambda)$ names a *particular* world specifically associated with α. Such terms will be used whenever Kripke semantic notions of the form "there exists a possible world ..." need to be formalised. In contrast, ground terms of the form $box_\alpha(\lambda)$ can be thought of as referring to any *arbitrary* world specifically associated with α. These terms will be used whenever Kripke semantic notions of the form "for all possible worlds ..." need to be expressed. However, formally speaking, $f_\alpha(\lambda)$ and $box_\beta(\lambda)$ are both just labels – within a particular model they might even refer to the same possible world.

Different classes of Kripke frames are formalised in a PMLD system by a first–order axiomatisation, called *labelling algebra*, written in the language $\mathcal{L}_L$. For example, for the T–PMLD system, the labelling algebra $\mathcal{A}_T$ is the set $\{\forall x R(x,x)\}$ and for the $S4$–PMLD system, $\mathcal{A}_{S4}$ is the set of axioms $\{\forall x R(x,x), \forall x,y,z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))\}$. Table 1 shows the labelling algebra associated with some example PMLD systems. Each

TABLE 1. Labelling algebrae

PMLD system S	Labelling algebra $\mathcal{A}_S$
K	$\{\}$
T	$\{\forall x R(x,x)\}$
$K4$	$\{\forall x,y,z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))\}$
$K45$	$\{\forall x,y,z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$, $\forall x,y(R(x,y) \rightarrow R(y,x))\}$
KD	$\{\forall x R(x, succ(x))\}$
KB	$\{\forall x,y(R(x,y) \rightarrow R(y,x))\}$

set $\mathcal{A}_S$ of axioms reflects van Benthem's correspondence theory shown in [15]. However, to reflect the "pre–skolemized" approach given by the use of the function symbols f_{α_i} and box_{α_i}, the seriality property of the D class of systems is formalised by the axiom $\forall x R(x, succ(x))$ instead of the axiom

$\forall x \exists y R(x, y)$. It is clear that for each possible world λ the term $succ(\lambda)$ names one accessible world.

2.2. SYNTAX

The PMLD language facilitates the formalisation of two types of information, (i) what holds at particular possible worlds and (ii) which worlds are in relation with each other and which are not. As exemplified above, (i) and (ii) are captured within the syntax of a PMLD system by two different types of syntactic entity, the *declarative unit* and the *R–literal*. A declarative unit is defined as a pair separated by a colon – "*label:modal formula*" – expressing that a modal formula is true at a possible world. The label component is a ground term of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$ and the modal formula is a wff of the modal language $\mathcal{L}_M$. Other examples of declarative units are $W_0\!:\!p$, $f_{\Diamond q}(W_3)\!:\!p$ and $f_q(W_1)\!:\!p \rightarrow r$. An R–literal is any ground literal in the language $Func(\mathcal{L}_L, \mathcal{L}_M)$ of the form $R(\lambda_1, \lambda_2)$ or $\neg R(\lambda_1, \lambda_2)$, where λ_1 and λ_2 are labels, expressing that λ_2 is or is not accessible from λ_1. Examples of R–literals are $R(W_1, W_2)$, $\neg R(W_2, box_p(W_2))$ and $R(f_p(W_0), box_q(f_p(W_0)))$. For each R-literal Δ, the *conjugate* of Δ, written $\overline{\Delta}$, is the opposite in sign of Δ (i.e $\neg R(\lambda_1, \lambda_2)$, if $\Delta = R(\lambda_1, \lambda_2)$ and $R(\lambda_1, \lambda_2)$ if $\Delta = \neg R(\lambda_1, \lambda_2)$).

As mentioned above, this combined aspect of the PMLD syntax yields a definition of a PMLD theory more general than the traditional notion of a modal theory ([8], [3]). Informally, a PMLD theory, called a *configuration*, is composed of two sets, a set of R–literals and a set of declarative units. An example of a PMLD theory is the pair of sets $\{R(W_0, W_1), R(W_0, W_2), R(W_1, f_p(W_1))\}$ and $\{W_0 : \Box(p \rightarrow q), W_0 : \Box r, W_1 : \Diamond p, f_p(W_1) : p, W_2 : q\}$. The formal definition of a configuration is as follows.

Definition 2.2 Given a PMLD language, a configuration is a tuple $\langle \mathcal{D}, \mathcal{F} \rangle$ where $\mathcal{D}$ is a set of R–literals and $\mathcal{F}$ is a function from the set of ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ to the set PW(wff($\mathcal{L}_M$)) of sets of wffs of $\mathcal{L}_M$.

