Modal Logics
as
Labelled Deductive Systems

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Abstract

This thesis shows how Gabbay's general methodology of Labelled Deductive Systems can be used to reformulate and generalise a class of normal modal logics, and to develop types of modal reasoning for which axiomatizations are not definable. In particular, it shows how standard modal logics can be extended to enable reasoning about structures of actual worlds, where each world has an arbitrary associated modal theory. In the predicate case, the method gives rise to a uniform deductive procedure for arbitrary varying domain semantics.

In Part I, a family of labelled deductive systems called propositional *Modal Labelled Deductive Systems* (MLDS) is described. These logics combine the standard syntax of propositional modal logic with a simple subset of first–order predicate logic, called a *labelling algebra*, to allow syntactic reference to a Kripke–like structure of possible worlds. Propositional MLDSs are a generalisation of normal propositional modal logic in that they facilitate reasoning about what is true at different points in a (possibly singleton) structure of actual worlds, called a *configuration*. Moreover, they can be extended to classes of Kripke frames which are not-modally definable. A model–theoretic semantics, based on first–order logic, is provided and its equivalence to Kripke semantics for the case of normal propositional modal logic is shown, whenever the initial configuration is a single point. A sound and complete natural deduction style proof system is also described, which shows how different "local" modal theories can interact with each other. Unlike traditional proof systems for modal logics, this system is uniform in the sense that every deduction rule is applicable to (the generalisation of) each normal modal logic, and to "new" modal logics obtainable by considering non modally definable classes of frames.

Part II extends the formalism developed in Part I to the predicate case. The resulting predicate MLDSs thus combine standard predicate modal logics with extensions of the labelling algebra given in Part I, and allow syntactic reference to (possibly different) elements existing in different actual worlds. The proof system of Part I is extended with natural deduction rules for quantifiers, and with *visa* rules for reasoning about relationships between objects in different accessible worlds. This allows semantic distinctions between different nestings of quantifiers and modal operators to be preserved within the deductive system. The first–order model–theoretic semantics described in Part I is also adopted here – different types of varying domain semantics are expressed in terms of many sorted first–order classical theories. Results are given showing that the predicate MLDSs are sound and complete with respect to the proposed semantics and that the latter is equivalent to varying domain Kripke semantics for a sub–class of systems which use only singleton initial configurations.

Ai miei genitori

To my parents

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Contents

Preface

The use of logics in applied research has, in the last few years, brought about an increasing demand for logical systems able to represent and to reason about *structures* of information (see [Gab92] for a discussion and examples). Gabbay has argued that neither classical nor non-classical logics are very intuitive when used in applications requiring structures of data more complex than just sets. To support this claim he gives the example of a temporal database. Here information is structured according to an underlying flow of time (e.g. the natural or real numbers). A classical formalisation would need to use a two-sorted logic in order to include the additional parameter of *time*, with which to express the time-dependency of data and the assumed time structure. Gabbay argues that this leads to an unintuitive representation and unnecessary redundancy in the language. On the other hand, temporal logics' characteristic of referring to a single time point, "now", with respect to which notions of "future" and "past" are evaluated, turns out to be restrictive in situations where information relative to specific instants of time, different from "now", needs to be explicitly represented. This characteristic is true in general for modal logics, whose theories are evaluated with respect to one single possible world, the *actual world*.

Gabbay [Gab94] has therefore proposed a new methodology called *Labelled Deductive Systems* (LDS), both for the theoretical study of logics and for the development of logical systems suitable for the needs of specific applications (a notion of an algebraic LDS is also given in [Gab92]). In the LDS approach, a basic unit of information is not just a formula but a *labelled formula*, where the label belongs to a given *labelling algebra*. Additional information, for example regarding the overall structure of the data, can be explicitly embodied, via the labels, in the object language of the system. Derivation rules act on both the labels and the formulae, according to certain fixed rules of propagation. The addition of this new separate "object-level" dimension (i.e. labels and labelling algebra) helps overcome the limitations mentioned above – first-order logic can be used directly without being extended to a two-sorted logic, and non-classical logic theories can be explicitly associated with different elements within a structure.

The general proposals in [Gab92, Gab94] have acted as a manifesto or starting point for a wide programme of research. For example, more detailed investigations have already been undertaken into the benefits of using the LDS methodology to reformulate substructural logics [DG94] and intuitionistic modal logics [Sim93]. This thesis substantiates Gabbay's intuitions by demonstrating how LDS techniques can be used to reformulate and generalise a large class of propositional and predicate modal logics.

In the *Modal Labelled Deductive Systems* (MLDS) described in this thesis, explicit reference can be made to (possibly singleton) "structures" of actual worlds, to which arbitrary modal theories can be *locally* associated. Deduction processes are then defined on the overall structure, allowing the inference of what is true at different points. This work contributes to the theoretical study of modal logics in the following ways. (1) It describes a uniform natural deduction system for propositional and predicate modal logics, including those whose accessibility relations satisfy conditions which are not, in van Benthem's terminology, "modally definable" [vB84]. With the exception of Hilbert systems, existing proof systems for modal logic are not uniform – the inference rules for modal operators change according to the particular modal logic. Moreover, and in contrast to the system described here, none of the existing proof systems can deal with *all* accessibility relations which are first–order axiomatisable [vB84]. (2) It extends the scope of propositional and predicate modal logic in order to deal with domains containing not just one, but a structure of one or more local or actual worlds. A proof is given that existing propositional and predicate modal systems are subsumed by the MLDSs described here. (3) In the case of predicate modal logics, it provides a uniform natural deduction system which is sound and complete with respect to arbitrary varying domain semantics. This is achieved without requiring any of the additional conditions on semantic structures which are imposed by existing first–order modal systems [HC68, Fit83]. (4) It provides an alternative approach to modal reasoning whose increased syntactic expressivity and deductive power contributes towards the long term aim of constructing logical formalisms more suitable to the needs of applications (in the sense described in [Gab92]).

Part I of the thesis describes a wide family of propositional MLDSs which includes generalisations of normal propositional modal logics built from the set of axioms $\{\mathbf{T}, \mathbf{D}, \mathbf{B}, \mathbf{4}, \mathbf{5}\}$. After a brief introduction to modal logic and various deduction systems, Chapter 1 gives the formal definition of a propositional MLDS. Here a labelling algebra is defined as a binary first-order theory which axiomatises the Kripke semantic notion of an accessibility relation. The basic building blocks of the MLDS language are *declarative units*, i.e. labelled formulae of the form $\lambda : \alpha$ interpreted as "the modal formula α holds at the possible world λ", and *R-literals* of the form $R(\lambda_i, \lambda_j)$ or $\neg R(\lambda_i, \lambda_j)$ where λ_i and λ_j are labels. They respectively facilitate the formalisation of two types of information, (i) what holds at particular possible worlds and (ii) which worlds are in relation with each other and which are not. This combined aspect of the syntax of a MLDS yields the definition of a theory, called a *configuration*, as a structure of arbitrary local modal theories. It is in this sense that a configuration can be seen as a generalisation of the traditional notion of a modal theory ([Fit83]). In Chapter 2, a uniform natural deduction system is described, in which inference rules are applied to configurations. It is shown that the same set of deduction rules can be applied to (the generalisation of) each well-known propositional normal modal logic. The difference between one modal logic and another is captured entirely by the labelling algebra.

The first-order nature of the labelling algebra implies that a MLDS can be constructed which is equivalent to any one of the modal logics whose axiomatisations define first-order relational conditions. But this thesis will show that the MLDS methodology provides even greater flexibility. Results from Correspondence Theory [vB84] show that some simple first-order relational conditions, such as $\forall x \neg R(x,x)$ (irreflexivity) and $\exists x \exists y R(x,y)$, are not modally definable, which means that they do not have a corresponding complete modal axiomatisation. In a MLDS, the explicit declaration of any first-order condition on the accessibility relation makes the need for a corresponding modal formula (axiom) unnecessary. Therefore modal reasoning involving any such type of classical structure can be treated. In Chapter 2, it will be shown, as an example, that the same modal labelled deductive proof system for "well behaved" normal modal logics can be applied in the case of an irreflexive accessibility relation. A model-theoretic semantics, based on a translation method into classical logic, is also provided. In Chapter 3, the natural deduction system is proved to be sound and complete, and it is shown that the semantics is equivalent to Kripke semantics whenever the initial configuration is a single point, and the underlying classical structure is modally definable.

In the propositional case, modal logics are uniquely and well defined – in particular, normal modal logics have been widely studied and discussed in the literature. In summary, Part I of this thesis further expands this horizon by proposing a generalisation of these systems and a new modal proof theoretical approach for certain types of classical mathematical structures which would otherwise be modally untreatable. Part II contributes instead to the theoretical study of quantified (predicate) modal logics.

The addition of quantifiers to modal logics is a complex and controversial process. One of the main questions is whether to assume that all the possible worlds contain the same objects, and if not, what the relation is between objects that belong to different possible worlds. Semantics which include the first assumption are known in the literature as *fixed domain semantics*. In contrast, *varying domain semantics* may describe increasing (or "nested"), decreasing or arbitrary domains for a possible world accessible from another. Defining a predicate modal logic in which a classical interpretation of quantifiers is adopted imposes the assumption of fixed domains. Garson [Gar84] has pointed out that, although fixed domains are satisfying from a formal point of view, the varying domain semantics is a more powerful semantic approach which turns out to be very useful in applications such as natural language. However, existing modal logics based on a varying domain semantics and on an interpretation of quantifiers which is world-relative have limitations. The relative predicate Hilbert modal systems [HC68] are sound and complete only with respect to increasing domains, and thus, in the case of a symmetric accessibility relation, only with respect to a fixed domain semantics. Moreover, systems based on other proof theoretical approaches, such as Fitting's prefixed tableaux, even though they allow the general case of arbitrary varying domains, require that every ground term of the "core" language must refer to an object which exists in every possible world. This is a significant restriction – only those objects which cannot be explicitly referred to (by a ground term) in the "core" language may not exist in some possible world. Garson has observed that the reason for such limitations lies in the fact that the semantics still validates the classical axioms for quantifiers, which are themselves incompatible with the world-relative interpretation of quantifiers. In Garson's words:

> ... There is little reason for attempt to preserve the classical rules in formulating systems with the objectual interpretation and world-relative domains ... The world-relative interpretation of quantifiers virtually demands the adoption of free logics. [Gar84]

Part II contributes to the theoretical investigation of quantified modal logics by providing a natural deduction system for predicate modal logics, which is sound and complete with respect to Kripke varying domain semantics [Kri63b], and which does not have any of the limitations described above. This is achieved by introducing rules for quantifiers which are somewhat similar to the derivation rules for quantifiers in free logic. In Chapter 4, predicate MLDSs are defined as quantified extensions of the propositional MLDSs given in Part I. The quantified MLDS language includes an additional entity, the *declarative term*, which expresses either that a term exists or that a term does not exist in a possible world. The natural deduction system described in Chapter 5 includes rules that can utilise this type of information. Four different classes of quantified MLDSs are described. The first one (the most basic and in a sense the most general) is obtained by adding introduction and elimination rules for quantifiers. It allows modal reasoning based on arbitrary varying domain semantics, and allows for the possibility of local empty domains. The second and the third classes facilitate reasoning on increasing and decreasing varying domain semantics. They extend the first class of MLDSs by including deduction rules which facilitate the "navigation" of terms between possible worlds accessible to each others. Using Gabbay's terminology [Gab92], these are called *increasing* and *decreasing visa rules* respectively. MLDSs for constant domains are instead obtained by adding a third visa rule, called the *constant visa rule*, to the first class of MLDSs. The model-theoretic semantics given in Chapter 5 is a straightforward predicate extension of the semantics given in Part I. The soundness and completeness of the quantified MLDSs with respect to this semantics is proved in Chapter 6. Finally, appropriate equivalence results are also proved between quantified MLDSs with singleton initial configurations and the Kripke semantics described in [Kri63b].

The explicit object-level declaration of the accessibility relation and of the existence of terms in possible worlds has facilitated the use of an interesting technique in the proofs of completeness (in both the propositional and predicate cases). These proofs are based on the standard Henkin methodology [HC68]. However, only the construction of one maximal consistent set is required instead of the usual sets of maximal consistent sets of canonical Kripke models. Moreover, unlike the usual completeness proof for quantified modal logics, in which more complex constructions are needed for a canonical model than the one needed for propositional modal logics, it is shown in Chapter 6 that the approach used in Part I for propositional MLDS is directly applicable to the proof of completeness of quantified MLDS. This is an attractive technical advantage of the labelled approach described here.

The thesis concludes with a summary and discussion, and some remarks about future work.

Notation and Terminology

Specific notation is introduced as and when necessary throughout the thesis. However, the reader might like to bear the following in mind. Two different languages will be used, namely a "labelling language" and a "logical language". In general, constant and predicate symbols will begin with an upper-case letter, whereas variables and function symbols will begin with a lower-case letter. Meta-variables of the form $\lambda_1, \lambda_2, \ldots$ and $A, B, C, \ldots$ will be adopted to refer respectively to terms and to wffs of the labelling language. Meta-variables of the form $t_1, t_2, \ldots$ and $\alpha, \beta, \gamma, \ldots$ will refer, respectively, to terms and wffs of the logical language. Larger entities such as structures, sets, theories and languages will often be symbolised in calligraphic font, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$

Part I

Propositional MLDS

Chapter 1

Introduction to Modal Logics and Labelled Deductive Systems

1.1 An Overview of Propositional Modal Logics

The history of modal logic can be summarised in terms of the development of three different traditions or schools of thought. The oldest, called the *syntactic* tradition, goes back to 1912, when a first intuitive understanding of logical necessity was proposed by Lewis, in the form of "intensional implication", as an alternative to classical implication (for a discussion and examples see [BS84, Lem77]). However, the first attempt to semantically define the notions of "necessity" and "possibility" only appeared in the next phase, called the *algebraic* tradition, when extensions of classical boolean algebraic structures were suggested in order to capture the meaning of modal operators in algebraic terms (see [JT51]). Despite the rigour and precision of its formalism, this approach did not gain widespread popularity, and for many years the investigation into modal logics was mainly focused on proof theoretic aspects. It was only around the 1960s that the so called *model theoretic* tradition of modal logic began with the introduction of relational semantics. Although similar ideas were proposed independently by Hintikka [Hin61], this semantics, also known as *possible worlds semantics*, is commonly associated with Saul Kripke. He defined [Kri59, Kri63a] a model theory based on "possible worlds" and explicit "accessibility relations" on possible worlds for a wide class of modal logics.

With the introduction of possible world semantics, new logics were developed as *non-classical extensions* of classical logic. These are defined by adding non-classical operators to the traditional syntax of classical logic – i.e. operators whose semantics does not reflect the truth-functional property of classical connectives. A wide variety of modal logics have been proposed using this methodology. A class of these, called *normal modal logics* stand out as being well-behaved and intuitively motivated, and have been much discussed in the literature (see [HC68] for a survey on modal logics). A brief description of normal modal logics, limited to the propositional case, is given below. (The predicate case is discussed in Part II.) This is to introduce some basic concepts, and to underline some of the limitations of these logics which are overcome by the new approach of propositional MLDS described in Part I of this thesis.

In the standard approach, known also as the *implicit* approach, propositional modal logic extends propositional logic with two modal operators, the $\Box$ and the $\Diamond$ operators. This approach is briefly described in this chapter from the model-theoretic prospective. The language can be defined as follows.

Definition 1.1 {Propositional modal language ($\mathcal{L}_M$) } A *propositional modal language* $\mathcal{L}_M$ is composed of:

- $p, q, r, p_1, q_1, r_1, \ldots$ propositional letters
- $\neg, \wedge, \vee \rightarrow$ logical connectives
- $\Box, \Diamond$ modal operators

$\Box$

The syntax is based on the standard notion of a well formed formula (wff) [HC68]. The set of wffs is the smallest set satisfying the following conditions: (i) propositional letters are wffs; (ii) if α is a wff and $\sharp$ is a unary operator (either classical or modal) then $\sharp\alpha$ is a wff; (iii) if α and β are wffs and $\sharp$ is a binary operator (only classical in this case) then $(\alpha\sharp\beta)$ is a wff.

The two operators $\Box$ and $\Diamond$ are often understood as expressing the notions "is necessary that" and "is possible that" respectively. These are interdefinable. Intuitively, to say that a proposition is possible, or possibly true, is to say that it is not necessarily false. Thus $\Diamond\alpha$ can be regarded as an abbreviation of $\neg\Box\neg\alpha$. Equally, to say that a proposition is necessary, or necessarily true, is to deny that its negation is possible. Thus $\Box\alpha$ can be regarded as an abbreviation of $\neg\Diamond\neg\alpha$. However, for reasons of clarity, in this thesis both operators will be considered as primitive operators of the language.

Necessity and possibility are qualifying (i.e. not functional) attributes of the notion of truth, and as such are often referred to as non–classical operators. In the case of classical connectives the truth value is a function of the truth values of the sentences to which the connectives are applied. For the modal operators, the definition of truth does not necessary satisfy this condition. For example, suppose that α is a contingent truth such as "Hume is a famous philosopher". The truth value of $\Box\alpha$ cannot be determined only on the basis of this truth because there might be other "circumstances" (or possible worlds) in which Hume is not or has not been a famous philosopher. The sentence "It is necessary that Hume is a famous philosopher" would therefore be false. On the other hand, it is obvious that sentences of the form "All famous philosophers are philosophers" necessarily hold. The interpretation of the modalities $\Box$ and $\Diamond$ depends then not only on the truth of the sentences they are applied to, but also on a "quantification" over entities called "possible worlds". These are elements of a mathematical structure, called a *frame*, usually defined as follows.

Definition 1.2 {Frame} A *frame* is a pair $\langle W, R\rangle$ where W is a non-empty set and R is a binary relation on W. Members of W are referred to as *possible worlds*, and for any given two worlds w,$w' \in W$, if wRw' it is said that w' is *accessible* from w. $\Box$

Possible worlds and accessibility relations have been interpreted in different ways in the literature, and are not restricted to logics with the $\Box$ and $\Diamond$ operators only. Different interpretations lead to different (possible worlds) semantics and therefore to different logics. For example, in the case of intuitionistic logic, possible worlds are seen as "pieces of information" which "prove" formulae, and the accessibility relation as the operator of set-inclusion. In the case of modal logics, possible worlds are simply considered to be elements of a structure to which the notion of truth is relative. This interpretation, which will also be adopted here, is formally expressed via the definition of a "*valuation* in a frame" given below.

Definition 1.3 {Valuation} Let $\langle W, R\rangle$ be a frame. A *valuation* in $\langle W, R\rangle$ is a function v mapping each propositional letter to a set of possible worlds (subset of W). That is, for every propositional letter p, $v(p) \subseteq W$ is the set of possible worlds where the proposition p holds. $\Box$

Intuitively, "$w \in v(p)$" says that p is true at the possible world w. Other equivalent definitions of a valuation are also in common use. For example, the function v is sometimes defined as mapping possible worlds and propositional letters to truth values [HC68], or alternatively as mapping possible worlds to sets of propositional letters, those holding at the possible world [Che80]. The above definition has been chosen here because it facilitates a proof of equivalence between the standard possible world semantics and the model-theoretic semantics for MLDS given in Chapter 2.

A frame and a valuation function together define a *model*.

Definition 1.4 {Model} A *model* is a triple $M = \langle W, R, v \rangle$, where $\langle W, R \rangle$ is a frame and v is a valuation in it. Moreover, w is said to be a possible world of M if $w \in W$. $\square$

Models and frames are also referred to as "Kripke models" (or "semantic structures") and "Kripke frames" respectively. In this thesis these terms will be used interchangebly. Moreover, models obtained by defining valuation functions in a given Kripke frame $\langle W, R \rangle$, are said to be models *based on* the frame $\langle W, R \rangle$. The notion of a model in modal logic is different from the notion of a model in classical logic. The latter (see [Sco81, Tar94] for introduction to classical logic) is strictly related to the notion of truth – a model is an interpretation which satisfies a given set of formulae – whereas a semantic structure does not necessarily imply the truth of any particular sentence. In fact, as pointed out before, in modal logic a sentence is true or false only with respect to a particular possible world. A wff α is said to be true at a possible world w of a model M, written $M, w \Vdash \alpha$, if one of the following satisfiability conditions holds.

Definition 1.5 {Satisfiability conditions} Let $\mathcal{L}_M$ be a propositional modal language, let $M = \langle W, R, v \rangle$ be a model, let w be a possible world of M and let α and β be two wffs of $\mathcal{L}_M$. The satisfiability relation $\Vdash$ is uniquely defined as follows:

1. $M, w \Vdash p$ iff $w \in v(p)$ (for a propositional letter p)
2. $M, w \Vdash \neg\alpha$ iff not $M, w \Vdash \alpha$
3. $M, w \Vdash \alpha \wedge \beta$ iff $M, w \Vdash \alpha$ and $M, w \Vdash \beta$
4. $M, w \Vdash \alpha \vee \beta$ iff $M, w \Vdash \alpha$ or $M, w \Vdash \beta$
5. $M, w \Vdash \alpha \rightarrow \beta$ iff $M, w \Vdash \beta$ or not $M, w \Vdash \alpha$
6. $M, w \Vdash \Box\alpha$ iff, for all $w' \in W$, if wRw' then $M, w' \Vdash \alpha$
7. $M, w \Vdash \Diamond\alpha$ iff there exists a $w' \in W$ such that wRw' and $M, w' \Vdash \alpha$

$\square$

Note that the definition of satisfiability for classical connectives (the first five clauses of Definition 1.5) reflects the definition of satisfiability used in classical logic. Therefore, at each possible world, the classical fragment (i.e. set of wffs whose principal connectives are classical) of modal logic behaves semantically as classical logic. From this point of view, a model of modal logic can be seen as a "structure" of classical models (one for each possible world) which are interdependent on each other according to the satisfiability of modal formulae. (This is one of the reasons why modal logic is considered to be an extension of classical logic.) The notion of validity in modal logic is also more general than the one given in classical logic. A model is by definition based on a given frame. Different valuation functions, given in the same frame $\langle W, R\rangle$, generate a *class* $\mathbf{C}$ of models associated with $\langle W, R\rangle$. Validity can be defined either with respect to a given model or with respect to a given class $\mathbf{C}$ of models.

Definition 1.6 {Validity} Let $M = \langle W, R, v\rangle$ be a model and let α be a modal wff. α is *valid* in the model M, written $M \models \alpha$, if for every possible world w of W, $M, w \Vdash \alpha$ (i.e. α is true at every possible world of the model). Moreover, α is *valid* in a non-empty class $\mathbf{C}$ of models generated by $\langle W, R\rangle$, written α is $\mathbf{C}$-valid, if for every model $M' \in \mathbf{C}$, $M' \models \alpha$ (i.e. α is valid in every model of $\mathbf{C}$). □

Note that the terminology adopted in the literature is not always uniform. Goldblatt [Gol87] uses the notion of validity only with respect to a class of models (for which there is a better association between valuations and interpretation functions of classical logics). The validity in a model, $M \models \alpha$, is instead called *truth* in a model. But this is just a matter of different terminology since the mathematical definition and the mathematical denotation are identical. Furthermore, in Fitting's terminology [Fit93], formulae which are true in all the possible worlds of a model are called *logical* or *global truths* whereas formulae which are true only in some possible worlds are called *factual* or *local* truths (with respect to those possible worlds). This distinction is important when dealing with modal assumptions, and will be used in Chapter 3 of this thesis.

In Definition 1.6, the class $\mathbf{C}$ of models is based on a given frame $\langle W, R\rangle$. A wider collection of models can be obtained by varying the underlining frame structure. In this case, the notion of validity is usually expressed with respect to the notion of a (non-empty) collection $\mathbf{F}$ of frames.

Definition 1.7 {$\mathbf{F}$-validity} Let $\mathbf{F}$ be a non-empty collection of frames and let α be a modal wff. α is *valid* in $\mathbf{F}$, written α is $\mathbf{F}$-valid, if for each $F \in \mathbf{F}$, α is $\mathbf{C}_F$-valid, where $\mathbf{C}_F$ is the collection of all models based on the frame F. □

The notion of $\mathbf{F}$-validity is used here to give the formal definition of a propositional modal logic that will be used throughout this thesis. Simply, a *propositional modal logic* is the set of all $\mathbf{F}$-valid wffs for some non-empty collection $\mathbf{F}$ of frames. Strictly speaking, this definition only covers *frame logics* (to use Fitting's terminology [Fit93]). But it nevertheless identifies a very broad class of system, including, as Fitting remarks, virtually all of the modal logics that have ever found any applications.

Other purely syntactic definitions of the notion of a modal logic, or of particular classes of modal logics, can be found in the literature. In particular, a *normal modal logic* is sometimes defined as follows.

Definition 1.8 {Normal modal logics} A *normal* propositional modal logic is a set $\mathbf{L}$ of wffs which, for any arbitrary wffs α and β, includes

- all classical tautologies
- all formulae of the form $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$
- the formula β whenever it includes formulae of the form α and $\alpha \rightarrow \beta$
- the formula of the form $\Box\alpha$ whenever it includes α
- the formula β whenever it includes a formula α, and β can be obtained by replacing the propositional letters in α uniformly by formulae.

$\Box$

It is easy to see from the definitions above that every frame logic is a normal modal logic. (See [Fit93] for the appropriate theorem and proof.) *Non-normal modal logics* are sometimes (syntactically) defined as logics which replace the schema $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$ in the definition above by a weaker condition. These logics can only be given a possible world type semantics by altering Definition 1.2 of a frame and Definition 1.5 of the satisfiability relation (in particular, conditions (6) and (7)). Such systems will not be considered at all in this thesis. Hence from now on, the term "propositional modal logic" should be understood as meaning "frame logic".

In subsequent discussion, a particular modal logic will sometimes be (unambiguously) identified by a collection of frames $\mathbf{F}$ for which it is the set of $\mathbf{F}$-valid formulae. Given such a set $\mathbf{F}$, it is possible to define a notion of semantic entailment, $\models_{\mathbf{F}}$, between "theories" written in (the modal logic associated with) $\mathbf{F}$ and formulae. In the most general case, modal theories can be defined as a pair $\langle S, U\rangle$, where S is a (possibly empty) set of *global assumptions*, and U is a (possibly empty) set of *local assumptions*. Intuitively, global assumptions are those sentences temporarily assumed to be true in every possible world, whereas local assumptions are sentences temporarily assumed to be true in the actual world. The definition of semantic entailment (given here using Fitting's notation [Fit83, Fit93]) takes this distinction into account as follows.

Definition 1.9 {Semantic entailment} Let $\mathbf{F}$ be a class of frames, let $\langle S, U\rangle$ be a theory and let α be a modal formula. α is *entailed* from $\langle S, U\rangle$, written $S \models_{\mathbf{F}} U \Rightarrow \alpha$, if for every frame $F \in \mathbf{F}$ and for every model M based on F in which all members of S are valid, α is true at every possible world $w \in M$ at which all members of U are true. $\Box$

Entailed formulae α are in general *local consequences* except when the given theory is of the form $\langle S, \{\}\rangle$ for which they are *global consequences*. In practice, the set S is usually empty, in which case a simpler and more conventional notation can be adopted.

Notation 1.1 Given a class of frames $\mathbf{F}$, a theory $\langle\{\}, U\rangle$ and a modal formula α, the expression $\{\} \models_{\mathbf{F}} U \Rightarrow \alpha$ will sometimes be written as $U \models_{\mathbf{F}} \alpha$. $\lhd$

Kripke frames can be considered to be essentially classically definable structures, so that certain classes $\mathbf{F}$ of frames can be identified as the set of models of particular (first or higher order) classical theories, written using the predicate constants R and $=$. In particular, various modal logics whose classes of frames have certain properties which can be defined in first-order classical logic (e.g. reflexivity, symmetry and transitivity) are very common in the literature. Results in *Correspondence Theory* [vB84] have shown a correspondence between such properties and particular modal formulae. For example, it has been shown that the largest collection of frames in which the modal formula $\Box p \to p$ is valid is exactly the class of frames whose relation R is reflexive, i.e. exactly the set of classical models of the first-order sentence $\forall x R(x, x)$. More generally, it has been proved [vB84] that there exists a set M_1 of modal formulae such that for each $\Phi \in M_1$ there exists a first-order sentence A (expressed in terms of the predicates R and $=$) such that for any frame F

$$\Phi \text{ is } \{F\}\text{-valid} \qquad \text{iff} \qquad F \Vdash A \tag{1.1}$$

The expression "$F \Vdash A$" in (1.1) should be read as "F is a classical model of A". Clause (1.1) implies that Φ is a modal formula valid in the class of frames which is exactly the class of classical models of the formula A. The modal formulae belonging to M_1 are called "characteristic" axioms. For further discussion and a proof of (1.1) see for example [vB84].

Table 1.1 gives some examples of axiom schemas whose instances Φ are included in M_1, together with their associated sentences A. Standard notation for the modal logics given by combining zero or more of the first-order conditions of Table 1.1 will be adopted in this thesis. Hence K ("K" for "Kripke") stands

Modal formulae		First-order conditions
T	$\Box\alpha \rightarrow \alpha$	$\forall x R(x, x)$
4	$\Box\alpha \rightarrow \Box\Box\alpha$	$\forall x, y, z(R(x, y) \wedge R(y, z)) \rightarrow R(x, z)$
B	$\alpha \rightarrow \Box\Diamond\alpha$	$\forall x, y(R(x, y) \rightarrow R(y, x))$
D	$\Box\alpha \rightarrow \Diamond\alpha$	$\forall x \exists y R(x, y)$
5	$\Diamond\alpha \rightarrow \Box\Diamond\alpha$	$\forall x, y, z(R(x, y) \wedge R(x, z)) \rightarrow R(y, z)$

Table 1.1: Modal formulae and associated first-order conditions

for the "smallest" modal logic (consisting of the set of $\mathbf{F}_K$-valid formulae, where $\mathbf{F}_K$ is the set of all frames). To give another example, $S5$ stands for the set of formulae valid in all frames which are reflexive, symmetric and transitive, i.e. which are characterised by the axiom schemas **T**, **4** and **B**. In addition, the following notation will sometimes be adopted for simplicity.

Notation 1.2 Let $\{\mathbf{T}, \mathbf{4}, \mathbf{D}, \mathbf{B}, \mathbf{5}\}$ be the set of modal axiom schemas given in Table 1.1 and let $\partial \subseteq \{\mathbf{T}, \mathbf{4}, \mathbf{D}, \mathbf{B}, \mathbf{5}\}$. $K\partial$ will denote the modal logic characterised by the set of axiom schemas in ∂. $\triangleleft$

For example, $K\{\mathbf{T}, \mathbf{4}, \mathbf{B}\}$ is an alternative notation for $S5$.

Modal logics can be defined whose associated classes of frames are based on first-order conditions other than those in Table 1.1. In addition, some modal logics are characterised by axiom schemas whose associated class of frames is identified by second-order conditions on the accessibility relation (e.g. Lob's axiom). Although such logics are not considered in detail in this thesis, a brief discussion on how to develop propositional (and predicate) MLDSs for these logics is given at the end of this chapter.

In the context of this thesis, an important point is the following. Correspondence Theory has shown [vB84] that it is only when the set of frames identified by a particular first-order sentence satisfies certain mathematical properties that there exists a corresponding characteristic modal formula. For example, the set of frames must be closed under p-morphism[1]. The set of irreflexive frames, i.e. classical models of the sentence $\forall x \neg R(x, x)$, is not closed under p-morphism[2]. Hence no axiomatic system or proof theory has been developed for the modal logic corresponding to the class of irreflexive frames. This thesis remedies this situation. A MLDS will be defined for irreflexive frames, thus illustrating how such a system can be defined for *any* arbitrary first-order axiomatisable class of frames.

1.1.1 Hilbert Systems for Modal Logics

In this section, a proof-theoretic presentation based on a Hilbert system is given for each modal logic $K\partial$, together with the standard syntactic definition of a consequence relation, and remarks as to how it relates to the semantic description of the previous section. These notions will be used in Chapter 3, where it will be shown that propositional MLDSs are syntactically more expressive then traditional modal logics by proving that each $K\partial$-MLDS subsumes the corresponding $K\partial$ axiomatic system.

Traditional axiomatisations of modal logics extend standard axiomatic systems for propositional classical logic with specific axiom schemas and inference rules for modal operators. As mentioned above, the "smallest" modal logic is the logic K for which $\partial = \{\}$. Axiomatisations of K and its extensions $K\partial$ are given here, all based on Hilbert's axiomatisation of classical propositional logic [Hil68]. Other axiomatisations of modal logics can be found in the literature, but the once described here have the advantage that most of their axiom schemas have a straightforward connection with natural deduction rules. This will facilitate a proof of the equivalence between these systems and their associated propositional MLDSs in Chapter 3.

Definition 1.10 {Axiomatic system for K} Let $\mathcal{L}_M$ be a propositional modal language and let α, β and γ be wffs of $\mathcal{L}_M$. The axiomatic system for the modal logic K in the language $\mathcal{L}_M$, written K_{Ax}, is defined as the following set of axiom

[1]Given two frames $F_1 = \langle W_1, R_1 \rangle$ and $F_2 = \langle W_2, R_2 \rangle$ a p-morphism from F_1 to F_2 is a function $f : W_1 \longrightarrow W_2$ which satisfies the conditions (i) sR_1t implies $f(s)R_2f(t)$ and (ii) $f(s)R_2u$ implies $\exists t$, $t \in W_1$ such that sR_1t and $f(t) = u$.

[2]Consider the frames $F_1 = \langle \{1\}, < \rangle$ and $F_2 = \langle \{0, 1\}, \{(0, 0)\} \rangle$. F_2 is a p-morphic image of F_1, in which the relational condition $\forall x \neg R(x, x)$ does not hold (i.e. F_2 does not belong to the class of irreflexive frames).

schemas and rules.

Axioms about $\rightarrow$

$\alpha \rightarrow (\beta \rightarrow \alpha)$ [A1]
$(\alpha \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow \beta)$ [A2]
$(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma))$ [A3]
$(\beta \rightarrow \gamma) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$ [A4]

Axioms about $\wedge$ and $\vee$

$\alpha \wedge \beta \rightarrow \alpha$ [A5]
$\alpha \wedge \beta \rightarrow \beta$ [A6]
$\alpha \rightarrow (\beta \rightarrow \alpha \wedge \beta)$ [A7]
$\alpha \rightarrow \alpha \vee \beta$ [A8]
$\beta \rightarrow \alpha \vee \beta$ [A9]
$((\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma)$ [A10]

Axioms about $\neg$

$(\alpha \rightarrow \beta \wedge \neg\beta) \rightarrow \neg\alpha$ [A11]
$\neg\neg\alpha \rightarrow \alpha$ [A12]

Axioms about $\Box$ and $\Diamond$

$\Box\alpha \rightarrow \neg\Diamond\neg\alpha$ [A13]
$\neg\Diamond\neg\alpha \rightarrow \Box\alpha$ [A14]
$\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$ [K]

Modus Ponens Rule

Conclude β from α and $\alpha \rightarrow \beta$ [MP]

Necessitation Rule

Conclude $\Box\alpha$ from α [Nec]

□

The above axiom schemas are global truths – any instantiation holds at each possible world of a given semantic structure. In particular, the set of all instantiations of the axiom schemas [A1]–[A12] together with the rule [MP] gives the first set of wffs described in Definition 1.8 (i.e. all classical tautologies), and the instantiations of [K] correspond to the second set of these formulae.

The distinction between local and global assumptions is also reflected within the syntactic definition of a consequence relation (*derivability relation*). It is important to notice that, in the definition of a derivation given below, the rule [Nec] can only be applied to global assumptions, whereas the rule [MP] can equally be applied to global or local assumptions. This is because, in order to guarantee the soundness of the [Nec] rule, a formula $\Box\alpha$ can only be inferred if α is true at every possible world (i.e. if α is a global assumption).

Definition 1.11 {Derivation} Let K_{Ax} be the axiomatic system for modal logic K, let $\langle S, U\rangle$ be a modal theory and let α be a modal formula. A K_{Ax} *derivation* of α from $\langle S, U\rangle$ is a sequence of formulae, ending with α, which is composed of two parts. The first (*global*) part consists of formulae which are either instantiations of the axiom schemas, or members of S, or are formulae inferred by previous terms by the application of the two rules [MP] and [Nec]. The second (*local*) part consists of formulae which are either instantiations of axiom schemas, or members of $U \cup S$, or are obtained from previous terms (in either part of the sequence) by the [MP] rule (not by the [Nec] rule). $\Box$

Definition 1.12 {Derivability of K_{Ax}} Let K_{Ax} be the axiomatic system for modal logic K, let $\langle S, U\rangle$ be a modal theory and let α be a modal formula. α is *derivable* from $\langle S, U\rangle$ in K_{Ax}, written $S \vdash_{K_{Ax}} U \Rightarrow \alpha$, if there exists a derivation of α from $\langle S, U\rangle$. $\Box$

Because of the soundness of the axiomatic system K_{Ax}, formulae derived using only global assumptions ($U = \{\}$) are valid in the class $\mathbf{C}$ of models identified by the set S and, because of the completeness, the closure of the derivability relation generates the whole set of $\mathbf{F}$-valid formulae of the modal logic K. On the other hand, if $U \neq \{\}$, the derived formulae are local truths – they hold at the possible worlds where the local assumptions hold.

Proofs of soundness and completeness theorems have been described in many places in the literature (see e.g. [HC68, Che80, Fit93]). In particular, given the Kripke semantics defined in the previous section and the above notions of derivability relation and semantic entailment, the following equivalence holds for the K_{Ax} axiomatic system, where α is an arbitrary formula and $\langle S, U\rangle$ is an arbitrary theory.

$$S \vdash_{K_{Ax}} U \Rightarrow \alpha \ \text{ iff } \ S \models_K U \Rightarrow \alpha \tag{1.2}$$

When S is the empty set, the above equivalence can be rewritten in a simplier way as follows:

$$U \vdash_{K_{Ax}} \alpha \ \text{ iff } \ U \models_K \alpha \tag{1.3}$$

Since this (implicit) approach to modal logic does not allow syntactic reference to possible worlds, the set U is often considered to hold at a particular possible world, called the *actual world* (w_a). Therefore any local assumption ϕ true at a certain world w for which there exist $w_1, w_2, \ldots, w_{n-1}$ possible worlds such that $w_a R w_1, w_1 R w_2, \ldots, w_{n-1} R w$ is represented in U by a modal formula of the form $\overbrace{\Diamond\Diamond\cdots\Diamond}^{n}\phi$. Moreover, some information about the underlying structure of possible worlds can be captured by appropriate local assumptions. For example, formulae of the form $(\Diamond\alpha)\wedge\neg\alpha$ state the existence of a second world different from the actual one. However, there are some types of information which cannot be represented within the implicit approach of modal logic. For example, assumptions like "there exists a second world different from the actual one where an identical set of formulae holds" cannot be captured by any initial local assumption. This is because the implicit approach does not allow explicit syntactic reference to possible worlds, so that the only way of formalising information like the existence of different possible worlds is by assuming local sets of formulae which are somehow contradictory with each other. This is a limitation of the expressive power of modal logics, which can be overcome by the MLDS approach described in this thesis.

Other modal logics (frame logics) well known in the literature are obtained by extending the above axiomatisation with the associated characteristic axiom schemas. For example, the axiomatic system for T is defined as follows.

Definition 1.13 {Axiomatic system for T} Let $\mathcal{L}_M$ be a propositional modal language and let K_{Ax} be the axiomatic system for the modal logic K in the language $\mathcal{L}_M$ as defined previously. Let α be a modal formula of $\mathcal{L}_M$. The axiomatic system for T in the language $\mathcal{L}_M$, written T_{Ax}, is $K_{Ax} \cup \{\mathbf{T}\}$, where $\mathbf{T}$ is the axiom schema $\Box\alpha \rightarrow \alpha$. $\Box$

Any instantiation of the axiom schema $\mathbf{T}$ is a characteristic formula of the class of frames whose accessibility relation satisfies the *reflexivity* condition, i.e. $\forall x R(x,x)$. The addition of this formula to the K_{Ax} system as a global assumption restricts the class $\mathbf{F}$ of frames to those which are reflexive.

Definition 1.13 is just an instance (in which $\partial = \{\mathbf{T}\}$) of the more general definition of the axiomatic system for $K\partial$ given below.

Definition 1.14 {Axiomatic system for $K\partial$} Let $\mathcal{L}_M$ be a propositional modal language, let $\partial \subseteq \{\mathbf{T}, \mathbf{4}, \mathbf{D}, \mathbf{B}, \mathbf{5}\}$ and let K_{Ax} be the axiomatic system for the modal logic K in the language $\mathcal{L}_M$ as defined previously. Let α be a modal formula of $\mathcal{L}_M$. The axiomatic system for $K\partial$ in the language $\mathcal{L}_M$, written $K\partial_{Ax}$, is $K_{Ax} \cup \partial$. □

The definition of derivability can be extended appropriately:

Definition 1.15 {Derivability of $K\partial_{Ax}$} Let $K\partial_{Ax}$ be the axiomatic system for modal logic $K\partial$, let $\langle S, U\rangle$ be a modal theory and let α be a formula. α is derivable from $\langle S, U\rangle$ in $K\partial_{Ax}$, written $S \vdash_{K\partial_{Ax}} U \Rightarrow \alpha$, if $\Sigma_\partial \cup S \vdash_{K_{Ax}} U \Rightarrow \alpha$, where Σ_∂ is the set of all instantiations of all the axiom schemas in ∂. □

Each of these systems is sound and complete (see for example [HC68]) in the following sense.

$$S \vdash_{K\partial_{Ax}} U \Rightarrow \alpha \quad \text{iff} \quad S \models_{K\partial} U \Rightarrow \alpha \tag{1.4}$$

Note that the axiomatic approach to modal logic described above is *uniform* in that the set of inference rules ([MP] and [Nec] rules) is common to each of the modal logics $K\partial$ – the difference between one logic and another is captured entirely by the set ∂ of characteristic axiom schemas.

MLDSs will be defined in Chapter 2 corresponding to each of the modal logics $K\partial$, and equivalences will be shown between restricted forms of these MLDSs and the above axiomatic systems. The proof of these equivalences uses a particular property of the semantic consequence relation of these modal logics, which can be stated as a *local deduction theorem.* In general, as stated in [Fit93], there are two types of deduction theorems, called respectively *global* and *local* deduction theorems, covering the two cases in which the appropriate property is stated for either global or local assumptions. Only the statement of the local deduction theorem is given here, since this will be used in Chapter 3. For proof and further details, the reader is referred to [Fit83, Fit93].

Theorem 1.1 {Local Deduction Theorem} Let $K\partial$ be a modal logic, let $\langle S, U\rangle$ be a modal theory and let α and β be two modal formulae. Then

$$S \models_{K\partial} U \cup \{\alpha\} \Rightarrow \beta \quad \text{iff} \quad S \models_{K\partial} U \Rightarrow \alpha \rightarrow \beta$$

In the simple case when S is the empty set the above property can be rewritten as

$$U \cup \{\alpha\} \models_{K\partial} \beta \quad \text{iff} \quad U \models_{K\partial} \alpha \rightarrow \beta \tag{1.5}$$

To conclude the description of axiomatic systems for frame logics $K\partial$, an example derivation is given below.

Example 1.1 Let $K\{\mathbf{B},\mathbf{4}\}_{Ax}$ be the axiomatic system for the modal logic $K\{\mathbf{B},\mathbf{4}\}$, let $\langle S, U\rangle$ be the modal theory given by $S = \{\}$ and $U = \{\Box\Box p \rightarrow r\}$ and let α be the modal formula $\Box(p \wedge q) \rightarrow \Box\Diamond r$. Then

$$U \vdash_{K\{\mathbf{B},\mathbf{4}\}_{Ax}} \alpha \tag{1.6}$$

By Definition 1.15, the statement (1.6) is equivalent to $\Sigma_{\{\mathbf{B},\mathbf{4}\}} \vdash_{K_{Ax}} U \Rightarrow \alpha$, where $\Sigma_{\{\mathbf{B},\mathbf{4}\}}$ is the set of all instantiations of the two axiom schemas **4**, $\Box\alpha \rightarrow \Box\Box\alpha$, and **B**, $\alpha \rightarrow \Box\Diamond\alpha$. Below is a shortened derivation of (1.6), (with the trivial steps omitted).

$$\begin{array}{rll}
\text{Global:}\left\{\begin{array}{l} \\ \\ \\ \\ \end{array}\right. & \begin{array}{l}(1)\\(2)\\(3)\\(4)\end{array} & \begin{array}{l}(p \wedge q) \rightarrow p\\ \Box((p \wedge q) \rightarrow p)\\ \Box(p \wedge q) \rightarrow \Box p\\ \Box(p \wedge q) \rightarrow \Box\Box p\end{array}\\
\text{Local:}\left\{\begin{array}{l} \\ \\ \end{array}\right. & \begin{array}{l}(5)\\(6)\end{array} & \begin{array}{l}\Box(p \wedge q) \rightarrow r\\ \Box(p \wedge q) \rightarrow \Box\Diamond r\end{array}
\end{array}$$

In the above derivation, (1) is an instantiation of the axiom schema [A5], (2) is obtained from (1) by the application of the [Nec] rule, and (3) is from (2) and the instantiation $\Box((p \wedge q) \rightarrow p) \rightarrow (\Box(p \wedge q) \rightarrow \Box p)$ of the axiom schema K by the application of [MP] rule. (4) is obtained from (3), the instantiation $\Box p \rightarrow \Box\Box p$ of the axiom schema **4**, and the axiom schema [A4] (rewritten in the equivalent form $((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma)$). Similar instantiations of [A4] are used to get (5) from (4) and the local assumption in U, and to get (6) from (5) and the instantiation $r \rightarrow \Box\Diamond r$ of the axiom schema **B**. •

1.1.2 Other Proof Theoretical Approaches

In the previous section an axiomatic system has been described for a family of modal logics, which is sound and complete with respect to Kripke possible world semantics. This approach has the advantage of being a uniform proof system in the sense that its inference rules are equally applicable to any of the modal logics taken into consideration. However, other non uniform proof theories have been developed for these logics, such as Natural Deduction [Pra65, Fit83] and Semantic Tableaux [Fit83, Mas94] systems. The latter are well known for allowing an easy search for proofs of particular formulae[3], whereas the former are a natural and

[3]This is basically due to the fact that semantic tableaux satisfy the *subformula property*, i.e. the only formulae needed to prove a given formula α are its subformulae.

intuitive way of formalising modal reasoning. Some remarks on existing natural deduction systems for modal logics are given here, as the proof theory for MLDS systems will also be given in this style.

Prawitz [Pra65] has described a natural deduction system for the modal logics $S4$ and $S5$. This extends Gentzen's natural deduction for classical logics [Gen34] with two inference rules, [$\Box$E] and [$\Diamond$I], and two "improper" (Gentzen's terminology) *deduction* rules, the [$\Box$I] and the [$\Diamond$E]. These two rules are improper in the sense that they have sub-derivations as their premises and additional conditions need to be specified for defining the set of *discharged* assumptions and the type of formulae that can be used within the sub-derivations. These additional conditions differ from logic to logic.

