Preface

The use of logics in applied research has, in the last few years, brought about an increasing demand for logical systems able to represent and to reason about *structures* of information (see [Gab92] for a discussion and examples). Gabbay has argued that neither classical nor non-classical logics are very intuitive when used in applications requiring structures of data more complex than just sets. To support this claim he gives the example of a temporal database. Here information is structured according to an underlying flow of time (e.g. the natural or real numbers). A classical formalisation would need to use a two-sorted logic in order to include the additional parameter of *time*, with which to express the time-dependency of data and the assumed time structure. Gabbay argues that this leads to an unintuitive representation and unnecessary redundancy in the language. On the other hand, temporal logics' characteristic of referring to a single time point, "now", with respect to which notions of "future" and "past" are evaluated, turns out to be restrictive in situations where information relative to specific instants of time, different from "now", needs to be explicitly represented. This characteristic is true in general for modal logics, whose theories are evaluated with respect to one single possible world, the *actual world*.

Gabbay [Gab94] has therefore proposed a new methodology called *Labelled Deductive Systems* (LDS), both for the theoretical study of logics and for the development of logical systems suitable for the needs of specific applications (a notion of an algebraic LDS is also given in [Gab92]). In the LDS approach, a basic unit of information is not just a formula but a *labelled formula*, where the label belongs to a given *labelling algebra*. Additional information, for example regarding the overall structure of the data, can be explicitly embodied, via the labels, in the object language of the system. Derivation rules act on both the labels and the formulae, according to certain fixed rules of propagation. The addition of this new separate "object-level" dimension (i.e. labels and labelling algebra) helps overcome the limitations mentioned above – first-order logic can be used

directly without being extended to a two-sorted logic, and non-classical logic theories can be explicitly associated with different elements within a structure.

The general proposals in [Gab92, Gab94] have acted as a manifesto or starting point for a wide programme of research. For example, more detailed investigations have already been undertaken into the benefits of using the LDS methodology to reformulate substructural logics [DG94] and intuitionistic modal logics [Sim93]. This thesis substantiates Gabbay's intuitions by demonstrating how LDS techniques can be used to reformulate and generalise a large class of propositional and predicate modal logics.

In the *Modal Labelled Deductive Systems* (MLDS) described in this thesis, explicit reference can be made to (possibly singleton) "structures" of actual worlds, to which arbitrary modal theories can be *locally* associated. Deduction processes are then defined on the overall structure, allowing the inference of what is true at different points. This work contributes to the theoretical study of modal logics in the following ways. (1) It describes a uniform natural deduction system for propositional and predicate modal logics, including those whose accessibility relations satisfy conditions which are not, in van Benthem's terminology, "modally definable" [vB84]. With the exception of Hilbert systems, existing proof systems for modal logic are not uniform – the inference rules for modal operators change according to the particular modal logic. Moreover, and in contrast to the system described here, none of the existing proof systems can deal with *all* accessibility relations which are first–order axiomatisable [vB84]. (2) It extends the scope of propositional and predicate modal logic in order to deal with domains containing not just one, but a structure of one or more local or actual worlds. A proof is given that existing propositional and predicate modal systems are subsumed by the MLDSs described here. (3) In the case of predicate modal logics, it provides a uniform natural deduction system which is sound and complete with respect to arbitrary varying domain semantics. This is achieved without requiring any of the additional conditions on semantic structures which are imposed by existing first–order modal systems [HC68, Fit83]. (4) It provides an alternative approach to modal reasoning whose increased syntactic expressivity and deductive power contributes towards the long term aim of constructing logical formalisms more suitable to the needs of applications (in the sense described in [Gab92]).

Part I of the thesis describes a wide family of propositional MLDSs which includes generalisations of normal propositional modal logics built from the set of axioms $\{\mathbf{T}, \mathbf{D}, \mathbf{B}, \mathbf{4}, \mathbf{5}\}$. After a brief introduction to modal logic and various deduction systems, Chapter 1 gives the formal definition of a propositional MLDS. Here a labelling algebra is defined as a binary first-order theory which axiomatises the Kripke semantic notion of an ac-

cessibility relation. The basic building blocks of the MLDS language are *declarative units*, i.e. labelled formulae of the form $\lambda : \alpha$ interpreted as "the modal formula α holds at the possible world λ", and *R-literals* of the form $R(\lambda_i, \lambda_j)$ or $\neg R(\lambda_i, \lambda_j)$ where λ_i and λ_j are labels. They respectively facilitate the formalisation of two types of information, (i) what holds at particular possible worlds and (ii) which worlds are in relation with each other and which are not. This combined aspect of the syntax of a MLDS yields the definition of a theory, called a *configuration*, as a structure of arbitrary local modal theories. It is in this sense that a configuration can be seen as a generalisation of the traditional notion of a modal theory ([Fit83]). In Chapter 2, a uniform natural deduction system is described, in which inference rules are applied to configurations. It is shown that the same set of deduction rules can be applied to (the generalisation of) each well-known propositional normal modal logic. The difference between one modal logic and another is captured entirely by the labelling algebra.