The $\mathcal{D}$ component of a configuration $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$ will sometimes be referred to as a *diagram* and set membership statements of the form $\alpha \in \mathcal{F}(\lambda)$ will sometimes be written as $\lambda : \alpha \in \mathcal{C}$. Configurations can be infinite. This occurs either when the diagram $\mathcal{D}$ is infinite (i.e. the configuration contains an infinite number of R–literals), or when for some label λ of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$, the value of $\mathcal{F}(\lambda)$ is an infinite set of formulae of $\mathcal{L}_M$ (i.e. the configuration contains an infinite number of declarative units referring to λ), or when $\mathcal{F}(\lambda) \neq \emptyset$ for an infinite number of terms λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$.

In the next section, a natural deduction style proof system for an arbitrary PMLD system S is given, in which inference rules and a derivability relation are defined between configurations. A set $\mathcal{R}$ of such deduction rules together with a PMLD language $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$ and a labelling algebra $\mathcal{A}$ uniquely define a PMLD system (i.e. for any PMLD system S, $S = \langle \langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R} \rangle$).

3. A Natural Deduction System

As mentioned above, explicit syntactic reference to a Kripke–like accessibility relation facilitates reasoning about structures of actual worlds (i.e. configurations). A natural deduction style proof system is described in this section whose inference rules and derivability relation are defined between configurations. This system allows reasoning about both the declarative units and the R–literals within a given configuration. Elimination and introduction rules are given for each classical connective and modal operator. For the R–literals, three additional rules are defined. One of these (the $\mathcal{I}_{R-A}$ rule) facilitates first–order derivations of relationships between possible worlds using the labelling algebra $\mathcal{A}$. This rule captures entirely the difference between one modal system and another, allowing all other deduction rules, including those for the modal operators, to be equally applicable to any PMLD system S. It is in this sense that the PMLD systems described here are *uniform*. The basic notion of an inference rule is defined as follows.

Definition 3.1 An inference rule $\mathcal{I}$ is a set of pairs of configurations, where each such pair is written as $\mathcal{C}/\mathcal{C}'$. If $\mathcal{C}/\mathcal{C}' \in \mathcal{I}$ then we say $\mathcal{C}$ is an *antecedent configuration* of $\mathcal{I}$, and $\mathcal{C}'$ is an *inferred (or consequence) configuration of $\mathcal{I}$ with respect to $\mathcal{C}$.*

All the rules except one have the effect of expanding the antecedent configuration. These rules can extend an antecedent configuration $\mathcal{C}$ with either a declarative unit, or with an R–literal or with both. However, configurations equal or smaller than the antecedent one can also be inferred. This is facilitated by an inference rule called the $\mathcal{C}$–Reduction ($\mathcal{I}_{C-R}$) rule (see Table 4). Informally, a *proof* is a non empty sequence of configurations, $\mathcal{C}_0, \ldots, \mathcal{C}_n$, where, for each $0 < i \leq n$, $\mathcal{C}_i$ is obtained from $\mathcal{C}_{i-1}$ by the application of an inference rule. A configuration $\mathcal{C}'$ is said to be *derivable* from a configuration $\mathcal{C}$, written $\mathcal{C} \vdash_S \mathcal{C}'$, if and only if there exists a proof $\mathcal{C}, \ldots, \mathcal{C}'$.

Because of space limitations, only a schematic representation of the inference rules is given here. A complete mathematical formalisation of the proof system can be found in [13]. Table 2 summarises the set of natural deduction rules for the classical connectives, Table 3 describes the natural deduction rules for the modal operators and in Table 4 additional rules for R–literals are given. However, some remarks are essential to clarify this schematic representation. For any configuration $\mathcal{C}$, the informal notation

$\mathcal{C}\langle\lambda:\alpha\rangle$ (respectively $\mathcal{C}\langle\Delta\rangle$) denotes that $\mathcal{C}$ includes a declarative unit $\lambda:\alpha$ (respectively R–literal Δ). Declarative units and R–literals contained in square brackets (see e.g. the $\mathcal{I}_{\vee E}$ rule) are assumptions introduced within a derivation that are subsequently discharged. The notation $\mathcal{C}'\langle\psi\rangle$ represents that the inferred configuration $\mathcal{C}'$ is $\mathcal{C}$ extended with the declarative unit or R–literal ψ. In the $\mathcal{I}_{\vee E}$ rule, $\tilde{\mathcal{C}}$ and $\overline{\mathcal{C}}$ are the configurations derived in