Fitting [Fit83] has developed a simpler way of defining natural deduction rules based on the technique of introducing "boxes" to delimit the sub-derivations used as premises in the Gentzen's improper rules. A full description of the natural deduction rules for classical connectives and for modal operators can be found in [Fit83]. Only those aspects and features of this approach to which reference will be made in Chapter 2 are given here. Using Fitting's box technique, the assumptions which appear at the first line within a box are assumptions that need to be discharged. For example, in the [$\rightarrow$I] rule given below the formula α is a new assumption added within the sub-derivation and discharged with the closure of the box.

$$\begin{array}{c} \boxed{\begin{array}{c}\alpha\\ \vdots\\ \beta\end{array}} \quad [\rightarrow\mathrm{I}] \\ \alpha \rightarrow \beta \end{array}$$

Moreover, formulae inferred within a box or a sub-derivation cannot be used outside the box. However, it is possible to repeat formulae outside (above) a box inside the box, so allowing the use of already inferred formulae or of more global assumptions. This is done by specifying an *iteration rule*, which is somewhat similar to the Gentzen's side conditions for the improper rules. The proof within a box is considered to be a *subordinate* derivation in the sense that it provides the condition for the overlying inference rule to be applied. The same principle is applied to the inference rules for modal operators, following the idea of Fitch's *strict subordinate derivations* [Fit52]. Fitting defines a particular type of box called a *strict box*. Inference rules for modal operators are given as rules for "creating" and "closing" boxes and for reiterating formulae within the box. Strict boxes can be thought of as deductions which occur within an (alternative) accessible world.

Therefore no dischargable assumptions are introduced. Two different types of strict boxes have been defined corresponding respectively to the interpretation of an *arbitrary* accessible world and of a *particular* accessible world. However, they have to be used separately, generating two different types of natural deduction systems for modal logics, called the A-style and the I-style respectively. In the A-style natural deduction system, a strict box can be opened at any point of a derivation to capture the idea of a [$\Box$I] rule. In fact, a rule called the *Strict Box Closure* allows the inference of formulae of the form $\Box\alpha$ whenever a box of this kind ends with a formula α. Moreover, a rule called the *Strict Iteration Rule* is also defined to specify the conditions under which formulae outside the box can be copied inside. For the basic modal logic K a formula α can be added inside a strict box if a formula of the form $\Box\alpha$ occurs outside it. This can be seen as a sort of [$\Box$E] rule. But different versions of the strict iteration rule are given for each type of modal logic. For example, in the case of $S4$ the iteration condition also includes the property of adding inside a strict box formulae of the form $\Box\alpha$ whenever they have already occurred outside the box. This is to capture the transitivity property of the accessibility relation. Moreover, for some modal logics additional *special* rules need to be given. Specifically,

$$\mathbf{B}, \mathbf{T}, \mathbf{S4} \textbf{ and } \mathbf{S5} \quad \frac{\Box\alpha}{\alpha} \qquad\qquad \mathbf{D}, \mathbf{DB} \textbf{ and } \mathbf{D4} \quad \frac{\Box\alpha}{\neg\Box\neg\alpha}$$

Therefore it is easy to see that this type of proof system, although reasonably intuitive, does not provide a uniform approach to the whole family of frame logics.

The following example is a reformulation of the derivation given in Example 1.1 within the A-style natural deduction system described above. In this system "$U \vdash_{A(K\{\mathbf{B},\mathbf{4}\})} \alpha$ " states that there exists an A-style natural deduction proof of the formula α, where the local assumptions in U are used in any point of the proof except in strict boxes. Strict boxes are here represented with a top double line (adopting Fitting's notation), and use is made of the strict iteration rule associated with $K\{\mathbf{B}, \mathbf{4}\}$. (No additional special rules are necessary for this class of logics.)

Example 1.2 Consider the modal logic $K\{\mathbf{B}, \mathbf{4}\}$. Let $\langle S, U\rangle$ be the modal theory given by $S = \{\}$ and $U = \{\Box\Box p \rightarrow r\}$ and let α be the modal formula $\Box(p \wedge q) \rightarrow \Box\Diamond r$. Then

$$U \vdash_{A(K\{\mathbf{B},\mathbf{4}\})} \alpha \tag{1.7}$$

An outline derivation of (1.7) is given below, with easy steps omitted and some equivalence rewriting of formulae included for simplicity. It may be helpful to bear in mind that the strict iteration rule associated with the logic under consideration is of the following form.

Strict Iteration Rule. If S is the set of formulae occurring above, but at the same level as a strict box[4], then any member of the set

$$S^{\sharp} = \{\alpha \mid \Box\alpha \in S\} \cup \{\Box\alpha \mid \Box\alpha \in S\} \cup \{\neg\Box\alpha \mid \neg\alpha \in S\}$$

can be added to the strict box.

(1)	$\Box(p \wedge q)$
(2)	$\Box(p \wedge q)$
(3)	$p \wedge q$
(4)	p
(5)	$\Box p$
(6)	$\Box\Box p$
(7)	$\Box\Box p \rightarrow r$
(8)	r
(9)	$\neg\Box\neg r$
(10)	$\Diamond r$
(11)	$\Box\Diamond r$
(12)	$\Box(p \wedge q) \rightarrow \Box\Diamond r$

In the above derivation, (1) is a new assumption introduced using the classical natural deduction rule [$\rightarrow$I]. (2) is obtained by applying the strict iteration rule to (1) in the first strict box, whereas (3) is obtained by applying the strict iteration rule to (2) in the (inner) second strict box. (4) is obtained from (3) using the classical rule [$\wedge$E] [Fit83], and (5) is obtained using the closure rule on the innermost strict box. (6) is obtained using the closure rule on the external strict box, (7) is the given assumption in U, (8) is obtained by applying the classical rule [$\rightarrow$E]. Next, another strict box is opened in which (9) is added by applying the strict iteration rule to (8). (10) is a simple equivalence rewriting of (9). The

[4] A formula α and a (strict) box are said to be *at the same level* if they occur inside the same box.

closure of this last strict box gives (11) and finally (12) is obtained by applying the rule [$\rightarrow$I] to (1) and (11). •

The I-style natural deduction system has the same disadvantage as the A-style system – it is not uniform. Here, strict boxes are introduced within a proof only after the occurrence of a formula of the form $\Diamond\alpha$ (or $\neg\Box\alpha$), acting therefore as sub-proofs in a *particular* accessible world. Different strict boxes need to be opened for different $\Diamond$-formulae, and a closure occurs whenever $\bot$ is proved in the box, reiterating it also outside. The same set of strict iteration rules and special rules as for the A-style system are used here, still varying according to the type of modal logic under consideration. To show the differences between this and the A-style method, an I-style natural deduction proof is given for the same example as above.

Example 1.3 Consider the modal logic $K\{\mathbf{B},\mathbf{4}\}$. Let $\langle S, U\rangle$ be the modal theory given by $S = \{\}$ and $U = \{\Box\Box p \rightarrow r\}$ and let α be the modal formula $\Box(p \wedge q) \rightarrow \Box\Diamond r$. Then

$$U \vdash_{I(K\{\mathbf{B},\mathbf{4}\})} \alpha \tag{1.8}$$

Again, an outline derivation of (1.8) is given in Figure 1.1, with easy steps omitted and some equivalence rewriting of formulae. In this derivation, (1) is a new assumption, introduced using the classical rule [$\rightarrow$I], (2) is an equivalence rewriting of the formulae assumed in U and (3) is the new assumption introduced using the classical rule [$\neg$I]. (4) is obtained by applying the rule of "creation" of a strict box to (3), whereas (5) and (6) are obtained by applying the strict iteration rule to (2) and (1) respectively. Next, the rule of creation of a strict box is applied again to (5), inferring the formula (7), whereas (8), (9) and (10) are obtained by applying the strict iteration rule twice to (6) and to (4) respectively. (11) is obtained using the classical rule [$\rightarrow$ E] on (7) and (10), allowing a new strict box to be opened in which (12) is added by the rule of creation and (13) is added by the rule of strict iteration. Because of the formula (12), another strict box is opened where (14) and (15) are added for reasons similar to those given for (12) and (13). Finally, the application of the classical rule [$\wedge$E] allows (16) to be inferred, which together with (14) gives the inconsistency (17). Once an inconsistency has been derived, the inner-most box can be closed reiterating the inconsistency outside the box (18). (19), (20) and (21) are also derived in this way. Hence (22) and (23) are obtained by applying [$\neg$I] to (3) and (21), and by applying [$\rightarrow$I] to (1) and (22). •

The proof of soundness and completeness of these two natural deduction systems with respect to Kripke semantics can be found in [Fit83].

In summary, no uniform natural deduction proof system exists for the family of logics of the form $K\partial$. This situation is remedied in Chapter 2 of this thesis, where a uniform natural deduction system is given for a class of propositional MLDSs which includes the logics $K\partial$ as special cases.

(1) $\Box(p \wedge q)$
(2) $\neg(\Box\Box p \wedge \neg r)$
(3) $\neg\Box\Diamond r$
(4) $\neg\Diamond r$
(5) $\neg\Box(\Box\Box p \wedge \neg r)$
(6) $\Box(p \wedge q)$
(7) $\neg(\Box\Box p \wedge \neg r)$
(8) $p \wedge q$
(9) $\Box(p \wedge q)$
(10) $\neg r$
(11) $\neg\Box\Box p$
(12) $\neg\Box p$
(13) $\Box(p \wedge q)$
(14) $\neg p$
(15) $p \wedge q$
(16) p
(17) $\perp$
(18) $\perp$
(19) $\perp$
(20) $\perp$
(21) $\perp$
(22) $\Box\Diamond r$
(23) $\Box(p \wedge q) \rightarrow \Box\Diamond r$

Figure 1.1: Derivation of Example 1.3

1.1.3 Explicit Formalisation of Modal Logics

The deductive systems described so far are all so called "implicit" systems. The modal language is as in Definition 1.1 and the syntax does not allow explicit reference to possible worlds, or to the accessibility relation R. However, the need of developing efficient theorem provers for various applications has recently motivated new approaches to modal logics based on translation methods into classical logic. Within these "explicit" approaches, a modal theory is translated into a first-order theory by replacing modal operators with first-order expressions that mirror their semantic meanings. This allows standard classical theorem provers to be applied directly to the translation theory to infer classical theorems which can then be re-translated back into modal formulae. Several translation methods have been developed which can be classified as *relational* [Moo80, Ohl91], *functional* [Ohl91, ARS94] and *semi-functional* [Non93] translations. These are in general applicable to any logic based on possible world semantics, whose connectives have semantic definitions that can be expressed in terms of classical logic.

In the case of relational translations, the modal logic's possible world structure is explicitly formalised by introducing in the classical language a distinguished binary predicate R, to represent the accessibility relation. The semantics of modal operators is reflected in the *translation rules.* For instance, the satisfiability condition of the $\Box$ operator

$$w \Vdash \Box p \quad \text{iff} \quad \forall v \, wRv \quad \text{implies} \quad v \Vdash p \tag{1.9}$$

is reflected in the translation rule

$$trans(\Box p, w) = \forall v R(w, v) \rightarrow trans(p, v)$$

where *trans* is a function which takes a modal formula and a possible world, and returns a classical formula. Local assumptions are considered to hold at the initial world 0, whereas global assumptions are appropriately universally quantified. So, to give an example, the modal axiom **D**,

$$\Box p \rightarrow \Diamond p$$

is translated into the following first-order sentence

$$\forall x[\forall y(R(x, y) \rightarrow P(y)) \rightarrow \exists y(R(x, y) \wedge P(y))]$$

where P is the unary predicate symbol associated with the propositional letter p of the modal language. This type of translation has the disadvantage of generating (exponentially) large formulae. Some modal formulae with n modal operators are translated into first-order formulae with at least 2^n literals.

An alternative, the functional translation method proposed by Ohlbach [Ohl91], adopts a different formalisation of the possible worlds structure. Information about accessible worlds is represented using a set of unary function symbols. These functions map each world to a world accessible in "one step". Chains of accessible worlds are then syntactically expressed via a concatenation operator on these functions. So for example, $f \circ g$ maps each world to a world accessible in two steps. Properties on the accessibility relation are captured by imposing conditions on the $\circ$ operator or by imposing particular properties on the function symbols. For instance, reflexivity is expressed by requiring the existence of an identity function, symmetry by requiring the existence of an inverse function, and transitivity by imposing that the set of functions is closed under the operation $\circ$. In the latter case, functions of the form $f \circ g$ will also be functions that map worlds to worlds accessible in one step. Informally, the axiom **4** for transitivity

$$\Box p \rightarrow \Box\Box p$$

is translated into

$$\forall x[\forall f P(f(x))) \rightarrow (\forall g, h P(g \circ h(x)))]$$

which can be classically proved by binding $g \circ h$ to f. This method, although having the advantage that the translated formulae have the same complexity (i.e. number of atomic subformulae) as the modal formula, requires suitable unification procedures for each type of modal logic.

Nonnengart [Non93] has suggested a hybrid approach called the *semi-translation* method. This combines the two techniques described above by translating the semantics of the $\Box$ operator relationally and the semantics of the $\Diamond$ operator functionally, and by introducing particular axioms, called *Simulator axioms*, which link the relational and functional representation of accessible worlds. Very roughly, these two axioms are

$$\forall s, t(R(s,t) \rightarrow \exists f(f(s) = t)) \qquad\qquad \forall s \forall f R(s, f(s))$$

Of course this approach requires that the accessibility relation satisfies the seriality property (i.e. for each possible world there exists an accessible one). This is also a limitation of the purely functional translation method. Some methods for removing this limitation have been suggested, (see [Ohl91] and [Non93] for a discussion), but these have the effect of reducing the efficiency of the systems, thus moving away from the main goal of the explicit approach to modal logics.

Among the three methods described above, it is clear that the relational one is the most natural, since the possible world semantics is already based on classical relational structures (frames). But unfortunately, it is also the most inefficient. This is mainly due to the fact that the approach uses (for implementation reasons) a "complete" translation of a given modal theory into a classical theory. A interesting question is whether a "partial" translation could be used in order to give the same syntactic expressive power of the relational translation but simpler translated theories. The MLDS framework described in this thesis provides an affirmative answer to this question. It shows that much simpler but equally expressive modal theories can be formalised by *combining* a first-order relational representation of the possible worlds structure with the standard (implicit) modal language. From the point of view of syntactic formalisation of modal logic, this thesis offers a new alternative approach which overcomes many of the limitations of the implicit and explicit approaches, whilst inheriting many of their advantages.

1.2 Formalising Modal Logics as Labelled Deductive Systems

In the previous section, a brief introduction and discussion of existing formalisms for modal logics has been given, highlighting some of their respective advantages and limitations. In summary, and focusing attention on syntactic expressivity, the following conclusions may be drawn. The implicit approaches, although they provide a concise language, cannot always capture within their syntax the desired underlying semantic structure of possible worlds (e.g. an irreflexive structure). This limitation is overcome by the explicit approach but with techniques which are either syntactically expensive and opaque, or which are not applicable to all modal (frame) logics.

A new approach to modal logic is described in this thesis, applicable to any frame logic, which generates a class of formalisms each of which is syntactically more expressive than its traditional implicit counterpart, and each of which describes its associated semantic structure in a natural way. These systems are called Modal Labelled Deductive Systems (MLDS). The approach is inspired by Gabbay's ongoing work in developing a very general unifying "LDS" framework, which will eventually be used to study the similarities and the relationships between many different styles of logics.

1.2.1 Introducing Labelled Deductive Systems

Gabbay's proposals [Gab94] for developing Labelled Deductive Systems are based largely on the observation that most logics differ from each other only in "small" ways, related to minor variations either in their proof theories or in their semantics. For example, within the family of substructural logics [Dôs93], differences between logics, like relevance and linear logics, lie mainly in the structural properties of the derivability relation – e.g. only permutativity for linear logic, permutativity and contractive properties for relevance logic. These are "metalevel" aspects of the proof theory, in the sense that they impose conditions on the application of inference rules like [MP]. But the underlying set of rules is basically identical (see [DG94] for a wide discussion). The same is true for modal logics. It has been shown in the previous section that their differences are dictated only by different properties of the accessibility relation. These also effectively impose extra conditions on the use of the basic [Nec] rule.

The LDS methodology facilitates a way of representing these types of variations explicitly inside the logic, to allow a basic unifying framework in which different logics can find a common formalisation. This is achieved by pairing logical information (i.e. wffs) with *labels* which "encode" information about the "metalevel" properties explicitly into the object language. Besides the logical theory, a *labelling algebra* is defined which represents the properties that differentiate one logic from another. Inference rules act on both the two syntactic components, logical formulae and labels, according to the desired properties of the connectives and of the labelling algebra. Recent results have already shown that this approach does indeed facilitate the development of a uniform proof system for a wide family of substructural logics [DG94], in which appropriate modifications of the labelling algebra imply different substructural logics, leaving the overall set of inference rules unchanged. Similar results are shown in this thesis for the family of modal logics. Here, the type of information captured in the labelling algebra is information about the semantic accessibility relation between possible worlds.

1.2.2 The Modal Labelled Deductive Language

As mentioned above, one of the limitations of the implicit formalisation of modal logics is the fact that assumptions (or modal theories in general) are evaluated at only one world, the *actual* world. MLDS generalises the notion of a single actual world. Formulae are associated with *labels* explicitly denoting possible worlds. Labels are terms of a "labelling language", and formulae are expressed in a standard propositional modal language $\mathcal{L}_M$ (see Definition 1.1). These two languages together define the basic *modal labelled deductive language.*

Definition 1.16 {Labelling language $\mathcal{L}_L$} A *labelling language* $\mathcal{L}_L$ is a first-order language composed of:

- $W_0, W_1, \ldots, W_n, \ldots$ a countable set of constant symbols
- $x, y, z, x_1, y_1, z_1, \ldots$ a countable set of variables
- R binary predicate symbol
- $\neg, \wedge, \vee, \rightarrow, \equiv$ logical connectives
- $\forall, \exists$ quantifiers

□

Constants and variables of $\mathcal{L}_L$ denote possible worlds and the binary predicate R formalises the accessibility relation between possible worlds. The above language allows Kripke semantic notions of frames and possible world structures to be expressed syntactically. Logical information is instead expressed in the propositional modal language $\mathcal{L}_M$. The combination of these two languages gives a modal labelled deductive language.

Definition 1.17 {Modal labelled deductive language (MLDL)} Given the labelling language $\mathcal{L}_L$ and the propositional modal language $\mathcal{L}_M$, a *propositional modal labelled deductive language* (propositional MLDL) is the ordered pair

$$\langle \mathcal{L}_L, \mathcal{L}_M \rangle$$

□

For proof-theoretic purposes, the labelling language is extended with additional sets of terms, which will be used only in derivations. Specifically, two sets of "skolem" function symbols are defined together with a function symbol *succ*. The resulting language is called a *semi–extended labelling language.*

Definition 1.18 {Semi-extended labelling language $Func(\mathcal{L}_L, \mathcal{L}_M)$ } Let $\mathcal{L}_L$ be a labelling language and $\mathcal{L}_M$ a propositional modal language. Let $\alpha_1, \ldots, \alpha_n, \ldots$ be the ordered set of all wffs of $\mathcal{L}_M$[5]. The first–order language $Func(\mathcal{L}_L, \mathcal{L}_M)$ is

[5]A canonical ordering is assumed. That such an ordering exists follows from the normal inductive definition of a wff in a modal language $\mathcal{L}_M$.

defined as the language $\mathcal{L}_L$ extended with the function symbol *succ* and the two sets of unary function symbols

$$\{f_{\alpha_1}, \ldots, f_{\alpha_n}, \ldots\} \qquad \text{and} \qquad \{box_{\alpha_1}, \ldots, box_{\alpha_n}, \ldots\}$$

□

The terms of the labelling language $\mathcal{L}_L$ constructed by using the function symbols f_{α_i} and box_{α_i} play particular roles. For each possible world λ and modal formula α, $f_\alpha(\lambda)$ names a *particular* world specifically associated with α. Such terms will be used whenever Kripke semantic notions of the form "there exists a possible world ..." need to be formalised. In contrast, terms of the form $box_\alpha(\lambda)$ can be though of as referring to any *arbitrary* world specifically associated with α. These terms will be used whenever Kripke semantic notions of the form "for all possible worlds ..." need to be expressed. The function *succ* is mainly used for modal logics which include the seriality property. It is a total function which maps each possible world into an "existing" one. However, formally speaking $f_\alpha(\lambda)$, $box_\alpha(\lambda)$ and $succ(\lambda)$ are just terms of $\mathcal{L}_L$ and within a particular model might even refer to the same possible world. The whole set of ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ defines the set of *labels*. Since they denote actual and accessible worlds, the expressions "labels" and "possible worlds" will be used interchangebly.

The MLDL facilitates the formalisation of two types of information, (i) what holds at particular possible worlds and (ii) which worlds are in relation with each other and which are not. Two different syntactic entities are defined to capture these two aspects of the language, called respectively *declarative units* and *R-literals*. A declarative unit is a pair separated by colon of the form "*label:modal formula*", expressing that a modal formula is true at a possible world. In a very general sense, the symbol ":" between the two components can be regarded as a sort of "*Holds*" predicate. (This interpretation will be reflected in the semantics.) The label component is a ground term of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$ and the modal component is a wff of the propositional modal language $\mathcal{L}_M$. This is the only syntactic entity which combines the two entities of the modal labelled deductive language, and it is formally defined as follows.

Definition 1.19 {Declarative unit} Given the language $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$, a *declarative unit* is a pair $\lambda : \alpha$ where λ is a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and α is a wff of $\mathcal{L}_M$. □

A declarative unit is said to be *atomic*, if the modal wff component is an atomic formula. In the propositional case, considered here, this means that the wff is just a propositional letter. Examples of atomic declarative units are $W_0 : p$ and $f_{\Diamond q}(W_3) : p$. Arbitrary declarative units include arbitrary modal wffs, e.g. $W_1 : \Diamond p$ and $f_q(W_1) : p \rightarrow r$.

An R-literal is any *ground* literal in the language $Func(\mathcal{L}_L, \mathcal{L}_M)$ of the form $R(\lambda_1, \lambda_2)$ or $\neg R(\lambda_1, \lambda_2)$, where λ_1 and λ_2 are labels, expressing that λ_2 is or is not accessible from λ_1. Examples of R–literals are $R(W_1, W_2)$, $\neg R(W_2, box_p(W_2))$ and $R(W_0, f_p(W_2))$. To distinguish an R-literal from its opposite in sign, the notion of the *conjugate* of an R-literal is also introduced.

Definition 1.20 {R–literal} Given the language $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$, an R–literal is a literal of the form $R(\lambda_1, \lambda_2)$ or $\neg R(\lambda_1, \lambda_2)$, where λ_1 and λ_2 are ground terms of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$. □

Definition 1.21 {Conjugate of an R-literal} Let Δ be an R-literal. The *conjugate* of Δ, written $\overline{\Delta}$, is defined as

- $\neg R(\lambda_1, \lambda_2)$ if $\Delta = R(\lambda_1, \lambda_2)$
- $R(\lambda_1, \lambda_2)$ if $\Delta = \neg R(\lambda_1, \lambda_2)$

□

1.2.3 Modal Labelled Deductive Theories

The syntax of MLDS allows arbitrary sets of modal formulae to be associated with (different) labels, describing not only one initial set of local assumptions (as in the implicit approach) but allowing for several (distinct) local initial modal theories to be specified. With the addition of R-literals, these local theories can be stated to be either independent (using negative R-literals) or interacting with each other (using positive R-literals). This yields a definition of a modal labelled deductive theory more general than the traditional notion of a modal theory given in Section 1.1, This contributes towards the long term aim of providing a modal formalism closer to the needs of applications (see discussions in the Preface and Conclusions). A modal labelled deductive theory, called a *configuration*, is composed of two sets of information, (i) a set of R-literals and (ii) a set of declarative units. Sets of declarative units having the same labels denote local modal theories associated with that label, whereas declarative units with different labels express modal formulae belonging to (possibly) different local actual worlds. The first set (i), consisting of R-literals, is called a *diagram* and it provides the "structural" information about a modal labelled deductive theory.

Definition 1.22 {Diagram} Given the language $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$, a *diagram* $\mathcal{D}$ is a set of R-literals, whose arguments are ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$. $\square$

For example, the set $\{R(W_0, W_1), R(W_0, W_2), \neg R(W_1, W_2)\}$ is a diagram which can be partially[6] represented graphically as in Figure 1.2. Notice that this diagram does not tell us, for example, whether W_0 is accessible from W_1 or not. In general, complete or incomplete information about any arbitrary directed graph can be formalised as a diagram. A local modal theory can be assigned to each node

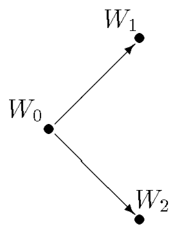

Figure 1.2: A Diagram

of the diagram by adding appropriate declarative units. For example, consider the set $\{W_0 : \square(p \rightarrow q), W_0 : \square r, W_1 : \Diamond p, W_1 : r \rightarrow s, W_2 : q, W_2 : \Diamond p\}$ together with the diagram given in Figure 1.2. The resulting theory (configuration) can be graphically represented as shown in Figure 1.3. A formal definition of a configuration is given below.

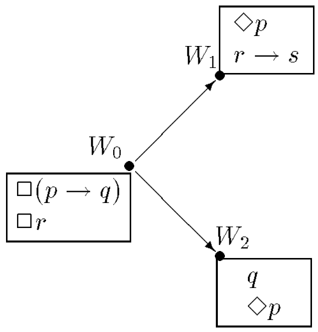

Figure 1.3: A Configuration

[6]Such illustrations do not include representations of R-literals of the form $\neg R(\lambda_i, \lambda_j)$.

Definition 1.23 {Configuration} Given a MLDL, a *configuration* is a tuple

$$\langle \mathcal{D}, \mathcal{F} \rangle$$

where:
- $\mathcal{D}$ is a diagram
- $\mathcal{F}$ is a function from the set of ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ to the set $\mathcal{P}(\mathrm{wff}(\mathcal{L}_M))$ of sets of wffs of $\mathcal{L}_M$

$\square$

The function $\mathcal{F}$ is a total function which assigns an empty set to the labels for which there is not information and a non-empty local modal theory to the rest. The formal description of the configuration in Figure 1.3 is given in Table 1.2. Configurations can be infinite theories. This occurs either when the diagram

$$\begin{array}{rcll}
\mathcal{D} & = & \{R(W_0, W_1), R(W_0, W_2), \neg R(W_1, W_2)\} & \\
\\
\mathcal{F}(\lambda) & = & \{\Box(p \rightarrow q), \Box r\} & \text{if } \lambda = W_0 \\
 & & \{\Diamond p, r \rightarrow s\} & \text{if } \lambda = W_1 \\
 & & \{q, \Diamond p\} & \text{if } \lambda = W_2 \\
 & & \{\} & \text{otherwise}
\end{array}$$

Table 1.2: Formal definition of configuration in Figure 1.3

$\mathcal{D}$ is infinite (i.e. the configuration contains an infinite number of R-literals), or when for some label λ of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$, the value of $\mathcal{F}(\lambda)$ is an infinite set of formulae of $\mathcal{L}_M$ (i.e. the configuration contains an infinite number of declarative units referring to λ), or when $\mathcal{F}(\lambda) \neq \{\}$ for an infinite number of terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$. To express what information a configuration contains, the following notation will be useful.

Notation 1.3 Given a configuration $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$, the R-literal Δ is said to be *member of* C, written $\Delta \in \mathcal{C}$, if $\Delta \in \mathcal{D}$, whereas a declarative unit $\lambda : \alpha$ is said to be *member of* C, written $\lambda : \alpha \in \mathcal{C}$, if $\alpha \in \mathcal{F}(\lambda)$. $\triangleleft$

As mentioned before, function symbols of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$ have been introduced for proof-theoretical reasons. Therefore any user specified modal labelled deductive theory (*initial* configuration) will usually (initially) contain only constant symbols of $Func(\mathcal{L}_L, \mathcal{L}_M)$ as labels, whereas configurations containing general ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ will mainly appear in proof-derivations as inferred configurations. This will become clearer in the next chapter where the description of a natural deduction proof system for a MLDS is given.

So far, notions of a modal labelled deductive language, syntax and theory have been defined as "basic" concepts of a MLDS, in the sense that they are common to all of the propositional MLDSs described in this part of the thesis. However, there is another essential component which needs to be defined. The notion of a class of Kripke frames is captured by a first-order axiomatization, called a *labelling algebra*, written in the language $Func(\mathcal{L}_L, \mathcal{L}_M)$.

Definition 1.24 {Labelling algebra $\mathcal{A}$} A *labelling algebra* $\mathcal{A}$ is a (possibly empty) first-order theory, written in the semi-extended labelling language $Func(\mathcal{L}_L, \mathcal{L}_M)$, given by any consistent[7] subset of the following axiom set:

$$\forall x(R(x,x)) \qquad (\mathbf{T})$$

$$\forall x,y,z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z)) \qquad (\mathbf{4})$$

$$\forall x,y(R(x,y) \rightarrow R(y,x)) \qquad (\mathbf{B})$$

$$\forall x(R(x,succ(x))) \qquad (\mathbf{D})$$

$$\forall x,y,z((R(x,y) \wedge R(x,z)) \rightarrow R(y,z)) \qquad (\mathbf{5})$$

$$\forall x \neg R(x,x) \qquad (\mathbf{Irr})$$

□

Labelling algebrae involving only the first five axioms define classes of frames which are modally definable, i.e. for which there is a characteristic axiom schema. Each of these identifies a particular propositional MLDS corresponding[8] to the related standard modal logic. In contrast, labelling algebrae which include the axiom (**Irr**), identify the class of frames whose accessibility relation satisfies the irreflexivity property. As pointed out before, this property is not modally definable. There is no modal axiom schema which can characterise this class of frames, but by introducing the (**Irr**) axiom into the labelling algebra, it is

[7]In the standard sense of classical logic (so that it cannot include both (**T**) and (**Irr**)).

[8]A particular notion of "correspondence" is formally defined and proved to hold in Chapter 3.

possible to define propositional MLDSs which facilitate modal reasoning over irreflexive classes of frames. These systems thus generate a new set of modal logics. The following notation will be adopted to distinguish between different labelling algebrae. Note that in Definition 1.24 the axioms which are modally definable are denoted with the same names as their corresponding modal axiom schemas, but are written in parentheses ().

Notation 1.4 Let $\{(\mathbf{T}), (\mathbf{4}), (\mathbf{D}), (\mathbf{B}), (\mathbf{5}), (\mathbf{Irr})\}$ be the set of axioms given in Definition 1.24 and let $\delta \subseteq \{(\mathbf{T}), (\mathbf{4}), (\mathbf{D}), (\mathbf{B}), (\mathbf{5}), (\mathbf{Irr})\}$. δ will sometimes be written as $\mathcal{A}_{K\delta}$, with braces and parentheses omitted from the subscripted δ. $\triangleleft$

However, for simplicity the subscript $K\delta$ will be replaced with the traditional name of the associated modal logic whenever possible. For example, $\mathcal{A}_{K\{\mathbf{T},\mathbf{4}\}}$ will sometimes be written $\mathcal{A}_{\mathbf{S4}}$. Table 1.3 shows some examples of the labelling algebrae considered in this thesis.

Names	Labelling Algebrae
$\mathcal{A}_{\mathbf{T}}$	$\{\forall x R(x,x)\}$
$\mathcal{A}_{\mathbf{Irr}}$	$\{\forall x \neg R(x,x)\}$
$\mathcal{A}_{\mathbf{S4}}$	$\{\forall x R(x,x),$ $\forall x,y,z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))\}$
$\mathcal{A}_{K\mathbf{4},\mathbf{Irr}}$	$\{\forall x \neg R(x,x),$ $\forall x,y,z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))\}$
$\mathcal{A}_{K\mathbf{4}}$	$\{\forall x,y,z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))\}$
$\mathcal{A}_{K\mathbf{D}}$	$\forall x R(x, succ(x))\}$
$\mathcal{A}_{K\mathbf{D},\mathbf{Irr}}$	$\{\forall x \neg R(x,x),$ $\forall x R(x, succ(x))\}$

Table 1.3: Examples of Labelling Algebrae

To give a full definition of a propositional MLDS it is necessary to specify a set of inference rules together with the language and the particular labelling algebra. This is formally stated as follows.

Definition 1.25 {A modal labelled deductive system} Given a propositional MLDL=$\langle \mathcal{L}_L, \mathcal{L}_M \rangle$, a propositional *modal labelled deductive system* (propositional MLDS) is a tuple of the form

$$\langle \langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}, \mathcal{R} \rangle$$

where:
- $\mathcal{A}$ is a labelling algebra written in $Func(\mathcal{L}_L, \mathcal{L}_M)$
- $\mathcal{R}$ is a set of inference rules which 'generate' one configuration from another

□

From the above definition it is clear that, given a set of rules, different propositional MLDSs can be obtained by considering different labelling algebrae. The following notation will be adopted to indicate each particular propositional MLDS considered in the subsequent chapters.

Notation 1.5 Let $\mathcal{A}_{K\delta}$ be a particular labelling algebra. $K\delta$-MLDS will denote the associated propositional MLDS, i.e. $K\delta$-MLDS=$\langle \langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}_{K\delta}, \mathcal{R} \rangle$. ◁

For each $K\delta$-MLDS, the inference rules are defined as relations over the set of $K\delta$ configurations. The form which such relations take is described in the next chapter.

1.3 Discussion

Propositional MLDSs are clearly hybrid systems of modal logic, combining first-order relational theories (labelling algebrae) with a standard modal language. From a purely syntactic point of view, this approach can be seen as a compromise between the implicit and explicit approaches described previously. Syntactic representations of possible worlds and accessibility relations are incorporated without requiring a full translation into first-order logic, so allowing logical information to be expressed within a concise modal syntax. As a result, MLDSs' declarative expressivity is comparable to the expressivity offered by the relational translation technique, with the additional advantage of not resulting in sentences which are exponentially long compared to their implicit modal counterparts.

When a MLDS configuration has only a single label W_0, it is clearly analogous to the traditional notion of a set of local assumptions. In this case, it is also close to Fitting's system of *prefixed* tableaux [Fit83]. In Fitting's system formulae are *prefixed* with terms (called *prefixes*) which name possible worlds. So terms of the form $\sigma\alpha$ express that the formula α is true at the possible world σ. Prefixed formulae are therefore analogous to the MLDS notion of declarative units, and prefixes to the notion of labels[9].

However, there is a fundamental difference between Fitting's propositional prefixed tableaux and propositional MLDSs. In the former, the semantic notion of the accessibility relation is embedded in the prefix "machinery". Prefixes are in general defined as sequences of characters. Particular conditions are introduced to identify which prefixes are "accessible from" which. These conditions are purely syntactic and they depend on the particular type of modal logics being represented. For instance, in the simplest case of the modal logic K, one prefix is defined to be *accessible from* another if the latter (sequence of characters) is included in the former (sequence of characters) – e.g. the prefix $[1, 1]$ is accessible from the prefix $[1]$. The proof theory defined for this "semi-implicit" way of representing Kripke frames is not uniform. In the MLDS approach, both possible worlds and the accessibility relation are represented declaratively as part of the logic.

Semantically, there is no reason why modal logics whose characteristic axiom schemas correspond to higher order relational conditions (e.g. Lob's logic whose characteristic axiom schema $\mathbf{G}$ is $\Box(\Box\alpha \rightarrow \alpha) \rightarrow \Box\alpha)$ should not be represented within an MLDS whose labelling algebra is written in an appropriate higher-order classical logic.

Finally, the introduction of labels and R–literals generalises the notion of a modal theory (as a set of local assumptions), since it effectively allows the existence of more than one actual world. Moreover, global assumptions can be incorporated simply by including them in $\mathcal{F}(\lambda)$ for every label λ of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$.

1.4 Summary

In this chapter, several standard formalisms for modal logic have been described. These fall into two categories, implicit and explicit. A brief introduction to the Kripke possible worlds semantics has been given together with the definition of

[9]The use of prefixes (or labels) can actually be traced back to [Fit66].

semantic entailment for the family of frame logics. Sound and complete Hilbert systems for the class of standard propositional modal logics have been described, together with the notion of derivability. Finally the basic notion and syntax of a propositional MLDS have been introduced.

Chapter 2

Derivability and Semantic Entailment

In the previous chapter, the basic syntax of a propositional MLDS has been given. To complete the description of a propositional MLDS as a logical framework the syntactic and semantic notions of a consequence relation have to be defined. This is the topic of this chapter. In the following section a derivability relation $\vdash_{\mathrm{MLDS}}$ is given which is applicable to any $K\delta$-MLDS, together with the (uniform) set R of inference rules. In the second part of this chapter (Section 2.2) a model-theoretic semantics for propositional MLDSs is described, together with the notion of semantic entailment $\models_{\mathrm{MLDS}}$.

2.1 A Natural Deduction System

Propositional MLDSs are generalisations of implicit formalisms of modal logic – their syntax facilitates the representation of (possibly singleton) structures of local actual modal theories. This main characteristic is also preserved within the MLDS proof theory in the sense that inference rules and the derivability relation are defined *between configurations*. This differs from the (implicit and explicit) systems described previously, for which the syntactic consequence relation (e.g. see Definition 1.12 in Section 1.1.1) and each inference rule (see e.g. [MP] rule) are generally defined between a theory (or a set of formulae) and a formula. In a propositional MLDS, the "structural" aspect of a theory has stimulated the idea of defining deductive processes that describe how configurations can "evolve" by reasoning within or between local modal theories or by reasoning about diagrams. An inference rule of a propositional MLDS can be generally defined[1] as follows.

[1] This definition is also applicable to the family of predicate MLDSs described in Part II of this thesis.

Definition 2.1 {Inference Rule} An *inference rule* $\mathcal{I}$ is a set of pairs of configurations, where each such pair is written as $\mathcal{C}/\mathcal{C}'$. If $\mathcal{C}/\mathcal{C}' \in \mathcal{I}$ then $\mathcal{C}$ is said to be an *antecedent configuration* of $\mathcal{I}$, and $\mathcal{C}'$ an *inferred* (or *consequence*) configuration of $\mathcal{I}$ with respect to $\mathcal{C}$. It is also said that $\mathcal{I}$ *generates* $\mathcal{C}'$ from $\mathcal{C}$, or that $\mathcal{I}$ *infers* $\mathcal{C}'$ from $\mathcal{C}$. □

Note that this definition holds also for inference rules that require sub-derivations as their premises. Therefore Prawitch's distinction between inference rules and improper deduction rules, as well as Fitting technique of using both boxes and strict boxes, are not needed. This allows a simpler formalisation of a natural deduction style proof system in which no rules are necessary apart from those required by the logical and structural information of a configuration.

Remark 2.1 For the remainder of Part I of this thesis, a MLDL=$\langle \mathcal{L}_L, \mathcal{L}_M \rangle$ is assumed. $K\delta$-MLDS will refer to any MLDS=$\langle \langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}_{K\delta}, \mathcal{R} \rangle$ given by any labelling algebra $\mathcal{A}_{K\delta} \subseteq \{(\mathbf{T}), (\mathbf{4}), (\mathbf{D}), (\mathbf{B}), (\mathbf{5}), (\mathbf{Irr})\}$, unless otherwise specified. All terms referred to are ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and all wffs referred to are wffs of $\mathcal{L}_M$.

The definition of a natural deduction *derivation* (or *proof*) for a $K\delta$-MLDS is as follows.

Definition 2.2 {Proof} Given a $K\delta$-MLDS, a *proof* is a pair $\langle \mathcal{P}, m \rangle$, where $\mathcal{P}$ is a sequence of configurations $\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}$ (of $K\delta$-MLDS), with $n > 0$, and m is a mapping from the set $\{0, \ldots, n-1\}$ to $\mathcal{R}$ such that for each i, $0 \leq i < n$, $\mathcal{C}_i/\mathcal{C}_{i+1} \in m(i)$. □

Definition 2.3 {Derivability} Let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations. $\mathcal{C}'$ is *derivable* from $\mathcal{C}$ in $K\delta$-MLDS, written $\mathcal{C} \vdash_{K\delta-\mathrm{MLDS}} \mathcal{C}'$, if there exists a proof $\langle \{\mathcal{C}, \ldots, \mathcal{C}'\}, m \rangle$. □

Where the choice of $K\delta$ is arbitrary, the derivability symbol $\vdash_{K\delta-\mathrm{MLDS}}$ will often be written $\vdash_{\mathrm{MLDS}}$. It is easy to see that Definition 2.3 is most similar to the standard notion of derivation (given in Definitions 1.12 and 1.15) when the *derivable* configuration $\mathcal{C}'$ is reduced to a single declarative unit with label W_0 and the given (initial) configuration $\mathcal{C}$ (only) includes global assumptions in $\mathcal{F}(\lambda)$, for every label λ of the language, and local assumptions in $\mathcal{F}(W_0)$. A derivability relation between configurations and single units of information (declarative units or R-literals) can be expressed in terms of the symbol $\vdash_{\mathrm{MLDS}}$ by adopting the following notation.

Notation 2.1 Given a configuration $\mathcal{C}$, a declarative unit $\lambda : \alpha$ and an R-literal Δ,

$$\mathcal{C} \vdash_{\text{MLDS}} \lambda : \alpha$$

if there exists a configuration $\mathcal{C}'$ such that $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$ and $\lambda : \alpha \in \mathcal{C}'$. Similarly

$$\mathcal{C} \vdash_{\text{MLDS}} \Delta$$

if there exists a configuration $\mathcal{C}'$ such that $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$ and $\Delta \in \mathcal{C}'$. Moreover

$$\mathcal{C} \vdash_{\text{MLDS}} \lambda : \perp$$

if there exists a term λ and a wff α such that $\mathcal{C} \vdash_{\text{MLDS}} \lambda : \alpha \wedge \neg\alpha$. $\triangleleft$

Configurations are pairs of sets. Therefore standard operations on sets, such as union, difference and inclusion, can also be applied to configurations. The following series of definitions and remarks about notation specify the particular operations that will be used throughout this thesis.

Notation 2.2 Given a configuration $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$, a declarative unit $\lambda : \alpha$ and an R-literal Δ, then

1. $\mathcal{C} + [\lambda : \alpha]$ is the configuration $\langle \mathcal{D}, \mathcal{F}' \rangle$, such that
   - $\mathcal{F}'(\lambda) = \mathcal{F}(\lambda) \cup \{\alpha\}$
   - $\mathcal{F}'(\lambda') = \mathcal{F}(\lambda')$ for each ground term $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$
2. $\mathcal{C} + [\Delta]$ is the configuration $\langle \mathcal{D}', \mathcal{F} \rangle$, such that
   - $\mathcal{D}' = \mathcal{D} \cup \{\Delta\}$

$\triangleleft$

A similar notation is adopted in the case of "subtraction" of declarative units and R–literals from configurations.

Notation 2.3 Given a configuration $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$, a declarative unit $\lambda : \alpha$ and an R–literal Δ, then

1. $\mathcal{C} - [\lambda : \alpha]$ is the configuration $\langle \mathcal{D}, \mathcal{F}' \rangle$, such that
   - $\mathcal{F}'(\lambda) = \mathcal{F}(\lambda) - \{\alpha\}$[2]
   - $\mathcal{F}'(\lambda') = \mathcal{F}(\lambda')$ for each ground term $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$
2. $\mathcal{C} - [\Delta]$ is the configuration $\langle \mathcal{D}', \mathcal{F} \rangle$, such that
   - $\mathcal{D}' = \mathcal{D} - \{\Delta\}$

$\triangleleft$

The above notation can be appropriately extended to cover the cases of addition and subtraction of finite (non singleton) sets from configurations. This will be shown later on in the thesis. Operations of union, difference and containment between configurations are given below.

Definition 2.4 {Configuration union}[3] Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$ and $\mathcal{C}' = \langle \mathcal{D}', \mathcal{F}' \rangle$, be two configurations. The *configuration union* $\mathcal{C} \cup \mathcal{C}'$ is defined as the configuration $\langle \mathcal{D}_u, \mathcal{F}_u \rangle$ where

- $\mathcal{D}_u = \mathcal{D} \cup \mathcal{D}'$
- $\mathcal{F}_u(\lambda) = \mathcal{F}(\lambda) \cup \mathcal{F}'(\lambda)$ for each ground term λ in $Func(\mathcal{L}_L, \mathcal{L}_M)$

$\square$

Definition 2.5 {Configuration difference} Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$ and $\mathcal{C}' = \langle \mathcal{D}', \mathcal{F}' \rangle$ be two configurations. The configuration difference $\mathcal{C}' - \mathcal{C}$ is defined as the configuration $\langle \mathcal{D}_{diff}, \mathcal{F}_{diff} \rangle$ where

- $\mathcal{D}_{diff} = \mathcal{D}' - \mathcal{D}$
- $\mathcal{F}_{diff}(\lambda) = \mathcal{F}'(\lambda) - \mathcal{F}(\lambda)$ for each ground term $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$

$\square$

[2] The standard set difference $A - B$ for sets A and B is assumed. $e \in A - B$ iff$_{\text{def}}$ $e \in A$ and $e \notin B$.

[3] A union of configurations, in which one is finite, can be also expressed via "+" (Notation 2.2).

Definition 2.6 {Configuration containment} Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$ and $\mathcal{C}' = \langle \mathcal{D}', \mathcal{F}' \rangle$ be two configurations. $\mathcal{C}'$ *contains* $\mathcal{C}$, written $\mathcal{C} \subseteq \mathcal{C}'$, if

- $\mathcal{D} \subseteq \mathcal{D}'$
- $\mathcal{F}(\lambda) \subseteq \mathcal{F}'(\lambda)$ for each ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$

□

As already mentioned in the previous chapter, configurations can be either finite or infinite theories. A definition of finiteness of a configuration is given below, assuming that configurations are infinite if they are *not* finite. (This definition will be used mainly in the next chapter.)

Definition 2.7 {Finite configuration} Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$ be a configuration. $\mathcal{C}$ is *finite* if

- $\mathcal{D}$ is finite, and
- there exists a finite set $\mathcal{T}$ of ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that, for each ground term $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$, if $\lambda \in \mathcal{T}$ then $\mathcal{F}(\lambda)$ is finite, and if $\lambda \notin \mathcal{T}$ then $\mathcal{F}(\lambda) = \{\}$

□

As described in Definition 2.1, inference rules of a propositional MLDS are generally applied to configurations to infer "new" configurations. The main question is of course how an inferred configuration is generated. Given an antecedent configuration $\mathcal{C}$, three types of reasoning step can occur. Those of the first type are "classical", and occur within any particular local modal theory included in $\mathcal{C}$, respecting standard notions of inference for classical connectives. Those of the second type are "modal" and concern the interaction between different local modal theories in $\mathcal{C}$. Both these two types of reasoning step (classical and modal) are based on the logical (classical and modal) information (wffs) incorporated in the declarative units that belong to $\mathcal{C}$. Inferred configurations are mainly "logical expansions" of (i.e. additions of declarative units to) the antecedent configurations. The third type of reasoning step is instead related to the diagram information in $\mathcal{C}$ and to the "interaction" between the diagram and the declarative units. In this case, inferred configurations are often "structural expansions" of (i.e. additions of R-literals to) the antecedent configurations.