The first-order nature of the labelling algebra implies that a MLDS can be constructed which is equivalent to any one of the modal logics whose axiomatisations define first-order relational conditions. But this thesis will show that the MLDS methodology provides even greater flexibility. Results from Correspondence Theory [vB84] show that some simple first-order relational conditions, such as $\forall x \neg R(x, x)$ (irreflexivity) and $\exists x \exists y R(x, y)$, are not modally definable, which means that they do not have a corresponding complete modal axiomatisation. In a MLDS, the explicit declaration of any first-order condition on the accessibility relation makes the need for a corresponding modal formula (axiom) unnecessary. Therefore modal reasoning involving any such type of classical structure can be treated. In Chapter 2, it will be shown, as an example, that the same modal labelled deductive proof system for "well behaved" normal modal logics can be applied in the case of an irreflexive accessibility relation. A model-theoretic semantics, based on a translation method into classical logic, is also provided. In Chapter 3, the natural deduction system is proved to be sound and complete, and it is shown that the semantics is equivalent to Kripke semantics whenever the initial configuration is a single point, and the underlying classical structure is modally definable.

In the propositional case, modal logics are uniquely and well defined – in particular, normal modal logics have been widely studied and discussed in the literature. In summary, Part I of this thesis further expands this horizon by proposing a generalisation of these systems and a new modal proof theoretical approach for certain types of classical mathematical structures which would otherwise be modally untreatable. Part II contributes instead to the theoretical study of quantified (predicate) modal logics.

The addition of quantifiers to modal logics is a complex and controver-

sial process. One of the main questions is whether to assume that all the possible worlds contain the same objects, and if not, what the relation is between objects that belong to different possible worlds. Semantics which include the first assumption are known in the literature as *fixed domain semantics*. In contrast, *varying domain semantics* may describe increasing (or "nested"), decreasing or arbitrary domains for a possible world accessible from another. Defining a predicate modal logic in which a classical interpretation of quantifiers is adopted imposes the assumption of fixed domains. Garson [Gar84] has pointed out that, although fixed domains are satisfying from a formal point of view, the varying domain semantics is a more powerful semantic approach which turns out to be very useful in applications such as natural language. However, existing modal logics based on a varying domain semantics and on an interpretation of quantifiers which is world-relative have limitations. The relative predicate Hilbert modal systems [HC68] are sound and complete only with respect to increasing domains, and thus, in the case of a symmetric accessibility relation, only with respect to a fixed domain semantics. Moreover, systems based on other proof theoretical approaches, such as Fitting's prefixed tableaux, even though they allow the general case of arbitrary varying domains, require that every ground term of the "core" language must refer to an object which exists in every possible world. This is a significant restriction – only those objects which cannot be explicitly referred to (by a ground term) in the "core" language may not exist in some possible world. Garson has observed that the reason for such limitations lies in the fact that the semantics still validates the classical axioms for quantifiers, which are themselves incompatible with the world-relative interpretation of quantifiers. In Garson's words:

> ... There is little reason for attempt to preserve the classical rules in formulating systems with the objectual interpretation and world-relative domains ... The world-relative interpretation of quantifiers virtually demands the adoption of free logics. [Gar84]

Part II contributes to the theoretical investigation of quantified modal logics by providing a natural deduction system for predicate modal logics, which is sound and complete with respect to Kripke varying domain semantics [Kri63], and which does not have any of the limitations described above. This is achieved by introducing rules for quantifiers which are somewhat similar to the derivation rules for quantifiers in free logic. In Chapter 4, predicate MLDSs are defined as quantified extensions of the propositional MLDSs given in Part I. The quantified MLDS language includes an additional entity, the *declarative term*, which expresses either that a term exists

or that a term does not exist in a possible world. The natural deduction system described in Chapter 5 includes rules that can utilise this type of information. Four different classes of quantified MLDSs are described. The first one (the most basic and in a sense the most general) is obtained by adding introduction and elimination rules for quantifiers. It allows modal reasoning based on arbitrary varying domain semantics, and allows for the possibility of local empty domains. The second and the third classes facilitate reasoning on increasing and decreasing varying domain semantics. They extend the first class of MLDSs by including deduction rules which facilitate the "navigation" of terms between possible worlds accessible to each others. Using Gabbay's terminology [Gab92], these are called *increasing* and *decreasing visa rules* respectively. MLDSs for constant domains are instead obtained by adding a third visa rule, called the *constant visa rule*, to the first class of MLDSs. The model-theoretic semantics given in Chapter 5 is a straightforward predicate extension of the semantics given in Part I. The soundness and completeness of the quantified MLDSs with respect to this semantics is proved in Chapter 6. Finally, appropriate equivalence results are also proved between quantified MLDSs with singleton initial configurations and the Kripke semantics described in [Kri63].

The explicit object-level declaration of the accessibility relation and of the existence of terms in possible worlds has facilitated the use of an interesting technique in the proofs of completeness (in both the propositional and predicate cases). These proofs are based on the standard Henkin methodology [HC68]. However, only the construction of one maximal consistent set is required instead of the usual sets of maximal consistent sets of canonical Kripke models. Moreover, unlike the usual completeness proof for quantified modal logics, in which more complex constructions are needed for a canonical model than the one needed for propositional modal logics, it is shown in Chapter 6 that the approach used in Part I for propositional MLDS is directly applicable to the proof of completeness of quantified MLDS. This is an attractive technical advantage of the labelled approach described here.

The thesis concludes with a summary and discussion, and some remarks about future work.

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