TABLE 2. Natural deduction rules for classical connectives

$$\frac{\mathcal{C}\langle\lambda:\alpha\wedge\beta\rangle}{\mathcal{C}'\langle\lambda:\alpha\rangle}\;\mathcal{I}_{\wedge E}\qquad\qquad \frac{\mathcal{C}\langle\lambda:\alpha,\lambda:\beta\rangle}{\mathcal{C}'\langle\lambda:\alpha\wedge\beta\rangle}\;\mathcal{I}_{\wedge I}$$

$$\frac{\mathcal{C}\langle\lambda:\alpha\vee\beta\rangle\quad \begin{array}{c}\mathcal{C}\langle[\lambda:\alpha]\rangle\\ \vdots\\ \tilde{\mathcal{C}}\langle\lambda:\gamma\rangle\end{array}\quad \begin{array}{c}\mathcal{C}\langle[\lambda:\beta]\rangle\\ \vdots\\ \overline{\mathcal{C}}\langle\lambda:\gamma\rangle\end{array}}{\mathcal{C}'\langle\lambda:\gamma\rangle}\;\mathcal{I}_{\vee E}\qquad\qquad \frac{\mathcal{C}\langle\lambda:\alpha\rangle}{\mathcal{C}'\langle\lambda:\alpha\vee\beta\rangle}\;\mathcal{I}_{\vee I}$$

$$\frac{\mathcal{C}\langle\lambda:\alpha\rightarrow\beta,\lambda:\alpha\rangle}{\mathcal{C}'\langle\lambda:\beta\rangle}\;\mathcal{I}_{\rightarrow E}\qquad\qquad \frac{\begin{array}{c}\mathcal{C}\langle[\lambda:\alpha]\rangle\\ \vdots\\ \tilde{\mathcal{C}}\langle\lambda:\beta\rangle\end{array}}{\mathcal{C}'\langle\lambda:\alpha\rightarrow\beta\rangle}\;\mathcal{I}_{\rightarrow I}$$

$$\frac{\mathcal{C}\langle\lambda:\neg\neg\alpha\rangle}{\mathcal{C}'\langle\lambda:\alpha\rangle}\;\mathcal{I}_{\neg E}\qquad\qquad \frac{\begin{array}{c}\mathcal{C}\langle[\lambda:\alpha]\rangle\\ \vdots\\ \tilde{\mathcal{C}}\langle\lambda':\perp\rangle\end{array}}{\mathcal{C}'\langle\lambda:\neg\alpha\rangle}\;\mathcal{I}_{\neg I}$$

subderivations after adding to the antecedent configuration $\mathcal{C}$ the assumptions $\lambda:\alpha$ and $\lambda:\beta$ respectively. For the $\mathcal{I}_{\rightarrow I}$, $\tilde{\mathcal{C}}$ is the configuration derived after adding the assumption $\lambda:\alpha$ to the antecedent configuration, to get the declarative unit $\lambda:\beta$. In the $\mathcal{I}_{\neg I}$ rule, the symbol $\perp$ is a short–hand for any wff of $\mathcal{L}_M$ of the form $\alpha\wedge\neg\alpha$. $\tilde{\mathcal{C}}$ is, in this case, the configuration derived after adding the assumption $\lambda:\alpha$ to $\mathcal{C}$ to get a contradiction – the declarative unit $\lambda':\perp$. In this case the label λ' can be different from λ because contradictions might occur in any possible world of a configuration[2]. When $\lambda' = \lambda$, the rules in Table 2 faithfully reflect the natural deduction proof system for propositional logic defined in [12]. In fact, since these rules do not define new labels, they can simply be considered as "local natural deduction rules" for propositional logic[3].

The natural deduction rules for modal operators are somewhat different. As shown in Table 3, these rules involve declarative units with different labels. They express the interaction between the local modal theories within a configuration. They refer explicitly to relationships between points (R–literals) to infer new formulae in accessible worlds, and allow the introduction of "new particular accessible worlds". The $\mathcal{I}_{\Box E}$ and $\mathcal{I}_{\Diamond I}$ rules clearly reflect the Kripke semantic truth definitions of the two modal operators. If at a local actual point λ_1, $\Box\alpha$ holds, this implies that for any point accessible from λ_1, the formula α must hold. Therefore, if a given configuration $\mathcal{C}$ contains a declarative unit $\lambda_1 : \Box\alpha$ and a relationship between λ_1 and another local actual point λ_2 (i.e. $R(\lambda_1, \lambda_2)$), then it is obvious to extend this configuration with the declarative unit $\lambda_2 : \alpha$. Similarly for the $\mathcal{I}_{\Diamond I}$ rule. The $\mathcal{I}_{\Diamond E}$ and $\mathcal{I}_{\Box I}$ rules show how ground terms of the form $f_\alpha(\lambda)$