Three different classes of inference rules can therefore be identified, which correspond to the three types of reasoning step mentioned above. A formal description of each is given below, followed by some derivation examples. For each inference rule a graphical representation is also given, based on the following key notation. For easy reference, all the inference rules introduced here and in Part II of this thesis are also listed in Appendix A.3.

Key notation An inferred configuration is always denoted by $\mathcal{C}'$. For the natural deduction rules which have sub-proofs as premises, the configurations derived within these sub-proofs are denoted by $\tilde{\mathcal{C}}$. The "temporary" addition of an assumption π (where π is either a declarative unit or an R-literal which needs to be discharged) to a configuration $\mathcal{C}$ is graphically represented as $\mathcal{C}\langle[\pi]\rangle$. Finally, for any configuration $\mathcal{C}$ and any declarative unit or R-literal π, the graphical notation $\mathcal{C}\langle\pi\rangle$ signifies that $\pi \in \mathcal{C}$.

2.1.1 Classical Rules

In this section *elimination* and *introduction* rules are given for each of the classical connectives included in the logical language $\mathcal{L}_M$.

Definition 2.8 $\{\wedge$–Elimination, $\mathcal{I}_{\wedge E}\}$ For all configurations $\mathcal{C}$, terms λ and wffs α and β, $\mathcal{C}/\mathcal{C} + [\lambda : \alpha]$ and $\mathcal{C}/\mathcal{C} + [\lambda : \beta]$ are members of the inference rule $\wedge$–Elimination (sometimes written $\mathcal{I}_{\wedge E}$) if

- $\lambda : \alpha \wedge \beta \in \mathcal{C}$

This is sometimes represented graphically as

$$\mathcal{I}_{\wedge E}\ \frac{\mathcal{C}\langle\lambda : \alpha \wedge \beta\rangle}{\mathcal{C}'\langle\lambda : \alpha\rangle} \qquad\qquad \frac{\mathcal{C}\langle\lambda : \alpha \wedge \beta\rangle}{\mathcal{C}'\langle\lambda : \beta\rangle}\ \mathcal{I}_{\wedge E}$$

□

Definition 2.9 $\{\wedge$–Introduction, $\mathcal{I}_{\wedge I}\}$ For all configurations $\mathcal{C}$, terms λ and wffs α and β, $\mathcal{C}/\mathcal{C} + [\lambda : \alpha \wedge \beta]$ is a member of the inference rule $\wedge$-Introduction (sometimes written $\mathcal{I}_{\wedge I}$) if

- $\lambda : \alpha \in \mathcal{C}$
- $\lambda : \beta \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda : \alpha, \lambda : \beta \rangle}{\mathcal{C}'\langle \lambda : \alpha \wedge \beta \rangle}\ \mathcal{I}_{\wedge I}$$

□

Definition 2.10 $\{\vee$–Elimination, $\mathcal{I}_{\vee E}\}$ For all configurations $\mathcal{C}$, terms λ and wffs γ, $\mathcal{C}/\mathcal{C} + [\lambda : \gamma]$ is a member of the inference rule $\vee$–Elimination (sometimes written $\mathcal{I}_{\vee E}$) if there exist wffs α and β such that

- $\lambda : \alpha \vee \beta \in \mathcal{C}$
- $\mathcal{C} + [\lambda : \alpha] \vdash_{\mathrm{MLDS}} \lambda : \gamma$
- $\mathcal{C} + [\lambda : \beta] \vdash_{\mathrm{MLDS}} \lambda : \gamma$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda : \alpha \vee \beta \rangle \quad \begin{array}{c} \mathcal{C}\langle [\lambda : \alpha] \rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle \lambda : \gamma \rangle \end{array} \quad \begin{array}{c} \mathcal{C}\langle [\lambda : \beta] \rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle \lambda : \gamma \rangle \end{array}}{\mathcal{C}'\langle \lambda : \gamma \rangle}\ \mathcal{I}_{\vee E}$$

□

Definition 2.11 $\{\vee$–Introduction, $\mathcal{I}_{\vee I}\}$ For all configurations $\mathcal{C}$, terms λ and wffs α and β, $\mathcal{C}/\mathcal{C} + [\lambda : \alpha \vee \beta]$ and $\mathcal{C}/\mathcal{C} + [\lambda : \beta \vee \alpha]$ are members of the inference rule $\vee$–Introduction (sometimes written $\mathcal{I}_{\vee I}$) if

- $\lambda : \alpha \in \mathcal{C}$

This is sometimes represented graphically as

$$\mathcal{I}_{\vee I}\ \frac{\mathcal{C}\langle \lambda : \alpha \rangle}{\mathcal{C}'\langle \lambda : \alpha \vee \beta \rangle} \qquad\qquad \frac{\mathcal{C}\langle \lambda : \alpha \rangle}{\mathcal{C}'\langle \lambda : \beta \vee \alpha \rangle}\ \mathcal{I}_{\vee I}$$

□

Definition 2.12 $\{\rightarrow$–Elimination, $\mathcal{I}_{\rightarrow E}\}$ For all configurations $\mathcal{C}$, terms λ and wffs β, $\mathcal{C}/\mathcal{C} + [\lambda : \beta]$ is a member of the inference rule $\rightarrow$–Elimination (sometimes written $\mathcal{I}_{\rightarrow E}$) if for some wff α,

- $\lambda : \alpha \rightarrow \beta \in \mathcal{C}$
- $\lambda : \alpha \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda : \alpha \rightarrow \beta, \lambda : \alpha \rangle}{\mathcal{C}'\langle \lambda : \beta \rangle}\ \mathcal{I}_{\rightarrow E}$$

$\square$

Definition 2.13 $\{\rightarrow$–Introduction, $\mathcal{I}_{\rightarrow I}\}$ For all configurations $\mathcal{C}$, terms λ and wffs α and β, $\mathcal{C}/\mathcal{C} + [\alpha \rightarrow \beta]$ is a member of the inference rule $\rightarrow$–Introduction (sometimes written $\mathcal{I}_{\rightarrow I}$) if

- $\mathcal{C} + [\lambda : \alpha]\ \vdash_{\mathrm{MLDS}}\ \lambda : \beta$

This is sometimes represented graphically as

$$\frac{\begin{array}{c}\mathcal{C}\langle [\lambda : \alpha] \rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle \lambda : \beta \rangle\end{array}}{\mathcal{C}'\langle \lambda : \alpha \rightarrow \beta \rangle}\ \mathcal{I}_{\rightarrow I}$$

$\square$

Definition 2.14 $\{\neg$–Elimination, $\mathcal{I}_{\neg E}\}$ For all configurations $\mathcal{C}$, terms λ and wffs α, $\mathcal{C}/\mathcal{C} + [\lambda : \alpha]$ is a member of the inference rule $\neg$–Elimination (sometimes written $\mathcal{I}_{\neg E}$) if

- $\lambda : \neg\neg\alpha \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda : \neg\neg\alpha \rangle}{\mathcal{C}'\langle \lambda : \alpha \rangle}\ \mathcal{I}_{\neg E}$$

$\square$

Definition 2.15 $\{\neg$–Introduction, $\mathcal{I}_{\neg I}\}$ For all configurations $\mathcal{C}$, terms λ and wffs α, $\mathcal{C}/\mathcal{C}+[\lambda:\neg\alpha]$ is a member of the inference rule $\neg$–Introduction (sometimes written $\mathcal{I}_{\neg I}$) if for some term λ'

- $\mathcal{C}+[\lambda:\alpha] \vdash_{\text{MLDS}} \lambda':\perp$

This is sometimes represented graphically as

$$\frac{\begin{array}{c}\mathcal{C}\langle[\lambda:\alpha]\rangle\\ \vdots\\ \tilde{\mathcal{C}}\langle\lambda':\perp\rangle\end{array}}{\mathcal{C}'\langle\lambda:\neg\alpha\rangle}\ \mathcal{I}_{\neg I}$$

□

Each of the above rules has the effect of expanding the antecedent configuration with a new declarative unit. With the exception of the $\mathcal{I}_{\neg I}$ rule, both the added declarative unit and the declarative unit(s) involved in the premise refer to the same label. This shows that the reasoning allowed by these rules takes place entirely within the scope of the local actual world under consideration. This characteristic is semantically motivated by the fact that, as pointed out in the previous chapter, the classical fragment of a modal logic behaves like a classical logic "locally" associated with any particular possible world. But for the $\mathcal{I}_{\neg I}$ rule, this is not always the case.

According to the classical interpretation of the $\neg$ connective, the negation of a formula can in general be proved by showing that the assumption of its positive form leads to a contradiction. In modal reasoning, contradictions can arise in possible worlds which are different from the current actual world. The semantics of a particular example is discussed here. Consider a modal language composed only of the propositional letter p, and the model $M = \langle W, R, v\rangle$, where $W = \{w_0, w_1\}$, $R = \{(W_0, W_1)\}$ and $v(p) = \{w_1\}$. To show $M, w_0 \Vdash \neg\Box\neg p$, it is sufficient to prove that $M, w_0 \not\Vdash \Box\neg p$. This is given by the fact that the assumption $M, w_0 \Vdash \Box\neg p$ implies $M, w_1 \Vdash \neg p$ which is in contradiction with the fact that $M, w_1 \Vdash p$ (given by the definition of v). Here the contradiction arises in a possible world (w_1) different from the current one (w_0). Analogously, in the $\mathcal{I}_{\neg I}$ rule given above the sub-derivation may involve reasoning about possible worlds different from the current one λ, in which contradictions may arise. Therefore, in order to capture these cases a different meta-symbol λ' is used, which may or may not be equal to λ.

A similar approach has been adopted by Fitting in his I-style natural deduction system, where the proof of a contradiction within a strict box (i.e. accessible possible world) causes the closure of the box and the inference of the inconsistency at the current world (see Example 1.8), so allowing the [¬I] rule to be applied. In the case of $\mathcal{I}_{\neg I}$ rule, this is equivalent to imposing the restriction that λ' is a world accessible from λ. In fact, in these particular circumstances $\mathcal{I}_{\neg I}$ could also be thought of as equivalent to the following three rules.

$$\frac{\begin{array}{c}\mathcal{C}\langle \lambda : \alpha \rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle \lambda : \bot \rangle\end{array}}{\mathcal{C}'\langle \lambda : \neg\alpha \rangle} \qquad \frac{\mathcal{C}\langle \lambda' : \bot, R(\lambda, \lambda') \rangle}{\mathcal{C}'\langle \lambda : \bot \rangle} \qquad \frac{\mathcal{C}\langle \lambda' : \bot, R(\lambda', \lambda) \rangle}{\mathcal{C}'\langle \lambda : \bot \rangle}$$

However, the $\mathcal{I}_{\neg I}$ rule in fact covers a more general case, for which the label λ' refers to a possible world different and possibly not accessible from λ. This is because, given the more general type of theories (configurations), inconsistency can also arise in local actual worlds which are different and not in relation with the possible world under consideration. In this case, $\mathcal{I}_{\neg I}$ reflects the classical principle "any formula can be inferred from a contradiction". Later it is shown that the same principle holds for inconsistencies caused by R-literals.

2.1.2 Modal Rules

In this section, a set of natural deduction rules concerning the modal operators $\Box$ and $\Diamond$ of $\mathcal{L}_M$ is given. Each of these rules describes how sets of information belonging to different possible worlds in relation with each other can interact.

Definition 2.16 {$\Diamond$–Elimination, $\mathcal{I}_{\Diamond E}$ } For all configurations $\mathcal{C}$, terms λ and wffs α, $\mathcal{C}/\mathcal{C} + [f_\alpha(\lambda) : \alpha] + [R(\lambda, f_\alpha(\lambda))]$ is a member of the inference rule $\Diamond$–Elimination (sometimes written $\mathcal{I}_{\Diamond E}$) if

- $\lambda : \Diamond\alpha \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda : \Diamond\alpha \rangle}{\mathcal{C}'\langle f_\alpha(\lambda) : \alpha, R(\lambda, f_\alpha(\lambda)) \rangle}$$

$\Box$

$\mathcal{I}_{\Diamond E}$ can be seen (informally) as a "skolemization" of the existential quantifier over possible worlds which is semantically implied by the formula $\Diamond\alpha$ in the premise. The term $f_\alpha(\lambda)$ defines a particular possible world uniquely associated with the formula α, and inferred to be accessible from the possible world λ (i.e. $R(\lambda, f_\alpha(\lambda))$). It is clear from the definition that this rule has the effect of expanding both the two components (diagram and set of declarative units) of the antecedent configuration. A graphical representation of the application of the $\mathcal{I}_{\Diamond E}$ rule is given in Figure 2.1.

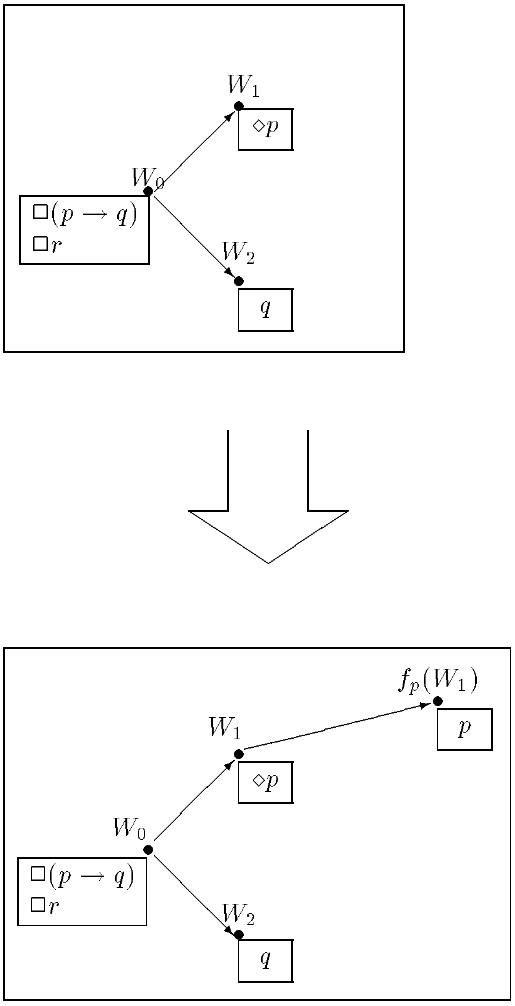

Figure 2.1: $\Diamond$–Elimination rule

Definition 2.17 $\{\Diamond$–Introduction, $\mathcal{I}_{\Diamond I}$ $\}$ For all configurations $\mathcal{C}$, terms λ_1 and λ_2 and wffs α, $\mathcal{C}/\mathcal{C} + [\lambda_1 : \Diamond\alpha]$ is a member of the inference rule $\Diamond$–Introduction (sometimes written $\mathcal{I}_{\Diamond I}$) if

- $\lambda_2 : \alpha \in \mathcal{C}$
- $R(\lambda_1, \lambda_2) \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda_2 : \alpha, R(\lambda_1, \lambda_2)\rangle}{\mathcal{C}'\langle \lambda_1 : \Diamond\alpha\rangle}$$

$\square$

Definition 2.18 $\{\square$–Elimination, $\mathcal{I}_{\square E}$ $\}$ For all configurations $\mathcal{C}$, terms λ_1 and λ_2 and wffs α, $\mathcal{C}/\mathcal{C} + [\lambda_2 : \alpha]$ is a member of the inference rule $\square$–Elimination (sometimes written $\mathcal{I}_{\square E}$) if

- $\lambda_1 : \square\alpha \in \mathcal{C}$
- $R(\lambda_1, \lambda_2) \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda_1 : \square\alpha, R(\lambda_1, \lambda_2)\rangle}{\mathcal{C}'\langle \lambda_2 : \alpha\rangle}$$

$\square$

The rules $\mathcal{I}_{\Diamond I}$ and $\mathcal{I}_{\square E}$ are somehow symmetric. They both rely on the structural information about the antecedent configuration (i.e. relations between possible worlds).

Definition 2.19 $\{\square$–Introduction, $\mathcal{I}_{\square I}$ $\}$ For all configurations $\mathcal{C}$, terms λ and wffs α, $\mathcal{C}/\mathcal{C}+[\lambda : \square\alpha]$ is a member of the inference rule $\square$–Introduction (sometimes written $\mathcal{I}_{\square I}$) if

- $\mathcal{C} + [R(\lambda, box_\alpha(\lambda))] \vdash_{\text{MLDS}} box_\alpha(\lambda) : \alpha$

This is sometimes represented graphically as

$$\begin{array}{c} \mathcal{C}\langle [R(\lambda, box_\alpha(\lambda))]\rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle box_\alpha(\lambda) : \alpha\rangle \\ \hline \mathcal{C}'\langle \lambda : \square\alpha\rangle \end{array}$$

$\square$

In the above rule the temporary assumption should be read as "given an arbitrary accessible world $box_\alpha(\lambda)$". Although, from a syntactical point of view, the use of a term is often seen as a way of naming particular objects (possible worlds), in this case it is adopted to refer to an arbitrary possible world. This is to ensure that labels are always ground terms. The role of the function symbol box_α will become clearer in the second part of this chapter where a semantics is given.

The above classical and modal rules are effectively generalisations of Fitting's natural deduction rules described in the previous chapter. Specifically, the $\mathcal{I}_{\Diamond E}$ and $\mathcal{I}_{\Box I}$ generalise the creation of strict boxes in the I-style system and the creation of strict boxes in the A-style system respectively, whereas the $\mathcal{I}_{\Box E}$ is somewhat similar to the basic strict iteration rule for the modal logic K. (The $\mathcal{I}_{\Diamond I}$ rule is not directly comparable.)

Furthermore, under the limitation of allowing only initial configurations composed of declarative units that refer to the single initial actual world W_0, the above two sets of rules provide a labelled proof system for the modal logic K. To give an example, formal and graphical derivations are given below in which the indeterdefinability of the two modal operators $\Box$ and $\Diamond$ is obtained by using only the above two sets of rules.

Example Derivations. In the formal versions of the following derivations, the number included as a superscript on each configuration symbol corresponds to the number of new assumptions introduced from the initial configuration $\mathcal{C}$. The subscript index (1, 2, etc.) corresponds to each step in each "sub-proof" related to a new assumption.

Example 2.1 $\{\mathcal{C}\langle W_0 : \Box p\rangle \vdash_{\text{MLDS}} W_0 : \neg\Diamond\neg p\ \}$ Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F}\rangle$ where $\mathcal{D} = \{\}$, $\mathcal{F}(W_0) = \{\Box p\}$ and $\mathcal{F}(\lambda) = \{\}$ for any $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $\lambda \neq W_0$. Then

$$\mathcal{C} \vdash_{\text{MLDS}} W_0 : \neg\Diamond\neg p$$

Derivation:

Let $\mathcal{C}^1 = \mathcal{C} + [W_0 : \Diamond\neg p]$ and let $\mathcal{C}^1_1 = \mathcal{C}^1 + [f_{\neg p}(W_0) : \neg p] + [R(W_0, f_{\neg p}(W_0))]$.

Then $\Diamond$–Elimination generates $\mathcal{C}^1_1$ from $\mathcal{C}^1$ (i)

Let $\mathcal{C}^1_2 = \mathcal{C}^1_1 + [f_{\neg p}(W_0) : p]$

Then $\Box$–Elimination generates $\mathcal{C}^1_2$ from $\mathcal{C}^1_1$ (ii)

Let $\mathcal{C}^1_3 = \mathcal{C}^1_2 + [f_{\neg p}(W_0) : p \wedge \neg p]$

Then $\wedge$–Introduction generates $\mathcal{C}_3^1$ from $\mathcal{C}_2^1$ (iii)

By (i), (ii) and (iii), $\mathcal{C}^1, \mathcal{C}_1^1, \mathcal{C}_2^1, \mathcal{C}_3^1$ is a proof, so that by (iii)

$$\mathcal{C} + [W_0 : \Diamond\neg p] \vdash_{\text{MLDS}} f_{\neg p}(W_0) : \bot \tag{iv}$$

Hence by (iv), $\neg$–Introduction generates $\mathcal{C} + [W_0 : \neg\Diamond\neg p]$ from $\mathcal{C}$, so that $\mathcal{C}, \mathcal{C} + [W_0 : \neg\Diamond\neg p]$ is a proof and therefore $\mathcal{C} \vdash_{\text{MLDS}} W_0 : \neg\Diamond\neg p$.

The above derivation can be represented graphically as follows:

$\underline{\mathcal{C}\langle W_0 : \Box p\rangle}$

$\underline{\mathcal{C}\langle W_0 : \Box p, [W_0 : \Diamond\neg p]\rangle}$ (assumption)

$\underline{\tilde{\mathcal{C}}_1\langle W_0 : \Box p, R(W_0, f_{\neg p}(W_0)), f_{\neg p}(W_0) : \neg p\rangle}$ ($\Diamond$–E)

$\underline{\tilde{\mathcal{C}}_2\langle f_{\neg p}(W_0) : p, f_{\neg p}(W_0) : \neg p\rangle}$ ($\Box$–E)

$\underline{\tilde{\mathcal{C}}_3\langle f_{\neg p}(W_0) : \bot\rangle}$ ($\wedge$–I)

$\mathcal{C}'\langle W_0 : \neg\Diamond\neg p\rangle$ ($\neg$–I)

(The configuration on line 2 corresponds to $\mathcal{C}^1$, line 3 to $\mathcal{C}_1^1$, line 4 to $\mathcal{C}_2^1$ and line 5 to $\mathcal{C}_3^1$.) •

Example 2.2 $\{\mathcal{C}\langle W_0 : \neg\Diamond\neg p\rangle \vdash_{\text{MLDS}} W_0 : \Box p\}$ Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F}\rangle$, where $\mathcal{D} = \{\}$, $\mathcal{F}(W_0) = \{\neg\Diamond\neg p\}$ and $\mathcal{F}(\lambda) = \{\}$ for any $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $\lambda \neq W_0$. Then

$$\mathcal{C} \vdash_{\text{MLDS}} W_0 : \Box p$$

Derivation:

Let $\mathcal{C}^1 = \mathcal{C} + [R(W_0, box_p(W_0))]$, let $\mathcal{C}^2 = \mathcal{C}^1 + [box_p(W_0) : \neg p]$, let $\mathcal{C}_1^2 = \mathcal{C}^2 + [W_0 : \Diamond\neg p]$ and let $\mathcal{C}_2^2 = \mathcal{C}_1^2 + [W_0 : (\Diamond\neg p) \wedge (\neg\Diamond\neg p)]$

Then $\wedge$–Introduction generates $\mathcal{C}_2^2$ from $\mathcal{C}_1^2$ and $\Diamond$–Introduction generates $\mathcal{C}_1^2$ from $\mathcal{C}^2$, so that $\mathcal{C}^2, \mathcal{C}_1^2, \mathcal{C}_2^2$ is a proof. So

$$\mathcal{C}^1 + [box_p(W_0) : \neg p] \vdash_{\text{MLDS}} W_0 : \bot \tag{i}$$

Let $\mathcal{C}^1_1 = \mathcal{C}^1 + [box_p(W_0) : \neg\neg p]$

By (i), $\neg$–Introduction generates $\mathcal{C}^1_1$ from $\mathcal{C}^1$ (ii)

Let $\mathcal{C}^1_2 = \mathcal{C}^1_1 + [box_p(W_0) : p]$

By (ii), $\neg$–Elimination generates $\mathcal{C}^1_2$ from $\mathcal{C}^1_1$ (iii)

Hence $\mathcal{C}^1, \mathcal{C}^1_1, \mathcal{C}^1_2$ is a proof and $\mathcal{C} + [R(W_0, box_p(W_0))] \vdash_{\text{MLDS}} box_p(W_0) : p$.

Let $\mathcal{C}_1 = \mathcal{C} + [W_0 : \Box p]$

By (iii), $\Box$–Introduction generates $\mathcal{C}_1$ from $\mathcal{C}$

Hence $\mathcal{C}, \mathcal{C}_1$ is a proof and therefore $\mathcal{C} \vdash_{\text{MLDS}} W_0 : \Box p$.

A graphical representation of the above derivation is given below.

$\underline{\mathcal{C}\langle W_0 : \neg\Diamond\neg p\rangle}$

$\underline{\mathcal{C}\langle W_0 : \neg\Diamond\neg p, [R(W_0, box_p(W_0))]\rangle}$ (assumption)

$\underline{\mathcal{C}\langle W_0 : \neg\Diamond\neg p, [box_p(W_0) : \neg p]\rangle}$ (assumption)

$\underline{\tilde{\mathcal{C}}_1\langle W_0 : \neg\Diamond\neg p, W_0 : \Diamond\neg p\rangle}$ ($\Diamond$–I)

$\underline{\tilde{\mathcal{C}}_2\langle W_0 : \bot\rangle}$ ($\wedge$–I)

$\underline{\tilde{\mathcal{C}}_3\langle box_p(W_0) : \neg\neg p\rangle}$ ($\neg$–I)

$\underline{\tilde{\mathcal{C}}_4\langle box_p(W_0) : p\rangle}$ ($\neg$–E)

$\mathcal{C}'\langle W_0 : \Box p\rangle$ ($\Box$–I)

•

2.1.3 Structural Rules

To allow reasoning about arbitrary configurations and to capture the different $K\delta$-MLDSs, a third set of inference rules needs to be included as part of a propositional MLDS. These rules facilitate reasoning about the diagram of a configuration, using the particular labelling algebra $\mathcal{A}_{K\delta}$ under consideration, and infer R-literals and declarative units which are not implied by the modal rules.

Definition 2.20 $\{R$–Assertion, $\mathcal{I}_{R-A}\}$

For all configurations $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$, and R-literals Δ, $\mathcal{C}/\mathcal{C} + [\Delta]$ is a member of the inference rule R–Assertion (sometimes written $\mathcal{I}_{R-A}$) if

- $\mathcal{D}, \mathcal{A} \vdash_{FOL} \Delta$

where $\mathcal{A}$ is a labelling algebra. $\square$

In the previous chapter, it has been emphasised that different labelling algebrae define different propositional MLDSs. Proof-theoretically these differences are imposed by the $\mathcal{I}_{R-A}$ rule. This rule facilitates the inference of new R-literals according to the properties of the accessibility relation given by a particular labelling algebra $\mathcal{A}$. So for example, if the labelling algebra $\mathcal{A}_{K\{\mathbf{D}\}}$ is considered, the above rule allows the inference of R-literals of the form $R(\lambda, succ(\lambda))$, for any label λ, thus embedding the seriality property of the accessibility relation in the derivation process. This enables the derivation of declarative units of the form $\lambda : \Box\alpha \rightarrow \Diamond\alpha$, for any arbitrary label λ, hence characterising the MLDS as a $K\{\mathbf{D}\}$-MLDS. In this way, the set of modal rules remains unchanged for the whole family of propositional MLDSs, making the natural deduction system *uniform.*

The following two rules formalise additional forms of interaction between R-literals and declarative units. The $\mathcal{I}_{\neg I}$ rule described previously, detects inconsistent information (i.e. $\lambda : \bot$) in configurations. But the question is, how can an inconsistency arise? In a standard modal theory a contradiction is generally due to assumptions (initial or temporary) which negate each other. In a MLDS theory contradictory assumptions can be either an R-literal and its conjugate, or a declarative unit and its negation[4]. Whereas the latter are captured by the $\neg$–Introduction rule (see previous discussion), the former are identified by the following $\bot$-Introduction rule, which again reflects the classical principle of any formula being derived from a contradiction.

[4]The negation of a declarative unit $\lambda : \alpha$ is a declarative unit of the form $\lambda : \neg\alpha$.

Definition 2.21 $\{\bot$–Introduction, $\mathcal{I}_{\bot I}\}$ For all configurations $\mathcal{C}$, any R–literals Δ, and any declarative unit $\lambda:\alpha$, $\mathcal{C}/\mathcal{C}+[\lambda:\alpha]$ is a member of the inference rule $\bot$–Introduction (sometimes written $\mathcal{I}_{\bot I}$) if

- $\Delta \in \mathcal{C}$
- $\overline{\Delta} \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \Delta, \overline{\Delta}\rangle}{\mathcal{C}'\langle \lambda:\alpha\rangle}$$

□

An example showing why this rule is included and how it is used is given below.

Example 2.3 Let $\mathcal{A}$ be the labelling algebra given by the set $\{\forall x \forall y \neg R(x,y)\}$. This formalises the class of frames in which possible worlds are all independent from each other – i.e. for each possible world, there is no world accessible from it. In this class of frames, any declarative unit of the form $\lambda:\Box\alpha$ is provable. A graphical derivation is given below which shows in particular that the declarative unit $W_0:\Box\alpha$ (where α is any arbitrary wff) is derivable from the configuration $\mathcal{C}_{\{\}} = \langle\{\}, \mathcal{F}_{\{\}}\rangle$, where $\mathcal{F}_{\{\}}(\lambda) = \{\}$ for any term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$.

$$\mathcal{C}_{\{\}}$$

$$\mathcal{C}_1'\langle \neg R(W_0, box_\alpha(W_0))\rangle \qquad (R\text{-assertion})$$

$$\mathcal{C}_1'\langle \neg R(W_0, box_\alpha(W_0)), [R(W_0, box_\alpha(W_0))]\rangle \qquad (\text{assumption})$$

$$\tilde{\mathcal{C}}_1\langle box_\alpha(W_0):\alpha\rangle \qquad (\bot\text{–I})$$

$$\mathcal{C}_2'\langle W_0:\Box\alpha\rangle \qquad (\Box\text{–I})$$

•

So far, only declarative units are derivable from inconsistent information (using the $\neg$–Introduction and $\bot$–Introduction rules). However, a similar type of reasoning can be applied to the R–literals. A second form of interaction between R–literals and declarative units is captured by the following inference rule, in which R–literals are derived whenever a logical inconsistency arises within a configuration.

Definition 2.22 $\{R$–Introduction, $\mathcal{I}_{R-I}\}$

For all configurations $\mathcal{C}$, and R-literals Δ, $\mathcal{C}/\mathcal{C}+[\Delta]$ is a member of the inference rule R–Introduction (sometimes written $\mathcal{I}_{R-I}$) if for some term λ'

- $\mathcal{C}+[\overline{\Delta}] \vdash_{\mathrm{MLDS}} \lambda' : \bot$

This is sometimes represented graphically as

$$\frac{\begin{array}{c}\mathcal{C}\langle\overline{\Delta}\rangle\\ \vdots \\ \tilde{\mathcal{C}}\langle\lambda' : \bot\rangle\end{array}}{\mathcal{C}'\langle\Delta\rangle}$$

□

An example showing when and how this rule is used is the following. Let $\mathcal{C}$ be the configuration $\langle\{\}, \mathcal{F}\rangle$, where $\mathcal{F}(W_0) = \{\Box p\}$, $\mathcal{F}(W_1) = \{\neg p\}$ and for any label λ different from W_0 and from W_1, $\mathcal{F}(\lambda) = \{\}$. Then $\mathcal{C} \vdash_{\mathrm{MLDS}} \neg R(W_0, W_1)$. This is obtained by applying the $\mathcal{I}_{R-I}$ rule and the $\mathcal{I}_{\Box E}$ rule.

The structural rules described above and the classical and modal rules given in the previous subsections all have the effect of expanding their antecedent configurations. For completeness, the following rule is included in every propositional MLDS, which simply allows the derivation of any configuration contained in the antecedent one.

Definition 2.23 $\{\mathcal{C}$–Reduction, $\mathcal{I}_{C-R}\}$ For all configurations $\mathcal{C}$ and $\mathcal{C}'$, $\mathcal{C}/\mathcal{C}'$ is a member of the inference rule $\mathcal{C}$–Reduction (sometimes written $\mathcal{I}_{C-R}$) if

- $\mathcal{C}' \subseteq \mathcal{C}$

□

Two example derivations (graphical representation) are given below which describe the proofs shown in Examples 1.2 and 1.3 using the modal labelled natural deduction system given above. Some equivalence rewriting of declarative units is implicitly assumed (e.g. $box_{\Diamond r}(W_0):\neg\Diamond r$ is replaced with $box_{\Diamond r}(W_0):\Box\neg r$ and $W_0:\neg\Box\Box p$ is replaced with $W_0:\Diamond\Diamond\neg p$).

Example 2.4 Consider the MLDS $K\{\mathbf{B},\mathbf{4}\}$-MLDS. Let $\mathcal{C}$ be the configuration $\langle\mathcal{D},\mathcal{F}\rangle$ where $\mathcal{D}=\{\}$, $F(W_0)=\{\Box\Box p\rightarrow r\}$ and $\mathcal{F}(\lambda)=\{\}$ for each $\lambda\in Func(\mathcal{L}_L,\mathcal{L}_M)$ such that $\lambda\neq W_0$. Then

$$\mathcal{C}\vdash_{K\{\mathbf{B},\mathbf{4}\}-MLDS} W_0:\Box(p\wedge q)\rightarrow\Box\Diamond r$$

Derivation:

$\mathcal{C}\langle W_0:\Box\Box p\rightarrow r\rangle$

$\mathcal{C}\langle[W_0:\Box(p\wedge q)]\rangle$ (assumption)

$\mathcal{C}\langle[R(W_0, box_{\Diamond r}(W_0))]\rangle$ (assumption)

$\tilde{\mathcal{C}}\langle R(box_{\Diamond r}(W_0), W_0)\rangle$ (R–assertion)

$\tilde{\mathcal{C}}\langle[R(W_0, box_{\Box p}(W_0))]\rangle$ (assumption)

$\tilde{\mathcal{C}}\langle[R(box_{\Box p}(W_0), box_p(box_{\Box p}(W_0)))]\rangle$ (assumption)

$\tilde{\mathcal{C}}_1\langle R(W_0, box_p(box_{\Box p}(W_0)))\rangle$ (R-assertion)

$\tilde{\mathcal{C}}_2\langle box_p(box_{\Box p}(W_0)):p\wedge q\rangle$ ($\Box$–E)

$\tilde{\mathcal{C}}_3\langle box_p(box_{\Box p}(W_0)):p\rangle$ ($\wedge$-E)

$\tilde{\mathcal{C}}_4\langle box_{\Box p}(W_0):\Box p\rangle$ ($\Box$–I)

$\tilde{\mathcal{C}}_5\langle W_0:\Box\Box p\rangle$ ($\Box$–I)

$\tilde{\mathcal{C}}_6\langle W_0:r\rangle$ ($\rightarrow$–E)

$\tilde{\mathcal{C}}_7\langle box_{\Diamond r}(W_0):\Diamond r\rangle$ ($\Diamond$–I)

$\tilde{\mathcal{C}}_8\langle W_0:\Box\Diamond r\rangle$ ($\Box$–I)

$\mathcal{C}'\langle W_0:\Box(p\wedge q)\rightarrow\Box\Diamond r\rangle$ ($\rightarrow$–I)

The above proof is most similar to the derivation given in Example 1.2. An alternative derivation is given below which resembles more that given in Example 1.3.

Alternative derivation:

$$
\begin{array}{ll}
\underline{\mathcal{C}\langle W_0 : \Box\Box p \rightarrow r\rangle} & \\
\quad \underline{\mathcal{C}\langle [W_0 : \Box(p \wedge q)]\rangle} & \text{(assumption)} \\
\quad\quad \underline{\mathcal{C}\langle [R(W_0, box_{\Diamond r}(W_0))]\rangle} & \text{(assumption)} \\
\quad\quad\quad \underline{\mathcal{C}\langle [box_{\Diamond r}(W_0) : \neg\Diamond r]\rangle} & \text{(assumption)} \\
\quad\quad\quad \underline{\tilde{\mathcal{C}}_1\langle R(box_{\Diamond r}(W_0), W_0)\rangle} & (R\text{–assertion}) \\
\quad\quad\quad \underline{\tilde{\mathcal{C}}_2\langle W_0 : \neg r\rangle} & (\Box\text{–E}) \\
\quad\quad\quad \underline{\tilde{\mathcal{C}}_3\langle W_0 : \neg\Box\Box p\rangle} & (\rightarrow\text{–E}) \\
\quad\quad\quad \underline{\tilde{\mathcal{C}}_4\langle f_{\Diamond\neg p}(W_0) : \Diamond\neg p, R(W_0, f_{\Diamond\neg p}(W_0))\rangle} & (\Diamond\text{–E}) \\
\quad\quad\quad \underline{\tilde{\mathcal{C}}_5\langle f_{\neg p}(f_{\Diamond\neg p}(W_0)) : \neg p, R(f_{\Diamond\neg p}(W_0), f_{\neg p}(f_{\Diamond\neg p}(W_0)))\rangle} & (\Diamond\text{–E}) \\
\quad\quad\quad \underline{\tilde{\mathcal{C}}_6\langle R(W_0, f_{\neg p}(f_{\Diamond\neg p}(W_0)))\rangle} & (R\text{-assertion}) \\
\quad\quad\quad \underline{\tilde{\mathcal{C}}_7\langle f_{\neg p}(f_{\Diamond\neg p}(W_0)) : p \wedge q\rangle} & (\Box\text{–E}) \\
\quad\quad\quad \underline{\tilde{\mathcal{C}}_8\langle f_{\neg p}(f_{\Diamond\neg p}(W_0)) : p\rangle} & (\wedge\text{–E}) \\
\quad\quad\quad \underline{\tilde{\mathcal{C}}_9\langle f_{\neg p}(f_{\Diamond\neg p}(W_0)) : \bot\rangle} & (\wedge\text{–I}) \\
\quad\quad \underline{\tilde{\mathcal{C}}_{10}\langle box_{\Diamond r}(W_0) : \Diamond r\rangle} & (\neg\text{–I}) \\
\quad \underline{\tilde{\mathcal{C}}_{11}\langle W_0 : \Box\Diamond r\rangle} & (\Box\text{–I}) \\
\mathcal{C}'\langle W_0 : \Box(p \wedge q) \rightarrow \Box\Diamond r\rangle & (\rightarrow\text{–I})
\end{array}
$$

•

2.2 A Semantics

A propositional MLDS can be considered to be a "semi-translated" approach to modal logic – a Kripke-like accessibility relation is syntactically expressed, but without requiring the full translation of modal formulae into first-order sentences. Therefore, a model-theoretic semantics could be equally given in terms of the traditional Kripke possible worlds semantics (see discussion [Gab94]) or in terms of a first-order semantics, using some translation method. The second approach has been chosen in this thesis. (A discussion of the advantages related to this choice is given at the end of the chapter.) In the following section, a translation method of a propositional MLDS into a first-order logic is defined and the semantic notion of a consequence relation, $\models_{\mathrm{MLDS}}$, as well as the definition of a model and of satisfiability of a configuration, are given in terms of classical semantics.

As already pointed out, the declarative unit $\lambda : \alpha$ can be interpreted as *the formula α is true at the possible world λ*. In what follows, such Kripke semantic notions are expressed in terms of first-order statements of the form $[\alpha]^*(\lambda)$, where $[\alpha]^*$ is a unary predicate symbol. Hence the semi-extended labelling language $Func(\mathcal{L}_L, \mathcal{L}_M)$ is further expanded to include the predicate symbol $[\alpha]^*$ for each wff α of $\mathcal{L}_M$. The relationships between these predicates are constrained by a set of first-order axiom schemas which capture the satisfiability conditions (see Definition 1.5) of each type[5] of formula α. These axiom schemas extend the labelling algebra $\mathcal{A}$ of a propositional MLDS into a first-order theory called an *extended algebra*. Formal definitions are given below.

Definition 2.24 {Extended labelling language $Mon(\mathcal{L}_L, \mathcal{L}_M)$}

Let $Func(\mathcal{L}_L, \mathcal{L}_M)$ be a semi–extended labelling language. Let $\alpha_1, \ldots, \alpha_n, \ldots$ be the ordered set of all wffs of $\mathcal{L}_M$. The *extended labelling language* $Mon(\mathcal{L}_L, \mathcal{L}_M)$ is defined as the language $Func(\mathcal{L}_L, \mathcal{L}_M)$ extended with the following set of unary predicate symbols

$$\{[\alpha_1]^*, \ldots, [\alpha_n]^*, \ldots\}$$

□

Definition 2.25 {Extended algebra $\mathcal{A}^+$} Given an extended labelling language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and a labelling algebra $\mathcal{A}$ written in $Func(\mathcal{L}_L, \mathcal{L}_M)$, the *extended algebra* $\mathcal{A}^+$ is the first–order theory in $Mon(\mathcal{L}_L, \mathcal{L}_M)$ consisting of the following axiom schemas (Ax1)–(Ax8), together with the axioms of $\mathcal{A}$:

[5]The type of a wff is given by the main connective of the wff itself, e.g. the wff $\Diamond(p \rightarrow q)$ is a $\Diamond$-formula.

For any wffs α and β of $\mathcal{L}_M$:

$$\forall x([\alpha \wedge \beta]^*(x) \equiv ([\alpha]^*(x) \wedge [\beta]^*(x))) \tag{Ax1}$$

$$\forall x([\neg\alpha]^*(x) \equiv \neg[\alpha]^*(x)) \tag{Ax2}$$

$$\forall x([\alpha \vee \beta]^*(x) \equiv ([\alpha]^*(x) \vee [\beta]^*(x))) \tag{Ax3}$$

$$\forall x([\alpha \rightarrow \beta]^*(x) \equiv ([\alpha]^*(x) \rightarrow [\beta]^*(x))) \tag{Ax4}$$

$$\forall x([\Diamond\alpha]^*(x) \rightarrow (R(x, f_\alpha(x)) \wedge [\alpha]^*(f_\alpha(x)))) \tag{Ax5}$$

$$\forall x(\exists y(R(x,y) \wedge [\alpha]^*(y)) \rightarrow [\Diamond\alpha]^*(x)) \tag{Ax6}$$

$$\forall x((R(x, box_\alpha(x)) \rightarrow [\alpha]^*(box_\alpha(x))) \rightarrow [\Box\alpha]^*(x)) \tag{Ax7}$$

$$\forall x([\Box\alpha]^*(x) \rightarrow (\forall y(R(x,y) \rightarrow [\alpha]^*(y)))) \tag{Ax8}$$

□

The first four axiom schemas express the distributive properties of the logical connectives among the monadic predicates of $Mon(\mathcal{L}_L, \mathcal{L}_M)$. They cover the Kripke semantic definition of satisfiability of the logical connectives $\wedge$, $\neg$, $\vee$ and $\rightarrow$ respectively (see Definition 1.5). (Ax5) forces the accessibility relation R on the labels generated by the application of function symbols f_{α_i} of $Mon(\mathcal{L}_L, \mathcal{L}_M)$. Axiom schemas (Ax5)–(Ax6) together cover the Kripke semantic definition of the modal operator $\Diamond$. (i.e. the statement $\forall x(\exists y(R(x,y) \wedge [\alpha]^*(y)) \equiv [\Diamond\alpha]^*(x))$ can be derived from (Ax5)–(Ax6)). Analogously axiom schemas (Ax7)–(Ax8) together cover the Kripke semantic definition of the modal operator $\Box$. (i.e. the statement $\forall x([\Box\alpha]^*(x) \equiv \forall y(R(x,y) \rightarrow [\alpha]^*(y)))$ can be derived from (Ax7)–(Ax8)). They confirm semantically that the $\Box$-Introduction rule works correctly. Therefore, the axiom schemas (Ax1) – (Ax8) of $\mathcal{A}^+$ reflect the Kripke semantic definition of satisfiability of modal wffs. This is easy to see, by interpreting the truth of $[\alpha]^*(x)$ as the truth of the modal formula α in the possible world x.

An alternative way of describing Kripke semantics in classical logic is to translate wffs of the language $\mathcal{L}_M$ into ground *terms* of the classical language and to use a *Holds* predicate to define the semantic satisfiability conditions of the classical and modal operators. An example is described in [ARS94] for belief logics. However, the extra syntactic complexity given by this approach is not necessary in the propositional case. On the other hand, for the predicate case the use of function symbols for representing wffs as terms inside a *Holds* predicate helps deal with quantifiers and variables, which would otherwise be embedded into the names of the monadic predicates, thus being rather "inaccessible". Hence, in Part II of the thesis, this alternative approach is adopted to describe the semantics for predicate MLDSs.

Note that since an extended algebra also includes the axioms of the accessibility relation, different labelling algebrae $\mathcal{A}$ determine different extended algebrae $\mathcal{A}^+$. Therefore, subscripts of the form $K\delta$ will sometimes be added to the symbol $\mathcal{A}^+$ to denote particular extended algebrae associated with particular $K\delta$-MLDS.

The following proposition shows that the standard interdefinability of $\Box$ and $\Diamond$ can be derived from any extended algebra $\mathcal{A}^+$.

Proposition 2.1 Given the extended language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and an associated extended algebra $\mathcal{A}^+$, for any formula $\alpha \in \mathcal{L}_M$ the following equivalence is satisfied:

$$\mathcal{A}^+ \vdash_{FOL} \forall x([\Box\alpha]^*(x) \equiv [\neg\Diamond\neg\alpha]^*(x))$$

Proof:
It is sufficient to prove the following two results for an arbitrary ground term λ.

$$\begin{array}{ll}
\text{(a)} & \mathcal{A}^+ \vdash_{FOL} [\Box\alpha]^*(\lambda) \rightarrow [\neg\Diamond\neg\alpha]^*(\lambda) \\
\text{(b)} & \mathcal{A}^+ \vdash_{FOL} [\neg\Diamond\neg\alpha]^*(\lambda) \rightarrow [\Box\alpha]^*(\lambda)
\end{array}$$

$[\Box\alpha]^*(\lambda)$	(i)
$\forall y(R(\lambda, y) \rightarrow [\alpha]^*(y))$	(ii)
$\forall y(R(\lambda, y) \rightarrow \neg[\neg\alpha]^*(y))$	(iii)
$\forall y\neg(R(\lambda, y) \wedge [\neg\alpha]^*(y))$	(iv)
$\neg(R(\lambda, f_{\neg\alpha}(x)) \wedge [\neg\alpha]^*(f_{\neg\alpha}(\lambda)))$	(v)
$\neg[\Diamond\neg\alpha]^*(\lambda)$	(vi)
$[\neg\Diamond\neg\alpha]^*(\lambda)$	(vii)

Proof of (a)

(i) is a starting assumption, (ii) is derived from (i) and (Ax8) by applying modus ponens, (iii) is obtained from (ii) and (Ax2), (iv) is obtained by equivalence rewriting of (iii), (v) is derived from (iv) using classical rules for universal quantification and (vi) is obtained from (v) and the contrapositive of (Ax5) using modus ponens. Finally, (vii) is derived from (vi) and (Ax2).

$[\neg\Diamond\neg\alpha]^*(\lambda)$	(i)
$\neg\exists y(R(\lambda, y) \wedge [\neg\alpha]^*(y))$	(ii)
$\forall y(\neg R(\lambda, y) \vee \neg[\neg\alpha]^*(y))$	(iii)
$\forall y(R(\lambda, y) \rightarrow [\alpha]^*(y))$	(iv)
$R(\lambda, box_\alpha(\lambda)) \rightarrow [\alpha]^*(box_\alpha(\lambda))$	(v)
$[\Box\alpha]^*(\lambda)$	(vi)

Proof of (b)

(i) is a starting assumption, (ii) is derived from (i), (Ax2) and (Ax6), (iii) is obtained from (ii) by equivalence rewriting, (iv) is obtained from (iii) and (Ax2), (v) is derived from (iv) using classical rules for universal quantification and finally (vi) is derived from (v) and (Ax7) by modus ponens. ⋆

A translation method is defined next. It associates syntactic expressions of a propositional MLDL with sentences of the first-order language $Mon(\mathcal{L}_L, \mathcal{L}_M)$, and hence associates theories (configurations) with first-order theories in the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$. Each declarative unit $\lambda : \alpha$ is translated into the sentence $[\alpha]^*(\lambda)$, and R-literals are translated as themselves. Therefore, the first-order translation of a configuration is a first-order theory including the R–literals, which are present in the diagram of the configuration, and the set of monadic formulae $[\alpha]^*(\lambda)$ that correspond to the declarative units present in the configuration. A formal definition is given below.

Definition 2.26 {First-order translation of a configuration} Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$ be a configuration. The *first-order translation* of $\mathcal{C}$, written $FOT(\mathcal{C})$, is the theory written in $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and defined by the expression:

$$FOT(\mathcal{C}) = \mathcal{D} \cup \mathcal{DU}$$

where $\mathcal{DU} = \{[\alpha]^*(\lambda) \mid \alpha \in \mathcal{F}(\lambda),\ \lambda \text{ is a ground term of } Func(\mathcal{L}_L, \mathcal{L}_M)\}$.

□

Note that, since labels can only be ground terms of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$, the first-order translation of a configuration is a set of *ground literals* of the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$.

Notions of models, satisfiability and semantic entailment are given in terms of classical semantics using the above definitions.

Definition 2.27 {Semantic structure of a $K\delta$-MLDS} Given a $K\delta$-MLDS and the associated extended algebra $\mathcal{A}^+_{K\delta}$, $\mathcal{M}$ is a *semantic structure* of $K\delta$-MLDS if $\mathcal{M}$ is a model of $\mathcal{A}^+_{K\delta}$. □

Different classes of semantic structures can be obtained by considering different underlying labelling algebrae. Therefore the symbol $\mathcal{M}$ will sometimes be written with the subscript $K\delta$ (i.e. $\mathcal{M}_{K\delta}$) to identify models of a particular class.

Definition 2.28 {Satisfiability of a declarative unit} Let $\lambda : \alpha$ be a declarative unit. $\lambda : \alpha$ *is satisfiable* (with respect to $K\delta$-MLDS) if there exists a semantic structure $\mathcal{M}_{K\delta}$ such that $\mathcal{M}_{K\delta} \Vdash_{FOL} [\alpha]^*(\lambda)$. In this case $\mathcal{M}_{K\delta}$ is said to *satisfy* $\lambda : \alpha$, written as $\mathcal{M}_{K\delta} \Vdash_{K\delta-\mathrm{MLDS}} \lambda : \alpha$. □

Note that the symbol $\Vdash_{FOL}$ signifies the standard classical notion of satisfiability, and the expression $\mathcal{M} \Vdash_{FOL} [\alpha]^*(\lambda)$ signifies that $\mathcal{M}$ is a classical model of the sentence $[\alpha]^*(\lambda)$.

Definition 2.29 {Satisfiability of an R–literal} Let Δ be an R–literal. Δ is *satisfiable* (with respect to $K\delta$-MLDS) if there exists a semantic structure $\mathcal{M}_{K\delta}$ such that $\mathcal{M}_{K\delta} \Vdash_{FOL} \Delta$. In this case $\mathcal{M}_{K\delta}$ is said to *satisfy* Δ, written as $\mathcal{M}_{K\delta} \Vdash_{K\delta-\text{MLDS}} \Delta$. □

The two definitions above establish the truth value of each of the syntactic entities of a propositional MLDS. Now the definition of satisfiability of a MLDS theory can be given.

Definition 2.30 {Satisfiability of a configuration} Let $\mathcal{C}$ be a configuration. $\mathcal{C}$ is *satisfiable* (with respect to $K\delta$-MLDS) if there exists a semantic structure $\mathcal{M}_{K\delta}$ such that for each π, $\pi \in \mathcal{C}$, $\mathcal{M}_{K\delta}$ satisfies π, (where π may be a declarative unit or an R–literal). In this case $\mathcal{M}_{K\delta}$ is said to *satisfy* $\mathcal{C}$, written $\mathcal{M}_{K\delta} \Vdash_{K\delta-\text{MLDS}} \mathcal{C}$. □

Finally the notion of semantic entailment for a propositional MLDS is given in terms of classical semantic entailment.

Definition 2.31 {Semantic entailment in a MLDS} Let $\mathcal{A}^+_{K\delta}$ be the extended algebra of a MLDS $K\delta$-MLDS. Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$ and $\mathcal{C}' = \langle \mathcal{D}', \mathcal{F}' \rangle$ be two configurations, and $FOT(\mathcal{C}) = \mathcal{D} \cup \mathcal{DU}$ and $FOT(\mathcal{C}') = \mathcal{D}' \cup \mathcal{DU}'$ their respective first-order translations. The configuration $\mathcal{C}$ *semantically entails* $\mathcal{C}'$, written $\mathcal{C} \models_{K\delta-\text{MLDS}} \mathcal{C}'$, if

- $\mathcal{A}^+_{K\delta} \cup FOT(\mathcal{C}) \models_{FOL} \Delta$ for each $\Delta \in \mathcal{D}'$
- $\mathcal{A}^+_{K\delta} \cup FOT(\mathcal{C}) \models_{FOL} [\alpha]^*(\lambda)$ for each $[\alpha]^*(\lambda) \in \mathcal{DU}'$ □

As for derivability ($\vdash_{\text{MLDS}}$), where $K\delta$ is arbitrary, the symbol $\models_{K\delta-\text{MLDS}}$ will sometimes be abbreviated to $\models_{\text{MLDS}}$. The following notation is based on the standard definitions of the classical derivability relation and classical semantic entailment.

Notation 2.4 Given two configurations $\mathcal{C}$ and $\mathcal{C}'$ and their first-order translations $FOT(\mathcal{C}) = \mathcal{D} \cup \mathcal{DU}$ and $FOT(\mathcal{C}') = \mathcal{D}' \cup \mathcal{DU}'$, the expression $\mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$ signifies that:

$$\begin{array}{ll} \mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{FOL} \Delta & \text{for each } \Delta \in \mathcal{D}' \\ \mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{FOL} [\alpha]^*(\lambda) & \text{for each } [\alpha]^*(\lambda) \in \mathcal{DU}' \end{array}$$

and analogously $\mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}')$ if

$$\begin{array}{ll} \mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{FOL} \Delta & \text{for each } \Delta \in \mathcal{D}' \\ \mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{FOL} [\alpha]^*(\lambda) & \text{for each } [\alpha]^*(\lambda) \in \mathcal{DU}' \end{array}$$

$\triangleleft$

The above notation allows the definition of semantic entailment between configurations to be alternatively stated as:

$$\mathcal{C} \models_{\text{MLDS}} \mathcal{C}' \quad \text{iff} \quad \mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}') \tag{2.1}$$

2.2.1 Extending Modal Labelled Deductive Systems

The MLDS framework described in this and in the previous chapter provides a methodology general enough to encompass the class of "standard" modal logics K, T, $S4$, etc., and it is this basic MLDS schema which is the main focus of this thesis. However, many extensions of the framework suggest themselves for future study, and it is worth briefly mentioning two in particular, which preliminary investigations show to be natural and unproblematical. Both involve extending the labelling language $\mathcal{L}_L$ with other binary predicate symbols, the first to encompass "multi-modal logics", and the second to include labelling algebrae with the equality predicate.

In multi-modal logics, the underlying semantics includes not just one, but a collection of accessibility relations. Each relation R_i has an associated pair of modal operators $\Box_i$ and $\Diamond_i$. For example, Dynamic Logic [Har79], has a necessity operator $[A]$ corresponding to each "action" A, and the formula $[A]\alpha$ is intended to mean "α is always true after the action A". Extending MLDS with multiple modalities is straightforward. The labelling language $\mathcal{L}_L$ is extended with a binary relation R_i for each modality i, and the labelling algebra includes extra axioms or axiom schemas involving each or all of these relations, analogous to those in Table 1.3. The semi-extended language $Func(\mathcal{L}_L, \mathcal{L}_M)$ includes function symbols f^i_α and box^i_α for each wff α and modality i, and the modal and structural inference rules (Sections 2.1.2 and 2.1.3) are appropriately duplicated for each modality. It is easy to see that the extended algebra $\mathcal{A}^+$ can also be appropriately extended, so that the definition of semantic entailment is also easily generalised to the case of multi-modalities.

Extending the labelling algebra to include the equality predicate is similarly straightforward, but involves the inclusion of one specialised inference rule, which reflects an aspect of the equality theory which is added to the labelling algebra $\mathcal{A}$. Inclusion of $=$ would be useful to describe certain modal logics, whose characteristic axioms Correspondence Theory shows to represent frame properties described by first-order sentences containing both R and $=$. For example, the labelling algebra given by the first-order condition $\forall x \forall y[R(succ(x), y) \rightarrow y = succ(x)]$, together with the axioms (**T**), (**D**) and (**4**) of Definition 1.24, would represent the class of frames associated with the modal logic $S4.1$, which extends the modal logic $S4$ with the "McKinsey Axiom" $\Box\Diamond\alpha \rightarrow \Diamond\Box\alpha$. In the appropriately extended MLDS, diagrams include equality (and inequality) literals as well as R-literals, and the "structural rules" (Section 2.1.3) are regarded as applying to equality and inequality literals as well as to R-literals. In addition, the following rule has to be included, reflecting the substitutivity property of equal terms in declarative units.

$$\frac{\lambda_1 : \alpha \qquad \lambda_1 = \lambda_2}{\lambda_2 : \alpha} \; \mathcal{I}_{Sub}$$

As far as the semantics is concerned, the standard equality axiom schema (already included in the labelling algebra $\mathcal{A}$) is extended in $\mathcal{A}^+$ so as to cover each of the monadic predicates $[\alpha]^*$.

Although intuitively straightforward, the extensions described above are not considered in greater detail in this thesis. However, a brief description of modifications that would be needed to the soundness, completeness and correspondence proofs is given in Chapter 3.

2.3 Discussion

In this chapter a proof system and semantics has been described for a wide family of propositional MLDSs, and brief discussion of and comparison with related work has already been given. However, some general points need to be emphasized.

The main features of the proof system described above are that (i) it uses an explicit syntactic formalisation of the modal notion of truth relative to a particular "possible world" (i.e. formulae are labelled), (ii) it extends the standard notion of a modal theory from a single actual world to a configuration of local actual worlds, (iii) it incorporates different conditions on the accessibility relation in a uniform way and finally (iv) it is in natural deduction style. Although there is related proof theoretical work on propositional modal logic, there is no other deduction system in which all these four aspects co-exist.

The use of a syntactic representation of possible worlds in proof systems for modal logics is not completely new. It first appeared in [Fit52] where the modal language was extended to include symbols intended to name possible worlds in Kripke models. It has subsequently been adopted by Fitting to develop a prefixed tableaux system based on the notion of prefixed formulae. However, neither of these approaches gives a logical way of reasoning with possible worlds and with relations between possible worlds, as part of the modal system. It is this feature which allows the natural deduction system described here to be uniform, in contrast with the other forms of natural deduction for classical modal logics given in the literature (e.g. [Pra65, Fit83]).

Moreover, the MLDS approach has the advantage over Fitting's systems of allowing reasoning on both the two modal operators within the same natural deduction system. Although this might not be crucial for the family of classical modal logics (the two modal operators are interdefinable) it is essential for other families of modal logics, such as the intuitionistic modal logics, in which the non-classical semantic interpretation of negation often leads to modal extensions where the definitions of the modal operators are independent (i.e. the definition $\Box =_{\mathrm{def}} \neg\Diamond\neg$ does not hold) [PS86]. For these types of logics, the propositional MLDS approach could (with appropriate modifications to the inference rules) be a promising framework for developing a natural deduction system using both $\Box$ and $\Diamond$ operators, and with the generalised notion of an intuitionistic modal theory (configuration). From this point of view, a related approach can be found in the natural deduction system for intuitionistic modal logic developed by Simpson [Sim93]. This is analogous to the MLDSs proof system in that it is a uniform natural deduction style proof system based on a syntactic notion of relative truth. The rules for connectives reflect the semantics of intuitionistic logic, whereas the rules for modal operators are similar to those given here in that they rely on relational assumptions (i.e. positive R-literals). However, to capture different modal logics additional sets of rules are introduced which allow the inference of labelled formulae from different (sets of) relational assumptions according to the properties of the accessibility relation under consideration. The MLDS is in this respect more modular. Furthermore the MLDS allows the representation of structured modal theories with negative relations, thus covering a wider family of modal logics. It is likely that the application of the propositional MLDS methodology to the family of intuitionistic modal logics would generalise and further extend Simpson's framework.

Another related approach is Benevides' natural deduction system for modal logic

[BM90]. This is also a "constructive" approach based on extensional techniques to deal with modal operators (i.e. explicit representation of possible worlds) within the natural deduction paradigm. However the motivation behind Benevides's work is somewhat different from that here. In MLDS, the explicit representation of possible worlds and relations between possible worlds facilitates the definition of introduction and elimination rules for modal operators in the same way as classical connectives, i.e. without stating any particular side conditions on their "intentional" features[6]. As a consequence, modal operators and classical connectives can all be considered to be object level operators. In contrast, Benevides provides a constructive natural deduction system for propositional modal logic in which modal operators are presented as "higher order" connectives, fundamentally different from "object level" classical connectives. In his work a modal system is considered as a set of sets of formulae where one is the "initial set" and the others are sets generated from the initial one as "accessible sets of formulae", by the application of modal rules[7]. Each set s_i has an associated assertion sign, $\vdash_{s_i}$. "$\vdash_{s_i} A$" can be read as "the proposition A is asserted to be in the set s_i". The 'indexes' for the different sets of formulae are constructed using a function symbol $f_{\Box}$ (and also a function symbol $f_{\Diamond}$ in [Ben91]) applied repeatedly to the singleton set $\{s_0\}$ (initial world). Additional sets of rules are introduced to capture different modal systems. Again, this is an example with respect to which the propositional MLDS has the advantage of a uniform and more general (more than one initial actual world) natural deduction system for modal logic.

Another parameter for evaluation of the propositional MLDS is provided by the existing explicit approaches to modal logics. In particular, the relational translation has, in common with MLDS, the characteristic of using "binary" first-order logic to represent Kripke models. The main difference is that a MLDS combines this classical language together with a standard modal language. In this way, a complete translation into first-order logic is not required before applying a proof system. Although a detailed investigation of automated modal theorem proving is beyond the scope of this thesis, this might provide a solution to the inefficiency problem of existing [Ohl91] relational translation approaches. For example, consider the following modal theory composed of two formulae, in a modal logic K.

$$\Diamond\Diamond\Box[(\Box p \wedge q) \wedge s] \qquad (a)$$
$$\Box\neg\Diamond\Box[(\Box p \wedge q) \wedge s] \qquad (b)$$

[6]For example, given a formula of the form $\Diamond A$, its intentional feature is the fact that its meaning relies on the existence of an accessible world where A holds.

[7]The higher level of modal operators is represented by the fact that their rules define how to generate 'accessible sets of formulae'

The two formulae are inconsistent, so that it is possible to derive falsity, $\perp$. The derivation of $\perp$, in the case of a MLDS, is very straightforward, and is given by the following proof. For simplicity, the two formulae are written as (a) $\Diamond\alpha$ and (b) $\Box\neg\alpha$, where α is equal to $\Diamond\Box[(\Box p \wedge q) \wedge s]$.

$$\begin{array}{ll}
\mathcal{C}\langle W_0 : \Diamond\alpha, W_0 : \Box\neg\alpha\rangle & \Diamond\text{–E} \\
\hline
\mathcal{C}_1\langle f_\alpha(W_0) : \alpha, R(W_0, f_\alpha(W_0)), \Box\neg\alpha\rangle & \Box\text{–E} \\
\hline
\mathcal{C}_2\langle f_\alpha(W_0) : \alpha, f_\alpha(W_0) : \neg\alpha\rangle & \wedge\text{–I} \\
\hline
\mathcal{C}_3\langle f_\alpha(W_0) : \perp\rangle &
\end{array}$$

In Ohlbach's first-order translation (for example) the proof of $\perp$ is much more complex. The two formulae are first translated into first-order clauses and then an empty clause is proved from the resulting first-order theory. Only the translation into clauses is shown here. This gives an idea of the dimension and complexity of the proof. The formulae (a) and (b) generate the following sets of clauses respectively:

$$\begin{array}{ll}
\text{(a)} & R(0, a) \\
& R(a, b) \\
& \neg R(b, x) \vee S(x) \\
& \neg R(b, x) \vee Q(x) \\
& \neg R(b, x) \vee \neg R(x, y) \vee P(y)
\end{array}$$

$$\begin{array}{ll}
\text{(b)} & \neg R(0, x) \vee \neg R(x, y) \vee R(y, f(x, y)) \\
& \neg R(0, x) \vee \neg R(x, y) \neg S(f(x, y)) \\
& \neg R(0, x) \vee \neg R(x, y) \vee \neg Q(f(x, y)) \\
& \neg R(0, x) \vee \neg R(x, y) \vee R(y, g(x, y)) \\
& \neg R(0, x) \vee \neg R(x, y) \vee \neg P(g(x, y))
\end{array}$$

Hence, even for a small modal theory, the translation is complex. The disadvantage is that each resolution on a predicate different from R has to be accompanied by a chain of resolutions with the R–literals. It seems likely that direct implementations of the proof system described in this thesis could provide an more efficient automated proof system for modal logics based on a "semi" relational translation. Of course, it is not possible to draw firm conclusions from a few examples, but this does point the way to an interesting line of investigation in the future.

To conclude, the choice of a semantic approach based on a first-order translation into classical logic provides the following two advantages. Firstly, the methodology is easily applicable to any family of logics whose semantics is first-order axiomatisable. This might also facilitate the development of a model-theoretic semantics for "new" logics resulting from the "combination" of other existing logics. Moreover, from the point of view of defining an automated proof system for modal logics, the extended algebra provides an interesting alternative choice. Classical theorem provers could be developed to reason with the theory $\mathcal{A}^+$. These could exploit equivalence rewriting techniques, thus improving the systems based on relational translation.

2.4 Summary

In this chapter, a natural deduction style proof system has been defined for propositional MLDS, which is uniform in the sense that the individual deduction rules are independent from the particular labelling algebra. A semantics has been defined based on a translation into first-order logic.

Chapter 3

Soundness, Completeness and Correspondence Results

In the previous chapter the notions of a syntactic and semantic consequence relation have been defined, so completing the description of a propositional MLDS as a logical framework. Three important results are proved in this chapter which show that (i) the two notions of consequence relation are equivalent and (ii) the propositional MLDS really is a generalisation of the family $K\partial$ of standard modal logics. To show (i) it is proved that the derivability relation $\vdash_{\mathrm{MLDS}}$ is *sound* and *complete* with respect to the semantic entailment relation $\models_{\mathrm{MLDS}}$, whereas to show (ii) it is proved that any modal logic $K\partial$ is *strictly subsumed* by the corresponding propositional $K\delta$-MLDS.

Many of the propositions and theorems in this chapter are proved *reasoning by cases*. To make the proofs more readable, cases which are very similar to a previous case are omitted in the main body of the proof, and instead are listed in Appendix A.

3.1 Soundness

This section shows that the relation $\vdash_{\mathrm{MLDS}}$ implies the relation $\models_{\mathrm{MLDS}}$. In other words, it is proved that whenever there exists a natural deduction proof of a configuration $\mathcal{C}'$ from a configuration $\mathcal{C}$ then $\mathcal{C}$ semantically entails $\mathcal{C}'$. In general this type of theorem is proved by induction on the number of inference steps of the assumed derivation. However, the natural deduction feature of using temporary new assumptions in a derivation needs particular attention, since it temporarily alters the initial theory. Proofs of soundness described in the literature often adopt a particular technique, which is that of defining for each step of the derivation the notion of a *context* – i.e. the set of all the assumptions which have been introduced and not jet discharged. (For an example of a soundness proof see [Fit83], Chapter 4.) However a different technique is adopted here. The basic idea is to define the notions of size of an inference rule and of size of a proof, and apply induction on the size of the assumed derivation. In this way there is no difference (apart from the size) between the inference rules that introduce new assumptions and the ones that do not introduce new assumptions. At the inductive step the important consideration is the total size of the subproof under consideration. (This will become more clear once the formal proofs are given.)

The notions of *size of a member of an inference rule* and *size of a derivation* in a propositional MLDS are introduced below and will be used in Lemma 3.1.

Notation 3.1 In the modal labelled deductive proof system, inference rules can be classified into four categories. The first one is the singleton set

$$\mathcal{I}^{00} = \{\mathcal{I}_{C-R}\}$$

$\mathcal{I}_{C-R}$ is the only inference rule that does not infer new declarative units or new R–literals and does not use any MLDS subderivation as condition. The second category consists of inference rules that infer new declarative units and/or new R–literals without using any subderivations as conditions. This is

$$\mathcal{I}^0 = \{\mathcal{I}_{\wedge E}, \mathcal{I}_{\wedge I}, \mathcal{I}_{\vee I}, \mathcal{I}_{\rightarrow E}, \mathcal{I}_{\neg E}, \mathcal{I}_{\Diamond E}, \mathcal{I}_{\Diamond I}, \mathcal{I}_{\Box E}, \mathcal{I}_{R-A}, \mathcal{I}_{\bot I}\}$$

The third category consists of those inference rules that require one subderivation as a condition and it is defined as

$$\mathcal{I}^+ = \{\mathcal{I}_{\rightarrow I}, \mathcal{I}_{\neg I}, \mathcal{I}_{\Box I}, \mathcal{I}_{R-I}\}$$

The fourth category refers to inference rules that use two subderivations as conditions. This is the singleton set

$$\mathcal{I}^{++} = \{\mathcal{I}_{\vee E}\}$$

Note that $\mathcal{I}^{00} \cup \mathcal{I}^0 \cup \mathcal{I}^+ \cup \mathcal{I}^{++} = \mathcal{R}$. ◁

Definition 3.1 {Size of a member of an inference rule} Let $K\delta$-MLDS be an arbitrary MLDS, let $\mathcal{I}_i \in \mathcal{R}$ and let $\mathcal{C}/\mathcal{C}' \in \mathcal{I}_i$. The *size* of $\mathcal{C}/\mathcal{C}'$ with respect to $\mathcal{I}_i$, written $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_i)$, is defined as follows:

- If $\mathcal{I}_i \in \mathcal{I}^{00}$ then $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_i) = 0$

- If $\mathcal{I}_i \in \mathcal{I}^0$, then $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_i) = 1$.

- If $\mathcal{I}_i \in \mathcal{I}^+$, then $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_i) = 1 + l_1$ where l_1 is the smallest of the sizes of all subderivations (defined below) that can be used as a condition of the rule.

- If $\mathcal{I}_i \in \mathcal{I}^{++}$, then $l(\mathcal{C}/\mathcal{C}', \mathcal{I}_i) = 1 + l_1 + l_2$ where l_1 is the smallest of the sizes of all subderivations that can be used as its first condition and l_2 is the smallest of the sizes of all subderivations that can be used as its second condition.

□

Definition 3.2 {Size of a proof} Let $K\delta$-MLDS be an arbitrary MLDS. The *size of a proof* $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle$, written $l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle)$, is defined as follows:

$$l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle) = \sum_{k=0}^{n-1} l(\mathcal{C}_k/\mathcal{C}_{k+1}, m(k))$$

□

Given that the semantics is based on a first-order translation method, the proof of the soundness property of $\vdash_{\text{MLDS}}$ with respect to $\models_{\text{MLDS}}$ will be based on the soundness of the first-order classical derivability relation $\vdash_{FOL}$. The formal statement of the theorem is given below and a brief explanatory introduction and intermediate lemmas will precede the actual proof.

Theorem 3.1 {Soundness} Let $K\delta$-MLDS be a arbitrary MLDS and let $\mathcal{C}$, $\mathcal{C}'$ be two configurations of $K\delta$-MLDS. If $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$ then $\mathcal{C} \models_{\text{MLDS}} \mathcal{C}'$.

A graphical representation of the main steps of the proof is given in the following diagram (Figure 3.1). The soundness statement, which corresponds to the arrow labelled with (1), is proved by the composition of three main steps, arrows (2), (3) and (4) respectively. The first step (arrow (2)) proves that the hypothesis, $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$, implies that $\mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$. This is proved in Lemma 3.1 given below. This result implies (given the soundness of first-order logic) that $\mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}')$ (Proposition 3.1), which gives the second step of the proof (arrow (3)). Arrow (4) is given by the definition of the semantic entailment between configurations (Definition 2.31).

$$\begin{tikzcd}
\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}' \arrow[r, "(1)"] \arrow[d, "(2)"'] & \mathcal{C} \models_{\text{MLDS}} \mathcal{C}' \\
\mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}') \arrow[r, "(3)"'] & \mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}') \arrow[u, "(4)"']
\end{tikzcd}$$

Figure 3.1: Proof of Soundness Theorem

Proposition 3.1 {Classical soundness} Let $\mathcal{A}^+$ be the extended algebra of an arbitrary $K\delta$-MLDS and let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$ and $\mathcal{C}' = \langle \mathcal{D}', \mathcal{F}' \rangle$ be two configurations. If $\mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$ then $\mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}')$.

Proof:
The hypothesis $\mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$ means that $\mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{FOL} \Delta$ for each $\Delta \in \mathcal{D}'$ and $\mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{FOL} [\alpha]^*(\lambda)$ for each $[\alpha]^*(\lambda) \in \mathcal{DU}'$. Then by soundness of first-order logic, $\mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{FOL} \Delta$ (for each $\Delta \in \mathcal{D}'$) and $\mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{FOL} [\alpha]^*(\lambda)$ (for each $[\alpha]^*(\lambda) \in \mathcal{DU}'$). Hence $\mathcal{A}^+ \cup FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}')$. $\star$

Proposition 3.2 Let $\mathcal{A}^+$ be the extended algebra of an arbitrary $K\delta$-MLDS. Let $\langle \{\mathcal{C}_0, \ldots, \mathcal{C}_k, \ldots, \mathcal{C}_n\}, m \rangle$ be a proof where $k \geq 0$ and $n > k$. Let $m(j) = \mathcal{I}_{C-R}$ for all $k \leq j < n$ and let $\mathcal{A}^+ \cup FOT(\mathcal{C}_0) \vdash_{FOL} FOT(\mathcal{C}_k)$. Then $\mathcal{A}^+ \cup FOT(\mathcal{C}_0) \vdash_{FOL} FOT(\mathcal{C}_n)$.

Proof:
For all j, $k \leq j < n$, $\mathcal{C}_{j+1} \subseteq \mathcal{C}_j$, since $m(j) = \mathcal{I}_{C-R}$, then $\mathcal{C}_n \subseteq \mathcal{C}_k$ and, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+ \cup FOT(\mathcal{C}_k) \vdash_{FOL} FOT(\mathcal{C}_n)$. Moreover by hypothesis and reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+ \cup FOT(\mathcal{C}_0) \vdash_{FOL} \mathcal{A}^+ \cup FOT(\mathcal{C}_k)$. Hence, by transitivity of $\vdash_{FOL}$, $\mathcal{A}^+ \cup FOT(\mathcal{C}_0) \vdash_{FOL} FOT(\mathcal{C}_n)$. $\star$

Lemma 3.1 {Soundness with respect to translations} Let $\mathcal{A}^+$ be the extended algebra of an arbitrary $K\delta$-MLDS. Let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations and let $FOT(\mathcal{C})$ and $FOT(\mathcal{C}')$ be their respective first-order translations. If $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$ then $\mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$.

Proof:

Proof is by induction on the smallest size of derivations of the form $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, \overline{m}\rangle$, where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_n = \mathcal{C}'$. In what follows, $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle$ is a proof of this smallest size.

BASE CASE:
When $l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle) = 0$, $\mathcal{C}_n \subseteq \mathcal{C}_0$ (by Definition 3.2), and $FOT(\mathcal{C}_n) \subseteq FOT(\mathcal{C}_0)$. Hence $\mathcal{A}^+ \cup FOT(\mathcal{C}_0) \vdash_{FOL} FOT(\mathcal{C}_n)$.

INDUCTIVE STEP:
Suppose that $l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle) = L$, $L > 0$, and the lemma is true whenever there is a smallest derivation of size less than L. Without loss of generality, it can be assumed that $m(n-1) \neq \mathcal{I}_{C-R}$[1]. Then there are two cases to consider:

(*i*) $n = 1$. In this case $n - 1 = 0$ and $l(\mathcal{C}_{n-1}/\mathcal{C}_n, m(n-1)) = L$. It remains to show by cases that $\mathcal{A}^+ \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_n)$ for any rule $m(n-1) \in \mathcal{R} - \{\mathcal{I}_{C-R}\}$.

(*ii*) $n > 1$. In this case $0 < l(\mathcal{C}_{n-1}/\mathcal{C}_n, m(n-1)) \leq L$ and $0 \leq l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_{n-1}\}, m'\rangle) < L$ (where $m'(i) = m(i)$ for all $0 \leq i \leq n-2$). Hence by the inductive hypothesis, $\mathcal{A}^+ \cup FOT(\mathcal{C}_0) \vdash_{FOL} \mathcal{A}^+ \cup FOT(\mathcal{C}_{n-1})$. So as in case (*i*), it remains to show by cases that $\mathcal{A}^+ \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_n)$ for any rule $m(n-1) \in \mathcal{R} - \{\mathcal{I}_{C-R}\}$, since then, $\mathcal{A}^+ \cup FOT(\mathcal{C}_0) \vdash_{FOL} FOT(\mathcal{C}_n)$.

Case 1: $\wedge$–**Elimination** ($\mathcal{I}_{\wedge E}$)
In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\wedge E}$. Then there exists a declarative unit of the form $\lambda : \alpha \wedge \beta \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is either equal to $\mathcal{C}_{n-1} + [\lambda : \alpha]$ or to $\mathcal{C}_{n-1} + [\lambda : \beta]$. Only the first case is considered since the argument for the second case is analogous. Therefore, $[\alpha \wedge \beta]^*(\lambda) \in FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\}$. Since $\mathcal{A}^+ \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+ \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda)$. This is proved applying axiom schema ($Ax1$), as shown in the following derivation:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{A}^+ \\ \vdots \\ [\alpha \wedge \beta]^*(\lambda) \rightarrow ([\alpha]^*(\lambda) \wedge [\beta]^*(\lambda))\end{matrix} \qquad \begin{matrix}FOT(\mathcal{C}_{n-1}) \\ \vdots \\ [\alpha \wedge \beta]^*(\lambda)\end{matrix}}{[\alpha]^*(\lambda) \wedge [\beta]^*(\lambda)}}{[\alpha]^*(\lambda)}$$

[1]This is because Proposition 3.2 allows us to extend the end of any proof with any finite number of applications of $\mathcal{I}_{C-R}$ without affecting the statement of the lemma, or the size of the proof.

Case 2: $\diamond$–**Elimination** ($\mathcal{I}_{\diamond E}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\diamond E}$. Then there exists a declarative unit of the form $\lambda : \diamond\alpha \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [f_\alpha(\lambda) : \alpha] + [R(\lambda, f_\alpha(\lambda)]$ (where $f_\alpha(\lambda)$ is a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$). Therefore $\{[\diamond\alpha]^*(\lambda)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{R(\lambda, f_\alpha(\lambda)), [\alpha]^*(f_\alpha(\lambda))\}$.
Since $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} R(\lambda, f_\alpha(\lambda))$ and $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(f_\alpha(\lambda))$. These are proved applying axiom schema $(Ax6)$, as shown in the following derivation:

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{A}^+ \\ \vdots \\ \forall x([\diamond\alpha]^*(x) \rightarrow (R(x, f_\alpha(x)) \wedge [\alpha]^*(f_\alpha(x))))\end{array}}{([\diamond]^*(\lambda) \rightarrow (R(\lambda, f_\alpha(\lambda)) \wedge [\alpha]^*(f_\alpha(\lambda))))} \qquad \begin{array}{c} FOT(\mathcal{C}_{n-1}) \\ \vdots \\ \vdots \\ [\diamond\alpha]^*(\lambda)\end{array}}{R(\lambda, f_\alpha(\lambda)) \wedge [\alpha]^*(f_\alpha(\lambda))}
$$

Cases 3-8 ($\mathcal{I}_{\wedge I}$, $\mathcal{I}_{\vee I}$, $\mathcal{I}_{\rightarrow E}$, $\mathcal{I}_{\neg E}$, $\mathcal{I}_{\diamond I}$ and $\mathcal{I}_{\Box E}$) are analogous to Case 2. See Appendix A.1.

Case 9: $\perp$–**Introduction** ($\mathcal{I}_{\perp I}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\perp I}$. Then there exists an R–literal Δ such that $\Delta \in \mathcal{C}_{n-1}$ and $\neg\Delta \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \alpha]$ where $\lambda : \alpha$ is any declarative unit. Therefore $\{\Delta, \neg\Delta\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\}$. Since $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to prove that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda)$. Since $\{\Delta, \neg\Delta\} \subseteq FOT(\mathcal{C}_{n-1})$, $FOT(\mathcal{C}_{n-1}) \vdash_{FOL} \perp$. Hence, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda)$.

Case 10: R-**Assertion** ($\mathcal{I}_{R-A}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{R-A}$. Let $\mathcal{C}_{n-1} = \langle \mathcal{D}_{n-1}, \mathcal{F}_{n-1} \rangle$. Then there exists an R–literal Δ such that there exists a first–order derivation $\mathcal{D}_{n-1}, \mathcal{A} \vdash_{FOL} \Delta$ and such that $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\Delta]$. Therefore $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{\Delta\}$. Since $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} \Delta$. Since $\mathcal{D}_{n-1} \subseteq FOT(\mathcal{C}_{n-1})$, $\mathcal{A} \subseteq \mathcal{A}^+$ and $\mathcal{D}_{n-1}, \mathcal{A} \vdash_{FOL} \Delta$, by monotonicity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} \Delta$.

Case 11: $\Box$–**Introduction** ($\mathcal{I}_{\Box I}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\Box I}$. Then there exists an R–literal of the form $R(\lambda, box_\alpha(\lambda))$ and a wff α such that $\mathcal{C}_{n-1} + [R(\lambda, box_\alpha(\lambda))] \vdash_{\mathrm{MLDS}} box_\alpha(\lambda) : \alpha$

and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1}+[\lambda:\Box\alpha]$. Therefore $FOT(\mathcal{C}_n)=FOT(\mathcal{C}_{n-1})\cup\{[\Box\alpha]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+\cup FOT(\mathcal{C}_{n-1})\vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+\cup FOT(\mathcal{C}_{n-1})\vdash_{FOL}[\Box\alpha]^*(\lambda)$. Let $\langle\{\mathcal{C}_{n-1}+[R(\lambda,box_\alpha(\lambda))],\ldots,\tilde{\mathcal{C}}\},\tilde{m}\rangle$, with $box_\alpha(\lambda):\alpha\in\tilde{\mathcal{C}}$, be a proof of the smallest size of $\mathcal{C}_{n-1}+[R(\lambda,box_\alpha(\lambda))]\vdash_{\mathrm{MLDS}} box_\alpha(\lambda):\alpha$. By hypothesis of the inductive step, $0<l(\mathcal{C}_{n-1}/\mathcal{C}_n,\mathcal{I}_{\Box I})=1+l_1\leq L$. Then $0\leq l_1=l(\langle\{\mathcal{C}_{n-1}+[R(\lambda,box_\alpha(\lambda))],\ldots,\tilde{\mathcal{C}}\},\tilde{m}\rangle)<L$. By inductive hypothesis, $\mathcal{A}^+\cup FOT(\mathcal{C}_{n-1})\cup\{R(\lambda,box_\alpha(\lambda))\}\vdash_{FOL} FOT(\tilde{\mathcal{C}})$ and in particular $\mathcal{A}^+\cup FOT(\mathcal{C}_{n-1})\cup\{R(\lambda,box_\alpha(\lambda))\}\vdash_{FOL}[\alpha]^*(box_\alpha(\lambda))$. By the Deduction Theorem of first-order logic, $\mathcal{A}^+\cup FOT(\mathcal{C}_{n-1})\vdash_{FOL} R(\lambda,box_\alpha(\lambda))\rightarrow[\alpha]^*(box_\alpha(\lambda))$. So by axiom schema (Ax8), $\mathcal{A}^+, FOT(\mathcal{C}_{n-1})\vdash_{FOL}[\Box\alpha]^*(\lambda)$ as shown in the following derivation:

$$\dfrac{\begin{array}{c}\mathcal{A}^+\\ \vdots\\ (\mathrm{Ax8})\end{array}\qquad\begin{array}{c}\mathcal{A}^+\cup FOT(\mathcal{C}_{n-1})\\ \vdots\\ R(\lambda,box_\alpha(\lambda))\rightarrow[\alpha]^*(box_\alpha(\lambda))\end{array}}{[\Box\alpha]^*(\lambda)}$$

Cases 12-13 ($\mathcal{I}_{\rightarrow I}$ and $\mathcal{I}_{\neg I}$) are analogous to Case 11. See Appendix A.1.

Case 14: R–**Introduction** ($\mathcal{I}_{R-I}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n\in\mathcal{I}_{R-I}$. Then there exists an R–literal Δ such that $\mathcal{C}_{n-1}+[\overline{\Delta}]\vdash_{\mathrm{MLDS}}\lambda':\bot$ for some term λ', and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1}+[\Delta]$. Therefore $FOT(\mathcal{C}_n)=FOT(\mathcal{C}_{n-1})\cup\{\Delta\}$. Since $\mathcal{A}^+, FOT(\mathcal{C}_{n-1})\vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1})\vdash_{FOL}\Delta$. Let $\langle\{\mathcal{C}_{n-1}+[\overline{\Delta}],\ldots,\tilde{\mathcal{C}}\},\tilde{m}\rangle$, with $\lambda':\bot\in\tilde{\mathcal{C}}$, be a proof of the smallest size of $\mathcal{C}_{n-1}+[\overline{\Delta}]\vdash_{\mathrm{MLDS}}\lambda':\bot$. By hypothesis of the inductive step, $0<l(\mathcal{C}_{n-1}/\mathcal{C}_n,\mathcal{I}_{R-I})=1+l_1\leq L$. Then $0\leq l_1=l(\langle\{\mathcal{C}_{n-1}+[\overline{\Delta}],\ldots,\tilde{\mathcal{C}}\},\tilde{m}\rangle)<L$. By inductive hypothesis, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1})\cup\{\overline{\Delta}\}\vdash_{FOL} FOT(\tilde{\mathcal{C}})$ and in particular $\mathcal{A}^+, FOT(\mathcal{C}_{n-1})\cup\{\overline{\Delta}\}\vdash_{FOL}\bot$. By the Deduction Theorem of first-order logic, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1})\vdash_{FOL}\overline{\Delta}\rightarrow\bot$. Since $\overline{\Delta}\rightarrow\bot$ is equivalent to $\neg\overline{\Delta}$, which is also equivalent to Δ, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1})\vdash_{FOL}\Delta$.

Case 15: $\vee$–**Elimination** ($\mathcal{I}_{\vee E}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n\in\mathcal{I}_{\vee E}$. Then there exist declarative units of the form $\lambda:\alpha\vee\beta$, $\lambda:\alpha$, $\lambda:\beta$, and $\lambda:\gamma$, such that $\lambda:\alpha\vee\beta\in\mathcal{C}_{n-1}$, $\mathcal{C}_{n-1}+[\lambda:\alpha]\vdash_{\mathrm{MLDS}}\lambda:\gamma$, $\mathcal{C}_{n-1}+[\lambda:\beta]\vdash_{\mathrm{MLDS}}\lambda:\gamma$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1}+[\lambda:\gamma]$. Therefore $FOT(\mathcal{C}_n)=FOT(\mathcal{C}_{n-1})\cup\{[\gamma]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+\cup FOT(\mathcal{C}_{n-1})\vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+\cup FOT(\mathcal{C}_{n-1})\vdash_{FOL}[\gamma]^*(\lambda)$. Let $\langle\{\mathcal{C}_{n-1}+[\lambda:\alpha],\ldots,\tilde{\mathcal{C}}\},\tilde{m}\rangle$, with $\lambda:\gamma\in\tilde{\mathcal{C}}$, be a proof of the smallest size of $\mathcal{C}_{n-1} + [\lambda:\alpha] \vdash_{\text{MLDS}} \lambda:\gamma$ (the first condition of $\mathcal{I}_{\vee I}$). Analogously, let $\langle\{\mathcal{C}_{n-1} + [\lambda:\beta], \ldots, \tilde{\mathcal{C}'}\}, \tilde{m}'\rangle$, with $\lambda:\gamma \in \tilde{\mathcal{C}'}$, be a proof of the smallest size of $\mathcal{C}_{n-1} + [\lambda:\beta] \vdash_{\text{MLDS}} \lambda:\gamma$ (the second condition of $\mathcal{I}_{\vee I}$). By hypothesis of the inductive step, $0 < l(\mathcal{C}_{n-1}/\mathcal{C}_n, \mathcal{I}_{\vee I}) = 1 + l_1 + l_2 \leq L$. Then

$$0 \leq l_1 = l(\langle\{\mathcal{C}_{n-1} + [\lambda:\alpha], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle) < L$$
$$0 \leq l_2 = l(\langle\{\mathcal{C}_{n-1} + [\lambda:\beta], \ldots, \tilde{\mathcal{C}'}\}, \tilde{m}'\rangle) < L$$

By inductive hypothesis,

$$\mathcal{A}^+ \cup FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\} \vdash_{FOL} [\gamma]^*(\lambda)$$
$$\mathcal{A}^+ \cup FOT(\mathcal{C}_{n-1}) \cup \{[\beta]^*(\lambda)\} \vdash_{FOL} [\gamma]^*(\lambda)$$

By the Deduction Theorem of first-order logic,

$$\mathcal{A}^+ \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda) \rightarrow [\gamma]^*(\lambda)$$
$$\mathcal{A}^+ \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\beta]^*(\lambda) \rightarrow [\gamma]^*(\lambda)$$

So by axiom schema $(Ax3)$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\gamma]^*(\lambda)$ as shown in the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{cc}\mathcal{A}^+ \cup FOT(\mathcal{C}_{n-1}) & \mathcal{A}^+ \cup FOT(\mathcal{C}_{n-1}) \\ \vdots & \vdots \\ [\alpha]^*(\lambda) \rightarrow [\gamma]^*(\lambda) & [\beta]^*(\lambda) \rightarrow [\gamma]^*(\lambda)\end{array}}{((([\alpha]^*(\lambda) \rightarrow [\gamma]^*(\lambda)) \wedge ([\beta]^*(\lambda) \rightarrow [\gamma]^*(\lambda)))}}{([\alpha]^*(\lambda) \vee [\beta]^*(\lambda)) \rightarrow [\gamma]^*(\lambda)} \qquad \begin{array}{c}\mathcal{A}^+ \cup FOT(\mathcal{C}_{n-1}) \\ \vdots \\ \vdots \\ \vdots \\ [\alpha]^*(\lambda) \vee [\beta]^*(\lambda)\end{array}}{[\gamma]^*(\lambda)}$$

⋆

Proof of Theorem 3.1:

By hypothesis

$$\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$$

By Lemma 3.1

$$\mathcal{A}^+ \cup FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$$

where $\mathcal{A}^+$ is the extended algebra associated with the arbitrary $K\delta$-MLDS, and $FOT(\mathcal{C})$, $FOT(\mathcal{C}')$ are the respective first-order translations of $\mathcal{C}$ and $\mathcal{C}'$. By Proposition 3.1

$$\mathcal{A}^+, FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}')$$

Hence by Definition 2.31,

$$\mathcal{C} \models_{\text{MLDS}} \mathcal{C}'$$

⋆

3.2 Completeness

This section shows that the derivability relation $\vdash_{\text{MLDS}}$ is complete with respect to the semantic entailment $\models_{\text{MLDS}}$. In other words it is proved that given an arbitrary $K\delta$-MLDS, if a configuration $\mathcal{C}$ semantically entails a configuration $\mathcal{C}'$ then $\mathcal{C}'$ is derivable from $\mathcal{C}$. The proof is based on a Henkin style methodology (for a more general discussion of the method see [HC68]). Preliminary definitions, propositions and theorems are given in the next section, where the notion of *consistency* is described, and some useful associated properties are shown. Then in Section 3.2.3 the construction of a maximal consistent configuration is given and various properties of this particular configuration are proved. Finally, the main lemma (known in the literature as the *model existence lemma*) of the completeness theorem is proved together with the theorem itself.

3.2.1 Preliminary Notions

One of the basic notions regarding completeness is that of a *consistent* theory. In standard modal logic a theory is consistent if no contradiction can be derived from it. Analogously, in a propositional MLDS a configuration is consistent if no declarative unit of the form $\lambda : \perp$ is derivable from it, for any label λ of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$. A formal definition is given below.

Definition 3.3 {Inconsistent configuration} Let $K\delta$-MLDS be an arbitrary MLDS and let $\mathcal{C}$ be a configuration of $K\delta$-MLDS. $\mathcal{C}$ is *inconsistent* if $\mathcal{C} \vdash_{\text{MLDS}} \lambda : \perp$ for some ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$. □

Definition 3.4 {Consistent configuration} Let $K\delta$-MLDS be a MLDS and let $\mathcal{C}$ be a configuration of $K\delta$-MLDS. $\mathcal{C}$ is *consistent* if it is not inconsistent. □

In the definition of a configuration, labels are terms of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$. The semantics given in the previous chapter shows that the terms generated by the function symbols f_α and box_α are strictly related to the interpretation of the modal operators. This relationship implies that the occurrences of labels of the form $f_\alpha(\lambda)$ and $box_\alpha(\lambda)$ in a MLDS theory ought to satisfy certain conditions in order to provide a consistent configuration. This is shown in the following proposition.

Proposition 3.3 {Constraints on consistent configurations} Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F} \rangle$ be a consistent configuration of an arbitrary $K\delta$-MLDS. Let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let α be a wff of $\mathcal{L}_M$. The following two statements hold.

1. If $\Diamond\alpha \in \mathcal{F}(\lambda)$ then $\neg R(\lambda, f_\alpha(\lambda)) \notin \mathcal{D}$ and $\neg\alpha \notin \mathcal{F}(f_\alpha(\lambda))$
2. If $\neg\Box\alpha \in \mathcal{F}(\lambda)$ then $\neg R(\lambda, box_\alpha(\lambda)) \notin \mathcal{D}$ and $\alpha \notin \mathcal{F}(box_\alpha(\lambda))$

Proof:
The proof is by contradiction.