TABLE 3. Natural deduction rules for modal operators

$$\frac{\mathcal{C}\langle\lambda : \Diamond\alpha\rangle}{\mathcal{C}'\langle f_\alpha(\lambda) : \alpha, R(\lambda, f_\alpha(\lambda))\rangle}\ \mathcal{I}_{\Diamond E} \qquad \frac{\mathcal{C}\langle\lambda_2 : \alpha, R(\lambda_1, \lambda_2)\rangle}{\mathcal{C}'\langle\lambda_1 : \Diamond\alpha\rangle}\ \mathcal{I}_{\Diamond I}$$

$$\frac{\mathcal{C}\langle\lambda_1 : \Box\alpha, R(\lambda_1, \lambda_2)\rangle}{\mathcal{C}'\langle\lambda_2 : \alpha\rangle}\ \mathcal{I}_{\Box E} \qquad \frac{\begin{array}{c}\mathcal{C}\langle[R(\lambda, box_\alpha(\lambda))]\rangle\\ \vdots \\ \tilde{\mathcal{C}}\langle box_\alpha(\lambda) : \alpha\rangle\end{array}}{\mathcal{C}'\langle\lambda : \Box\alpha\rangle}\ \mathcal{I}_{\Box I}$$

and $box_\alpha(\lambda)$ are used within this proof system[4]. The consequence of the

[2] This is because a configuration might already have declarative units of the form $\lambda' : \alpha$ and $\lambda' : \neg\alpha$, or because applications of rules for modal operators might generate contradictions in some accessible world.

[3] Of course, strictly speaking, two $\wedge$–elimination rules and two $\vee$–introduction rules are included in any such system. For example, from $p \wedge q$, both p and q can be classically inferred.

[4] Intuitively, $f_\alpha(\lambda)$ represents a "new" possible world when introduced within a proof. Indeed, in the completeness proof described in the next section, a canonical model is defined over a Herbrand universe of such terms.

$\mathcal{I}_{\Diamond E}$ rule is equivalent to "there exists an accessible point where α holds", whereas the subderivation of the $\mathcal{I}_{\Box I}$ rule corresponds to checking whether "at any accessible world α holds".
The following table completes the description of the PMLD deduction system. The $\perp$–Introduction rule allows the inference of falsity (i.e. $\lambda : \perp$) whenever an R–literal and its negation are present in a configuration. This is necessary because since no compound classical formulae with R–literals can be inferred in a configuration, inconsistency of this form would not otherwise be captured. The R–Assertion rule, $\mathcal{I}_{R-A}$, expands the diagram of the antecedent configuration using the labelling algebra $\mathcal{A}$. Thus different R–literals are inferred for different PMLD systems. The R–Introduction rule, $\mathcal{I}_{R-I}$, is the equivalent of a $\neg$–Introduction rule for R-literals. Finally, with the C–reduction rule, $\mathcal{I}_{C-R}$, it is possible to infer any smaller configuration contained in an existing one. To guarantee completeness, this rule must be included because, as mentioned above, all the other rules have the effect of expanding their antecedent configurations.

TABLE 4. Other rules

$\mathcal{I}_{\perp I}$	$\dfrac{\mathcal{C}\langle \Delta, \overline{\Delta}\rangle}{\mathcal{C}'\langle \lambda : \alpha\rangle}$	$\mathcal{I}_{C-R}$	$\dfrac{\mathcal{C}}{\mathcal{C}'}$ where $\mathcal{C}' \subseteq \mathcal{C}$
$\mathcal{I}_{R-I}$	$\dfrac{\begin{array}{c}\mathcal{C}\langle [\,\overline{\Delta}\,]\rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle \lambda' : \perp\rangle\end{array}}{\mathcal{C}'\langle \Delta\rangle}$	$\mathcal{I}_{R-A}$	$\dfrac{\mathcal{C}}{\mathcal{C}'\langle \Delta\rangle}$ if $\mathcal{A} \cup \mathcal{D} \vdash_{FOL} \Delta$

Unfortunately, space limitations do not allow example derivations to be given in detail here (see [13] for these). However, the reader is invited to use the rules in Tables 2, 3 and 4 to check the following. (i) Any configuration of any PMLD system which contains the declarative units $W_0 : \Box(p \rightarrow q)$ and $W_0 : \Box p$ can be expanded to a configuration containing $W_0 : \Box q$ (use $\mathcal{I}_{\Box I}$, $\mathcal{I}_{\Box E}$ and $\mathcal{I}_{\rightarrow E}$). (ii) Any configuration of the $K4$–PMLD system containing the declarative units $W_0 : \Box p$, $W_2 : \Box(p \rightarrow q)$ and $W_3 : \neg q$, and the R–literals $R(W_0, W_1)$ and $R(W_1, W_3)$, can be expanded to a configuration containing the R–literal $\neg R(W_2, W_3)$ (use $\mathcal{I}_{R-I}$, $\mathcal{I}_{\Box E}$, $\mathcal{I}_{R-A}$, $\mathcal{I}_{\rightarrow E}$ and $\mathcal{I}_{\wedge I}$). Note that derivations such as (ii) have no obvious counterpart in traditional modal logic.