1. Suppose that $\Diamond\alpha \in \mathcal{F}(\lambda)$ and that $\neg R(\lambda, f_\alpha(\lambda)) \in \mathcal{D}$. Then $\mathcal{C} \vdash_{\text{MLDS}} \lambda \colon \bot$, as shown in the following derivation. Hence $\mathcal{C}$ is inconsistent which is in contradiction with the original hypothesis.

$$\begin{array}{ll} \mathcal{C}\langle \lambda \colon \Diamond\alpha, \neg R(\lambda, f_\alpha(\lambda))\rangle & \Diamond\text{-E} \\ \hline \mathcal{C}'_1\langle R(\lambda, f_\alpha(\lambda)), \neg R(\lambda, f_\alpha(\lambda)), f_\alpha(\lambda) \colon \alpha\rangle & \bot\text{-I} \\ \hline \mathcal{C}'_2\langle \lambda \colon \bot\rangle & \end{array}$$

Now suppose that $\Diamond\alpha \in \mathcal{F}(\lambda)$ and that $\neg\alpha \in \mathcal{F}(f_\alpha(\lambda))$. Then $\mathcal{C} \vdash_{\text{MLDS}} f_\alpha(\lambda) : \bot$ as shown in the following derivation. Hence $\mathcal{C}$ is inconsistent which is in contradiction with the original hypothesis.

$$\begin{array}{ll} \mathcal{C}\langle \lambda \colon \Diamond\alpha, f_\alpha(\lambda) \colon \neg\alpha\rangle & \Diamond\text{-E} \\ \hline \mathcal{C}'_1\langle R(\lambda, f_\alpha(\lambda)), f_\alpha(\lambda) \colon \alpha, f_\alpha(\lambda) \colon \neg\alpha\rangle & \wedge\text{-I} \\ \hline \mathcal{C}'_2\langle f_\alpha(\lambda) \colon \bot\rangle & \end{array}$$

2. Suppose that $\neg\Box\alpha \in \mathcal{F}(\lambda)$ and that $\neg R(\lambda, box_\alpha(\lambda)) \in \mathcal{D}$. Then $\mathcal{C} \vdash_{\text{MLDS}} \lambda \colon \bot$ as shown in the following derivation. Hence $\mathcal{C}$ is inconsistent which is in contradiction with the original hypothesis.

$$\begin{array}{ll} \mathcal{C}\langle \lambda \colon \neg\Box\alpha, \neg R(\lambda, box_\alpha(\lambda))\rangle & \\ \hline \quad \mathcal{C}\langle \neg R(\lambda, box_\alpha(\lambda))\rangle + [R(\lambda, box_\alpha(\lambda))] & \text{(assumption)} \\ \hline \quad \mathcal{C}_1\langle box_\alpha(\lambda) \colon \alpha\rangle & (\bot\text{–I}) \\ \hline \mathcal{C}'_1\langle \lambda \colon \neg\Box\alpha, \lambda \colon \Box\alpha\rangle & (\Box\text{–I}) \\ \hline \mathcal{C}'_2\langle \lambda \colon \bot\rangle & (\wedge\text{–I}) \end{array}$$

Now suppose that $\neg\Box\alpha \in \mathcal{F}(\lambda)$ and that $\alpha \in \mathcal{F}(box_\alpha(\lambda))$. Then $\mathcal{C} \vdash_{\text{MLDS}} \lambda \colon \bot$ as shown in the following derivation. Hence $\mathcal{C}$ is inconsistent which is in contradiction with the original hypothesis.

$$\begin{array}{ll} \mathcal{C}\langle \lambda : \neg\Box\alpha, box_\alpha(\lambda) : \alpha\rangle & \quad \Box\text{-I} \\ \hline \mathcal{C}_1'\langle \lambda : \Box\alpha, \lambda : \neg\Box\alpha\rangle & \quad \wedge\text{-I} \\ \hline \mathcal{C}_2'\langle \lambda : \bot\rangle & \end{array}$$

⋆

Some properties of the derivability relation $\vdash_{\text{MLDS}}$ are proved below, which will be used later in this chapter for the completeness theorem. In particular, it is shown that $\vdash_{\text{MLDS}}$ satisfies the standard properties of a classical derivability relation (i.e. reflexivity, monotonicity and transitivity) as well as the compactness property.

Proposition 3.4 {Monotonicity} Let $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ be three configurations of an arbitrary $K\delta$-MLDS such that $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$ and $\mathcal{C} \subseteq \mathcal{C}''$. Then $\mathcal{C}'' \vdash_{\text{MLDS}} \mathcal{C}'$.

Proof:
Let $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle$ (where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_n = \mathcal{C}'$) be a proof of $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$. By hypothesis $\mathcal{C} \subseteq \mathcal{C}''$, so $\mathcal{C}''/\mathcal{C} \in \mathcal{I}_{C-R}$. Let m' be a mapping from the set $\{0, \ldots, n\}$ to the set $\mathcal{R}$ such that $m'(0) = \mathcal{I}_{C-R}$ and for each i, $1 \leq i \leq n$, $m'(i) = m(i-1)$. Then the pair $\langle\{\mathcal{C}'', \mathcal{C} \ldots, \mathcal{C}'\}, m'\rangle$ is a proof in $K\delta$-MLDS. Hence $\mathcal{C}'' \vdash_{\text{MLDS}} \mathcal{C}'$. ⋆

Proposition 3.5 {Reflexivity} Let $\mathcal{C}$ be a configuration of an arbitrary $K\delta$-MLDS, then $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}$.

Proof:
By Definition 2.23, $\mathcal{C}/\mathcal{C} \in \mathcal{I}_{C-R}$. Then let m be a mapping from the set $\{0\}$ to the set $\mathcal{R}$ such that $m(0) = \mathcal{I}_{C-R}$. The pair $\langle\{\mathcal{C}, \mathcal{C}\}, m\rangle$ is a proof in $K\delta$-MLDS. Hence $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}$. ⋆

Proposition 3.6 {Transitivity} Let $\mathcal{C}$,$\mathcal{C}'$,$\mathcal{C}''$ be three configurations of an arbitrary $K\delta$-MLDS such that $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$ and $\mathcal{C}' \vdash_{\text{MLDS}} \mathcal{C}''$. Then $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}''$.

Proof:
Let the pair $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_h\}, m\rangle$ (where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_h = \mathcal{C}'$) be a proof of $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$ and the pair $\langle\{\mathcal{C}_0', \ldots, \mathcal{C}_k'\}, m'\rangle$ (where $\mathcal{C}_0' = \mathcal{C}'$ and $\mathcal{C}_k' = \mathcal{C}''$) be a proof of $\mathcal{C}' \vdash_{\text{MLDS}} \mathcal{C}''$. Let $\overline{m}$ be a mapping from the set $\{0, \ldots, h + k - 1\}$ to the set $\mathcal{R}$ such that for each i, $0 \leq i \leq h-1$, $\overline{m}(i) = m(i)$, and for each i, $h \leq i \leq h+k-1$, $\overline{m}(i) = m'(i-h)$. Then the pair $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_h, \ldots, \mathcal{C}_k'\}, \overline{m}\rangle$ is a proof in $K\delta$-MLDS. Hence $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}''$. ⋆

Note that monotonicity, reflexivity and transitivity properties of the derivability relation $\vdash_{\text{MLDS}}$ hold for both finite and infinite configurations.

In the previous chapter, notation has been introduced to capture the standard notion of a derivability relation between theories (configurations) and formulae (single units of information). This has been expressed in terms of the more general definition of the derivability relation between two configurations given by $\vdash_{\text{MLDS}}$. A "vice-versa" characterisation can be shown – a configuration $\mathcal{C}'$ is derivable from a configuration $\mathcal{C}$ if each unit of information of $\mathcal{C}'$ is derivable from $\mathcal{C}$. This result (stated in the theorem below) also provides a characterisation of non-derivability of a configuration from another, which will be used in the proof by contrapositive of the completeness theorem.

Lemma 3.2 Let $K\delta$-MLDS be an arbitrary MLDS, let $\mathcal{I}$ be an inference rule and $\mathcal{C}$, $\mathcal{C}'$ and $\overline{\mathcal{C}}$ be three configurations of $K\delta$-MLDS. Let $\mathcal{C}/\mathcal{C}' \in \mathcal{I}$. Then $(\mathcal{C} \cup \overline{\mathcal{C}})/(\mathcal{C}' \cup \overline{\mathcal{C}}) \in \mathcal{I}$.

Proof:
This follows directly from Definition 2.1 (definition of $\mathcal{I}$) and from Proposition 3.4 (monotonicity). $\star$

Theorem 3.2 {Characterisation of derivability} Let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations of an arbitrary $K\delta$-MLDS such that the configuration $\mathcal{C}' - \mathcal{C}$ is finite. $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$ if and only if for each $\pi \in \mathcal{C}' - \mathcal{C}$, $\mathcal{C} \vdash_{\text{MLDS}} \pi$, where π is either a declarative unit or an R–literal.

Proof:
("Only if" half:) By hypothesis $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$. If $\pi \in \mathcal{C}' - \mathcal{C}$ then $\pi \in \mathcal{C}'$. Hence by Notation 2.1 for each $\pi \in \mathcal{C}' - \mathcal{C}$, $\mathcal{C} \vdash_{\text{MLDS}} \pi$.

("If" half:) To prove that $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$, it is necessary to show that there exists a proof $\langle\{\mathcal{C}, \ldots, \mathcal{C}'\}, m\rangle$. Let $\pi_1, \pi_2, \pi_3, \ldots, \pi_n$ be a (possibly empty) enumeration of all the elements (declarative units and R–literals) of the configuration $\mathcal{C}' - \mathcal{C}$. The proof is by induction on n.

BASE CASE:
The base case is when $n = 0$ (empty enumeration). Then $\mathcal{C}' \subseteq \mathcal{C}$ which implies that $\mathcal{C}/\mathcal{C}' \in \mathcal{I}_{C-R}$. So, $\langle\{\mathcal{C}_0, \mathcal{C}_1\}, m\rangle$ is a proof, where $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{C}_1 = \mathcal{C}'$ and $m(0) = \mathcal{I}_{C-R}$. Hence, $\mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$.

INDUCTIVE STEP:
Assume by the inductive hypothesis that for any pair of configurations $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}'$, such that $\tilde{\mathcal{C}}' - \tilde{\mathcal{C}} = [\tilde{\pi}_1, \ldots, \tilde{\pi}_{n-1}]$, and such that for each $\tilde{\pi} \in \tilde{\mathcal{C}}' - \tilde{\mathcal{C}}$, $\tilde{\mathcal{C}} \vdash_{\mathrm{MLDS}} \tilde{\pi}$, then $\tilde{\mathcal{C}} \vdash_{\mathrm{MLDS}} \tilde{\mathcal{C}}'$.

Let $\overline{\mathcal{C}}$ be the configuration $\mathcal{C} + [\pi_1, \ldots, \pi_{n-1}]$ and let $\mathcal{C}'' = \overline{\mathcal{C}} + [\pi_n]$, so that for each $\pi \in \mathcal{C}'' - \mathcal{C}$, $\mathcal{C} \vdash_{\mathrm{MLDS}} \pi$. Notice that by the inductive hypothesis $\mathcal{C} \vdash_{\mathrm{MLDS}} \overline{\mathcal{C}}$, and, since $\mathcal{C}' \subseteq \mathcal{C}''$, $\mathcal{C}'' \vdash_{\mathrm{MLDS}} \mathcal{C}'$. Therefore, by the transitivity property of $\vdash_{\mathrm{MLDS}}$, in order to prove that $\mathcal{C} \vdash_{\mathrm{MLDS}} \mathcal{C}'$ it is sufficient to prove that $\overline{\mathcal{C}} \vdash_{\mathrm{MLDS}} \mathcal{C}''$. By the original hypothesis, $\mathcal{C} \vdash_{\mathrm{MLDS}} \pi_n$. Since $\mathcal{C} \subseteq \overline{\mathcal{C}}$, by the monotonicity property of $\vdash_{\mathrm{MLDS}}$, $\overline{\mathcal{C}} \vdash_{\mathrm{MLDS}} \pi_n$. Then there exists a configuration $\mathcal{C}_{\pi_n}$ such that $\overline{\mathcal{C}} \vdash_{\mathrm{MLDS}} \mathcal{C}_{\pi_n}$ and $\pi_n \in \mathcal{C}_{\pi_n}$. Let

$$\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_h\}, m\rangle \qquad \text{(i)}$$

be a proof of $\overline{\mathcal{C}} \vdash_{\mathrm{MLDS}} \mathcal{C}_{\pi_n}$, where $\mathcal{C}_0 = \overline{\mathcal{C}}$ and $\mathcal{C}_h = \mathcal{C}_{\pi_n}$ and m is a mapping from the set $\{0, \ldots, (h-1)\}$ to the set $\mathcal{R}$. A corresponding proof

$$\langle\{\tilde{\mathcal{C}}_0, \ldots, \tilde{\mathcal{C}}_h\}, m\rangle \qquad \text{(ii)}$$

can be constructed in the following way. $\tilde{\mathcal{C}}_0 = \mathcal{C}_0$ and for each $0 \leq i \leq (h-1)$, if $m(i) = \mathcal{I}_{C-R}$ then $\tilde{\mathcal{C}}_{i+1} = \tilde{\mathcal{C}}_i$, otherwise $\tilde{\mathcal{C}}_{i+1} = \tilde{\mathcal{C}}_i \cup \mathcal{C}_{i+1}$. By Lemma 3.2, (ii) is a proof. Moreover, since in any inference rule different from $\mathcal{I}_{C-R}$ the inferred configuration contains the antecedent configuration, it follows that $\mathcal{C}_0 \subseteq \tilde{\mathcal{C}}_h$. Since $\mathcal{C}_0 = \overline{\mathcal{C}}$ and $\pi_n \in \tilde{\mathcal{C}}_h$, then $\mathcal{C}'' \subseteq \tilde{\mathcal{C}}_h$. So $\tilde{\mathcal{C}}_h/\mathcal{C}' \in \mathcal{I}_{C-R}$. Then from the proof (ii) a final proof can be constructed

$$\langle\{\tilde{\mathcal{C}}_0, \ldots, \tilde{\mathcal{C}}_h, \tilde{\mathcal{C}}_{h+1}\}, \overline{m}\rangle \qquad \text{(iii)}$$

where $\tilde{\mathcal{C}}_{h+1} = \mathcal{C}''$ and $\overline{m}$ is a mapping from the set $\{0, \ldots, h\}$ to the set $\mathcal{R}$ such that for each i, $0 \leq i \leq (h-1)$, $\overline{m}(i) = m(i)$ and $\overline{m}(h) = \mathcal{I}_{C-R}$. Hence $\overline{\mathcal{C}} \vdash_{\mathrm{MLDS}} \mathcal{C}''$. $\star$

Two important properties of a consistent configuration are shown below.

Proposition 3.7 {Consistency of sub-configurations} Let $\mathcal{C}$ be a consistent configuration of an arbitrary $K\delta$-MLDS and let π be a declarative unit or an R–literal. If $\mathcal{C} + [\pi]$ is a consistent configuration then for any configuration $\mathcal{C}'$, $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{C}' + [\pi]$ is consistent too.

Proof:
The proof is by contradiction. Suppose $\mathcal{C}' + [\pi]$ is not consistent. Then by definition of inconsistency $\mathcal{C}' + [\pi] \vdash_{\mathrm{MLDS}} \lambda : \perp$, for some ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$. Since $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{C}' + [\pi] \subseteq \mathcal{C} + [\pi]$. Then, by the monotonicity

property of $\vdash_{\text{MLDS}}$, $\mathcal{C} + [\pi] \vdash_{\text{MLDS}} \lambda\colon \bot$. Hence $\mathcal{C} + [\pi]$ is inconsistent which is in contradiction with the original hypothesis. $\star$

Proposition 3.7 shows a property of the consistency of a configuration $\mathcal{C}$ with respect to the set of its "sub–configurations" $\mathcal{C}'$ ($\mathcal{C}' \subseteq \mathcal{C}$). No assumption is made about the finiteness of $\mathcal{C}'$. A second property of a consistent configuration $\mathcal{C}$, known as *compactness* property, is proved in Corollary 3.1 (see below) which shows that a configuration is consistent if every finite sub-configuration is consistent. This is a straightforward corollary of the following theorem, which states that, because of the finiteness of a MLDS proof, only a finite part of a given (possibly infinite) configuration is used within a derivation.

Theorem 3.3 {Finiteness} Let $\mathcal{C}$ be a configuration of an arbitrary $K\delta$-MLDS. Let π be a declarative unit or an R–literal. If $\mathcal{C} \vdash_{\text{MLDS}} \pi$ then there exists a configuration $\mathcal{C}'$ such that $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{C}' \vdash_{\text{MLDS}} \pi$ and $\mathcal{C}'$ is finite.

Proof:
It is only necessary to prove the case when $\mathcal{C}$ is not finite. By hypothesis $\mathcal{C} \vdash_{\text{MLDS}} \pi$. Then by Notation 2.1, there exists a configuration $\overline{\mathcal{C}}$ such that $\mathcal{C} \vdash_{\text{MLDS}} \overline{\mathcal{C}}$ and $\pi \in \overline{\mathcal{C}}$. The proof is by induction on the smallest size of derivations of the form $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, \overline{m}\rangle$, where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_n = \overline{\mathcal{C}}$. In what follows, $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle$ is a proof of the smallest size.

BASE CASE:
The base case is when $l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle) = 0$. Then $\mathcal{C}_n \subseteq \mathcal{C}_0$. Since $\pi \in \mathcal{C}_n$, then $\pi \in \mathcal{C}_0$. If π is an R–literal then let $\mathcal{C}' = \langle\mathcal{D}', \mathcal{F}'\rangle$, where $\mathcal{D}' = \{\pi\}$ and for each ground term $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{F}'(\lambda) = \{\}$. If π is a declarative unit, $\lambda : \alpha$, then let $\mathcal{C}' = \langle\mathcal{D}', \mathcal{F}'\rangle$, where $\mathcal{D}' = \{\}$, $\mathcal{F}'(\lambda) = \{\alpha\}$ and for each ground term $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda' \neq \lambda$, $\mathcal{F}'(\lambda') = \{\}$. In both cases $\mathcal{C}'$ is finite and $\mathcal{C}' \subseteq \mathcal{C}_0$. Notice that $\pi \in \mathcal{C}'$ and $\mathcal{C}'/\mathcal{C}' \in \mathcal{I}_{C-R}$. Then $\langle\{\mathcal{C}'_0, \mathcal{C}'_1\}, m'\rangle$ is a proof, where $\mathcal{C}'_0 = \mathcal{C}'_1 = \mathcal{C}'$ and $m'(0) = \mathcal{I}_{C-R}$. Then $\mathcal{C}' \vdash_{\text{MLDS}} \mathcal{C}'$, hence $\mathcal{C}' \vdash_{\text{MLDS}} \pi$.

INDUCTIVE STEP:
Assume, by the inductive hypothesis, that for any configuration $\mathcal{C}^*$ and π^* such that there exists a smallest derivation $\langle\{\mathcal{C}^*_0, \ldots, \mathcal{C}^*_n\}, m^*\rangle$ (where $\mathcal{C}^*_0 = \mathcal{C}^*$ and $\pi^* \in \mathcal{C}^*_n$) of size $l(\langle\{\mathcal{C}^*_0, \ldots, \mathcal{C}^*_n\}, m^*\rangle) < L$, then there exists a finite configuration $\mathcal{C}^*_f$ such that $\mathcal{C}^*_f \subseteq \mathcal{C}^*_0$ and $\mathcal{C}^*_f \vdash_{\text{MLDS}} \pi^*$.

Suppose that $l(\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle) = L$, $L > 0$. Assume without loss of generality that $m(0) \neq \mathcal{I}_{C-R}$ and that $\pi \notin \mathcal{C}_0$. Then $\pi \in \mathcal{C}_n - \mathcal{C}_0$. For all j, $0 \leq j \leq n-1$, and for all $m(j) \in \mathcal{R} - \mathcal{I}_{C-R}$, the symbol $[\ ]_j = \mathcal{C}_{j+1} - \mathcal{C}_j$, represents the new declarative unit(s) and/or R–literal(s) inferred at the step j of the proof. Assume also, without loss of generality, that $n > 1$. Then $0 < l(\mathcal{C}_0/\mathcal{C}_1, m(0)) \leq L$, and $0 \leq l(\langle\{\mathcal{C}_1,\ldots,\mathcal{C}_n\}, m'\rangle) < L$ (where $m'(i) = m(i)$ for all $1 \leq i \leq n-1$)[2]. By the inductive hypothesis there exists a finite configuration $\mathcal{C}_1' \subseteq \mathcal{C}_1$, such that $\mathcal{C}_1' \vdash_{\mathrm{MLDS}} \pi$. Notice that $\mathcal{C}_1 = \mathcal{C}_0 + [\ ]_0$. So if $\mathcal{C}_1' \subseteq \mathcal{C}_0$ then the theorem is proved. Suppose now that $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$. Then, by the transitivity property of $\vdash_{\mathrm{MLDS}}$, it remains to show that there exists a finite configuration $\mathcal{C}' \subseteq \mathcal{C}_0$ such that $\mathcal{C}' \vdash_{\mathrm{MLDS}} \mathcal{C}_1'$. This is proved by cases on $m(0)$.

Case 1: $\wedge$–**Elimination**

In this case $m(0) = \mathcal{I}_{\wedge-E}$. Then there exists a declarative unit of the form $\lambda:\alpha \wedge \beta \in \mathcal{C}_0$ and $\mathcal{C}_1$ is either equal to $\mathcal{C}_0 + [\lambda:\alpha]$ or equal to $\mathcal{C}_0 + [\lambda:\beta]$. The first case only is considered, since the argument for the second case is analogous. $[\ ]_0 = [\lambda:\alpha]$. By the inductive hypothesis, $\mathcal{C}_1' \subseteq \mathcal{C}_1$ and $\mathcal{C}_1'$ is finite. By assumption $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$. So $[\lambda:\alpha] \in \mathcal{C}_1'$, but it is not necessarily the case that $\lambda:\alpha\wedge\beta \in \mathcal{C}_1'$. Then let $\mathcal{C}'$ be the configuration $(\mathcal{C}_1' - [\lambda:\alpha]) + [\lambda:\alpha\wedge\beta]$. Then $\mathcal{C}'$ is a finite configuration and $\langle\{\mathcal{C}', \mathcal{C}' + [\lambda:\alpha], \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{\wedge E}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence $\mathcal{C}' \vdash_{\mathrm{MLDS}} \mathcal{C}_1'$.

Case 2: R–**Assertion**

In this case $m(0) = \mathcal{I}_{R-A}$. Let $\mathcal{C}_0 = \langle\mathcal{D}_0, \mathcal{F}_0\rangle$. Then there exists an R–literal Δ such that $\mathcal{D}_0, \mathcal{A} \vdash_{FOL} \Delta$. $\mathcal{C}_1 = \mathcal{C}_0 + [\Delta]$. Let $\Gamma \subseteq \mathcal{D}_0$ be the set of R–literal assumptions used in the first–order derivation $\mathcal{D}_0, \mathcal{A} \vdash_{FOL} \Delta$. Since a proof in classical logic is a finite sequence of inference rules and each inference rule uses a finite set of assumptions, then Γ is finite. Notice now that $[\ ]_0 = [\Delta]$. By the inductive hypothesis, $\mathcal{C}_1' \subseteq \mathcal{C}_1$ and $\mathcal{C}_1'$ is finite. By assumption $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$. So $[\Delta] \in \mathcal{C}_1'$, but it is not necessary the case that $\Gamma \subseteq \mathcal{C}_1'$. Let $\mathcal{C}' = (\mathcal{C}_1' - [\Delta]) + \Gamma$. Then $\mathcal{C}'$ is a finite configuration and $\langle\{\mathcal{C}', \mathcal{C}' + [\Delta], \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{R-A}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence $\mathcal{C}' \vdash_{\mathrm{MLDS}} \mathcal{C}_1'$.

For Cases 3-10 ($\mathcal{I}_{\wedge I}, \mathcal{I}_{\vee I}, \mathcal{I}_{\rightarrow E}, \mathcal{I}_{\neg E}, \mathcal{I}_{\perp I}, \mathcal{I}_{\diamond E}, \mathcal{I}_{\diamond I}, \mathcal{I}_{\Box I}$) the argument is analogous to Case 1.

Case 11: $\rightarrow$–**Introduction**

In this case $m(0) = \mathcal{I}_{\rightarrow I}$. Then there exist declarative units of the form $\lambda:\alpha$, $\lambda:\beta$, $\lambda:\alpha\rightarrow\beta$ such that $\mathcal{C}_0 + [\lambda:\alpha] \vdash_{\mathrm{MLDS}} \lambda:\beta$ and $\mathcal{C}_1 = \mathcal{C}_0 + [\lambda:\alpha\rightarrow\beta]$. Then

[2]For $n = 1$ it is always possible to extend a smallest size proof $\langle\{\mathcal{C}_0, \mathcal{C}_1\}, m(0)\rangle$, where $m(0) \in \mathcal{R}$, to a proof of the same size of the form $\langle\{\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_1\}, m'\rangle$, where $m'(0) = m(0)$ and $m'(1) = \mathcal{I}_{C-R}$

$[\ ]_0 = [\lambda : \alpha \rightarrow \beta]$. By the inductive hypothesis $\mathcal{C}_1' \subseteq \mathcal{C}_1$ and $\mathcal{C}_1'$ is finite. Moreover, by assumption, $\mathcal{C}_1' \nsubseteq \mathcal{C}_0$, so $[\lambda : \alpha \rightarrow \beta] \in \mathcal{C}_1'$. Let $\langle\{\mathcal{C}_0 + [\lambda : \alpha], \ldots, \tilde{\mathcal{C}}_n\}, \tilde{m}\rangle$ (with $\lambda : \beta \in \tilde{\mathcal{C}}_n$) be a proof of the smallest size of $\mathcal{C}_0 + [\lambda : \alpha] \vdash_{\text{MLDS}} \lambda : \beta$. Since by hypothesis $0 < l(\mathcal{C}_0/\mathcal{C}_1, \mathcal{I}_{\rightarrow I}) \leq L$, then $0 \leq l(\langle\{\mathcal{C}_0 + [\lambda : \alpha], \ldots, \tilde{\mathcal{C}}_n\}, \tilde{m}\rangle) < L$. Then by the inductive hypothesis, there exists a finite configuration $\mathcal{C}_0^* \subseteq \mathcal{C}_0 + [\lambda : \alpha]$ such that $\mathcal{C}_0^* \vdash_{\text{MLDS}} \lambda : \beta$. Let $\mathcal{C}'$ be the configuration $(\mathcal{C}_1' - [\lambda : \alpha \rightarrow \beta]) \cup (\mathcal{C}_0^* - [\lambda : \alpha])$. Then $\mathcal{C}'$ is a finite configuration and $\mathcal{C}' \subseteq \mathcal{C}_0$. Since $\mathcal{C}_0^* \subseteq \mathcal{C}' + [\lambda : \alpha]$, by monotonicity $\mathcal{C}' + [\lambda : \alpha] \vdash_{\text{MLDS}} [\lambda : \beta]$. So $\mathcal{C}'/\mathcal{C}' + [\lambda : \alpha \rightarrow \beta] \in \mathcal{I}_{\rightarrow I}$. Therefore $\langle\{\mathcal{C}', \mathcal{C}' + [\lambda : \alpha \rightarrow \beta], \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{\rightarrow I}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence $\mathcal{C}' \vdash_{\text{MLDS}} \mathcal{C}_1'$.

For Cases 12-14 ($\mathcal{I}_{\Box I}$, $\mathcal{I}_{\neg I}$, $\mathcal{I}_{R-I}$) the argument is analogous to Case 11.

Case 15: $\vee$–**Elimination**

In this case $m(0) = \mathcal{I}_{\vee E}$. Then there exist declarative units of the form $\lambda : \alpha$, $\lambda : \beta$, $\lambda : \alpha \vee \beta$ and $\lambda : \gamma$ such that $\lambda : \alpha \vee \beta \in \mathcal{C}_0$, $\mathcal{C}_0 + [\lambda : \alpha] \vdash_{\text{MLDS}} \lambda : \gamma$, $\mathcal{C}_0 + [\lambda : \beta] \vdash_{\text{MLDS}} \lambda : \gamma$ and $\mathcal{C}_1$ is equal to $\mathcal{C}_0 + [\lambda : \gamma]$. Then $[\ ]_0 = [\lambda : \gamma]$. By the inductive hypothesis $\mathcal{C}_1' \subseteq \mathcal{C}_1$ and $\mathcal{C}_1'$ is finite. By assumption $\mathcal{C}_1' \nsubseteq \mathcal{C}_0$, so that $[\lambda : \gamma] \in \mathcal{C}_1'$. Let $\langle\{\mathcal{C}_0 + [\lambda : \alpha], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle$ (with $\lambda : \gamma \in \tilde{\mathcal{C}}$) be a proof of the smallest size of $\mathcal{C}_0 + [\lambda : \alpha] \vdash_{\text{MLDS}} \lambda : \gamma$. Analogously let $\langle\{\mathcal{C}_0 + [\lambda : \beta], \ldots, \tilde{\mathcal{C}}'\}, \tilde{m}'\rangle$ (with $\lambda : \gamma \in \tilde{\mathcal{C}}'$) be a proof of the smallest size of $\mathcal{C}_0 + [\lambda : \beta] \vdash_{\text{MLDS}} \lambda : \gamma$. Since by hypothesis $0 < l(\mathcal{C}_0/\mathcal{C}_1, \mathcal{I}_{\vee I}) \leq L$, by Definition 3.1

$$0 \leq l(\langle\{\mathcal{C}_0 + [\lambda : \alpha], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle) < L$$
$$0 \leq l(\langle\{\mathcal{C}_0 + [\lambda : \beta], \ldots, \tilde{\mathcal{C}}'\}, \tilde{m}'\rangle) < L$$

By the inductive hypothesis, there exist finite configurations $\mathcal{C}^* \subseteq \mathcal{C}_0 + [\lambda : \alpha]$ and $\tilde{\mathcal{C}}^* \subseteq \mathcal{C}_0 + [\lambda : \beta]$ such that $\mathcal{C}^* \vdash_{\text{MLDS}} \lambda : \gamma$ and $\tilde{\mathcal{C}}^* \vdash_{\text{MLDS}} \lambda : \gamma$. Let

$$\mathcal{C}' = ((\mathcal{C}_1' - [\lambda : \gamma]) + [\lambda : \alpha \vee \beta]) \cup (\mathcal{C}^* - [\lambda : \alpha]) \cup (\tilde{\mathcal{C}}^* - [\lambda : \beta])$$

Then $\mathcal{C}'$ is a finite configuration, $\mathcal{C}' \subseteq \mathcal{C}_0$ and $\lambda : \alpha \vee \beta \in \mathcal{C}'$. Moreover since $\mathcal{C}^* \subseteq \mathcal{C}' + [\lambda : \alpha]$, by monotonicity $\mathcal{C}' + [\lambda : \alpha] \vdash_{\text{MLDS}} \lambda : \gamma$. Analogously, since $\tilde{\mathcal{C}}^* \subseteq \mathcal{C}' + [\lambda : \beta]$, by monotonicity $\mathcal{C}' + [\lambda : \beta] \vdash_{\text{MLDS}} \lambda : \gamma$. Then $\mathcal{C}'/\mathcal{C}' + [\lambda : \gamma] \in \mathcal{I}_{\vee E}$ and $\langle\{\mathcal{C}', \mathcal{C}' + [\lambda : \gamma], \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{\vee E}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence, $\mathcal{C}' \vdash_{\text{MLDS}} \mathcal{C}_1'$. ⋆

Corollary 3.1 {Compactness} Let $\mathcal{C}$ be a configuration of an arbitrary $K\delta$-MLDS. If for any finite configuration $\mathcal{C}'$, $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{C}'$ is consistent then $\mathcal{C}$ is consistent.

Proof:
The contrapositive statement is proved. Suppose that $\mathcal{C}$ is inconsistent. By definition of inconsistency, $\mathcal{C} \vdash_{\text{MLDS}} \lambda : \perp$, for some ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$. Hence, by Theorem 3.3, there exists a finite configuration $\mathcal{C}' \subseteq \mathcal{C}$ such that $\mathcal{C}' \vdash_{\text{MLDS}} \lambda : \perp$, so that $\mathcal{C}'$ is inconsistent. $\star$

3.2.2 Description of the Completeness Proof

So far the notion of consistency has been described together with its relevant properties. These results, which hold for any arbitrary $K\delta$-MLDS, lay down the basis for proving the completeness of the derivability relation $\vdash_{\text{MLDS}}$ with respect to the semantic entailment relation $\models_{\text{MLDS}}$. Before going into the details of the formal proof, a brief description of the main steps is given to make the rest of this chapter easier to follow. Firstly, the statement of the theorem.

Theorem 3.4 {Completeness} Let $K\delta$-MLDS be an arbitrary propositional MLDS. Let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations of $K\delta$-MLDS such that the configuration difference $\mathcal{C}' - \mathcal{C}$ is finite.

$$\text{If } \ \mathcal{C} \models_{\text{MLDS}} \mathcal{C}' \ \text{ then } \ \mathcal{C} \vdash_{\text{MLDS}} \mathcal{C}'$$

The condition of finiteness of $\mathcal{C}' - \mathcal{C}$ in the above statement does not constitute a significant restriction on the completeness property of $\vdash_{\text{MLDS}}$. On the contrary, it allows $\mathcal{C}$ and $\mathcal{C}'$ to be infinite, even though derivations are finite.

The theorem is proved by contrapositive as shown by the diagram given in Figure 3.2. The proof is given by the composition of two main steps, arrows (2) and (3). Arrow (3) is straightforwardly given by Definition 2.31 whereas arrow (2) constitutes the main part of the completeness theorem.

In particular, the assumption $\mathcal{C} \not\vdash_{\text{MLDS}} \mathcal{C}'$ implies that there exists a $\pi \in \mathcal{C}' - \mathcal{C}$ (where π is a declarative unit or an R–literal) such that $\mathcal{C} \not\vdash_{\text{MLDS}} \pi$. (This has already been proved by Theorem 3.2.) So, $\mathcal{C} + [\neg\pi]$ is a consistent configuration (see Proposition 3.33). This implies by the *Model Existence Lemma* that the configuration $\mathcal{C} + [\neg\pi]$ is satisfiable. Therefore, there exists a semantic structure $\mathcal{M}_{K\delta}$ of $K\delta$-MLDS which satisfies $\mathcal{C}$ and that also satisfies $\neg\pi$. Then, as shown below, this semantic structure $\mathcal{M}_{K\delta}$ does not satisfy π. Since $\pi \in \mathcal{C}'$, by the definition of satisfiability of a configuration, $\mathcal{M}_{K\delta}$ does not satisfy $\mathcal{C}'$ too. Hence, $\mathcal{A}^{+}_{K\delta} \cup FOT(\mathcal{C}) \not\models_{FOL} FOT(\mathcal{C}')$

$$
\begin{tikzcd}
\mathcal{C} \nvdash_{\mathrm{MLDS}} \mathcal{C}' \arrow[r, "(1)"] \arrow[dr, "(2)"] & \mathcal{C} \not\models_{\mathrm{MLDS}} \mathcal{C}' \\
& \mathcal{A}^+ \cup FOT(\mathcal{C}) \not\models_{FOL} FOT(\mathcal{C}') \arrow[u, "(3)"']
\end{tikzcd}
$$

Figure 3.2: Proof of the Completeness Theorem

The above informal description shows that the main part of this Henkin-style proof of completeness is the model existence lemma. This links together the proof-theoretic and the semantic notions of consistency – i.e. any consistent theory is satisfiable[3]. It is proved in the following two steps. Firstly, it is shown that for any consistent configuration $\mathcal{C}$ it is possible to construct a *maximal consistent configuration* $\mathcal{C}_{mcc}$ which satisfies properties (see Propositions 3.9-3.26) about the units of information π (declarative units and R-literals) that belong to it. Secondly, it is shown that it is possible to construct a semantic structure $\mathcal{M}_{K\delta}(\mathcal{C}_{\mathrm{mcc}})$ which satisfies $\mathcal{C}_{mcc}$. Since the maximal consistent configuration contains the starting consistent configuration $\mathcal{C}$, this semantic structure $\mathcal{M}_{K\delta}(\mathcal{C}_{\mathrm{mcc}})$ also satisfies $\mathcal{C}$.

3.2.3 Formal Proofs

This section is composed of two main parts. In the first part, the notion of a maximal consistent configuration is given together with the proofs of its properties. Most of these are common to the whole family of $K\delta$-MLDSs considered in this thesis, but a few are instead specific to particular $K\delta$-MLDS's. In the second part, the construction of a canonical semantic structure for a K-MLDS is described and it is proved that this is a model of a maximal consistent configuration relative to K-MLDS. These steps are then extended to semantic structures associated with any $K\delta$-MLDS by considering the specific labelling algebrae $\mathcal{A}_{K\delta}$.

Maximal consistent configurations

[3]Semantically, a theory is consistent if there exists a model which satisfies it [HC68].

Definition 3.5 {Maximal Consistent Configuration, $\mathcal{C}_{mcc}$} Let $K\delta$-MLDS be an propositional MLDS. $\mathcal{C}_{\mathrm{mcc}}$ is a *maximal consistent configuration relative to* $K\delta$-MLDS, if it is a consistent configuration and if for any $\pi \notin \mathcal{C}_{\mathrm{mcc}}$ (where π is a declarative unit or an R–literal), the configuration $\mathcal{C}_{\mathrm{mcc}} + [\pi]$ is not consistent. □

It is shown now that given a consistent configuration $\mathcal{C}$ of a $K\delta$-MLDS, it is always possible to construct a maximal consistent configuration $\mathcal{C}_{\mathrm{mcc}}$ which contains it. To do this it is assumed that the set of all declarative units and R–literals of $K\delta$-MLDS is ordered so that it is possible to speak about the 1st, 2nd, 3rd, ..., nth, ..., etc. element π of $K\delta$-MLDS (where π is a declarative unit or an R–literal). Using this assumption, in Definition 3.6 and Proposition 3.8 it is described how to expand an initial consistent configuration $\mathcal{C}$ into a maximal consistent configuration $\mathcal{C}_{\mathrm{mcc}}$. Informally, the construction is based on the following procedure. Starting from the configuration $\mathcal{C}$, and going through all the elements π_i of $K\delta$-MLDS (where π_i is a declarative unit or an R–literal) in a certain chosen ordering, each in turn is added to $\mathcal{C}$ if and only if it can be done so consistently. This is formally defined as follows.

Definition 3.6 {Construction of $MCC(\mathcal{C})$} Let $K\delta$-MLDS be an arbitrary MLDS. Let $\pi_1, \pi_2, \pi_3, \ldots, \pi_n, \ldots$ be an ordering on the set of all declarative units and R–literals of $K\delta$-MLDS. Let $\mathcal{C}$ be a consistent configuration of $K\delta$-MLDS. Let $\mathcal{C}_0 = \mathcal{C}$. Consider the first element π_1 in the chosen ordering. If $\mathcal{C}_0 + [\pi_1]$ is consistent then let $\mathcal{C}_1 = \mathcal{C}_0 + [\pi_1]$, otherwise let $\mathcal{C}_1 = \mathcal{C}_0$. Then take the second element π_2 of the chosen ordering. If $\mathcal{C}_1 + [\pi_2]$ is consistent then let $\mathcal{C}_2 = \mathcal{C}_1 + [\pi_2]$, otherwise let $\mathcal{C}_2 = \mathcal{C}_1$. Then apply the same process on each element π_i of S in turn according to the chosen ordering.

$$
\begin{array}{lll}
\mathcal{C}_0 = & \mathcal{C} & \\
\mathcal{C}_1 = & \mathcal{C}_0 + [\pi_1] & \text{if } \mathcal{C}_0 + [\pi_1] \text{ is consistent} \\
\mathcal{C}_1 = & \mathcal{C}_0 & \text{otherwise} \\
\ldots & \ldots & \\
\mathcal{C}_n = & \mathcal{C}_{n-1} + [\pi_n] & \text{if } \mathcal{C}_{n-1} + [\pi_n] \text{ is consistent} \\
\mathcal{C}_n = & \mathcal{C}_{n-1} & \text{otherwise} \\
\ldots & \ldots &
\end{array}
$$

Let $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n, \ldots$ be the sequence of configurations constructed above. $MCC(\mathcal{C})$ is the configuration which contains all the elements π_i (declarative units and R–literals) which are in any configuration $\mathcal{C}_i$. It will sometimes be written as $\mathcal{C}_{\mathrm{mcc}}$.

$$MCC(\mathcal{C}) = \bigcup_{i \geq 0} \mathcal{C}_i$$

□

Remark 3.1 The sequence of configurations $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$, $\ldots$, $\mathcal{C}_n$, $\ldots$, described in Definition 3.6, is such that for each $i \geq 0$, $\mathcal{C}_i \subseteq \mathcal{C}_{\text{mcc}}$. Moreover, the configuration $\mathcal{C}_0$ is consistent by assumption ($\mathcal{C}_0 = \mathcal{C}$), and for each $i \geq 0$, if $\mathcal{C}_i$ is consistent then by "construction" also $\mathcal{C}_{i+1}$ is consistent. Hence for each $i \geq 0$, $\mathcal{C}_i$ is a consistent configuration.

The following proposition shows that the configuration $\mathcal{C}_{\text{mcc}}$, described in Definition 3.6, is indeed a maximal consistent configuration.

Proposition 3.8 Let $K\delta$-MLDS be an arbitrary MLDS, let $\mathcal{C}$ be a consistent configuration and let $\mathcal{C}_{\text{mcc}}$ be the configuration specified in Definition 3.6. The two following statements hold

1. $\mathcal{C}_{\text{mcc}}$ is consistent
2. $\mathcal{C}_{\text{mcc}}$ is maximal

Proof:

1. Proof is by contradiction. Suppose that $\mathcal{C}_{\text{mcc}}$ is not consistent. Then by Definition 3.3, for some ground term $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$ $\mathcal{C}_{\text{mcc}} \vdash_{\text{MLDS}} \lambda\colon \bot$. Then by Theorem 3.3 there exists a configuration $\mathcal{C}'$ such that $\mathcal{C}'$ is finite, $\mathcal{C}' \subseteq \mathcal{C}_{\text{mcc}}$ and $\mathcal{C}' \vdash_{\text{MLDS}} \lambda : \bot$. Enumerate all the elements (declarative units and R–literals) of $\mathcal{C}'$ according to the ordering chosen to constructed $\mathcal{C}_{\text{mcc}}$. Let π_n be the last element of $\mathcal{C}'$. Then $\mathcal{C}' \subseteq \mathcal{C}_n$. By monotonicity $\mathcal{C}_n \vdash_{\text{MLDS}} \lambda\colon \bot$ which is in contradiction with Remark 3.1.

2. To prove that $\mathcal{C}_{\text{mcc}}$ is a maximal configuration, it is sufficient to prove, by Definition 3.5, that for any π of $K\delta$-MLDS (where π is either a declarative unit or an R–literal), which is consistent with $\mathcal{C}_{\text{mcc}}$, $\pi \in \mathcal{C}_{\text{mcc}}$. Let π_n be a declarative unit or an R–literal of $K\delta$-MLDS, in the chosen ordering specified to construct $\mathcal{C}_{\text{mcc}}$, such that π_n is consistent with $\mathcal{C}_{\text{mcc}}$. Since $\mathcal{C}_{\text{mcc}}$ is a consistent configuration, by Proposition 3.7, for any configuration $\mathcal{C}' \subseteq \mathcal{C}_{\text{mcc}}$, $\mathcal{C}' + [\pi_n]$ is also a consistent configuration. Let $\mathcal{C}_{n-1}$ be the configuration at step n of the construction of $\mathcal{C}_{\text{mcc}}$, then $\mathcal{C}_{n-1} \subseteq \mathcal{C}_{\text{mcc}}$, by Remark 3.1. So, $\mathcal{C}_{n-1} + [\pi_n]$ is a consistent configuration. Hence, since $\mathcal{C}_{n-1} + [\pi_n] = \mathcal{C}_n$, and $\mathcal{C}_n \subseteq \mathcal{C}_{\text{mcc}}$, $\pi_n \in \mathcal{C}_{\text{mcc}}$.

$\star$

So far it has been shown that given a consistent configuration $\mathcal{C}$ of a propositional $K\delta$-MLDS, it is always possible to construct a maximal consistent configuration $\mathcal{C}_{\rm mcc}$ relative to $K\delta$-MLDS such that $\mathcal{C} \subseteq \mathcal{C}_{\rm mcc}$. Some properties of maximal consistent configurations are now proved.

Proposition 3.9 {Consistency w.r.t. declarative units} Let $K\delta$-MLDS be an arbitrary MLDS and let $\mathcal{C}_{\rm mcc}$ be a maximal consistent configuration relative to $K\delta$-MLDS. Then for any declarative unit $\lambda:\alpha$, $\lambda:\alpha$ and $\lambda:\neg\alpha$ are not both in $\mathcal{C}_{\rm mcc}$.

Proof:
The proof follows directly from the definition of $\wedge$–Introduction rule. $\star$

Proposition 3.10 {Consistency w.r.t. R-literals} Let $K\delta$-MDLS be an arbitrary MLDS and let $\mathcal{C}_{\rm mcc}$ be a maximal consistent configuration relative to $K\delta$-MLDS. Then for any R–literal Δ, Δ and $\neg\Delta$ are not both in $\mathcal{C}_{\rm mcc}$.

Proof:
The proof follows directly from the definition of $\perp$–Introduction rule. $\star$

The above two propositions guarantee that no contradictory information is in a maximal consistent configuration. In the extensions of propositional MLDSs that reason with equalities (inequalities) between labels, Proposition 3.10 will also include equality and inequality literals, since the $\perp$-introduction rule will be regarded as applying to these types of unit of information as well (see Section 2.2.1 in Chapter 2).

Proposition 3.11 {Maximality on declarative units} Let $K\delta$-MLDS be an arbitrary MLDS and let $\mathcal{C}_{\rm mcc}$ be a maximal consistent configuration relative to $K\delta$-MLDS. For any declarative unit $\lambda:\alpha$, either $\lambda:\alpha \in \mathcal{C}_{\rm mcc}$ or $\lambda:\neg\alpha \in \mathcal{C}_{\rm mcc}$.

Proof:
The proof is by contradiction. Suppose that $\lambda:\alpha \notin \mathcal{C}_{\rm mcc}$ and that $\lambda:\neg\alpha \notin \mathcal{C}_{\rm mcc}$. Then, since $\mathcal{C}_{\rm mcc}$ is a maximal consistent configuration, $\mathcal{C}_{\rm mcc} + [\lambda : \alpha]$ and $\mathcal{C}_{\rm mcc} + [\lambda:\neg\alpha]$ are inconsistent configurations. Therefore, $\mathcal{C}_{\rm mcc} + [\lambda:\alpha] \vdash_{\rm MLDS} \lambda' : \perp$, and $\mathcal{C}_{\rm mcc} + [\lambda:\neg\alpha] \vdash_{\rm MLDS} \lambda'' : \perp$, for some ground terms λ', λ'' of $Func(\mathcal{L}_L, \mathcal{L}_M)$. By Theorem 3.3, there exist two finite configurations $\mathcal{C}_1 \subseteq \mathcal{C}_{\rm mcc}$ and $\mathcal{C}_2 \subseteq \mathcal{C}_{\rm mcc}$, such that $\mathcal{C}_1 + [\lambda : \alpha] \vdash_{\rm MLDS} \lambda' : \perp$, and $\mathcal{C}_2 + [\lambda : \neg\alpha] \vdash_{\rm MLDS} \lambda'' : \perp$. Let $\mathcal{C}'$ be

the configuration $\mathcal{C}' = \mathcal{C}_1 \cup \mathcal{C}_2$. By monotonicity, $\mathcal{C}' + [\lambda:\alpha] \vdash_{\mathrm{MLDS}} \lambda' : \bot$. So $\mathcal{C}'/\mathcal{C}' + [\lambda:\neg\alpha] \in \mathcal{I}_{\neg I}$ and $\langle\{\mathcal{C}', \mathcal{C}' + [\lambda:\neg\alpha]\}, m\rangle$ is a proof, where $m(0) = \mathcal{I}_{\neg I}$. So $\mathcal{C}' \vdash_{\mathrm{MLDS}} \mathcal{C}' + [\lambda:\neg\alpha]$. Moreover, by monotonicity, $\mathcal{C}' + [\lambda:\neg\alpha] \vdash_{\mathrm{MLDS}} \lambda'' : \bot$. Then by transitivity $\mathcal{C}' \vdash_{\mathrm{MLDS}} \lambda'' : \bot$. Since $\mathcal{C}' \subseteq \mathcal{C}_{\mathrm{mcc}}$, by monotonicity $\mathcal{C}_{\mathrm{mcc}} \vdash_{\mathrm{MLDS}} \lambda'' : \bot$. Hence $\mathcal{C}_{\mathrm{mcc}}$ is inconsistent which is in contradiction with the original hypothesis. $\star$

Proposition 3.12 {Maximality on R-literals} Let $K\delta$-MLDS be an arbitrary MLDS and let $\mathcal{C}_{\mathrm{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS. For any R–literal Δ either $\Delta \in \mathcal{C}_{\mathrm{mcc}}$ or $\neg\Delta \in \mathcal{C}_{\mathrm{mcc}}$.

Proof:
The proof is by contradiction. Suppose that $\Delta \notin \mathcal{C}_{\mathrm{mcc}}$ and that $\neg\Delta \notin \mathcal{C}_{\mathrm{mcc}}$. Since $\mathcal{C}_{\mathrm{mcc}}$ is a maximal consistent configuration, $\mathcal{C}_{\mathrm{mcc}} + [\Delta]$ and $\mathcal{C}_{\mathrm{mcc}} + [\neg\Delta]$ are inconsistent configurations. Then $\mathcal{C}_{\mathrm{mcc}} + [\Delta] \vdash_{\mathrm{MLDS}} \lambda' : \bot$ and $\mathcal{C}_{\mathrm{mcc}} + [\neg\Delta] \vdash_{\mathrm{MLDS}} \lambda'' : \bot$, for some ground terms λ', λ'' of $Func(\mathcal{L}_L, \mathcal{L}_M)$. By Theorem 3.3, there exist two finite configurations $\mathcal{C}_1 \subseteq \mathcal{C}_{\mathrm{mcc}}$ and $\mathcal{C}_2 \subseteq \mathcal{C}_{\mathrm{mcc}}$, such that $\mathcal{C}_1 + [\Delta] \vdash_{\mathrm{MLDS}} \lambda' : \bot$, and $\mathcal{C}_2 + [\neg\Delta] \vdash_{\mathrm{MLDS}} \lambda'' : \bot$. Let $\mathcal{C}'$ be the configuration $\mathcal{C}' = \mathcal{C}_1 \cup \mathcal{C}_2$. By monotonicity $\mathcal{C}' + [\Delta] \vdash_{\mathrm{MLDS}} \lambda' : \bot$. Then $\mathcal{C}'/\mathcal{C}' + [\neg\Delta] \in \mathcal{I}_{R-I}$ and $\langle\{\mathcal{C}', \mathcal{C}' + [\neg\Delta]\}, m\rangle$ is a proof, where $m(0) = \mathcal{I}_{R-I}$. Then $\mathcal{C}' \vdash_{\mathrm{MLDS}} \mathcal{C}' + [\neg\Delta]$. Moreover by monotonicity $\mathcal{C}' + [\neg\Delta] \vdash_{\mathrm{MLDS}} \lambda'' : \bot$. Then by transitivity $\mathcal{C}' \vdash_{\mathrm{MLDS}} \lambda'' : \bot$. Since $\mathcal{C}' \subseteq \mathcal{C}_{\mathrm{mcc}}$, by monotonicity $\mathcal{C}_{\mathrm{mcc}} \vdash_{\mathrm{MLDS}} \lambda'' : \bot$. Hence $\mathcal{C}_{\mathrm{mcc}}$ is inconsistent which is in contradiction with the original hypothesis. $\star$

Proposition 3.13 {Property for $\wedge$-formulae} Let $K\delta$-MLDS be an arbitrary MLDS and let $\mathcal{C}_{\mathrm{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS. Let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let α and β be two wffs of $\mathcal{L}_M$. Then $\lambda:\alpha \wedge \beta \in \mathcal{C}_{\mathrm{mcc}}$, if and only if $\lambda:\alpha \in \mathcal{C}_{\mathrm{mcc}}$ and $\lambda:\beta \in \mathcal{C}_{\mathrm{mcc}}$.

Proof:
("Only if" half:) The proof is by contradiction. Assume that either $\lambda:\alpha \notin \mathcal{C}_{\mathrm{mcc}}$ or $\lambda:\beta \notin \mathcal{C}_{\mathrm{mcc}}$. Only the first case is considered since the argument for the second case is analogous. If $\lambda:\alpha \notin \mathcal{C}_{\mathrm{mcc}}$, then by Proposition 3.11 $\lambda:\neg\alpha \in \mathcal{C}_{\mathrm{mcc}}$. Then $\mathcal{C}_{\mathrm{mcc}}$ is inconsistent as shown in the following derivation. This is in contradiction with the original hypothesis.

$$\begin{array}{ll}
\underline{\mathcal{C}_{\mathrm{mcc}}\langle\lambda:\alpha \wedge \beta, \lambda:\neg\alpha\rangle} & \wedge\text{–E} \\
\underline{\mathcal{C}'\langle\lambda:\alpha, \lambda:\neg\alpha\rangle} & \wedge\text{–I} \\
\mathcal{C}_1'\langle\lambda:\bot\rangle &
\end{array}$$

("If" half:) By hypothesis $\lambda : \alpha \in \mathcal{C}_{\text{mcc}}$ and $\lambda : \beta \in \mathcal{C}_{\text{mcc}}$. Assume that $\lambda : \alpha \wedge \beta \notin \mathcal{C}_{\text{mcc}}$. Then, by Proposition 3.11 $\lambda : \neg(\alpha \wedge \beta) \in \mathcal{C}_{\text{mcc}}$. Hence $\mathcal{C}_{\text{mcc}}$ is inconsistent, as shown in the following derivation, which is in contradiction with the original hypothesis.

$$\begin{array}{ll} \mathcal{C}_{\text{mcc}}\langle \lambda : \neg(\alpha \wedge \beta), \lambda : \alpha, \lambda : \beta \rangle & \wedge\text{–I} \\ \hline \mathcal{C}'\langle \lambda : \alpha \wedge \beta, \lambda : \neg(\alpha \wedge \beta)\rangle & \wedge\text{–I} \\ \hline \mathcal{C}'_1\langle \lambda : \perp \rangle & \end{array}$$

⋆

Proposition 3.14 {Property for $\vee$-formulae} Let $K\delta$-MLDS be an arbitrary MLDS and let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS. Let λ be any ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let α, β be two wffs of $\mathcal{L}_M$. Then $\lambda : \alpha \vee \beta \in \mathcal{C}_{\text{mcc}}$, if and only if $\lambda : \alpha \in \mathcal{C}_{\text{mcc}}$ or $\lambda : \beta \in \mathcal{C}_{\text{mcc}}$.

Proof:
("Only if" half:) The proof is by contradiction. Assume that $\lambda : \alpha \notin \mathcal{C}_{\text{mcc}}$ and that $\lambda : \beta \notin \mathcal{C}_{\text{mcc}}$. Then by Proposition 3.11, $\lambda : \neg\alpha \in \mathcal{C}_{\text{mcc}}$ and $\lambda : \neg\beta \in \mathcal{C}_{\text{mcc}}$. Therefore $\mathcal{C}_{\text{mcc}} + [\lambda : \alpha] \vdash_{\text{MLDS}} \lambda : \perp$ and $\mathcal{C}_{\text{mcc}} + [\lambda : \beta] \vdash_{\text{MLDS}} \lambda : \perp$. So, by the $\vee$–Elimination rule, $\mathcal{C}_{\text{mcc}} \vdash_{\text{MLDS}} \lambda : \perp$. Hence $\mathcal{C}_{\text{mcc}}$ is inconsistent which is in contradiction with the original hypothesis.

("If" half:) The proof is by contradiction. By hypothesis either $\lambda : \alpha \in \mathcal{C}_{\text{mcc}}$ or $\lambda : \beta \in \mathcal{C}_{\text{mcc}}$. Only the first case is considered since the argument for the second case is analogous. Assume that $\lambda : \alpha \vee \beta \notin \mathcal{C}_{\text{mcc}}$. Then by Proposition 3.11 $\lambda : \neg(\alpha \vee \beta) \in \mathcal{C}_{\text{mcc}}$. Hence $\mathcal{C}_{\text{mcc}}$ is inconsistent as shown in the following derivation.

$$\begin{array}{ll} \mathcal{C}_{\text{mcc}}\langle \lambda : \neg(\alpha \vee \beta), \lambda : \alpha \rangle & \vee\text{–I} \\ \hline \mathcal{C}'\langle \lambda : \alpha \vee \beta, \lambda : \neg(\alpha \vee \beta)\rangle & \wedge\text{–I} \\ \hline \mathcal{C}'_1\langle \lambda : \perp \rangle & \end{array}$$

⋆

Proposition 3.15 {Property for $\rightarrow$-formulae (i)} Let $K\delta$-MDLS be an arbitrary MLDS and let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS. Let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and α, β be two wffs of $\mathcal{L}_M$. If $\lambda : \neg\alpha \in \mathcal{C}_{\text{mcc}}$ or $\lambda : \beta \in \mathcal{C}_{\text{mcc}}$ then $\lambda : \alpha \rightarrow \beta \in \mathcal{C}_{\text{mcc}}$.

Proof:
Proof is by contradiction. Assume that $\lambda : \alpha \rightarrow \beta \notin \mathcal{C}_{\text{mcc}}$. Then by Proposition 3.11, $\lambda : \neg(\alpha \rightarrow \beta) \in \mathcal{C}_{\text{mcc}}$. Hence $\mathcal{C}_{\text{mcc}}$ is inconsistent as shown in the following derivation.

$$
\begin{array}{ll}
\underline{\mathcal{C}_{\text{mcc}}\langle \lambda:\neg(\alpha\to\beta), \lambda:\neg\alpha\rangle} & \\
\underline{\mathcal{C}'\langle \lambda:\neg(\alpha\to\beta), \lambda:\neg\alpha\vee\beta\rangle} & (\vee\text{–I}) \\
\quad\underline{\mathcal{C}'\langle \lambda:\neg\alpha\vee\beta, [\lambda:\alpha]\rangle} & (\text{assumption}) \\
\qquad\underline{\mathcal{C}'\langle \lambda:\neg\alpha\vee\beta, [\lambda:\alpha], [\lambda:\neg\beta]\rangle} & (\text{assumption}) \\
\qquad\underline{\tilde{\mathcal{C}}\langle \lambda:\neg\alpha\vee\beta, [\lambda:\alpha], [\lambda:\neg\beta], \lambda:\bot\rangle} & (\vee\text{–E}) \\
\quad\underline{\tilde{\mathcal{C}}_1\langle \lambda:\neg\alpha\vee\beta, [\lambda:\alpha], \lambda:\neg\neg\beta\rangle} & (\neg\text{–I}) \\
\quad\underline{\tilde{\mathcal{C}}_2\langle \lambda:\neg\alpha\vee\beta, [\lambda:\alpha], \lambda:\beta\rangle} & (\neg\text{–E}) \\
\underline{\tilde{\mathcal{C}}_3\langle \lambda:\neg(\alpha\to\beta), \lambda:\alpha\to\beta\rangle} & (\to\text{–I}) \\
\mathcal{C}_1'\langle \lambda:\bot\rangle & (\wedge\text{–I})
\end{array}
$$

Now suppose that $\lambda:\beta \in \mathcal{C}_{\text{mcc}}$. Assume by contradiction that $\lambda:\alpha\to\beta \notin \mathcal{C}_{\text{mcc}}$. Then by Proposition 3.11, $\lambda:\neg(\alpha\to\beta) \in \mathcal{C}_{\text{mcc}}$. Hence $\mathcal{C}_{\text{mcc}}$ is inconsistent as shown in the following derivation.

$$
\begin{array}{ll}
\underline{\mathcal{C}_{\text{mcc}}\langle \lambda:\neg(\alpha\to\beta), \lambda:\beta\rangle} & \\
\quad\underline{\mathcal{C}_{\text{mcc}}\langle \lambda:\neg(\alpha\to\beta), \lambda:\beta, [\lambda:\alpha]\rangle} & (\text{assumption}) \\
\quad\underline{\tilde{\mathcal{C}}\langle \lambda:\beta\rangle} & (\text{C-R}) \\
\underline{\mathcal{C}'\langle \lambda:\neg(\alpha\to\beta), \lambda:\alpha\to\beta\rangle} & (\to\text{–I}) \\
\mathcal{C}_1'\langle \lambda:\bot\rangle & (\wedge\text{–I})
\end{array}
$$

⋆

Proposition 3.16 {Property for $\to$-formulae (ii)} Let $K\delta$-MLDS be an arbitrary MLDS and let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS. Let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let α, β, be two wffs of $\mathcal{L}_M$. If $\lambda:\alpha \in \mathcal{C}_{\text{mcc}}$ and $\lambda:\alpha\to\beta \in \mathcal{C}_{\text{mcc}}$ then $\lambda:\beta \in \mathcal{C}_{\text{mcc}}$.

Proof:
The proof is by contradiction. Suppose that $\lambda:\beta \notin \mathcal{C}_{\text{mcc}}$. Then by Proposition 3.11 $\lambda:\neg\beta \in \mathcal{C}_{\text{mcc}}$. Hence $\mathcal{C}_{\text{mcc}} \vdash_{\text{MLDS}} \lambda:\bot$ as shown in the following derivation.

$$
\begin{array}{ll}
\underline{\mathcal{C}_{\text{mcc}}\langle \lambda:\alpha, \lambda:\alpha\to\beta, \lambda:\neg\beta\rangle} & \to\text{–E} \\
\underline{\mathcal{C}'\langle \lambda:\beta, \lambda:\neg\beta\rangle} & \wedge\text{–I} \\
\mathcal{C}_1'\langle \lambda:\bot\rangle &
\end{array}
$$

⋆

Proposition 3.17 {Property for $\Diamond$-formulae (i)} Let $K\delta$-MLDS be an arbitrary MLDS, let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS and let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let α be a wff of $\mathcal{L}_M$. If $\lambda : \Diamond\alpha \in \mathcal{C}_{\text{mcc}}$ then $f_\alpha(\lambda) : \alpha \in \mathcal{C}_{\text{mcc}}$ and $R(\lambda, f_\alpha(\lambda)) \in \mathcal{C}_{\text{mcc}}$.

Proof:
By hypothesis $\lambda : \Diamond\alpha \in \mathcal{C}_{\text{mcc}}$, Then by Proposition 3.3, $f_\alpha(\lambda) : \neg\alpha \notin \mathcal{C}_{\text{mcc}}$ and $\neg R(\lambda, f_\alpha(\lambda)) \notin \mathcal{C}_{\text{mcc}}$. Hence by Proposition 3.11, $f_\alpha(\lambda) : \alpha \in \mathcal{C}_{\text{mcc}}$ and by Proposition 3.12, $R(\lambda, f_\alpha(\lambda)) \in \mathcal{C}_{\text{mcc}}$. ⋆

Proposition 3.18 {Property for $\Box$-formulae (i)} Let $K\delta$-MLDS be an arbitrary MLDS, let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS, let λ, λ' be two ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let α be a wff of $\mathcal{L}_M$. If $\lambda : \Box\alpha \in \mathcal{C}_{\text{mcc}}$ and $R(\lambda, \lambda') \in \mathcal{C}_{\text{mcc}}$, then $\lambda' : \alpha \in \mathcal{C}_{\text{mcc}}$.

Proof:
The proof is by contradiction. Suppose that $\lambda' : \alpha \notin \mathcal{C}_{\text{mcc}}$. Then by Proposition 3.11, $\lambda' : \neg\alpha \in \mathcal{C}_{\text{mcc}}$. Then $\mathcal{C}_{\text{mcc}} \vdash_{\text{MLDS}} \lambda' : \bot$ as shown in the following derivation.

$$\begin{array}{ll} \mathcal{C}_{\text{mcc}}\langle \lambda : \Box\alpha, R(\lambda, \lambda'), \lambda' : \neg\alpha \rangle & \Box\text{–E} \\ \hline \mathcal{C}'\langle \lambda' : \alpha, \lambda' : \neg\alpha \rangle & \wedge\text{–I} \\ \hline \mathcal{C}'_1\langle \lambda' : \bot \rangle & \end{array}$$

⋆

Proposition 3.19 {Property for $\Diamond$-formulae (ii)} Let $K\delta$-MLDS be an arbitrary MLDS, let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS, let λ_1 and λ_2 be two ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let α be a wff of $\mathcal{L}_M$. If $R(\lambda_1, \lambda_2) \in \mathcal{C}_{\text{mcc}}$ and $\lambda_2 : \alpha \in \mathcal{C}_{\text{mcc}}$ then $\lambda_1 : \Diamond\alpha \in \mathcal{C}_{\text{mcc}}$.

Proof:
The proof is by contradiction. Suppose that $\lambda_1 : \Diamond\alpha \notin \mathcal{C}_{\text{mcc}}$. Then by Proposition 3.11, $\lambda_1 : \neg\Diamond\alpha \in \mathcal{C}_{\text{mcc}}$. Then $\mathcal{C}_{\text{mcc}} \vdash_{\text{MLDS}} \lambda_1 : \bot$ as shown in the following derivation.

$$\begin{array}{ll} \mathcal{C}_{\text{mcc}}\langle R(\lambda_1, \lambda_2), \lambda_2 : \alpha, \lambda_1 : \neg\Diamond\alpha \rangle & \Diamond\text{–I} \\ \hline \mathcal{C}'\langle \lambda_1 : \Diamond\alpha, \lambda_1 : \neg\Diamond\alpha \rangle & \wedge\text{–I} \\ \hline \mathcal{C}'_1\langle \lambda_1 : \bot \rangle & \end{array}$$

⋆

Proposition 3.20 {Property for $\Box$-formulae (ii)} Let $K\delta$-MLDS be an arbitrary MLDS, let $\mathcal{C}_{\rm mcc}$ be a maximal consistent configuration relative to $K\delta$-MLDS, let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M))$ and let α be a wff of $\mathcal{L}_M$. If $\neg R(\lambda, box_\alpha(\lambda)) \in \mathcal{C}_{\rm mcc}$ or $box_\alpha(\lambda) : \alpha \in \mathcal{C}_{\rm mcc}$ then $\lambda : \Box\alpha \in \mathcal{C}_{\rm mcc}$.

Proof:
Proof is by contradiction. Assume that $\lambda : \Box\alpha \notin \mathcal{C}_{\rm mcc}$. Then by Proposition 3.11, $\lambda : \neg\Box\alpha \in \mathcal{C}_{\rm mcc}$ which implies, by Proposition 3.3, that $box_\alpha(\lambda) : \alpha \notin \mathcal{C}_{\rm mcc}$ and $\neg R(\lambda, box_\alpha(\lambda)) \notin \mathcal{C}_{\rm mcc}$. Hence, by Proposition 3.11 $box_\alpha(\lambda) : \neg\alpha \in \mathcal{C}_{\rm mcc}$ and by Proposition 3.12 $R(\lambda, box_\alpha(\lambda)) \in \mathcal{C}_{\rm mcc}$. $\star$

The properties described above (Propositions 3.9–3.20) hold for any maximal consistent configuration relative to any arbitrary $K\delta$-MLDS. However, in the construction of a maximal consistent configuration, different sets of R–literals are added for different labelling algebrae $\mathcal{A}_{K\delta}$ under consideration. Therefore, for each of the $K\delta$-MLDS additional properties need to be proved about the R-literals contained in the associated maximal consistent configuration. Because each labelling algebra is a combination of the basic axioms (**T**), (**4**), (**D**), (**B**), (**5**), (**Irr**) it is sufficient to consider the properties associated with these individual axioms. Appropriate combinations of these properties will cover the cases of all the other arbitrary labelling algebrae $\mathcal{A}_{K\delta}$.

Proposition 3.21 {Property for reflexive algebrae} Let $K\delta$-MLDS be a MLDS such that $\{\forall x R(x,x)\} \subseteq \mathcal{A}_{K\delta}$. Let $\mathcal{C}_{\rm mcc}$ be a maximal consistent configuration relative to $K\delta$-MLDS. Then for each ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$, the R–literal $R(\lambda, \lambda) \in \mathcal{C}_{\rm mcc}$.

Proof:
The proof is by contradiction. Let λ' be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $R(\lambda', \lambda') \notin \mathcal{C}_{\rm mcc}$. Then by Proposition 3.12, $\neg R(\lambda', \lambda') \in \mathcal{C}_{\rm mcc}$. Then $\mathcal{C}_{\rm mcc} \vdash_{K\delta-\rm MLDS} \lambda'' : \bot$, for some ground term $\lambda'' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, as shown in the following derivation. Hence $\mathcal{C}_{\rm mcc}$ is inconsistent which is in contradiction with the original hypothesis.

$$\dfrac{\dfrac{\mathcal{C}_{\rm mcc}\langle \neg R(\lambda', \lambda')\rangle}{\mathcal{C}'\langle \neg R(\lambda', \lambda'), R(\lambda', \lambda')\rangle}\ R\text{–Assertion}}{\mathcal{C}'_1\langle \lambda'' : \bot\rangle}\ \bot\text{–I}$$

$\star$

Proposition 3.22 {Property for irreflexive algebrae} Let $K\delta$-MLDS be a MLDS such that $\{\forall x \neg R(x,x)\} \subseteq \mathcal{A}_{K\delta}$. Let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS. Then for each ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$, the R–literal $\neg R(\lambda, \lambda) \in \mathcal{C}_{\text{mcc}}$.

Proof:
The proof is by contradiction. Let λ' be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $\neg R(\lambda', \lambda') \notin \mathcal{C}_{\text{mcc}}$. Then by Proposition 3.12, $R(\lambda', \lambda') \in \mathcal{C}_{\text{mcc}}$. Then $\mathcal{C}_{\text{mcc}} \vdash_{K\delta-\text{MLDS}} \lambda'' : \bot$, for some ground term $\lambda'' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, as shown in the following derivation. Hence $\mathcal{C}_{\text{mcc}}$ is inconsistent which is in contradiction with the original hypothesis.

$$\begin{array}{ll}
\mathcal{C}_{\text{mcc}}\langle R(\lambda', \lambda')\rangle & \\
\hline
\mathcal{C}'\langle R(\lambda', \lambda'), \neg R(\lambda', \lambda')\rangle & R\text{–Assertion} \\
\hline
\mathcal{C}'_1\langle \lambda'' : \bot\rangle & \bot\text{–I}
\end{array}$$

⋆

Proposition 3.23 {Property for transitive algebrae} Let $K\delta$-MLDS be a MLDS such that $\{\forall x, y, z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))\} \subseteq \mathcal{A}_{K\delta}$. Let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS. Let λ_1,λ_2,λ_3 be three ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $R(\lambda_1, \lambda_2) \in \mathcal{C}_{\text{mcc}}$ and $R(\lambda_2, \lambda_3) \in \mathcal{C}_{\text{mcc}}$. Then $R(\lambda_1, \lambda_3) \in \mathcal{C}_{\text{mcc}}$.

Proof:
The proof is by contradiction. Suppose that $R(\lambda_1, \lambda_3) \notin \mathcal{C}_{\text{mcc}}$. Then by Proposition 3.12, $\neg R(\lambda_1, \lambda_3) \in \mathcal{C}_{\text{mcc}}$. Then $\mathcal{C}_{\text{mcc}} \vdash_{K\delta-\text{MLDS}} \lambda' : \bot$, for some ground term $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, as shown in the following derivation. Hence $\mathcal{C}_{\text{mcc}}$ is inconsistent.

$$\begin{array}{ll}
\mathcal{C}_{\text{mcc}}\langle \neg R(\lambda_1, \lambda_3)\rangle & \\
\hline
\mathcal{C}'\langle \neg R(\lambda_1, \lambda_3), R(\lambda_1, \lambda_3)\rangle & R\text{–Assertion} \\
\hline
\mathcal{C}'_1\langle \lambda' : \bot\rangle & \bot\text{–I}
\end{array}$$

⋆

Proposition 3.24 {Property for symmetric algebrae} Let $K\delta$-MLDS be a MLDS such that $\{\forall x, y R(x,y) \rightarrow R(y,x))\} \subseteq \mathcal{A}_{K\delta}$. Let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS. Let λ_1 and λ_2 be two ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $R(\lambda_1, \lambda_2) \in \mathcal{C}_{\text{mcc}}$. Then $R(\lambda_2, \lambda_1) \in \mathcal{C}_{\text{mcc}}$.

Proof:
The proof is by contradiction. Suppose that $R(\lambda_2, \lambda_1) \notin \mathcal{C}_{\text{mcc}}$. Then by Proposition 3.12, $\neg R(\lambda_2, \lambda_1) \in \mathcal{C}_{\text{mcc}}$. Then $\mathcal{C}_{\text{mcc}} \vdash_{K\delta-\text{MLDS}} \lambda' : \perp$, for some ground term $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, as shown in the following derivation. Hence $\mathcal{C}_{\text{mcc}}$ is inconsistent.

$$\begin{array}{ll}
\underline{\mathcal{C}_{\text{mcc}}\langle R(\lambda_1, \lambda_2), \neg R(\lambda_2, \lambda_1)\rangle} & \\
\underline{\mathcal{C}'\langle \neg R(\lambda_2, \lambda_1), R(\lambda_2, \lambda_1)\rangle} & R\text{–Assertion} \\
\mathcal{C}_1'\langle \lambda' : \perp\rangle & \perp\text{–I}
\end{array}$$

⋆

Proposition 3.25 {Property for euclidean algebrae} Let $K\delta$-MLDS be a MLDS such that $\{\forall x, y, z(R(x, y) \wedge R(x, z)) \rightarrow R(y, z))\} \subseteq \mathcal{A}_{K\delta}$. Let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS. Let λ_1, λ_2 and λ_3 be three ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such that $R(\lambda_1, \lambda_2) \in \mathcal{C}_{\text{mcc}}$ and $R(\lambda_1, \lambda_3) \in \mathcal{C}_{\text{mcc}}$. Then $R(\lambda_2, \lambda_3) \in \mathcal{C}_{\text{mcc}}$.

Proof:
The proof is by contradiction. Suppose that $R(\lambda_2, \lambda_3) \notin \mathcal{C}_{\text{mcc}}$. Then by Proposition 3.12, $\neg R(\lambda_2, \lambda_3) \in \mathcal{C}_{\text{mcc}}$. Then $\mathcal{C}_{\text{mcc}} \vdash_{K\delta-\text{MLDS}} \lambda' : \perp$, for some ground term $\lambda' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, as shown in the following derivation. Hence $\mathcal{C}_{\text{mcc}}$ is inconsistent.

$$\begin{array}{ll}
\underline{\mathcal{C}_{\text{mcc}}\langle R(\lambda_1, \lambda_2), R(\lambda_1, \lambda_3) \neg R(\lambda_2, \lambda_3)\rangle} & \\
\underline{\mathcal{C}'\langle \neg R(\lambda_2, \lambda_3), R(\lambda_2, \lambda_3)\rangle} & R\text{–Assertion} \\
\mathcal{C}_1'\langle \lambda' : \perp\rangle & \perp\text{–I}
\end{array}$$

⋆

Proposition 3.26 {Property for serial algebrae} Let $K\delta$-MLDS be a MLDS such that $\{\forall x R(x, succ(x))\} \subseteq \mathcal{A}_{K\delta}$. Let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS. Then for each ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$, the R–literal $R(\lambda, succ(\lambda)) \in \mathcal{C}_{\text{mcc}}$.

Proof:
The proof is by contradiction. Let λ' be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ such

that $R(\lambda', succ(\lambda')) \notin \mathcal{C}_{\mathrm{mcc}}$. Then by Proposition 3.12, $\neg R(\lambda', succ(\lambda')) \in \mathcal{C}_{\mathrm{mcc}}$. Then $\mathcal{C}_{\mathrm{mcc}} \vdash_{K\delta-\mathrm{MLDS}} \lambda'' : \bot$, for some ground term $\lambda'' \in Func(\mathcal{L}_L, \mathcal{L}_M)$, as shown in the following derivation. Hence $\mathcal{C}_{\mathrm{mcc}}$ is inconsistent.

$$\dfrac{\dfrac{\mathcal{C}_{\mathrm{mcc}}\langle \neg R(\lambda', succ(\lambda'))\rangle}{\mathcal{C}'\langle \neg R(\lambda', succ(\lambda')), R(\lambda', succ(\lambda'))\rangle}\ R\text{–Assertion}}{\mathcal{C}'_1\langle \lambda'' : \bot\rangle}\ \bot\text{–I}$$

$\star$

Model Existence Lemma

This section shows that any consistent configuration is satisfiable. This is proved by showing that it is possible to construct for each $K\delta$-MLDS a semantic structure $\mathcal{M}_{K\delta}$ which satisfies a maximal consistent configuration relative to $K\delta$-MLDS.

Definition 3.7 {Canonical interpretation, $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}})$} Let $K\delta$-MLDS be an arbitrary MLDS. Let $\mathcal{C}_{\mathrm{mcc}} = \langle \mathcal{D}_{mcc}, \mathcal{F}_{mcc}\rangle$ be a maximal consistent configuration relative to $K\delta$-MLDS. Let $FOT(\mathcal{C}_{\mathrm{mcc}})$ be its first–order translation. $FOT(\mathcal{C}_{\mathrm{mcc}}) = \mathcal{D}_{mcc} \cup \mathcal{DU}_{mcc}$, where $\mathcal{DU}_{mcc} = \{[\alpha]^*(\lambda) \mid \lambda : \alpha \in \mathcal{C}_{\mathrm{mcc}}\}$. Let $\mathcal{U}$ be the Herbrand Universe of the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$. $I(\mathcal{C}_{\mathrm{mcc}})$ is the interpretation function on the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$, defined as follows.

- For each ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$

  $\| \lambda \|_{I(\mathcal{C}_{\mathrm{mcc}})} = \lambda.$

- For the binary predicate $R \in Mon(\mathcal{L}_L, \mathcal{L}_M)$

  $\| R \|_{I(\mathcal{C}_{\mathrm{mcc}})} = \{\langle \lambda_i, \lambda_j\rangle \mid R(\lambda_i, \lambda_j) \in FOT(\mathcal{C}_{mcc})\}$[4].

- For each monadic predicate $[\alpha]^* \in Mon(\mathcal{L}_L, \mathcal{L}_M)$

  $\| [\alpha]^* \|_{I(\mathcal{C}_{\mathrm{mcc}})} = \{\lambda_i \mid [\alpha]^*(\lambda_i) \in FOT(\mathcal{C}_{mcc})\}$

$\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}) = \langle \mathcal{U}, I(\mathcal{C}_{\mathrm{mcc}})\rangle$ is a *canonical interpretation*. $\square$

---

[4]Notice that $FOT(\mathcal{C}_{\mathrm{mcc}})$ contains only ground literals

Remark 3.2 Let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to a $K\delta$-MLDS, let $FOT(\mathcal{C}_{\text{mcc}})$ be its first–order translation and let $\mathcal{M}(\mathcal{C}_{\text{mcc}})$ be a canonical interpretation. Observe that for any ground atomic formula of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ of the form $R(\lambda_i, \lambda_j)$, $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \Vdash_{FOL} R(\lambda_i, \lambda_j)$ if and only if $R(\lambda_i, \lambda_j) \in FOT(\mathcal{C}_{\text{mcc}})$; analogously for any ground atomic formula of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ of the form $[\alpha]^*(\lambda)$, $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \Vdash_{FOL} [\alpha]^*(\lambda)$ if and only if $[\alpha]^*(\lambda) \in FOT(\mathcal{C}_{\text{mcc}})$. Now let V be a variable assignment from the set of variables of the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ to the Herbrand Universe $\mathcal{U}$. The truth value of any wff of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ is defined as follows.

- $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} R(x, y)$ if and only if $\langle V(x), V(y)\rangle \in \parallel R \parallel_{I(\mathcal{C}_{\text{mcc}})}$
- $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} [\alpha]^*(x)$ if and only if $V(x) \in \parallel [\alpha]^* \parallel_{I(\mathcal{C}_{\text{mcc}})}$
- for any wff ϕ of $Mon(\mathcal{L}_L, \mathcal{L}_M)$, the truth value of ϕ with respect to $\mathcal{M}(\mathcal{C}_{\text{mcc}})$ and V is defined in the usual way (see for example [Gen88]).

It is necessary to show now that given $\mathcal{C}_{\text{mcc}}$ relative to a particular $K\delta$-MLDS, the canonical interpretation $\mathcal{M}(\mathcal{C}_{\text{mcc}})$ is a semantic structure of $K\delta$-MLDS. This is done in two steps: (i) it is shown that $\mathcal{M}(\mathcal{C}_{\text{mcc}})$ satisfies the particular labelling algebra $\mathcal{A}_{K\delta}$ and (ii) it is shown that $\mathcal{M}(\mathcal{C}_{\text{mcc}})$ satisfies all the axiom schemas (Ax1)-(Ax8) of the associated extended algebra $\mathcal{A}^+_{K\delta}$. This second step is independent from the type of propositional MLDS (i.e. the type of δ) and therefore it will be proved first. The former step needs to take into account the different specific labelling algebrae as well as the specific additional properties for $\mathcal{C}_{\text{mcc}}$ proved above (Propositions 3.21-3.26).

For any variable assignment V and any variable x of the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$, $V(x)$ refers to some ground term in the Herbrand Universe $\mathcal{U}$. Hence, for simplicity, $V(x)$ will be used also to denote an arbitrary ground term in the arguments that follow.

Theorem 3.5 {Model existence for $\mathcal{A}^+ - \mathcal{A}$} Let $\mathcal{A}^+_{K\delta}$ be the extended algebra associated with a propositional $K\delta$-MLDS, let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS and let $FOT(\mathcal{C}_{\text{mcc}})$ be its first–order translation. Then $\mathcal{M}(\mathcal{C}_{\text{mcc}})$ is a model of $\mathcal{A}^+ - \mathcal{A}$.

Proof:
Let U, V be arbitrary variable assignments, and let α, β be two wffs of $\mathcal{L}_M$. The proof is by cases on each of the axiom schemas $(Ax1)$–$(Ax8)$.

Case 1: **(Ax1)** $\quad \forall x([\alpha \wedge \beta]^*(x) \equiv ([\alpha]^*(x) \wedge [\beta]^*(x))$

It is sufficient to prove that $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\alpha \wedge \beta]^*(V(x))$ if and only if $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\alpha]^*(V(x))$ and $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\beta]^*(V(x))$.

("Only if" half:) Assume that $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\alpha \wedge \beta]^*(V(x))$.
Then, $[\alpha \wedge \beta]^*(V(x)) \in FOT(\mathcal{C}_{\mathrm{mcc}})$. So, $V(x) : \alpha \wedge \beta \in \mathcal{C}_{\mathrm{mcc}}$, which implies by Proposition 3.13 that $V(x) : \alpha \in \mathcal{C}_{\mathrm{mcc}}$ and $V(x) : \beta \in \mathcal{C}_{\mathrm{mcc}}$. Therefore $V(x) \in \parallel [\alpha]^* \parallel_{I(\mathcal{C}_{\mathrm{mcc}})}$ and $V(x) \in \parallel [\beta]^* \parallel_{I(\mathcal{C}_{\mathrm{mcc}})}$. Hence $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\alpha]^*(V(x))$ and $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\beta]^*(V(x))$.

("If" half:) Assume that

$$\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\alpha]^*(V(x)) \text{ and that } \mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\beta]^*(V(x))$$

Then $[\alpha]^*(V(x)) \in FOT(\mathcal{C}_{\mathrm{mcc}})$ and $[\beta]^*(V(x)) \in FOT(\mathcal{C}_{\mathrm{mcc}})$, which implies that $V(x) : \alpha \in \mathcal{C}_{\mathrm{mcc}}$ and $V(x) : \beta \in \mathcal{C}_{\mathrm{mcc}}$. Therefore, by Proposition 3.13, $V(x) : \alpha \wedge \beta \in \mathcal{C}_{\mathrm{mcc}}$. Hence $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\alpha \wedge \beta]^*(V(x))$.

For Cases 2–4 (i.e. (**(Ax2)**, **(Ax3)**, **(Ax4)**) see Appendix A.2

Case 5: **(Ax5)** $\quad \forall x([\Diamond\alpha]^*(x) \rightarrow (R(x, f_\alpha(x)) \wedge [\alpha]^*(f_\alpha(x))))$.

It is sufficient to prove that if $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\Diamond\alpha]^*(V(x))$ then

$$\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} R(V(x), f_\alpha(V(x)))^5 \text{ and } \mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\alpha]^*(f_\alpha(V(x)))$$

Assume that $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\Diamond\alpha]^*(V(x))$. Then, $[\Diamond\alpha]^*(V(x)) \in FOT(\mathcal{C}_{\mathrm{mcc}})$ which implies that $V(x) : \Diamond\alpha \in \mathcal{C}_{\mathrm{mcc}}$. By Proposition 3.17, $f_\alpha(V(x)) : \alpha \in \mathcal{C}_{\mathrm{mcc}}$ and $R(V(x), f_\alpha(V(x))) \in \mathcal{C}_{\mathrm{mcc}}$. Hence $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} R(V(x), f_\alpha(V(x)))$ and $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\alpha]^*(f_\alpha(V(x)))$.

Case 6: **(Ax6)** $\quad \forall x(\exists y(R(x,y) \wedge [\alpha]^*(y)) \rightarrow [\Diamond\alpha]^*(x))$.

Consider the equivalent formula

$$\forall x \forall y(R(x,y) \wedge [\alpha]^*(y)) \rightarrow [\Diamond\alpha]^*(x))$$

It is sufficient to prove that

$$\text{if } \mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V, U \Vdash_{FOL} R(V(x), U(y)) \text{ and } \mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V, U \Vdash_{FOL} [\alpha]^*(U(y))$$
$$\text{then } \mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V, U \Vdash_{FOL} [\Diamond\alpha]^*(V(x))$$

[5] The definition of the interpretation function $I(\mathcal{C}_{\mathrm{mcc}})$ of a canonical interpretation $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}})$, allows us to write $f_\alpha(V(x))$ instead of $\parallel f_\alpha \parallel_{I(\mathcal{C}_{\mathrm{mcc}})} (V(x))$.

By assumption, $R(V(x), U(y)) \in FOT(\mathcal{C}_{\rm mcc})$ and $[\alpha]^*(U(y)) \in FOT(\mathcal{C}_{\rm mcc})$, which implies that $R(V(x), U(y)) \in \mathcal{C}_{\rm mcc}$ and $U(y){:}\,\alpha \in \mathcal{C}_{\rm mcc}$. Then by Proposition 3.19, $V(x){:}\,\Diamond\alpha \in \mathcal{C}_{\rm mcc}$. Hence $\mathcal{M}(\mathcal{C}_{mcc}), V, U \Vdash_{FOL} [\Diamond\alpha]^*(V(x))$.

Case 7: **(Ax7)** $\forall x(R(x, box_\alpha(x)) \rightarrow [\alpha]^*(box_\alpha(x))) \rightarrow [\Box\alpha]^*(x))$.
It is sufficient to prove that

$$\text{if } \mathcal{M}(\mathcal{C}_{\rm mcc}), V \Vdash_{FOL} \neg R(V(x), box_\alpha(V(x)))^6 \text{ or } \mathcal{M}(\mathcal{C}_{\rm mcc}), V \Vdash_{FOL} [\alpha]^*(box_\alpha(V(x)))$$

$$\text{then } \mathcal{M}(\mathcal{C}_{\rm mcc}), V \Vdash_{FOL} [\Box\alpha]^*(V(x))$$

By assumption,
either $\neg R(V(x), box_\alpha(V(x))) \in FOT(\mathcal{C}_{\rm mcc})$ or $[\alpha]^*(box_\alpha(V(x))) \in FOT(\mathcal{C}_{\rm mcc})$
This implies that
either $\neg R(V(x), box_\alpha(V(x))) \in \mathcal{C}_{\rm mcc}$ or $box_\alpha(V(x)){:}\,\alpha \in \mathcal{C}_{\rm mcc}$
Then, by Proposition 3.20, $V(x){:}\,\Box\alpha \in \mathcal{C}_{\rm mcc}$. Hence $\mathcal{M}(\mathcal{C}_{\rm mcc}), V \Vdash_{FOL} [\Box\alpha]^*(V(x))$.

Case 8: **(Ax8)** $\forall x([\Box\alpha]^*(x) \rightarrow (\forall y(R(x, y) \rightarrow [\alpha]^*(y)))$.
Consider the equivalent formula

$$\forall x \forall y(([\Box\alpha]^*(x) \wedge (R(x, y)) \rightarrow [\alpha]^*(y))$$

It is sufficient to prove that

$$\begin{array}{l} \text{if } \mathcal{M}(\mathcal{C}_{\rm mcc}), V, U \Vdash_{FOL} [\Box\alpha]^*(V(x)) \text{ and} \\ \quad \mathcal{M}(\mathcal{C}_{\rm mcc}), V, U \Vdash_{FOL} R(V(x), U(y)) \text{ then} \\ \quad\quad \mathcal{M}(\mathcal{C}_{\rm mcc}), V, U \Vdash_{FOL} [\alpha]^*(U(y)) \end{array}$$

By assumption $[\Box\alpha]^*(V(x)) \in FOT(\mathcal{C}_{\rm mcc})$, and $R(V(x), U(y)) \in FOT(\mathcal{C}_{\rm mcc})$. This implies that $V(x) : \Box\alpha \in \mathcal{C}_{\rm mcc}$ and $R(V(x), U(y)) \in \mathcal{C}_{\rm mcc}$. Then, by Proposition 3.18, $U(y){:}\,\alpha \in \mathcal{C}_{\rm mcc}$. Hence $\mathcal{M}(\mathcal{C}_{\rm mcc}), V, U \Vdash_{FOL} [\alpha]^*(U(y))$. ⋆

Proposition 3.27 {Satisfiability of (**T**)} Let $K\delta$-MLDS be a MLDS such that $\{\forall x R(x, x)\} \subseteq \mathcal{A}_{K\delta}$. Let $\mathcal{C}_{\rm mcc}$ be a maximal consistent configuration relative to $K\delta$-MLDS and let $FOT(\mathcal{C}_{\rm mcc})$ be its first–order translation. Then $\mathcal{M}(\mathcal{C}_{\rm mcc}) \Vdash_{FOL} \forall x R(x, x)$.

Proof:
Let V be an arbitrary variable assignment. It is sufficient to prove that $\mathcal{M}(\mathcal{C}_{\rm mcc}), V \Vdash_{FOL} R(V(x), V(x))$. By Proposition 3.21, $R(V(x), V(x)) \in \mathcal{C}_{\rm mcc}$

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[6]The definition of the interpretation function $I(\mathcal{C}_{\rm mcc})$ of a canonical interpretation $\mathcal{M}(\mathcal{C}_{\rm mcc})$, allows us to write $box_\alpha(V(x))$ instead of $\parallel box_\alpha \parallel_{I(\mathcal{C}_{\rm mcc})} (V(x))$.

and so $R(V(x),V(x)) \in FOT(\mathcal{C}_{\text{mcc}})$. Thus, $\langle V(x),V(x)\rangle \in\parallel R \parallel_{I(\mathcal{C}_{\text{mcc}})}$. Hence, $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} R(V(x),V(x))$. $\star$

Proposition 3.28 {Satisfiability of (**Irr**)} Let $K\delta$-MLDS be a MLDS such that $\{\forall x \neg R(x,x)\} \subseteq \mathcal{A}_{K\delta}$. Let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS and let $FOT(\mathcal{C}_{\text{mcc}})$ be its first–order translation. Then $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \Vdash_{FOL} \forall x \neg R(x,x)$.

Proof:
Let V be an arbitrary variable assignment. It is sufficient to prove that $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} \neg R(V(x),V(x))$. By Proposition 3.22, $\neg R(V(x),V(x)) \in \mathcal{C}_{\text{mcc}}$ and so $R(V(x),V(x)) \notin FOT(\mathcal{C}_{\text{mcc}})$. Hence, $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} \neg R(V(x),V(x))$. $\star$

Proposition 3.29 {Satisfiability of (**4**)} Let $K\delta$-MLDS be a MLDS such that $\{\forall x,y,z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))\} \subseteq \mathcal{A}_{K\delta}$. Let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS and let $FOT(\mathcal{C}_{\text{mcc}})$ be its first–order translation. Then $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \Vdash_{FOL} \forall x,y,z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$.

Proof:
Let V be an arbitrary variable assignment. It is sufficient to prove that

$$\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} (R(V(x),V(y)) \wedge R(V(y),V(z))) \rightarrow R(V(x),V(z))$$

Assume that $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} (R(V(x),V(y)) \wedge R(V(y),V(z)))$. This implies that $\langle V(x),V(y)\rangle \in\parallel R \parallel_{I(\mathcal{C}_{\text{mcc}})}$ and $\langle V(y),V(z)\rangle \in\parallel R \parallel_{I(\mathcal{C}_{\text{mcc}})}$. Therefore, $R(V(x),V(y)) \in FOT(\mathcal{C}_{\text{mcc}})$ and $R(V(y),V(z)) \in FOT(\mathcal{C}_{\text{mcc}})$.
Thus $R(V(x),V(y)) \in \mathcal{C}_{\text{mcc}}$ and $R(V(y),V(z)) \in \mathcal{C}_{\text{mcc}}$.
By Proposition 3.23, $R(V(x),V(z)) \in \mathcal{C}_{\text{mcc}}$. Then $R(V(x),V(z)) \in FOT(\mathcal{C}_{\text{mcc}})$.
Hence, $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} R(V(x),V(z))$. $\star$

Proposition 3.30 {Satisfiability of (**B**)} Let $K\delta$-MLDS be a MLDS such that $\{\forall x,y R(x,y) \rightarrow R(y,x)\} \subseteq \mathcal{A}_{K\delta}$. Let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS and let $FOT(\mathcal{C}_{\text{mcc}})$ be its first–order translation. Then $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \Vdash_{FOL} \forall x,y R(x,y) \rightarrow R(x,z)$.