4. Semantics, Soundness and Completeness

A PMLD system can be considered to be a "semi–translated" approach to modal logic — a Kripke–like accessibility relation is syntactically expressed, but without requiring the full translation of modal formulae into first–order sentences. Therefore, a model–theoretic semantics could be equally given in terms of the traditional Kripke possible worlds semantics (as discussed in [6]) or in terms of a first–order semantics using some translation method. The second approach has been chosen here. In this section, a translation method of PMLD systems into first–order logic is defined and the notions of model, satisfiability of a configuration and semantic entailment are then given in terms of classical semantics. Soundness and completeness results are stated which show that the derivability relation introduced in the previous section corresponds to the semantic entailment relation defined below. Finally, results on correspondences between PMLD systems and traditional axiomatisations of modal logic are given.

4.1. SEMANTICS

As pointed out above, a declarative unit $\lambda:\alpha$ represents that the formula α holds at the possible world λ. In what follows, such Kripke–style semantic notions are expressed in terms of first–order statements of the form $[\alpha]^*(\lambda)$, where $[\alpha]^*$ is a predicate symbol. The relationships between these predicates are constrained by a set of first–order axiom schemas which capture the satisfiability conditions of each type[5] of formula α. The *extended labelling language* $Mon(\mathcal{L}_L, \mathcal{L}_M)$ is an extension of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$ given by adding a monadic predicate symbol $[\alpha]^*$ for each wff α of $\mathcal{L}_M$. The *extended algebra* $\mathcal{A}^+$ is a first–order theory written in $Mon(\mathcal{L}_L, \mathcal{L}_M)$ which extends the labelling algebra $\mathcal{A}$ as follows.

Definition 4.1 Given an extended labelling language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and a labelling algebra $\mathcal{A}$ written in $Func(\mathcal{L}_L, \mathcal{L}_M)$, the extended algebra $\mathcal{A}^+$ is the first–order theory in $Mon(\mathcal{L}_L, \mathcal{L}_M)$ consisting of the following axiom schemas (Ax1)–(Ax8), together with the axioms of $\mathcal{A}$. For any wffs α and β of $\mathcal{L}_M$:

$$\forall x([\alpha \wedge \beta]^*(x) \equiv ([\alpha]^*(x) \wedge [\beta]^*(x))) \tag{Ax1}$$

$$\forall x([\neg\alpha]^*(x) \equiv \neg[\alpha]^*(x)) \tag{Ax2}$$

[5]The type of a wff is given by the main connective of the wff itself, e.g. the wff $\Diamond(p \rightarrow q)$ is a $\Diamond$–formula.

$$\forall x([\alpha \vee \beta]^*(x) \equiv ([\alpha]^*(x) \vee [\beta]^*(x))) \tag{Ax3}$$

$$\forall x([\alpha \rightarrow \beta]^*(x) \equiv ([\alpha]^*(x) \rightarrow [\beta]^*(x))) \tag{Ax4}$$

$$\forall x([\Diamond\alpha]^*(x) \rightarrow (R(x, f_\alpha(x)) \wedge [\alpha]^*(f_\alpha(x)))) \tag{Ax5}$$

$$\forall x(\exists y(R(x,y) \wedge [\alpha]^*(y)) \rightarrow [\Diamond\alpha]^*(x)) \tag{Ax6}$$

$$\forall x((R(x, box_\alpha(x)) \rightarrow [\alpha]^*(box_\alpha(x))) \rightarrow [\Box\alpha]^*(x)) \tag{Ax7}$$

$$\forall x([\Box\alpha]^*(x) \rightarrow (\forall y(R(x,y) \rightarrow [\alpha]^*(y)))) \tag{Ax8}$$

The first four axiom schemas express the distributive properties of the logical connectives among the monadic predicates of $Mon(\mathcal{L}_L, \mathcal{L}_M)$. (Ax5) and (Ax8) force the accessibility relation R on labels generated by the application of the function symbols f_α and box_α of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ whenever their respective antecedents hold. Clearly, all the axiom schemas (Ax1)–(Ax8) reflect the traditional Kripke semantic definition of satisfiability of modal wffs[6]. Note that a unique extended algebra $\mathcal{A}^+$ is associated with each PMLD system.