Proof:
Let V be an arbitrary variable assignment. It is sufficient to prove that if $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} (R(V(x),V(y))$ then $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} R(V(y),V(x))$. Assume that $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} (R(V(x),V(y))$. Then $\mathcal{M}(\mathcal{C}_{mcc}), V \Vdash_{FOL} R(V(x),V(y))$ which implies that $\langle V(x),V(y)\rangle \in\parallel R \parallel_{I(\mathcal{C}_{\text{mcc}})}$. Thus, $R(V(x),V(y)) \in FOT(\mathcal{C}_{\text{mcc}})$ and $R(V(x),V(y)) \in \mathcal{C}_{\text{mcc}}$ By Proposition 3.24, $R(V(y),V(x)) \in \mathcal{C}_{\text{mcc}}$. Hence, $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} R(V(y),V(x))$. $\star$

Proposition 3.31 {Satisfiability of (**D**)} Let $K\delta$-MLDS be a MLDS such that $\{\forall x R(x, succ(x))\} \subseteq \mathcal{A}_{K\delta}$. Let $\mathcal{C}_{\mathrm{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS and let $FOT(\mathcal{C}_{\mathrm{mcc}})$ be its first–order translation. Then $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}) \Vdash_{FOL} \forall x R(x, succ(x))$.

Proof:
Let V be an arbitrary variable assignment. It is sufficient to prove that

$$\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} R(V(x), succ(V(x)))$$

By Proposition 3.26, $R(V(x), succ(V(x))) \in \mathcal{C}_{\mathrm{mcc}}$ and so $R(V(x), succ(V(x))) \in FOT(\mathcal{C}_{\mathrm{mcc}})$. Thus, $\langle V(x), succ(V(x))\rangle \in \parallel R \parallel_{I(\mathcal{C}_{\mathrm{mcc}})}$.
Hence, $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} R(V(x), succ(V(x)))$. $\star$

Proposition 3.32 {Satisfiability of (**5**)} Let $K\delta$-MLDS be a MLDS such that $\{\forall x, y, z((R(x, y) \wedge R(x, z)) \rightarrow R(y, z))\} \subseteq \mathcal{A}_{K\delta}$. Let $\mathcal{C}_{\mathrm{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS and let $FOT(\mathcal{C}_{\mathrm{mcc}})$ be its first–order translation. Then $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}) \Vdash_{FOL} \forall x, y, z((R(x, y) \wedge R(x, z)) \rightarrow R(y, z))$.

Proof:
Let V be an arbitrary variable assignment. It is sufficient to prove that
$\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} (R(V(x), V(y)) \wedge R(V(x), V(z))) \rightarrow R(V(y), V(z))$.
Assume that $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} (R(V(x), V(y)) \wedge R(V(x), V(z)))$. This implies that $\langle V(x), V(y)\rangle \in \parallel R \parallel_{I(\mathcal{C}_{\mathrm{mcc}})}$ and $\langle V(x), V(z)\rangle \in \parallel R \parallel_{I(\mathcal{C}_{\mathrm{mcc}})}$.
Therefore, $R(V(x), V(y)) \in FOT(\mathcal{C}_{\mathrm{mcc}})$ and $R(V(x), V(z)) \in FOT(\mathcal{C}_{\mathrm{mcc}})$.
Thus $R(V(x), V(y)) \in \mathcal{C}_{\mathrm{mcc}}$ and $R(V(x), V(z)) \in \mathcal{C}_{\mathrm{mcc}}$.
By Proposition 3.25, $R(V(y), V(z)) \in \mathcal{C}_{\mathrm{mcc}}$. Hence, $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} R(V(y), V(z))$.
$\star$

So far it has been shown that for any propositional $K\delta$-MLDS, given a consistent configuration $\mathcal{C}$ of $K\delta$-MLDS there exists a maximal consistent configuration $\mathcal{C}_{\mathrm{mcc}}$ such that $\mathcal{C} \subseteq \mathcal{C}_{\mathrm{mcc}}$, and that there exists a canonical interpretation $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}})$ which is a semantic structure of $K\delta$-MLDS (i.e. it satisfies the extended algebra $\mathcal{A}^{+}_{K\delta}$). Now the model existence lemma can be proved. The following lemma shows that the semantic structure $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}})$ satisfies the maximal consistent configuration $\mathcal{C}_{\mathrm{mcc}}$ and therefore the initial consistent configuration $\mathcal{C}$.

Lemma 3.3 {Model Existence Lemma} Let $K\delta$-MLDS be an arbitrary MLDS and let $\mathcal{C}_{\mathrm{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS.

Then for any π (where π is a declarative unit or an R–literal) of $K\delta$-MLDS, $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \models_{\text{MLDS}} \pi$ if $\pi \in \mathcal{C}_{\text{mcc}}$, and $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \not\models_{\text{MLDS}} \pi$ if $\pi \notin \mathcal{C}_{\text{mcc}}$.

Proof:
There are two cases to consider.

1. π is a declarative unit of the form $\lambda : \alpha$.
   If $\lambda : \alpha \in \mathcal{C}_{\text{mcc}}$, then $[\alpha]^*(\lambda) \in FOT(\mathcal{C}_{\text{mcc}})$. Then $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \Vdash_{FOL} [\alpha]^*(\lambda)$. Hence,$\mathcal{M}(\mathcal{C}_{mcc}) \models_S \lambda : \alpha$. If $\lambda : \alpha \notin \mathcal{C}_{\text{mcc}}$ then $[\alpha]^*(\lambda) \notin FOT(\mathcal{C}_{\text{mcc}})$. Then, $\lambda \notin \| [\alpha]^* \|_{I(\mathcal{C}_{\text{mcc}})}$, and so by Remark 3.2, $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \not\models_{FOL} [\alpha]^*(\lambda)$. Hence, $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \not\models_S \lambda : \alpha$.

2. π is an R–literal.
   Then π can be either equal to $R(\lambda_i, \lambda_j)$, or equal to $\neg R(\lambda_i, \lambda_j)$, where λ_i, λ_j are ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$.

   - Suppose that π is of the form $R(\lambda_i, \lambda_j)$. If $R(\lambda_i, \lambda_j) \in \mathcal{C}_{\text{mcc}}$, then $R(\lambda_i, \lambda_j) \in FOT(\mathcal{C}_{\text{mcc}})$. So $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \Vdash_{FOL} R(\lambda_i, \lambda_j)$. Hence, $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \models_{\text{MLDS}} R(\lambda_i, \lambda_j)$. If $R(\lambda_i, \lambda_j) \notin \mathcal{C}_{\text{mcc}}$, then $R(\lambda_i, \lambda_j) \notin FOT(\mathcal{C}_{\text{mcc}})$. Then, $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \not\Vdash_{FOL} R(\lambda_i, \lambda_j)$. Hence, $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \not\models_{\text{MLDS}} R(\lambda_i, \lambda_j)$.
   - Suppose now that π is of the form $\neg R(\lambda_i, \lambda_j)$. If $\neg R(\lambda_i, \lambda_j) \in \mathcal{C}_{\text{mcc}}$ then by Proposition 3.12 $R(\lambda_i, \lambda_j) \notin \mathcal{C}_{\text{mcc}}$. Then $R(\lambda_i, \lambda_j) \notin FOT(\mathcal{C}_{\text{mcc}})$. So $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \not\Vdash_{FOL} R(\lambda_i, \lambda_j)$ and then $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \Vdash_{FOL} \neg R(\lambda_i, \lambda_j)$. Hence $\mathcal{M}(\mathcal{C}_{\text{mcc}}) \models_{\text{MLDS}} \neg R(\lambda_i, \lambda_j)$.

$\star$

Corollary 3.2 Let $K\delta$-MLDS be a propositional MLDS and let $\mathcal{C}$ be a consistent configuration of $K\delta$-MLDS. Then $\mathcal{M}(\mathcal{C}_{\text{mcc}})$ satisfies $\mathcal{C}$.

Proof:
The proof trivially follows from Lemma 3.3 and from the fact that $\mathcal{C} \subseteq \mathcal{C}_{\text{mcc}}$. $\star$

Proposition 3.33 Let $K\delta$-MLDS be a propositional MLDS and let $\mathcal{C}$ be a configuration of $K\delta$-MLDS. Let π be a declarative unit or an R–literal such that $\pi \notin \mathcal{C}$. If $\mathcal{C} \not\vdash_{\text{MLDS}} \pi$ then $\mathcal{C} + [\neg\pi]$ is a consistent configuration.

Proof:
There are two cases to consider: (i) π is a declarative unit and (ii) π is an

R–literal.

(i) Suppose that π is a declarative unit of the form $\lambda : \alpha$. The contrapositive of the proposition statement is proved. Assume that $\mathcal{C} + [\lambda : \neg\alpha]$ is not consistent. Then $\mathcal{C} + [\lambda : \neg\alpha] \vdash_{\mathrm{MLDS}} \lambda' : \perp$, for some ground term λ' of $Func(\mathcal{L}_L, \mathcal{L}_M)$. Let $\mathcal{C}' = \mathcal{C} + [\lambda : \neg\neg\alpha]$ and $\mathcal{C}^* = \mathcal{C}' + [\lambda : \alpha]$. The pair of configurations $\mathcal{C}/\mathcal{C}' \in \mathcal{I}_{\neg I}$ and the pair of configurations $\mathcal{C}'/\mathcal{C}^* \in \mathcal{I}_{\neg E}$. So $\langle\{\mathcal{C}, \mathcal{C}', \mathcal{C}^*\}, m\rangle$ is a proof, where $m(0) = \mathcal{I}_{\neg I}$ and $m(1) = \mathcal{I}_{\neg E}$. Since $\lambda : \alpha \in \mathcal{C}^*$, $\mathcal{C} \vdash_{\mathrm{MLDS}} \lambda : \alpha$.

(ii) Suppose that π is an R–literal Δ. The contrapositive of the proposition statement is proved. Assume that $\mathcal{C} + [\neg\Delta]$ is not consistent. $\mathcal{C} + [\neg\Delta] \vdash_{\mathrm{MLDS}} \lambda' : \perp$, for some ground term λ' of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and the pair of configurations $\mathcal{C}/\mathcal{C} + [\Delta] \in \mathcal{I}_{R-I}$. So $\langle\{\mathcal{C}, \mathcal{C} + [\Delta]\}, m\rangle$ is a proof, where $m(0) = \mathcal{I}_{R-I}$. Hence $\mathcal{C} \vdash_{\mathrm{MLDS}} \Delta$.

$\star$

The completeness theorem given at the beginning of this section can now be proved.

Proof of Completeness (Theorem 3.4)

The proof is by contrapositive. Assume that $\mathcal{C} \not\vdash_{\mathrm{MLDS}} \mathcal{C}'$. Then by Theorem 3.2 there exists a $\pi \in \mathcal{C}' - \mathcal{C}$, where π is a declarative unit or an R–literal, such that $\mathcal{C} \not\vdash_{\mathrm{MLDS}} \pi$. Then by Proposition 3.33, $\mathcal{C} + [\neg\pi]$ is a consistent configuration. By Corollary 3.2, $\mathcal{M}_m = \mathcal{M}(MCC(\mathcal{C} + [\neg\pi]))$ satisfies the configuration $\mathcal{C} + [\neg\pi]$. So by Definition 2.30, $\mathcal{M}_m \models_{\mathrm{MLDS}} \mathcal{C}$ and $\mathcal{M}_m \models_{\mathrm{MLDS}} \neg\pi$. There are two cases to consider.

1. π is either an R–literal. By Definition 2.29, $\mathcal{M}_m \Vdash_{FOL} \neg\pi$. So by satisfiability condition of first-order logic, $\mathcal{M}_m \not\Vdash_{FOL} \pi$. Then by Definition 2.27 and Definition 2.30, $\mathcal{A}^+ \cup FOT(\mathcal{C}) \not\Vdash_{FOL} \pi$. Hence by Definition 2.31, $\mathcal{C} \not\models_{\mathrm{MLDS}} \mathcal{C}'$.

2. π is a declarative unit of the form $\lambda : \alpha$. By Definition 2.28, $\mathcal{M}_m \models_{FOL} FOT(\neg\pi)$, where $FOT(\neg\pi) = [\neg\alpha]^*(\lambda)$. So $\mathcal{M}_m \models_{FOL} [\neg\alpha]^*(\lambda)$. Then by Theorem 3.5, $\mathcal{M}_m \models_{FOL} \neg[\alpha]^*(\lambda)$. So $\mathcal{M}_m \not\models_{FOL} [\alpha]^*(\lambda)$. which means that $\mathcal{A}^+, FOT(\mathcal{C}) \not\models_{FOL} \pi$. Hence $\mathcal{C} \not\models_{\mathrm{MLDS}} \mathcal{C}'$.

$\star$

3.3 Weak and Strong Correspondence

It was claimed previously that a propositional MLDS is a generalisation of propositional modal logic, in that it facilitates reasoning about structures of actual worlds, which may or may not be single points. This claim is substantiated in this section by showing (i) that there exists a correspondence between a MLDS and any sound and complete proof system for traditional modal logic, whenever certain restrictions are imposed on initial configurations and (ii) that the correspondence clearly fails if no restriction is imposed. As far as the first result is concerned, the restriction consists of identifying a particular constant symbol in $\mathcal{L}_L$, say W_0, and allowing initial configurations only of the form $\mathcal{C}_i = \langle \{\}, \mathcal{F}_i \rangle$ (i.e. no R–literals belong to $\mathcal{C}_i$) where for any ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda \neq W_0$, $\mathcal{F}_i(\lambda) = \{\}$. With this restriction, the only initial assumptions (if any) are modal formulae associated with the label W_0. This corresponds to the traditional notion of local assumptions in modal logic. In particular, the following two theorems show that any declarative unit of the form $W_0 : \alpha$ can be derived from an (empty) initial configuration of the form $\mathcal{C}_i$ if and only if its formula α is derivable, within the sound and complete axiomatic system for modal logic, given in Chapter 1, from the (empty) set of modal formulae that appear in $\mathcal{C}_i$.

Lemma 3.4 {Simple Correspondence for modal logic K}

Let K-MLDS be the propositional MLDS whose associated labelling algebra $\mathcal{A}_K = \{\}$, and let $\langle K_{Ax}, \vdash_{K_{Ax}} \rangle$ be the axiomatic system for modal logic K. Let $\mathcal{C}_{\{\}} = \langle \{\}, \mathcal{F}_{\{\}} \rangle$, where for any ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{F}_{\{\}}(\lambda) = \{\}$. Let α be a formula of $\mathcal{L}_M$. Then:

$$\vdash_{K_{Ax}} \alpha$$
$$\text{if and only if}$$
$$\text{for all ground terms } \lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)\ \mathcal{C}_{\{\}} \vdash_{K-\text{MLDS}} \lambda : \alpha$$

Proof:

("If" half:) The proof is by showing that the contrapositive statement holds – i.e. given a formula $\alpha \in \mathcal{L}_M$, if $\not\vdash_{K_{Ax}} \alpha$ then there exists a ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$ such that $\mathcal{C}_{\{\}} \not\vdash_{K-\text{MLDS}} \lambda : \alpha$. Since the two derivability relations $\vdash_{K_{Ax}}$ and $\vdash_{K-\text{MLDS}}$ are both sound and complete with respect to their semantics[7], it is sufficient to prove that if $\not\models_K \alpha$ then there exists a ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$ such that $\mathcal{C}_{\{\}} \not\models_{K-\text{MLDS}} \lambda : \alpha$. Suppose that $\not\models_K \alpha$. Since $\mathcal{C}_{\{\}}$ is an empty configuration, it is sufficient to show that there exists a semantic structure of K-MLDS (i.e. a model of $\mathcal{A}^+$) which satisfies the declarative unit

[7] $\vdash_{K_{Ax}}$ is complete with respect to the Kripke semantics and $\vdash_{K-\text{MLDS}}$ is complete with respect to the semantics defined in Chapter 2.

$\lambda : \neg\alpha$. By hypothesis and by the Kripke semantic definition of validity, there exists a Kripke model which satisfies the formula $\neg\alpha$ (i.e. a countermodel of the formula α). Let $M = \langle \widehat{W}, \hat{R}, v\rangle$ be such a model. Thus there exists some possible world $w \in \widehat{W}$ such that $M, w \Vdash_K \neg\alpha$. Assume a canonical well-ordering on the set $\widehat{W}$[8]. Let $Sat = \{w_i \mid w_i \in \widehat{W}$ and $M, w_i \Vdash_K \neg\alpha\}$ and let w' be the first element of Sat according to $\widehat{W}$'s canonical well-ordering relation[9]. Then $M, w' \Vdash_K \neg\alpha$. Let I be an interpretation function on the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$, whose universe of discourse is $\widehat{W}$, defined as follows.

- For each constant symbol W_i

  $\| W_i \|_I = w'$.

- For the function symbol $succ$,

  $\| succ \|_I = \widehat{succ}: \widehat{W} \longrightarrow \widehat{W}$ such that for each $w \in \widehat{W}$, if $Acc(w) = \{\tilde{w} \mid \tilde{w} \in \widehat{W}$ and $w\hat{R}\tilde{w}\}$ is empty then $succ(w) = w$, otherwise $succ(w) = \overline{w}$, where $\overline{w}$ is the first element of $Acc(w)$ with respect to the assumed canonical well-ordering of $\widehat{W}$.

- For each function symbol of the form f_β

  $\| f_\beta \|_I = \hat{f}_\beta : \widehat{W} \longrightarrow \widehat{W}$ such that for each $w \in \widehat{W}$

  - If $Sat_\beta(w) = \{\tilde{w} \mid \tilde{w} \in \widehat{W}, w\hat{R}\tilde{w}$ *and* $M, \tilde{w} \Vdash_K \beta\}$ is a non–empty set, then $\hat{f}_\beta(w) = \overline{w}$ where $\overline{w}$ is the first element of $Sat_\beta(w)$ with respect to $\widehat{W}$'s canonical well–ordering.
  - Otherwise $\hat{f}_\beta(w) = w_0$, where w_0 is the first element of $\widehat{W}$ according to its assumed canonical well-ordering.

- For each function symbol box_β

  $\| box_\beta \|_I = \widehat{box}_\beta : \widehat{W} \longrightarrow \widehat{W}$ such that for each $w \in \widehat{W}$

  - If for all $\tilde{w} \in \widehat{W}$ not $w\hat{R}\tilde{w}$, then $\widehat{box}_\beta(w) = w$
  - If for each $w_i \in Acc(w) = \{\tilde{w} \mid \tilde{w} \in \widehat{W}$ and $w\hat{R}\tilde{w}\}$, $M, w_i \Vdash_K \beta$, then $\widehat{box}_\beta(w) = \overline{w}'$, where $\overline{w}'$ is the first element of $Acc(w)$ with respect to the assumed canonical well-ordering of $\widehat{W}$.
  - Otherwise $\widehat{box}_\beta(w) = \overline{w}''$, where $\overline{w}''$ is the first element of the non–empty set $Sat_{\neg\beta} = \{\tilde{w} \mid \tilde{w} \in \widehat{W}, w\hat{R}\tilde{w}$ *and* $M, \tilde{w} \Vdash_K \neg\beta\}$, according to $\widehat{W}$'s canonical well-ordering.

[8]The existence of a well-ordering relation is guaranteed by Zermelo's *Well–Ordering Theorem.*

[9]The existence of this first element is guaranteed by the fact that Sat is a non empty subset of the well–ordered set $\widehat{W}$.

- For each monadic predicate $[\beta]^*$
  $\| [\beta]^* \|_I = \{w \mid w \in \widehat{W} \text{ and } M, m \Vdash_K \beta\}$

- For the binary predicate R
  $\| R \|_I = \widehat{R}$

It is shown now that $\langle \widehat{W}, I\rangle$ is a MLDS semantic structure. By Definition 2.27, this requires proving that $\langle \widehat{W}, I\rangle$ is a classical model of the extended algebra $\mathcal{A}^+$. Observe that in a K-MLDS, the labelling algebra $\mathcal{A} = \{\}$. Therefore the extended algebra $\mathcal{A}^+$ is given only by the set of axiom schemas (Ax1)–(Ax8). The proof is by cases on each of the axiom schemas (Ax1)–(Ax8). Let β, γ be two wffs of $\mathcal{L}_M$.

Case 1: **(Ax1)** $\forall x([\beta \wedge \gamma]^*(x) \equiv ([\beta]^*(x) \wedge [\gamma]^*(x))$

Let $w \in \widehat{W}$ be an arbitrary element. It is sufficient to prove that $w \in \| [\beta \wedge \gamma]^* \|_I$ if and only if $w \in \| [\beta]^* \|_I$ and $w \in \| [\gamma]^* \|_I$. This follows directly from the definition of $\| \beta \wedge \gamma \|_I$ and the Kripke semantic definition of satisfiability for $\wedge$–formulae.

The argument for Cases 2-4 (**(Ax2)**-**(Ax4)**) are analogous to Case 1 above.

Case 5: **(Ax5)** $\forall x([\Diamond\beta]^*(x) \rightarrow (R(x, f_\beta(x)) \wedge [\beta]^*(f_\beta(x))))$.

Let $w \in \widehat{W}$ be an arbitrary element. It is sufficient to prove that if $w \in \| [\Diamond\beta]^* \|_I$ then $\langle w, \hat{f}_\beta(w)\rangle \in \| R \|_I$ and $\hat{f}_\beta(w) \in \| [\beta]^* \|_I$. Assume that $w \in \| [\Diamond\beta]^* \|_I$. By definition of I, $M, w \Vdash_K \Diamond\beta$. Then, by Kripke semantic definition of satisfiability, there exists some possible world $\tilde{w} \in \widehat{W}$ such that $w\widehat{R}\tilde{w}$ and $M, \tilde{w} \Vdash_K \beta$. Therefore, by definition of I, the set $Sat_\beta(w) \neq \{\}$ and $\hat{f}_\beta(w) = \overline{w}$, where $\overline{w}$ is the first element of $Sat_\beta(w)$ according to $\widehat{W}$'s canonical well–ordering. Then $w\widehat{R}\hat{f}_\beta(w)$ and $M, \hat{f}_\beta(w) \Vdash_K \beta$. Hence by definition of I, the tuple $\langle w, \hat{f}_\beta(w)\rangle \in \| R \|_I$ and $\hat{f}_\beta(w) \in \| [\beta]^* \|_I$.

Case 6: **(Ax6)** $\forall x(\exists y(R(x, y) \wedge [\beta]^*(y)) \rightarrow [\Diamond\beta]^*(x))$.

Consider the equivalent formula

$$\forall x \forall y((R(x, y) \wedge [\beta]^*(y)) \rightarrow [\Diamond\beta]^*(x))$$

Let w, w' be two arbitrary elements of $\widehat{W}$. It is sufficient to prove that if the tuple $\langle w, w'\rangle \in \| R \|_I$ and $w' \in \| [\beta]^* \|_I$, then $w \in \| [\Diamond\beta]^* \|_I$. Assume that $\langle w, w'\rangle \in \| R \|_I$ and $w' \in \| [\beta]^* \|_I$. By definition of I, $w\widehat{R}w'$ and $M, w' \Vdash_K \beta$. Then by Kripke semantic definition of satisfiability of the $\Diamond$ operator, $M, w \Vdash_K \Diamond\beta$. Hence by definition of I, $w \in \| [\Diamond\beta]^* \|_I$.

Case 7: **(Ax7)** $\quad \forall x((R(x, box_\beta(x)) \rightarrow [\beta]^*(box_\beta(x))) \rightarrow [\Box\beta]^*(x))$.

Let w be an arbitrary element of $\widehat{W}$. It is sufficient to prove that if the tuple $\langle w, \widehat{box}_\beta(w)\rangle \notin \| R \|_I$ or $\widehat{box}_\beta(w) \in \| [\beta]^* \|_I$, then $w \in \| [\Box\beta]^* \|_I$. Suppose first, that the tuple $\langle w, \widehat{box}_\beta(w)\rangle \notin \| \widehat{R} \|_I$. This implies, by definition of $\| box_\beta \|_I$, that for all $w' \in \widehat{W}$, not $w\widehat{R}w'$. Then by Kripke semantic definition of satisfiability of the $\Box$ operator, $M, m \Vdash_K \Box\beta$. Hence by definition of I, $w \in \| [\Box\beta]^* \|_I$. Suppose now that $\widehat{box}_\beta(w) \in \| [\beta]^* \|_I$. Then by definition of $\| box_\beta \|_I$, there are two sub-cases to consider: (i) for all $w' \in \widehat{W}$, not $w\widehat{R}w'$. Then by Kripke semantic definition of satisfiability of the $\Box$ operator, $M, w \Vdash_K \Box\beta$. Hence by definition of I, $w \in \| [\Box\beta]^* \|_I$. (ii) For all $w' \in \widehat{W}$ such that $w\widehat{R}w'$, $M, w' \Vdash_K \beta$. Then again by Kripke semantic definition of satisfiability of the $\Box$ operator, $M, w \Vdash_K \Box\beta$. Hence by definition of I, $w \in \| [\Box\beta]^* \|_I$.

Case 8: **(Ax8)** $\quad \forall x([\Box\beta]^*(x) \rightarrow (\forall y(R(x,y) \rightarrow [\beta]^*(y))))$

Consider the equivalent formula

$$\forall x \forall y(([\Box\beta]^*(x) \wedge R(x,y)) \rightarrow [\beta]^*(y))$$

Let w, w' be two arbitrary elements of $\widehat{W}$ such that $w \in \| [\Box\beta]^* \|_I$ and $\langle w, w'\rangle \in \| R \|_I$. Then by definition of I, $M, w \Vdash_K \Box\beta$ and $w\widehat{R}w'$. By Kripke semantic satisfiability definition of the $\Box$ operator, $M, w' \Vdash_K \beta$. Hence by definition of I, $w' \in \| [\beta]^* \|_I$.

Therefore, the tuple $\langle \widehat{W}, I\rangle$ is a model of $\mathcal{A}^+$. Thus, since by hypothesis $M, w' \Vdash_K \neg\alpha$ and, by definition of I, $\| W_i \|_I = w'$, where W_i are constant symbols of $Mon(\mathcal{L}_L, \mathcal{L}_M)$, there exists a ground term λ, $\lambda = W_i$ for some $i \geq 0$, such that $\langle \widehat{W}, I\rangle \models_{FOL} [\neg\alpha]^*(\lambda)$. Hence, by definition of $\models_{K-\text{MLDS}}$, $\mathcal{C}_{\{\}} \not\models_{K-\text{MLDS}} \lambda : \alpha$.

(End of "If")

("Only if" half: i.e. if $\vdash_{K_{Ax}} \alpha$ then $\mathcal{C}_{\{\}} \vdash_{K-\text{MLDS}} \lambda : \alpha$.) Suppose that $\vdash_{K_{Ax}} \alpha$ and that $\alpha_1, \ldots, \alpha_m$, where $m \geq 1$ and $\alpha_m = \alpha$, is the shortest derivation of α with length (i.e. number of steps) $l \geq 1$[10]. Proof that $\mathcal{C}_{\{\}} \vdash_{K-\text{MLDS}} \lambda : \alpha$ is by induction on l.

BASE CASE:
The base case is when $l = 1$. Then α is an instantiation of one of the axiom schemas of K_{Ax}. It is shown that $\mathcal{C}_{\{\}} \vdash_{K-\text{MLDS}} \lambda : \alpha$ by cases considering each axiom schema of K_{Ax} in turn.

[10]Note that since there are no local assumptions the derivation is all global.

Case 1: [A1] Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and let $\mathcal{C}_0 = \mathcal{C}_{\{\}}$. Then $C_0 \vdash_{K-\text{MLDS}} \lambda : \alpha \rightarrow (\beta \rightarrow \alpha)$ as shown in the following derivation:

$$
\begin{array}{ll}
\mathcal{C}_0\langle\rangle & \\
\hline
\quad \mathcal{C}_0\langle[\lambda:\alpha]\rangle & \text{(assumption)} \\
\hline
\qquad \mathcal{C}_0\langle[\lambda:\alpha],[\lambda:\beta]\rangle & \text{(assumption)} \\
\hline
\qquad \mathcal{C}_1\langle[\lambda:\beta],\lambda:\alpha\rangle & \text{(C–R)} \\
\hline
\quad \mathcal{C}_2\langle[\lambda:\alpha],\lambda:\beta\rightarrow\alpha\rangle & (\rightarrow \text{I}) \\
\hline
\mathcal{C}'\langle\lambda:\alpha\rightarrow(\beta\rightarrow\alpha)\rangle & (\rightarrow \text{I})
\end{array}
$$

It is clear that the other axioms (Cases 2-4: [A2]-[A4]) concerning the $\rightarrow$ connective can be proved within the K-MLDS, by a sequence of ($\rightarrow$ I) and ($\rightarrow$ E) rules. The groups of axioms [A5]–[A6] (Cases 5 and 6) and [A8]–[A9] (Cases 8 and 9) can be derived in K-MLDS using the ($\wedge$ E) and the ($\vee$ I) rules, respectively. Analogously the axioms [A7] (Case 7) and [A10] (Case 10) can be proved in K-MLDS by using respectively the ($\wedge$ I) and the ($\vee$ E) rules. Finally, the axiom [A11] (Case 11) can be derived in K-MLDS by using the ($\neg$ I) rule and the axiom [A12] (Case 12) by using the ($\neg$ E) rule.

Case 13: [A13] Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and let $\mathcal{C}_0 = \mathcal{C}_{\{\}}$. Then $C_0 \vdash_{K-\text{MLDS}} \lambda : \Box\alpha \rightarrow \neg\Diamond\neg\alpha$, as shown in the following derivation:

$$
\begin{array}{ll}
\mathcal{C}_0\langle\rangle & \\
\hline
\quad \mathcal{C}_0\langle[\lambda:\Box\alpha]\rangle & \text{(assumption)} \\
\hline
\qquad \mathcal{C}_0\langle[\lambda:\Box\alpha],[\lambda:\Diamond\neg\alpha]\rangle & \text{(assumption)} \\
\hline
\qquad \mathcal{C}_1\langle R(\lambda, f_\alpha(\lambda)), f_\alpha(\lambda):\neg\alpha\rangle & (\Diamond \text{ E}) \\
\hline
\qquad \mathcal{C}_2\langle f_\alpha(\lambda):\neg\alpha, f_\alpha(\lambda):\alpha\rangle & (\Box \text{ E}) \\
\hline
\qquad \mathcal{C}_3\langle f_\alpha(\lambda):\bot\rangle & (\wedge \text{ I}) \\
\hline
\quad \mathcal{C}_4\langle[\lambda:\Box\alpha],\lambda:\neg\Diamond\neg\alpha\rangle & (\neg \text{ I}) \\
\hline
\mathcal{C}'\langle\lambda:\Box\alpha\rightarrow\neg\Diamond\neg\alpha\rangle & (\rightarrow \text{I})
\end{array}
$$

Case 14: [A14] Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and let $\mathcal{C}_0 = \mathcal{C}_{\{\}}$. Then $C_0 \vdash_{K-\text{MLDS}} \lambda : \neg\Diamond\neg\alpha \rightarrow \Box\alpha$, as shown in the following derivation:

$\mathcal{C}_0\langle\rangle$	
$\mathcal{C}_0\langle[\lambda : \neg\Diamond\neg\alpha]\rangle$	(assumption)
$\mathcal{C}_0\langle[\lambda : \neg\Diamond\neg\alpha], [R(\lambda, box_\alpha(\lambda))]\rangle$	(assumption)
$\mathcal{C}_0\langle[\lambda : \neg\Diamond\neg\alpha], [box_\alpha(\lambda) : \neg\alpha]\rangle$	(assumption)
$\mathcal{C}_1\langle[\lambda : \neg\Diamond\neg\alpha], \lambda : \Diamond\neg\alpha\rangle$	($\Diamond$ I)
$\mathcal{C}_2\langle\lambda : \bot\rangle$	($\wedge$ I)
$\mathcal{C}_3\langle box_\alpha(\lambda) : \neg\neg\alpha\rangle$	($\neg$ I)
$\mathcal{C}_4\langle box_\alpha(\lambda) : \alpha\rangle$	($\neg$ E)
$\mathcal{C}_5\langle\lambda : \Box\alpha\rangle$	($\Box$ I)
$\mathcal{C}'\langle\lambda : \neg\Diamond\neg\alpha \rightarrow \Box\alpha\rangle$	($\rightarrow$ I)

Case 15: [K] Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and let $\mathcal{C}_0 = \mathcal{C}_{\{\}}$. Then $C_0 \vdash_{K-\text{MLDS}} \lambda : \Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$, as shown in the following derivation:

$\mathcal{C}_0\langle\rangle$	
$\mathcal{C}_0\langle[\lambda : \Box(\alpha \rightarrow \beta)]\rangle$	(assumption)
$\mathcal{C}_0\langle[\lambda : \Box(\alpha \rightarrow \beta)], [\lambda : \Box\alpha],\rangle$	(assumption)
$\mathcal{C}_0\langle[R(\lambda, box_\alpha(\lambda))]\rangle$	(assumption)
$\mathcal{C}_1\langle box_\alpha(\lambda) : \alpha \rightarrow \beta\rangle$	($\Box$ E)
$\mathcal{C}_2\langle box_\alpha(\lambda) : \alpha \rightarrow \beta, box_\alpha(\lambda) : \alpha\rangle$	($\Box$ E)
$\mathcal{C}_3\langle box_\alpha(\lambda) : \beta\rangle$	($\rightarrow$ E)
$\mathcal{C}_4\langle[\lambda : \Box\alpha], \lambda : \Box\beta\rangle$	($\Box$ I)
$\mathcal{C}_5\langle[\lambda : \Box(\alpha \rightarrow \beta)], \lambda : \Box\alpha \rightarrow \Box\beta\rangle$	($\rightarrow$ I)
$\mathcal{C}'\langle\lambda : \Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)\rangle$	($\rightarrow$ I)

INDUCTIVE STEP:
Assume, by the inductive hypothesis, that for any formula α' such that $\vdash_{K_{Ax}} \alpha'$ and such that there exists a proof $\alpha'_1, \ldots, \alpha'_n$, where $\alpha'_n = \alpha'$, with $n > 0$, then for any ground term $\lambda' \in Mon(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{C}_{\{\}} \vdash_{K-\text{MLDS}} \lambda' : \alpha'$.

Suppose now that $\vdash_{K_{Ax}} \alpha$, with a shortest proof $\alpha_1, \ldots, \alpha_{n+1}$, where $\alpha_{n+1} = \alpha$, with $n > 0$. By Theorem 3.2, Theorem 3.3 and the inductive hypothesis, for any arbitrary label λ' there exists a configuration $\mathcal{C}_{\lambda'}$ such that $\lambda' : \alpha_i \in \mathcal{C}_{\lambda'}$ for all $1 \leq i \leq n$, and such that $\mathcal{C}_{\{\}} \vdash_{K-\text{MLDS}} \mathcal{C}_{\lambda'}$. Since $n + 1 > 1$, α ($= \alpha_{n+1}$) is not an instantiation of an axiom schema of K_{Ax}. Hence there are two cases to consider: (1) α is derived by an application of the [MP] rule, or (2) α is derived by an application of the [Nec] rule.

Case 1: [MP]

Suppose that the last step is given by the application of the [MP] rule. Then there exist two formulae of the form α_k and $\alpha_k \to \alpha$ in the sequence $\alpha_1, \ldots, \alpha_n$. Hence $\lambda : \alpha_k \in \mathcal{C}_\lambda$ and $\lambda : \alpha_k \to \alpha \in \mathcal{C}_\lambda$. Therefore by the definition of the $\mathcal{I}_{\to E}$ rule of MLDS, $\mathcal{C}_\lambda \vdash_{K-\mathrm{MLDS}} \lambda : \alpha$. Hence by transitivity of $\vdash_{K-\mathrm{MLDS}}$, $\mathcal{C}_{\{\}} \vdash_{K-\mathrm{MLDS}} \lambda : \alpha$.

Case 2: [Nec]

Suppose that the last step is given by the application of the [Nec] rule. Then α is of the form $\Box\alpha_k$ where α_k is a formula in the sequence $\alpha_1, \ldots, \alpha_n$. Hence $box_{\alpha_k}(\lambda) : \alpha_k \in \mathcal{C}_{box_{\alpha_k}(\lambda)}$ and $\mathcal{C}_{\{\}} \vdash_{K-\mathrm{MLDS}} \mathcal{C}_{box_{\alpha_k}(\lambda)}$. By reflexivity $\mathcal{C}_{box_{\alpha_k}(\lambda)} \vdash_{K-\mathrm{MLDS}} box_{\alpha_k}(\lambda) : \alpha_k$ and therefore by transitivity, $\mathcal{C}_{box_{\alpha_k}(\lambda)} + [R(\lambda, box_{\alpha_k}(\lambda))] \vdash_{K-\mathrm{MLDS}} box_{\alpha_k}(\lambda) : \alpha_k$, which implies by definition of $\mathcal{I}_{\Box I}$ that $\mathcal{C}_{box_{\alpha_k}(\lambda)} \vdash_{K-\mathrm{MLDS}} \lambda : \Box\alpha_k$. Hence by transitivity of $\vdash_{K-\mathrm{MLDS}}$, $\mathcal{C}_{\{\}} \vdash_{K-\mathrm{MLDS}} \lambda : \Box\alpha_k$. ⋆

The following theorem generalises the above result to each of the axiomatic modal logics given in Chapter 1, Definition 1.14.

Theorem 3.6 {Simple Correspondence for modal logic $K\partial$}

Let $\partial \subseteq \{\mathbf{T}, \mathbf{4}, \mathbf{D}, \mathbf{B}, \mathbf{5}\}$, let $\langle K\partial_{Ax}, \vdash_{K\partial_{Ax}}\rangle$ be the corresponding axiomatic system, let δ be the corresponding subset of $\{(\mathbf{T}), (\mathbf{4}), (\mathbf{D}), (\mathbf{B}), (\mathbf{5})\}$, and let $K\delta$-MLDS be the propositional MLDS whose labelling algebra is $\mathcal{A}_{K\delta}$. (See Notations 1.4 and 1.5 and Definition 1.14.). Let $\mathcal{C}_{\{\}} = \langle \{\}, \mathcal{F}_{\{\}} \rangle$, where $\mathcal{F}_{\{\}}(\lambda) = \{\}$ for any ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$. Let α be a formula of $\mathcal{L}_M$. Then:

$$\models_{K\partial_{Ax}} \alpha$$
$$\text{if and only if}$$
$$\text{for all ground terms } \lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)\ \mathcal{C}_{\{\}} \models_{K\delta-\mathrm{MLDS}} \lambda : \alpha$$

Proof:
("If" half:) The proof is identical to the proof of the "If" half of Lemma 3.4, augmented with the following additional argument to show that $\langle \widehat{W}, I \rangle$ is also a model of $\mathcal{A}_{K\delta}$

- If $\mathbf{T} \in \partial$ then by Correspondence Theory [vB84] $\widehat{R}$ is reflexive, i.e. $\langle \widehat{W}, I \rangle$ is a model of $(\mathbf{T})$.
- If $\mathbf{4} \in \partial$ then by Correspondence Theory [vB84] $\widehat{R}$ is transitive, i.e. $\langle \widehat{W}, I \rangle$ is a model of $(\mathbf{4})$.
- If $\mathbf{D} \in \partial$ then by Correspondence Theory [vB84] $\widehat{R}$ has the seriality property, i.e. $\langle \widehat{W}, I \rangle$ is a model of $(\mathbf{D})$.

- If $\mathbf{B} \in \partial$ then by Correspondence Theory [vB84] $\widehat{R}$ is symmetric, i.e. $\langle \widehat{W}, I \rangle$ is a model of ($\mathbf{B}$).
- If $\mathbf{5} \in \partial$ then by Correspondence Theory [vB84] $\widehat{R}$ has the euclidean property, i.e. $\langle \widehat{W}, I \rangle$ is a model of ($\mathbf{5}$).

("Only if" half:) The proof is identical to the proof of the "Only if" half of Lemma 3.4, with the base case augmented with the following additional optional cases:

Case 16: If $\mathbf{T} \in \partial$ then $(\mathbf{T}) \in \delta$. Hence $\mathcal{C}_{\{\}} \vdash_{K\delta\text{–MLDS}} \lambda : \Box\alpha \rightarrow \alpha$ as shown in the following derivation:

$$
\begin{array}{ll}
\mathcal{C}_0\langle\rangle & \\
\hline
\quad \mathcal{C}_0\langle[\lambda : \Box\alpha]\rangle & \text{(assumption)} \\
\quad \mathcal{C}_1\langle\lambda : \Box\alpha, R(\lambda, \lambda)\rangle & \text{(R–A)} \\
\quad \mathcal{C}_2\langle\lambda : \alpha\rangle & (\Box\ \text{E}) \\
\hline
\mathcal{C}'\langle\lambda : \Box\alpha \rightarrow \alpha\rangle & (\rightarrow \text{I})
\end{array}
$$

Case 17: If $\mathbf{4} \in \partial$ then $(\mathbf{4}) \in \delta$. Hence $\mathcal{C}_{\{\}} \vdash_{K\delta\text{–MLDS}} \lambda : \Box\alpha \rightarrow \Box\Box\alpha$ as shown in the following derivation:

$$
\begin{array}{ll}
\mathcal{C}_0\langle\rangle & \\
\hline
\quad \mathcal{C}_0\langle[\lambda : \Box\alpha]\rangle & \text{(assumption)} \\
\quad\quad \mathcal{C}_0\langle[R(\lambda, box_{\Box\alpha}(\lambda))]\rangle & \text{(assumption)} \\
\quad\quad\quad \mathcal{C}_0\langle[R(box_{\Box\alpha}(\lambda), box_{\alpha}(box_{\Box\alpha}(\lambda)))]\rangle & \text{(assumption)} \\
\quad\quad\quad \mathcal{C}_1\langle R(\lambda, box_{\alpha}(box_{\Box\alpha}(\lambda)))\rangle & \text{(R–A)} \\
\quad\quad\quad \mathcal{C}_2\langle box_{\alpha}(box_{\Box\alpha}(\lambda)) : \alpha\rangle & (\Box\ \text{E}) \\
\quad\quad \mathcal{C}_3\langle box_{\Box\alpha}(\lambda)) : \Box\alpha\rangle & (\Box\ \text{I}) \\
\quad \mathcal{C}_4\langle\lambda : \Box\Box\alpha\rangle & (\Box\ \text{I}) \\
\hline
\mathcal{C}'\langle\lambda : \Box\alpha \rightarrow \Box\Box\alpha\rangle & (\rightarrow \text{I})
\end{array}
$$

Case 18: If $\mathbf{D} \in \partial$ then $(\mathbf{D}) \in \delta$. Hence $\mathcal{C}_{\{\}} \vdash_{K\delta\text{–MLDS}} \lambda : \Box\alpha \rightarrow \Diamond\alpha$ as shown in the following derivation:

$$
\begin{array}{ll}
\mathcal{C}_0\langle\rangle & \\
\hline
\quad \mathcal{C}_0\langle[\lambda : \Box\alpha]\rangle & \text{(assumption)} \\
\quad \mathcal{C}_1\langle\lambda : \Box\alpha, R(\lambda, succ(\lambda))\rangle & \text{(R–A)} \\
\quad \mathcal{C}_2\langle succ(\lambda) : \alpha\rangle & (\Box\ \text{E}) \\
\quad \mathcal{C}_3\langle\lambda) : \Diamond\alpha\rangle & (\Diamond\ \text{I}) \\
\hline
\mathcal{C}'\langle\lambda : \Box\alpha \rightarrow \Diamond\alpha\rangle & (\rightarrow \text{I})
\end{array}
$$

Case 19: If $\mathbf{5} \in \partial$ then $(\mathbf{5}) \in \delta$. Hence $\mathcal{C}_{\{\}} \vdash_{K\delta\text{–MLDS}} \lambda : \Diamond\alpha \rightarrow \Box\Diamond\alpha$ as shown in the following derivation:

$\mathcal{C}_0\langle\rangle$	
$\mathcal{C}_0\langle[\lambda:\Diamond\alpha]\rangle$	(assumption)
$\mathcal{C}_1\langle R(\lambda, f_\alpha(\lambda)), f_\alpha(\lambda):\alpha\rangle$	($\Diamond$ E)
$\mathcal{C}_1\langle[R(\lambda, box_{\Diamond\alpha}(\lambda))]\rangle$	(assumption)
$\mathcal{C}_2\langle R(box_{\Diamond\alpha}(\lambda), f_\alpha(\lambda))\rangle$	(R–A)
$\mathcal{C}_3\langle box_{\Diamond\alpha}(\lambda):\Diamond\alpha\rangle$	($\Diamond$ E)
$\mathcal{C}_4\langle W_0:\Box\Diamond\alpha\rangle$	($\Box$ I)
$\mathcal{C}_1'\langle\lambda:\Diamond\alpha\rightarrow\Box\Diamond\alpha\rangle$	($\rightarrow$ I)

Case 20: If $\mathbf{B}\in\partial$ then $(\mathbf{B})\in\delta$. Hence $\mathcal{C}_{\{\}}\vdash_{K\delta\text{–MLDS}}\lambda:\alpha\rightarrow\Box\Diamond\alpha$ as shown in the following derivation:

$\mathcal{C}_0\langle\rangle$	
$\mathcal{C}_0\langle[\lambda:\alpha]\rangle$	(assumption)
$\mathcal{C}_0\langle[R(\lambda, box_{\Diamond\alpha}(\lambda))]\rangle$	(assumption)
$\mathcal{C}_1\langle R(box_{\Diamond\alpha}(\lambda), \lambda)\rangle$	(R–A)
$\mathcal{C}_2\langle box_{\Diamond\alpha}(\lambda):\Diamond\alpha\rangle$	($\Diamond$ I)
$\mathcal{C}_3\langle\lambda:\Box\Diamond\alpha\rangle$	($\Box$ I)
$\mathcal{C}'\langle\lambda:\ \alpha:\rightarrow\Box\Diamond\alpha\rangle$	($\rightarrow$ I)

$\star$

Theorem 3.6 shows how the Kripke semantics definition of validity of a modal formula α (i.e. $\models_{K\partial}\alpha$) can be reformulated in terms of MLDS semantics. To recap, given a modal logic $K\partial$ and its Kripke semantics, a formula α is valid if and only if it holds in all the possible worlds of each model of $K\partial$. Within a MLDS, this notion is equivalent, as demonstrated in Theorem 3.6, to the condition that, for any arbitrary $\lambda\in Mon(\mathcal{L}_L,\mathcal{L}_M)$, the declarative unit $\lambda:\alpha$ is satisfied in all MLDS semantic structures (i.e in all the models of $\mathcal{A}^+_{K\delta}$).

In Chapter 1 the notion of Kripke semantic entailment is defined in terms of global and local assumptions. Theorem 3.6 can easily be generalised to deal with both global and local assumptions. Given a configuration $\mathcal{C}=\langle\mathcal{D},\mathcal{F}\rangle$, a wff α is a global assumption α if and only if $\alpha\in\mathcal{F}(\lambda)$ for all $\lambda\in Mon(\mathcal{L}_L,\mathcal{L}_M)$. To represent local assumptions it is necessary to identify a particular constant symbol in $\mathcal{L}_L$, say W_0, and translate each local assumption β into the declarative unit $W_0:\beta$. The following two theorems generalise Theorem 3.6, first to derivability from global assumptions and second to derivability from both global and local assumptions (see Definitions 1.9 and 1.15).