A translation method is defined next. It associates syntactical expressions of a PMLD language with sentences of the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$, and PMLD theories (configurations) with first–order theories in the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$. Each declarative unit $\lambda : \alpha$ is associated with the sentence $[\alpha]^*(\lambda)$, and each R–literal Δ is associated with itself. The first–order translation of a configuration is defined as follows.

Definition 4.2 Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$ be a configuration written in $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$. The first order translation of $\mathcal{C}$, written $FOT(\mathcal{C})$, is the theory $\mathcal{D} \cup \mathcal{DU}$, where $\mathcal{DU} = \{[\alpha]^*(\lambda) \mid \alpha \in \mathcal{F}(\lambda),\ \lambda \text{ is a ground term of } Func(\mathcal{L}_L, \mathcal{L}_M)\}$.

Here a first–order translation of a given configuration is a set of ground literals of the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$. Notions of model, satisfiability and semantic entailment are given in terms of classical semantics using the above definitions, as follows (where "$\mathcal{M} \Vdash_{FOL} \psi$" signifies that the classical formula ψ is true in the classical model $\mathcal{M}$, according to the standard definition). Given a PMLD system S, the associated extended algebra $\mathcal{A}^+$, a declarative unit $\lambda : \alpha$ and a R–literal Δ,

$$\mathcal{M} \text{ is a semantic structure of } S \iff_{\text{def}} \mathcal{M} \text{ is a model of } \mathcal{A}^+ \tag{1}$$

[6]This is easily seen by interpreting the truth of $[\alpha]^*(x)$ as the truth of the modal formula α in the possible world x.

$$\mathcal{M} \Vdash_S \lambda : \alpha \quad \Leftrightarrow_{\text{def}} \quad \mathcal{M} \Vdash_{FOL} [\alpha]^*(\lambda) \tag{2}$$

$$\mathcal{M} \Vdash_S \Delta \quad \Leftrightarrow_{\text{def}} \quad \mathcal{M} \Vdash_{FOL} \Delta \tag{3}$$

(1) defines the class of models of a PMLD system S in terms of models of the extended algebra $\mathcal{A}^+$ associated with S. (2) and (3) define the satisfiability of declarative units and R–literals in terms of classical satisfiability of their associated first–order translations. A semantic structure $\mathcal{M}$ satisfies a configuration $\mathcal{C}$, written $\mathcal{M} \Vdash_S \mathcal{C}$, if and only if for each $\pi \in \mathcal{C}$ (where π may be a declarative unit or an R–literal), $\mathcal{M} \Vdash_S \pi$. A notion of semantic entailment in a PMLD system is given here as a relation between configurations. It is formally defined as follows.

Definition 4.3 Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{A}^+$ be the extended algebra of S. Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$ and $\mathcal{C}' = \langle \mathcal{D}', \mathcal{F}' \rangle$ be two configurations of S and $FOT(\mathcal{C}) = \mathcal{D} \cup \mathcal{DU}$ and $FOT(\mathcal{C}') = \mathcal{D}' \cup \mathcal{DU}'$ their respective first order translations. The configuration $\mathcal{C}$ semantically entails $\mathcal{C}'$, written $\mathcal{C} \models_S \mathcal{C}'$, iff for each $\Delta \in \mathcal{D}'$, $\mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{FOL} \Delta$, and for each $[\alpha]^*(\lambda) \in \mathcal{DU}'$, $\mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{FOL} [\alpha]^*(\lambda)$.

4.2. SOUNDNESS, COMPLETENESS AND CORRESPONDENCE

Soundness and completeness results are stated below. Their proofs, given in full in [13], take advantage of the soundness and completeness of first–order classical logic. The soundness result, Theorem 4.1, states that if there exists a natural deduction proof of a configuration $\mathcal{C}'$ from a configuration $\mathcal{C}$, then $\mathcal{C}$ semantically entails $\mathcal{C}'$.

Theorem 4.1 [Soundness] Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS and let $\mathcal{C}, \mathcal{C}'$ be two configurations of S. If $\mathcal{C} \vdash_S \mathcal{C}'$ then $\mathcal{C} \models_S \mathcal{C}'$.
Proof: See [13].

The proof of Theorem 4.1 is represented diagrammatically in Figure 1. The soundness statement, which corresponds to the arrow labelled with (1), is proved by the composition of three main steps, arrows (2), (3) and (4) respectively. The first step (arrow (2)) proves that the hypothesis, $\mathcal{C} \vdash_S \mathcal{C}'$, implies that $\mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$ (see [13], Lemma 3.1). This trivially implies (by soundness of first–order logic) that $\mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}')$, which gives the second step of the proof (arrow (3)). Arrow (4) is given by the definition of the semantic entailment between configurations.

The completeness result, Theorem 4.2, states that, given a MLDS S and two configurations $\mathcal{C}$ and $\mathcal{C}'$ of S, such that their configuration difference $\mathcal{C}' - \mathcal{C}$ is finite, if $\mathcal{C}'$ is semantically entailed from $\mathcal{C}$ then $\mathcal{C}'$ is also derived from $\mathcal{C}$. $\mathcal{C}' - \mathcal{C}$ (defined formally in [13]) is basically the set of declarative

$$\begin{tikzcd}
\mathcal{C} \vdash_S \mathcal{C}' \arrow[r, "(1)"] \arrow[d, "(2)"'] & \mathcal{C} \models_S \mathcal{C}' \\
\mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}') \arrow[r, "(3)"'] & \mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}') \arrow[u, "(4)"']
\end{tikzcd}$$

Figure 1. Proof of the Soundness Theorem

units and R–literals in $\mathcal{C}'$ but not in $\mathcal{C}$.[7] The proof of Theorem 4.2, given in [13], uses a Henkin–style methodology [8].

Theorem 4.2 [Completeness] Let $S = \langle\langle \mathcal{L}_L, \mathcal{L}_M\rangle, \mathcal{A}, \mathcal{R}\rangle$ be a MLDS. Let $\mathcal{A}^+$ be the associated extended algebra. Let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations of S such that the configuration difference $\mathcal{C}' - \mathcal{C}$ is finite. If $\mathcal{C} \models_S \mathcal{C}'$ then $\mathcal{C} \vdash_S \mathcal{C}'$.

Proof: See [13].

An informal description of the completeness proof, referring to its diagrammatical representation given in Figure 2, is as follows.

$$\begin{tikzcd}
\mathcal{C} \not\vdash_S \mathcal{C}' \arrow[r, "(1)"] \arrow[dr, "(2)"] & \mathcal{C} \not\models_S \mathcal{C}' \\
 & \mathcal{A}^+ \cup FOT(\mathcal{C}) \not\models_{FOL} FOT(\mathcal{C}') \arrow[u, "(3)"']
\end{tikzcd}$$

Figure 2. Proof of the Completeness Theorem

The proof is of the contrapositive statement of Theorem 4.2 (arrow (1)). It is proved by the composition of two main steps, arrows (2) and (3). Arrow

[7] Obviously, if the configuration difference $(\mathcal{C}' - \mathcal{C})$ were infinite, an infinite proof sequence would be required to prove $\mathcal{C}'$ from $\mathcal{C}$.

(3) is already given by Definition 4.3, while arrow (2) represents the main part of the completeness theorem. Its proof is based on the statement *if $\mathcal{C}$ is a consistent configuration then $\mathcal{C}$ is satisfiable*, known as the "Model Existence Lemma". The hypothesis that $\mathcal{C}'$ is not derivable from $\mathcal{C}$, $\mathcal{C} \nvdash_S \mathcal{C}'$, implies (see [13], Theorem 4.1) that there exists a $\pi \in \mathcal{C}' - \mathcal{C}$ (where π is a declarative unit or an R–literal) such that $\mathcal{C} \nvdash_S \pi$. So $\mathcal{C}$ extended with $\neg\pi$ (written $\mathcal{C} + [\neg\pi]$) is a consistent configuration[8]. This implies that the configuration $\mathcal{C} + [\neg\pi]$ is satisfiable (see [13], *Model Existence Lemma* and Corollary 4.2). Therefore, there exists a semantic structure $\mathcal{M}$ of S which satisfies $\mathcal{C}$ and that also satisfies $\neg\pi$. Hence, $\mathcal{M}$ does not satisfy π. Thus, since $\pi \in \mathcal{C}'$, by the definition of satisfiability of a configuration, $\mathcal{M}$ does not satisfy $\mathcal{C}'$. Hence $\mathcal{A}^+ \cup FOT(\mathcal{C}) \not\models_{FOL} FOT(\mathcal{C}')$.

It was claimed in the introduction that a PMLD system is a generalisation of normal propositional modal logic, in that it facilitates reasoning about structures of actual worlds, which may or may not be single points. This claim is substantiated in Theorem 4.3 below. It states that under the restriction of considering only singleton configurations, a PMLD system is equivalent to the corresponding normal propositional modal logic. The equivalence is stated with respect to traditional axiomatisations of normal propositional modal logics.

Theorem 4.3 Let N be any normal propositional modal logic obtained by extending K with combinations of the axiom schemas $T, 4, 5, D$ and B. Let $\langle N_{Ax}, \vdash_{N_{Ax}} \rangle$ be an axiomatic system for N which is sound and complete with respect to Kripke semantics. Let N-PMLD be the corresponding PMLD system. Let $\Psi = \{\alpha_1 \ldots, \alpha_n\}$ $(n \geq 0)$ be an arbitrary (possibly empty) set of wffs of N, and let $\mathcal{C}$ be the configuration of N-PMLD consisting only of all declarative units of the form $W_0 : \alpha_i$, $\alpha_i \in \Psi$, and containing no R–literals. Then for any formula γ of N, $\mathcal{C} \vdash_N W_0 : \gamma$ if and only if $\Psi \vdash_{N_{Ax}} \gamma$, and $\mathcal{C} \models_N W_0 : \gamma$ if and only if $\Psi \models_{N_{Ax}} \gamma$.

Proof: See [13].

Hence the semantics given here is equivalent to Kripke semantics whenever the initial configuration is a single point. However, PMLD systems allow for a more general form of initial configuration – wffs associated with different local actual worlds, and explicit relationships between possible worlds. These configurations cannot be represented within traditional modal logic, where the initial set of formulae must always refer to the same initial actual world.

[8] A configuration $\mathcal{C}$ is inconsistent if it derives a configuration $\mathcal{C}'$ which contains a declarative unit of the form $\lambda : \bot$. A configuration is consistent if it is not inconsistent.

5. Discussion

The research described above can be summarised as follows. In Section 2 a family of labelled deductive systems has been defined in which modal logic and first–order logic are combined to generalise the notion of a modal theory. In Section 3 a natural deduction style proof system has been described which reflects this more general notion (the derivability relation is defined between configurations), and which is uniform for a wide family of modal logics. In Section 4 a model–theoretic semantics has been given, based on a translation method from the combined formalism into first–order logic. The soundness and completeness of the system with respect to this semantics has been stated (Theorems 4.1 and 4.2), as well as the sense in which normal propositional modal logics can be regarded as special cases of PMLD systems (Theorem 4.3).

The PMLD approach is similar to that of Fitting's prefixed tableaux [3], in that, in both cases, modal formulae are labelled. However, whereas in Fitting's system the notion of an accessibility relation is embedded in the definition of "accessible prefixes", in a PMLD system it is made explicit. Consequently, the different side conditions imposed on the prefixed tableaux modal rules by different modal logics are replaced here by the R–Assertion rule, allowing the PMLD system to be uniform. In [14] Sympson has developed a natural deduction system for propositional intuitionistic modal logic. His work is similar to the approach described in this paper in that it is also built on the framework of LDS. Formulae are labelled with variables (lwffs), which denote possible worlds, and relationships between possible worlds are also made explicit (rwffs). However, rwffs are only used as assumptions in "geometric" rules. Consequently, different geometric rules are given for different intuitionistic modal logics, whereas the PMLD natural deduction rules are applicable to any normal modal logic. Other work related to the approach in this paper can be found in [4], [7], [9] and [2].

The translation method described in Section 3 is comparable with other existing relational translation approaches (e.g. [10] and [11]). It is hoped that the PMLD translation will result in a more compact first–order theory than that of Ohlbach (where an exponential number of clauses are generated) – this is the subject of further investigation. This could give an advantage when using existing classical theorem provers for modal logics. Moreover, in this case the need for expensive unification steps would be eliminated by use of the function symbols f_α and box_α.

The framework described in this paper is presently being extended to the predicate case. It is expected that this extension will provide a uniform proof system which is sound and complete with respect to varying domain semantics.

One reason for developing the PMLD framework has been to provide a logic based on modal reasoning which is closer to the needs of computing and A.I. applications (see [5] for a general discussion of this issue). Although the focus of this paper has not been on applications, it is perhaps worth briefly discussing an example. A distributed database system can be represented as a configuration, where each label is associated with a local database and connections between databases are described by R–literals. Here, modal operators would impose constraints which might affect the evolution of several local databases when an update is made at a particular node. This idea has also been discussed in [1].

Finally, this paper illustrates an important general point regarding work in Combining Logics. Most work in this area is concerned with showing how entirely *new* logics can be generated as hybrids of two or more existing logics. This paper makes a contribution in a slightly different direction. It shows how an existing class of logics can be *generalised* by combining them with another. It could be that such generalisations are possible on a much wider scale.

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