Theorem 3.7 {Restricted strong correspondence for modal logic $K\partial$}

Let $\partial \subseteq \{\mathbf{T}, \mathbf{4}, \mathbf{D}, \mathbf{B}, \mathbf{5}\}$, let $\langle K\partial_{Ax}, \vdash_{K\partial_{Ax}}\rangle$ be the corresponding axiomatic system, let δ be the corresponding subset of $\{(\mathbf{T}), (\mathbf{4}), (\mathbf{D}), (\mathbf{B}), (\mathbf{5})\}$, and let $K\delta$-MLDS be the propositional MLDS whose labelling algebra is $\mathcal{A}_{K\delta}$. Let S be a set of wffs of $\mathcal{L}_M$. Let $\mathcal{C}_S = \langle\{\}, \mathcal{F}_S\rangle$, where $\mathcal{F}_S(\lambda) = S$ for any ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)$. Let α be a formula of $\mathcal{L}_M$. Then:

$$S \models_{K\partial_{Ax}} \{\} \Rightarrow \alpha$$
$$\text{if and only if}$$
$$\text{for all ground terms } \lambda \in Mon(\mathcal{L}_L, \mathcal{L}_M)\ \mathcal{C}_S \models_{K\delta-\text{MLDS}} \lambda : \alpha$$

Proof:
("If" half:) Without loss of generality, it can be assumed that S is consistent. Therefore, the proof is identical to the "If" half of Theorem 3.6, but choosing the model $M = \langle\widehat{W}, \widehat{R}, v\rangle$ so that each of the assumptions in S are valid in M. In this case, by definition of the interpretation I, $\langle\widehat{W}, I\rangle$ is a model not only of $\mathcal{A}^+$ but also of $FOT(\mathcal{C}_S)$.

("Only if" half:) The proof is identical to the "Only if" half of Theorem 3.6, noting that, in the base case, if α is an instantiation of one of the global assumptions in S it is trivially derivable from $\mathcal{C}_S$. $\star$

Theorem 3.8 {Strong Correspondence for modal logic $K\partial$}

Let $\partial \subseteq \{\mathbf{T}, \mathbf{4}, \mathbf{D}, \mathbf{B}, \mathbf{5}\}$, let $\langle K\partial_{Ax}, \vdash_{K\partial_{Ax}}\rangle$ be the corresponding axiomatic system, let δ be the corresponding subset of $\{(\mathbf{T}), (\mathbf{4}), (\mathbf{D}), (\mathbf{B}), (\mathbf{5})\}$, and let $K\delta$-MLDS be the propositional MLDS whose labelling algebra is $\mathcal{A}_{K\delta}$. Let S and U be sets of wffs of $\mathcal{L}_M$. Let $\mathcal{C}_{SU} = \langle\{\}, \mathcal{F}_{SU}\rangle$, where $\mathcal{F}_{SU}(W_0) = S \cup U$ and $\mathcal{F}_{SU}(\lambda) = S$ for any ground term $\lambda \neq W_0$ of $Mon(\mathcal{L}_L, \mathcal{L}_M)$. Let α be a formula of $\mathcal{L}_M$. Then:

$$\mathcal{C}_{SU} \models_{K\delta-\text{MLDS}} W_0 : \alpha \text{ if and only if } \quad S \models_{K\partial_{Ax}} U \Rightarrow \alpha$$

Proof:
By Theorem 3.3 and the compactness property of propositional modal logic, it is sufficient to consider only the case where U is finite. Let $U = \{\alpha_1, \ldots, \alpha_n\}$. Proof is by induction on n.

BASE CASE:
The base case is when $n = 0$. Then the theorem holds by Theorem 3.7.

INDUCTIVE STEP:
Let $\mathcal{C}'_{SU} = \mathcal{C}_{SU} - [W_0 : \alpha_n]$ and $U' = U - \{\alpha_n\}$. By the deduction

theorem of first-order logic $\mathcal{A}^{+}_{K\delta} \cup FOT(\mathcal{C}_{SU}) \models_{FOL} [\alpha]^{*}(W_0)$ if and only if $\mathcal{A}^{+}_{K\delta} \cup FOT(\mathcal{C}'_{SU}) \models_{FOL} [\alpha_n]^{*}(W_0) \rightarrow [\alpha]^{*}(W_0)$. By axiom schema (Ax4) this can be written equivalently as $\mathcal{A}^{+}_{K\delta} \cup FOT(\mathcal{C}'_{SU}) \models_{FOL} [\alpha_n \rightarrow \alpha]^{*}(W_0)$. By definition of $\models_{K\delta\text{-MLDS}}$, this is equivalent to $\mathcal{C}'_{SU} \models_{K\delta\text{-MLDS}} W_0 : \alpha_n \rightarrow \gamma$. By the inductive hypothesis, this last statement is true if and only if $S \models_{K\partial} U' \Rightarrow \alpha_n \rightarrow \alpha$ which is equivalent to $S \models_{K\partial} U \Rightarrow \alpha$ by the local deduction theorem of modal logic $K\partial$. $\star$

The above theorem effectively provides a translation method from any modal theory $\langle S, U\rangle$ into an equivalent MLDS configuration, which preserves derivability and semantic entailment. However, it is clear that many configurations are not the translation of any modal theory. (For example, any configuration whose diagram $\mathcal{D}$ is not empty or whose function $\mathcal{F}$ differs at more than one label.) Hence, the information that such configurations encode cannot be represented within traditional modal logic.

3.4 Extending Soundness, Completeness and Correspondence Results

In Section 2.2.1 of Chapter 2, it was stated that the propositional MLDS framework could easily be extended to include labelling algebrae with equality. This section gives a brief description of how the proofs of soundness (Theorem 3.1), completeness (Theorem 3.4) and correspondence (Theorem 3.8) could be extended to cover a propositional MLDS whose labelling algebra includes axioms involving equalities and the standard equality theory. Only an outline is given here – it is obvious that many of the proofs in this chapter of lemmas, propositions and theorems relating to the "structural" derivation rules of Section 2.1.3 (e.g. Theorem 3.3, Lemma 3.3 and Proposition 3.33) are still valid for labelling algebrae with equality.

To show the soundness of the "extended" derivability relation $\vdash_{\text{MLDS}}$ (i.e. the relation based on deductions where reasoning about equality and inequality literals is allowed), an additional case is needed in Lemma 3.1 which proves essentially that the rule $\mathcal{I}_{Sub}$ given in Section 2.2.1 is sound.

In Section 3.2.1, Theorem 3.2 can straightforwardly be extended to the case of π being an equality or inequality literal, so showing that the derivability relation can also be characterised in terms of derivations of equalities and inequalities between possible worlds. In Theorem 3.3, an additional case needs to be included for the $\mathcal{I}_{Sub}$ rule, which is analogous to the reasoning used in Case 1, since $\mathcal{I}_{Sub}$ would be also part of the category $\mathcal{I}^{0}$ of MLDS rules.

The proof of the completeness theorem for extended MLDSs would be as follows. The construction of the maximal consistent configuration described in Definition 3.6 can be adopted without any modification. The consistency checking built into this construction, together with the maximality property, will guarantee that each equality literal, R-literal or declarative unit derivable by equality substitution or by the $\mathcal{I}_{Sub}$ rule is included in the maximal consistent configuration. Moreover, the maximal consistent configuration would in this case satisfy two additional properties, corresponding to Propositions 3.10 and 3.12 with Δ replaced by either an equality or an inequality literal, and the following proposition which guarantees that $\mathcal{C}_{mcc}$ is "closed" with respect to the $\mathcal{I}_{Sub}$ rule.

Proposition 3.34[11] Let $K\delta$-MLDS be an arbitrary MLDS and let $\mathcal{C}_{\text{mcc}}$ be a maximal consistent configuration relative to $K\delta$-MLDS. let λ_1 and λ_2 be two ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let α be a wff. If $\lambda_1 : \alpha \in \mathcal{C}_{\text{mcc}}$ and $\lambda_1 = \lambda_2 \in \mathcal{C}_{\text{mcc}}$ then $\lambda_2 : \alpha \in \mathcal{C}_{\text{mcc}}$ too.

In the construction of a canonical model (Definition 3.7 and Theorem 3.5), a Herbrand universe can still be used, and it can be shown that within this model, the $=$ relation is an equivalence relation, which satisfies all the standard equality axioms included in the extended labelling algebra. (The interpretation of equality as an equivalence relation is a standard technique in this type of completeness proof – see [CL73] for a discussion).

Finally, it is obvious that the strong correspondence result of Theorem 3.8 can readily be extended to cover frame logics whose characteristic axiom schema correspond to first-order conditions which use the equality predicate. For example, in Section 2.2.1 the example of the modal logic $S4.1$ has been given whose axiomatisation includes the schemas **T**, **D**, **4** and the McKinsey axiom $\Box\Diamond\alpha \rightarrow \Diamond\Box\alpha$. This logic can also be formalised within the MLDS framework by using the labelling algebra $\{(\mathbf{T}), (\mathbf{D}), (\mathbf{4}), \forall x, y[R(succ(x), y) \rightarrow y = succ(x)]\}$. A proof of a correspondence between the axiomatic system of $S4.1$ and this particular MLDS, could be easily obtained with appropriate modifications in Theorem 3.6. As a first step, it is shown below that McKinsey axiom is provable in a MLDS framework based on the above labelling algebra.

McKinsey axiom Let λ be an arbitrary ground term of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and let $\mathcal{C}_0 = \mathcal{C}_{\{\}}$. The following derivation shows that

$$\mathcal{C}_0 \vdash_{K\{(\mathbf{T}),(\mathbf{D}),(\mathbf{4}),(\mathbf{Mc})\}} \lambda : \Box\Diamond\alpha \rightarrow \Diamond\Box\alpha$$

[11] The proof is by contradiction using the $\mathcal{I}_{Sub}$ rule.

$$
\begin{array}{ll}
\mathcal{C}_0\langle\rangle & \\ \hline
\quad \mathcal{C}_0\langle[\lambda : \Box\Diamond\alpha]\rangle & \text{(assumption)} \\ \hline
\quad \tilde{\mathcal{C}}_1\langle R(\lambda, succ(\lambda))\rangle & \text{(R-Assertion)} \\ \hline
\qquad \tilde{\mathcal{C}}_2\langle[R(succ(\lambda), box_\alpha(succ(\lambda)))]\rangle & \text{(assumption)} \\ \hline
\qquad \tilde{\mathcal{C}}_3\langle succ(\lambda) : \Diamond\alpha\rangle & (\Box\ \mathrm{E}) \\ \hline
\qquad \tilde{\mathcal{C}}_4\langle f_\alpha(succ(\lambda)) : \alpha, R(succ(\lambda), f_\alpha(succ(\lambda)))\rangle & (\Diamond\ \mathrm{E}) \\ \hline
\qquad \tilde{\mathcal{C}}_5\langle f_\alpha(succ(\lambda)) = succ(\lambda)\rangle & \text{(R-Assertion)} \\ \hline
\qquad \tilde{\mathcal{C}}_6\langle succ(\lambda) : \alpha\rangle & (\mathcal{I}_{Sub}) \\ \hline
\qquad \tilde{\mathcal{C}}_7\langle box_\alpha(succ(\lambda)) = succ(\lambda)\rangle & \text{(R-Assertion)} \\ \hline
\qquad \tilde{\mathcal{C}}_8\langle box_\alpha(succ(\lambda)) : \alpha\rangle & (\mathcal{I}_{Sub}) \\ \hline
\quad \tilde{\mathcal{C}}_9\langle succ(\lambda) : \Box\alpha\rangle & (\Box\ \mathrm{I}) \\ \hline
\quad \tilde{\mathcal{C}}_{10}\langle \lambda : \Diamond\Box\alpha\rangle & (\Diamond\ \mathrm{I}) \\ \hline
\mathcal{C}'\langle \lambda : \Box\Diamond\alpha \rightarrow \Diamond\Box\alpha\rangle & (\rightarrow \mathrm{I})
\end{array}
$$

3.5 Discussion

One of the results shown in this chapter is that for an arbitrary $K\delta$-MLDS the derivability relation $\vdash_{\text{MLDS}}$ is complete with respect to the semantic entailment $\models_{\text{MLDS}}$. As a technical remark, the proof described here, although based on the Henkin methodology, turns out to be simpler than the standard completeness proofs of modal logics. The latter requires the construction of maximal consistent sets as *subordinate sets* (see for example [HC68]). This construction involves first defining the "initial world's maximal consistent set", then defining its subordinate (or accessible worlds') maximal consistent sets and repeating the same process for each one of them until no more subordinate maximal consistent sets can be constructed. This process varies for each type of modal logic, since the definition of a subordinate set depends on the properties of the accessibility relation (see for example [HC68]).

In the case of a $K\delta$-MLDS, there is no need for subordinate maximal consistent sets. The construction of a maximal consistent configuration requires only consistent addition of all the declarative units and the R–literals of a MLDL to an initial configuration. It does not impose any distinction between the initial maximal consistent set and subordinate sets. This simplification is due to the fact that in a declarative unit the association of modal formulae with a possible world is explicitly specified. The relations between possible worlds are also explicit. The construction of a maximal consistent configuration (Definition 3.5) is identical for each $K\delta$-MLDS. This simplification is due to the modularity of MLDSs. The properties of the accessibility relation are set aside in a labelling algebra $\mathcal{A}$ which defines different sets of R–literals in a maximal consistent configuration.

3.6 Summary

The results in this chapter can be summarised as follows. The natural deduction proof system of a $K\delta$-MLDS (see Chapter 2) is sound and complete with respect to the model-theoretic semantics (also given in Chapter 2) based on a translation method into classical logic. Finally, any arbitrary traditional modal axiomatic theory can be translated into an equivalent MLDS configuration. This translation preserves both derivability and traditional semantic entailment as defined in terms of Kripke semantics. MLDSs are more general than axiomatic systems in the sense that the inverse translation cannot be applied to any arbitrary configuration.

Part II

Predicate MLDS

Chapter 4

Introduction to Quantified Modal Logic

4.1 An Overview

Quantified modal logic is a controversial subject. In classical logic, there is a well established consensus as to how to define quantified extensions of propositional logic. This is not the case for modal logics. As pointed out in Chapter 1, within the propositional setting Kripke possible world semantics can be seen as a collection of different classical models associated with different possible worlds. The first question is, to what extent should the Kripke possible world semantics for a quantified modal logic also be thought of as a collection of various first order classical models? Different answers to this key point have given rise to a wide variety of semantic approaches to quantified modal logics. Although an exhaustive listing of these semantics is beyond the scope of the thesis, some basic aspects and characteristics of a representative cross-section are described in this chapter, to outline the main parameters of evaluation against which the quantified MLDS approach will be measured.

In the implicit approach, a quantified[1] modal language is traditionally defined by simply extending a standard first-order classical language with the two modal operators $\Box$ and $\Diamond$. A definition of the quantified modal language considered in this thesis is given below. Note that the symbol $\mathcal{L}_M$, introduced in Part I, now refers to this first-order modal language.

Definition 4.1 {Quantified modal language $\mathcal{L}_M$} The *quantified modal language* $\mathcal{L}_M$ is composed of:

[1]The terms "quantified", "predicate" and "first-order" will be used interchangeably in this second part of the thesis.

- u_1, v_1, u_2, v_2, ... variables
- F_1^n, F_2^n, F_3^n, ... a set of function symbols for each arity $n \geq 0$
- P_1^n, P_2^n, P_3^n, ... a set of predicate symbols for each arity $n \geq 0$
- $\neg$, $\wedge$, $\vee$, $\rightarrow$ logical connectives
- $\Box$, $\Diamond$ modal operators
- $\forall$, $\exists$ quantifiers

$\Box$

The language $\mathcal{L}_M$ thus includes function symbols of arbitrary arity. This is an extension of the usual definition – hence in surveying existing approaches to quantified modal logic in the first part of this chapter, it will be useful to also refer to the more restricted language $\mathcal{L}_M^r$, which contains only the function symbols F_1^0, F_2^0, ... of 0 arity (i.e. constant symbols).

In the remainder of this thesis, the symbols u and v will stand for arbitrary variables of $\mathcal{L}_M$, and the symbols P and F will stand for an arbitrary predicate and function symbol respectively, whose arity will be clear from the context.

The syntax of $\mathcal{L}_M$ is based on the standard classical definition of *terms*, *free variables* and *wffs* (see [Dav93]). The set of terms is defined (inductively) as consisting of all variables and function symbols F_1^0, F_2^0, ... of 0 arity, together with all expressions of the form $F_i^n(t_1, \ldots, t_n)$ where F_i^n is a function symbol of $\mathcal{L}_M$ of non-zero arity, and $t_1, \ldots, t_n$ are terms of $\mathcal{L}_M$. The set of *atomic formulae* is given by all expressions of the form $P_i^n(t_1, \ldots, t_n)$, where P_i^n is a predicate symbol of $\mathcal{L}_M$ and $t_1, \ldots, t_n$ are terms of $\mathcal{L}_M$, and the set of wffs is the smallest set satisfying the following conditions, in which the notion of a *free variable occurrence* is also given. (i) any atomic formula α is a wff and any variable occurring in α is a free variable; (ii) if α is a wff and $\sharp$ is a unary operator (either classical or modal) different from the quantifiers, then $\sharp\alpha$ is a wff and its free variable occurrences are those of α; (iii) If α and β are wffs and $\sharp$ is a binary operator ($\wedge$, $\vee$, $\rightarrow$), then $\alpha\sharp\beta$ is a wff and its free variable occurrences are the union of those of α with those of β; (iv) if α is a wff and v is a variable then $\forall v\alpha$ and $\exists v\alpha$ are wffs and their respective free variables occurrences are those of α except for the occurrences of v. A wff of $\mathcal{L}_M$ is said to be a *sentence* if it contains no free variable occurrences.

In the classical case, predicate extensions of the propositional semantics are always of a particular form. A model is usually given by a *universe of discourse* U and a function I which extends the propositional notion of an interpretation by assigning meanings to constant symbols and to predicate symbols. Each constant symbol is interpreted as an element of U, whereas each predicate symbol of arity

n is interpreted as a relation over U^n. Variables are "instantiated" by *variable assignment* functions, which assign *values* to them from the universe of discourse U (see [Dav93] for a formal discussion about first order logic). Consequently, each element of the *signature* (i.e. constant and predicate symbols) of a classical first-order language has a fixed semantic meaning.

In the case of modal logic, the definition of a quantified semantics is not so straightforward. Given a Kripke possible world semantics for propositional modal logic, different choices can be made in order to extend it to the predicate case and each one of them leads to a different form of quantified Kripke semantics. These choices concern mainly (i) the universe of discourse and (ii) the interpretation of the constant symbols (and functions if any) of the language. Kripke frames for quantified modal logics can be defined either by considering a unique universe of discourse, as the set of objects that exist in each possible world, or by assigning different universes of discourse to different possible worlds, so allowing different sets of objects to exist in different possible worlds. Semantics based on the first type of frames are usually called *fixed domain* semantics, whereas semantics built on the second type of frames are called *varying domain* semantics. The choice of definition of Kripke models based on any of these frames provides a further degree of freedom. In general, valuation functions of propositional possible world semantics are extended so as to assign meaning to constant symbols and to predicate symbols of a quantified modal language. As far as constant symbols are concerned, the interpretation can either depend or not depend on the possible worlds. If a valuation function assigns to a constant symbol different objects in different possible worlds, the resulting semantics is said to be based on *non-rigid designators*. On the other hand, if the interpretation of constant symbols is identical in each possible world, then the semantics is said to be based on *rigid designators*. In this thesis, the attention will be mainly focused on various forms of fixed and varying domain semantics based on rigid designators. Various quantified extensions of the proof system for propositional $K\delta$-MLDSs will be described which will be proved to be sound and complete with respect to the corresponding semantics. It will be also discussed how extensions for non-rigid designators could be formalised within the MLDS framework.

4.1.1 Fixed Domain Semantics

The aim of this section is to describe fixed domain semantics and the associated sound and complete axiomatic system, giving particular emphasis to its advantages and limitations. It summarises the system described for example by Hughes & Cressell in [HC68], and thus is based on the restricted language $\mathcal{L}_M^r$ (no function symbols except constants). As mentioned before, first-order frames of a fixed domain semantics extend propositional frames with a single set of objects, called the *global domain*, which is associated with each possible world of the structure. A formal definition is given below.

Definition 4.2 {First-order frame} Let $\langle W, R\rangle$ be a Kripke frame as given in Definition 1.2. A *first-order frame* for fixed domain semantics is a tuple $\langle W, R, Q\rangle$, where Q is a mapping[2] which assigns to each member w of W a (non-empty) set, called *local domain of* w, such that the following condition holds

(i) $Q(w) = Q(w')$ for any w, w' of W.

The union of each of the local domains is called *global domain* and it is denoted by D (i.e. $D = \bigcup_{w \in W} Q(w)$). □

Note that condition (i), called *fixed* or *constant* domain condition, guarantees that the same set of objects exists at each possible world (i.e. the local domains are all equal with each other). An alternative definition is also common in the literature [HC68] according to which a first-order frame for fixed domain semantics is given by adding to a tuple $\langle W, R\rangle$ instead of a mapping a non-empty global domain D (i.e. $\langle W, R, D\rangle$), and implicitly assuming that each local domain is given by D itself. Definition 4.2 has been chosen here because it is closer to the definition of first-order frames for varying domain semantics which will be given in the next section. Models based on first-order frames are also formalised as quantified extensions of models of propositional modal logic. A valuation function is replaced by a *rigid interpretation* whose definition is given below, adopting Fitting's notation [Fit93].

Definition 4.3 {Rigid interpretation} Let $\mathcal{L}^r_M$ be a (restricted) quantified modal language and let $\langle W, R, Q\rangle$ be a first-order frame. A *rigid interpretation* in $\langle W, R, Q\rangle$ is a pair $\langle \mathcal{F}_C, \mathcal{F}_P\rangle$, where

- $\mathcal{F}_C$ is a mapping which assigns to each constant symbol F^0_i of $\mathcal{L}^r_M$ an object $\mathcal{F}_C(F^0_i)$ in the global domain D.
- $\mathcal{F}_P$ is a mapping which assigns to each n-ary predicate symbol P^n_i of $\mathcal{L}^r_M$ and each possible world w of W, an n-place relation $\mathcal{F}_P(P^n_i, w) \subseteq Q(w)^n$.

□

[2]Here, as in many texts, the codomain of Q is left implicit.

The interpretation of the constant symbols is the same in each possible world and this is what gives the characteristic of being a semantics based on rigid designators. On the other hand, the interpretation of predicate symbols is related to possible worlds. This is analogous to the way a valuation assigns to each propositional letter different truth values in different possible worlds. Note that, since each local domain is identical to the global domain, the condition $\mathcal{F}_P(P_i^n, w) \subseteq Q(w)^n$ is equivalent to $\mathcal{F}_P(P_i^n, w) \subseteq D^n$. This will not be the case in varying domain semantics where local domains are generally different from a global domain, and the truth value of atomic formulae depends on which objects exist in a possible world (this will be shown in the following section). A first-order model is given by defining a rigid interpretation in a first-order frame.

Definition 4.4 {First-order model} Let $\mathcal{L}_M^r$ be a quantified modal language. A *first-order model* M is the tuple $\langle W, R, Q, \mathcal{F}_C, \mathcal{F}_P \rangle$, where $\langle W, R, Q \rangle$ is a first-order frame and $\langle \mathcal{F}_C, \mathcal{F}_P \rangle$ is a rigid interpretation in $\langle W, R, Q \rangle$. □

As in classical first-order logic, notions like variable assignment and term assignment are also needed in a quantified modal logic. These are defined in a standard way as follows.

Definition 4.5 {Variable assignment} Let $\mathcal{L}_M^r$ be a quantified modal language and let M be a first-order model. A variable *assignment* in M is a function s which assigns to each variable u of $\mathcal{L}_M^r$, an object $s(u)$ of the domain D. Moreover, given a variable u and an object a of the domain D, $s[{}^u_a]$ denotes the mapping such that $s[{}^u_a](u) = a$, and for all variables v different from u, $s[{}^u_a](v) = s(v)$. □

Definition 4.6 {Term assignment} Let $\mathcal{L}_M^r$ be a quantified modal language, let $M = \langle W, R, Q, \mathcal{F}_C, \mathcal{F}_P \rangle$ be a first-order model and let s be a variable assignment. A *term assignment* $T_{\mathcal{F}_C,s}$ *corresponding to* $\mathcal{F}_C$ *and* s is a mapping from the set of terms of $\mathcal{L}_M$ to the domain D defined as follows. For each term t of $\mathcal{L}_M^r$

- if t is a constant symbol, then $T_{\mathcal{F}_C,s}(t) = \mathcal{F}_C(t)$
- if t is a variable, then $T_{\mathcal{F}_C,s}(t) = s(t)$

□

Modal wffs, when not sentences, have free variable occurrences. The notion of truth of a formula α at a possible world w is therefore given with respect to a variable assignment which instantiates the free variables occurring in α. This is done in a way analogous to classical first-order logic, adopting the notation

$M, w \| \!\!- \alpha[s]$ to signify that a formula α is true at the possible world w of a model M when free variables of α are instantiated by s.

Definition 4.7 {First-order satisfiability} Let $\mathcal{L}^r_M$ be a quantified modal language, let $M = \langle W, R, Q, \mathcal{F}_C, \mathcal{F}_P \rangle$ be a first-order model and let s be an assignment. Given two wffs α and β of $\mathcal{L}^r_M$ and a possible world w of M, the satisfiability relation $\| \!\!-$ is uniquely defined as follows:

1. $M, w \| \!\!- P^n_i(t_1, \ldots, t_n)[s]$ iff $\langle T_{\mathcal{F}_C,s}(t_1), \ldots, T_{\mathcal{F}_C,s}(t_n) \rangle \in \mathcal{F}_P(P^n_i, w)$.
2. $M, w \| \!\!- \neg\alpha[s]$ iff not $M, w \| \!\!- \alpha[s]$
3. $M, w \| \!\!- \alpha \wedge \beta[s]$ iff $M, w \| \!\!- \alpha[s]$ and $M, w \| \!\!- \beta[s]$
4. $M, w \| \!\!- \alpha \vee \beta[s]$ iff $M, w \| \!\!- \alpha[s]$ or $M, w \| \!\!- \beta[s]$
5. $M, w \| \!\!- \alpha \rightarrow \beta[s]$ iff $M, w \| \!\!- \beta[s]$ or not $M, w \| \!\!- \alpha[s]$
6. $M, w \| \!\!- \Box\alpha[s]$ iff, for all $w' \in W$, if wRw' then $M, w' \| \!\!- \alpha[s]$
7. $M, w \| \!\!- \Diamond\alpha[s]$ iff there exists a $w' \in W$ such that wRw' and $M, w' \| \!\!- \alpha[s]$
8. $M, w \| \!\!- \forall u\alpha[s]$ iff $M, w \| \!\!- \alpha[s[^u_a]]$ for every object $a \in D$.
9. $M, w \| \!\!- \exists u\alpha[s]$ iff $M, w \| \!\!- \alpha[s[^u_a]]$ for some object $a \in D$.

□

The above definition shows that fixed domain semantics is a first-order extension of propositional possible world semantics which preserves the notions of truth for the modal operators and for the classical connectives. In fact, clauses (2)-(7) are identical to the corresponding satisfiability conditions given in Chapter 1 (see Definition 1.5) for propositional modal logic, with the meta-variables referring here to wffs of a quantified modal language. Clauses (1), (8) and (9) are instead identical to the standard classical first-order notions of truth for atomic formulae and for universal and existential quantifiers. This characteristic will be less evident in the case of varying domain semantics (see the following section) where local domains are different from each other and the interpretation of the quantifiers is "world-relative".

The definition of *validity* in a first-order model M extends the one given in Chapter 1 for propositional modal logic by including a notion of validity with respect to a variable assignment. In particular, a formula is valid in a model if and only if it is true under each variable assignment at every possible world.

Definition 4.8 {Validity in a first-order model} Let $\mathcal{L}_M$ be a quantified modal language, let M be a first-order model and let α be a wff of $\mathcal{L}_M$. $\alpha[s]$ is *valid* in M for a given variable assignment s, written $M \models \alpha[s]$, if $M, w \Vdash \alpha[s]$ for every possible world w of M. α is *valid* in M, written $M \models \alpha$, if for every variable assignment s, $M \models \alpha[s]$. □

Hence, a formula α is not valid in a model if there exists a variable assignment s and a possible world w where the truth value of α under the assignment s is false. The notions of validity in a class of models, validity in a frame and validity in a collection of frames are identical to those given in Part I for propositional modal logic. Therefore, a class $\mathcal{F}$ of first-order frames whose accessibility relations satisfy properties which are modally definable, is characterised by the same $\mathcal{F}$-valid modal schemas which characterise the corresponding propositional class $\mathcal{F}$ (see Table 1.1 in Chapter 1), except that the meta-variables refer to wffs of a quantified modal language. For example a quantified modal formula of the form $\Box P(v) \rightarrow P(v)$ is $\mathcal{F}$-valid in the class $\mathcal{F}$ of reflexive frames. Hence a *quantified modal logic* can be defined as the set of all $\mathcal{F}$-valid wffs written in a quantified modal language $\mathcal{L}_M$ for some non-empty collection $\mathcal{F}$ of first-order frames. (This definition is also applicable to varying domain semantics.)

Notation 4.1 Let $\mathcal{L}_M^r$ be a quantified modal language. $QK\partial$ will denote the quantified modal logic characterised by the set of axiom schemas in ∂ (see Notation 1.2 in Chapter 1) where the meta-variables refer to wffs of $\mathcal{L}_M^r$. ◁

A quantified modal theory is defined as in the propositional case by a pair $\langle S, U \rangle$ of global assumptions S and of local assumptions U[3]. Analogously, the notion of semantic entailment given in Chapter 1 for propositional modal logic is equally applicable to its first-order extension. The following is simply a restatement of this definition for a quantified modal language.

Definition 4.9 {Semantic entailment for quantified modal logic} Let $\mathcal{L}_M^r$ be a quantified modal language, let $\mathbf{F}$ be a class of first-order frames, let $\langle S, U \rangle$ be a theory in the language $\mathcal{L}_M^r$ and let α be a modal formula of $\mathcal{L}_M^r$. α is *entailed* from $\langle S, U \rangle$, written $S \models_{\mathbf{F}} U \Rightarrow \alpha$, if for every first-order frame $F \in \mathbf{F}$ and for every first-order model M based on F in which all members of S are valid, α is true at every possible world $w \in M$ at which all members of U are true. □

Fixed domain semantics is attractive from a purely formal point of view. Its model theory is easily obtained by extending the well established Kripke possible

[3] The notions of global and local assumptions are identical to those given in Chapter 1.

world semantics for propositional modal logic with standard semantic notions of classical first-order logic. Such an extension preserves all the semantic definitions and characterisations of propositional modal logic (e.g. validity and semantic entailment). Moreover, the process which leads to the definition of a quantified modal semantics from a propositional one is comparable to that adopted in classical logic, i.e. a universe of discourse is added to a propositional model and additional clauses are added to the propositional satisfiability conditions to express the truth values of atomic formulae and of quantifiers. A similar attractive correspondence is found in the various proof theories of quantified modal logic associated with fixed domain semantics. In the case of Hilbert systems, for example, given a sound and complete axiomatisation for a propositional modal logic (e.g. the $K\partial_{Ax}$ given in Chapter 1), a first-order axiomatic system, which is sound and complete with respect to the fixed domain semantics, can be obtained by simply adding standard first-order axioms and rules, together with a special axiom called the *Barcan Formula* (BF). This is formally described in the following section.

Axiomatisations of $QK\partial$ Modal Logics

In Chapter 1, axiomatic systems for $K\partial$ propositional modal logics have been described which extend a standard Hilbert axiomatisation of classical propositional logic with appropriate axiom schemas and rules for the modal operators. In this section, a definition of axiomatic systems for the quantified modal logics $QK\partial$ is given, which extends the axiomatic systems $K\partial_{Ax}$ with axioms and rules for the universal and existential quantifiers.

Notation 4.2 Let $\mathcal{L}_M$ be a quantified modal language. $\alpha(v)$ denotes that the wff α possibly contains free occurrences of the variable v. $\alpha(v/t)$ denotes the wff that is identical to $\alpha(v)$ except that all free occurrences of the variable v (if any) are substituted by the term t of $\mathcal{L}_M$. $\triangleleft$

Definition 4.10 $\{t$ free for v in $\alpha(v)\}$ Let t be term, v be a variable and $\alpha(v)$ be a wff. t is *free for v in $\alpha(v)$* if and only if v does not occur free in $\alpha(v)$ within the scope of any quantifier of any variable occurring in t. $\square$

The notion of a substitution being free for a variable guarantees that a variable is not replaced by any other variable which would become bound, but only by one which would remain free. For instance, given a formula of the form $\exists v_1 P_1^2(u_1, v_1)$, the substitution of v_1 for u_1 is not free for the variable u_1, since the latter appears in the scope of the quantifier $\exists v_1$, whereas the substitution of v_2 for u_1 is free for u_1. This notion is used in one of the first-order axioms given in the following definition.

Definition 4.11 {Axiomatic system for $QK\partial$} Let $\mathcal{L}^r_M$ be a quantified modal language, let $QK\partial$ be a quantified modal logic and let α and β be two wffs of $\mathcal{L}^r_M$. Let t be an arbitrary term of $\mathcal{L}^r_M$ and let v be an arbitrary variable. The axiomatic system $QK\partial_{Ax}$+[BF] for $QK\partial$ in the language $\mathcal{L}^r_M$ is defined by each axiom schema and rule of $K\partial_{Ax}$ given in Definition 1.14 (see Chapter 1) (with the meta-variables α and β referring to wffs of $\mathcal{L}^r_M$), together with the following axioms and rules:

Axioms about $\forall$ and $\exists$

$$\forall v\alpha \rightarrow \neg\exists v\neg\alpha \qquad [A15]$$
$$(\neg\exists v\neg\alpha) \rightarrow \forall v\alpha \qquad [A16]$$
$$\forall v\alpha(v) \rightarrow \alpha(v/t) \qquad [UQA1]$$
$$\forall v(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \forall v\beta) \qquad [UQA2]$$

where in the schema [UQA1] t is free for v in α and in the schema [UQA2] the variable v has no free occurrences in α.

Barcan Formula

$$\forall v\Box\alpha \rightarrow \Box\forall v\alpha \qquad [BF]$$

Universal Generalisation Rule

$$\text{Conclude } \forall v\alpha \text{ from } \alpha \qquad [UGen]$$

□

Notation 4.3 The axiom system $QK\partial_{Ax}$ refers to the axiomatic system $QK\partial_{Ax}$+[BF] but without the axiom schema [BF]. ◁

The Barcan Formula is the syntactic characterisation of constant domains. In fact it is proved to be *valid* in first-order models where the domain does not change from world to world. This is shown as follows, reasoning by contradiction. Suppose that $M = \langle W, R, Q, F_C, F_P\rangle$ is a fixed domain model based on rigid designators and that for a certain possible world $w \in W$ and an assignment

s, $M, w \Vdash \forall v \Box \alpha[s]$ and $M, w \nVdash \Box \forall v \alpha[s]$. The second assumption implies that there must be a possible world w' such that wRw' and $M, w' \nVdash \forall v \alpha[s]$. By the satisfiability condition of the universal quantifier, there is some object $a \in D(w')$ such that $M, w \nVdash \alpha[s[^v_a]]$. Now, the condition of fixed (or constant) domains implies that, since $Q(w) = Q(w')$, the object a exists also at the possible world w (i.e. $a \in Q(w)$). Hence, by the first assumption, $M, w \Vdash \Box \alpha[s[^v_a]]$ which gives $M, w' \Vdash \alpha[s[^v_a]]$ so leading to a contradiction.

In the next section it will be shown that this axiom can actually be validated by a semantic condition weaker than the requirement for constant domains – it is only necessary that domains be *decreasing*. However, the class of models based only on this condition would make the above axiomatization unsound. This is because the [Nec] rule together with the classical axioms and the classical rule of universal generalisation, allow the derivation of the formula $\Box \forall v \alpha(v) \rightarrow \forall v \Box \alpha(v)$, known as the *Converse Barcan Formula* [CBF]. This is shown in Table 4.1.

(1)	$\forall v \alpha(v) \rightarrow \alpha(v)$
(2)	$\Box(\forall v \alpha(v) \rightarrow \alpha(v))$
(3)	$\Box \forall v \alpha(v) \rightarrow \Box \alpha(v)$
(4)	$\forall v(\Box \forall v \alpha(v) \rightarrow \Box \alpha(v))$
(5)	$\Box \forall v \alpha(v) \rightarrow \forall v \Box \alpha(v)$

Table 4.1: Derivation of the Converse Barcan Formula

(1) is an instantiation of the axiom schema [UGA1], (2) is obtained from (1) using the [Nec] rule whereas (3) is obtained from (2) and the appropriate instantiation of the schema [K] using the rule [MP]. (4) is obtained by applying the [UGen] rule to (3), and finally (5) is derived by applying [MP] to (4) and the appropriate instantiation of the schema [UQA2]. This formula is valid in first-order models where objects persist from one possible world to those that are accessible from it (see [HC68, Fit93] for a proof). This condition, called the condition of *increasing domains*, is implicitly imposed by the axiomatisation $QK\partial_{Ax} + [BF]$, and together with the decreasing domain condition of the [BF] axiom, it gives the constant domain feature of the class of models described in Definition 4.4.

The notions of derivation and derivability are identical to those given in Chapter 1 (see Definitions 1.11 and 1.15 respectively). Therefore given a quantified modal theory $\langle S, U \rangle$ and a formula α of the quantified modal language $\mathcal{L}_M$, α is *derivable* from $\langle S, U \rangle$, written $S \vdash_{QK\partial_{Ax}+[BF]} U \Rightarrow \alpha$ if and only if there exists a derivation from the global assumptions S in the axiomatic system $QK\partial_{Ax} + [BF]$. (Note that $S \vdash_{QK\partial_{Ax}+[BF]} U \Rightarrow \alpha$ can be rewritten as $S \cup [BF] \vdash_{QK\partial_{Ax}} U \Rightarrow \alpha$.) It has been proved in the literature (e.g. [HC68] for further details) that each of the the axiomatic systems $QK\partial_{Ax} + [BF]$ is sound and complete with respect to the fixed domain semantics described in the previous section. Specifically, the following statement holds, where $\models_{QK\partial}$ is based on a fixed domain semantics.

$$S \vdash_{QK\partial_{Ax}+[BF]} U \Rightarrow \alpha \ \text{ iff } \quad S \models_{QK\partial} U \Rightarrow \alpha \tag{4.1}$$

In the next section, weaker classes of models will be defined for which only one or none of the two conditions of increasing and decreasing domains is imposed on the local domains. The semantics based on these types of models, called the *varying domain semantics*, is often preferred to the fixed domain semantics which, despite the advantages described above, is not general enough to fulfil the needs of many applications.

4.1.2 Varying Domain Semantics

In some applications, such as program specification [Har79] and natural language, it has been argued that the use of quantified modal logic ought to take into account the existence of different objects in different possible worlds. Garson [Gar84] gives the following example from natural language. The sentence "there is a man who signed the Declaration of Independence" is considered true if the term "there is" is read as "in 1777", but it is definitely false if it is interpreted in the present time. This is because in natural language the domain of discourse varies along dimensions such as time, context, etc. It is argued that to give a modal representation to domains like these, the Kripke semantics should allow domains to change from one possible world to another. This could be achieved within the fixed domain semantics described above by introducing additional predicates in the quantified modal language whose role in universally quantified sentences would be that of restricting the scope of universal quantifiers to specific classes of objects. However, this would cause unnecessary redundancy in the formalisation of these particular domains.

A better approach is that of *varying domain* semantics. The basic idea is to define semantic structures where different domains are associated with different possible worlds, and to restrict the domain of quantification to each individual *local* domain. This second condition is called (in Garson's terminology [Gar84]) *world-relative* interpretation of quantifiers. Three different types of varying domain semantics have been presented in the literature [HC68, Fit83, Kri63b], known as *increasing domain*, *decreasing domain* and *arbitrary domain* semantics. The first is given by first-order frames in which different local domains are associated with different worlds in such a way that objects persist from one world to those accessible from it. Frames of this type are defined by replacing condition (i) in Definition 4.2 with the following *increasing domain condition*:

(ii) $Q(w) \subseteq Q(w')$ for any w and w' of W such that wRw'.

Decreasing domain semantics is instead given by first-order frames in which the set of objects existing at a possible world w has to include the objects existing at each of the possible worlds accessible from it. Frames of this sort are defined by replacing condition (i) in Definition 4.2 with the following *decreasing domain condition*:

(iii) $Q(w') \subseteq Q(w)$ for any w and w' of W such that wRw'

Finally, arbitrary domain semantics is given by first-order frames where local domains are arbitrarily associated with possible worlds. Such frames are defined by replacing in Definition 4.2 condition (i) with the following most general condition:

(iv) $Q(w)$ is an arbitrary set for any $w \in W$.

Corresponding first-order models can be defined by considering in each of the above types of frames the same rigid interpretation given in Definition 4.3.

To capture the characteristic of varying domains in the truth value of quantified sentences, a *world-relative* interpretation needs to be defined for both the existential and universal quantifiers. This consists of restricting the domain of quantification to the local domain of the possible world where a quantifier is evaluated. This is usually done by replacing the satisfiability conditions (8) and (9) of Definition 4.7 with the following two statements:

$$8'.\ M, w \Vdash \forall v\alpha[s] \quad \text{iff} \quad M, w \Vdash \alpha[s[^v_a]] \ \text{ for every object } a \text{ in } Q(w) \tag{4.2}$$

$$9'.\ M, w \Vdash \exists v\alpha[s] \quad \text{iff} \quad M, w \Vdash \alpha[s[^v_a]] \ \text{ for some object } a \text{ in } Q(w) \tag{4.3}$$

As for the truth value of the other types of formulae (i.e. the satisfiability conditions for atomic formulae, classical connectives and modal operators), various approaches have been suggested in the literature [HC68, Fit83, Gab76, Kri63b] which all differ in some respects from the satisfiability conditions given in Definition 4.7. Before going into details, it is essential to explain which types of problems can arise when using the notions of truth given in Definition 4.7 together with the world-relative interpretation of quantifiers given above. In the rigid interpretation given in Definition 4.3, predicate symbols of $\mathcal{L}_M^r$ are interpreted locally at each possible world by assigning tuples of objects existing at that world, whereas constant symbols of $\mathcal{L}_M^r$ are associated with objects that belong to the global domain. Analogously, variables are instantiated with values also belonging to the global domain of a varying domain semantic structure. Given that local domains are different from world to world, it can easily be the case that the value assigned to a term t (constant symbol or variable) might not exist in the local domain of some possible world w, and therefore might be not included in the interpretation of any predicate symbol at the world w. The satisfiability condition of atomic formulae given in Definition 4.7 would make then any predicate which has such a term among its arguments, false at this possible world w. This diversity between the "local interpretations" of predicates and the "global assignments" of terms, together with the notions of truth given in Definition 4.7 and the world-relative quantifiers, has the semantic repercussion of invalidating the standard classical first-order principles. An example is given below

Example 4.1 {Falsification of the axiom schema [UQA1]} Consider the instantiation $\forall v P(v) \rightarrow P(F^0)$, where F^0 is a constant symbol of a quantified modal language $\mathcal{L}_M^r$. Let $M = \langle W, R, Q, \mathcal{F}_C, \mathcal{F}_P \rangle$ be a model with arbitrary domains (i.e. only condition (iv) is imposed on Q), given by $W = \{w, w'\}$, $R = \{\}$, $Q(w) = \{a\}$ and $Q(w') = \{b\}$, where the rigid interpretation $\langle \mathcal{F}_C, \mathcal{F}_P \rangle$ is such that $\mathcal{F}_P(P, w) = \{a\}$ and $\mathcal{F}_C(F^0) = b$. By the world-relative interpretation of the universal quantifier, $M, w \Vdash \forall v P(v)$ and $M, w \nVdash P(F^0)$ since $b \notin \mathcal{F}_P(P, w)$. Hence $\forall v P(v) \rightarrow P(F^0)$ is false at the possible world w. •

The above example shows that although the replacement of clauses (8) and (9) of Definition 4.7 with the equivalences (4.2) and (4.3) appears to be a natural way of extending propositional modal semantics to a quantified varying domain semantics[4], there is an incompatibility between the resulting semantics and some classical first-order principles. This is due mainly to the definition of a term assignment and to the notion of truth of atomic formulae. Therefore, appropriate modifications need to be made if the intention is to preserve as many standard first-order principles as possible.

A first solution could be that of making term assignments world dependent. This would force, in the above example, the constant symbol F to be interpreted in

[4]The satisfiability conditions for classical connectives and modal operators are identical to those given in Chapter 1 for propositional modal logic, and the satisfiability condition for atomic formulae is relative to possible worlds in the same way as the valuation of propositional letters is associated with each possible world.

the possible world w, as the object a, and the formula $\forall v P(v) \rightarrow P(F)$ to be true at w. But to preserve validity, $\forall v P(v) \rightarrow P(F)$ would have to be true at every possible world of a first-order model M. This can be obtained, in the above example, only if the local domain $Q(w')$ includes also the object a, since terms are considered to be rigid designators. Therefore, it is easy to see that in general the solution of world-relative term assignments, within varying domain semantics based on rigid designators, has the negative consequence of making the "denotations" of terms exist in every possible world of a varying domain semantic structure, and local domains vary only on objects which are never named by the language. This undercuts the whole point of introducing world-relative domains – which is in order to accommodate terms which refer to things that may not exist in other possible worlds – and therefore it is not an acceptable solution.

Other model-theoretic solutions have been suggested in the literature [HC68, Fit83, Gab76] which guarantee the validity of the classical first-order axioms regarding quantifies in a quantified modal logic. However, they all impose strong semantic restrictions both on the satisfiability conditions and on the class of varying domain models. A brief discussion of the most significant and most common quantified modal systems and their relative semantics is given below, with the aim of showing that classical axiomatisations do not "naturally" coexist with world-relative quantifiers and varying domain semantics, and that the only way of allowing the most general expressive power of varying domain semantics is by partially abandoning classical first-order principles. This point of view has also been taken by Garson in [Gar84], and finds its foundations in an early paper by Kripke about varying domain semantics [Kri63b], which is formally described in Section 4.2.

The Truth Value Gaps Semantics

Hughes and Cresswell [HC68] give an axiomatisation for any first-order modal logic $QK\partial$ which is equivalent to the axiomatic system $QK\partial_{Ax}$, i.e. the axiomatisation described in Definition 4.11 but without the Barcan Formula [BF]. A first characteristic of this system is that the classical axiom [UQA1], the universal generalisation rule [UGen] and the necessitation rule [Nec] facilitate the derivation of any instantiation of the Converse Barcan Formula [CBF], $\Box\forall v\alpha \rightarrow \forall v\Box\alpha$ (see Table 4.1 for an example derivation). This implies that the class of varying domain models with respect to which the system is sound and complete satisfies the increasing domain condition. The following reasoning by contrapositive shows in fact that CBF are valid only in increasing domain models. Let $\langle W, R, Q\rangle$ be a non-increasing frame. Then there is a $w, w' \in W$ and an object π such that $\pi \in Q(w)$, wRw' and $\pi \notin Q(w')$. Let $M = \langle W, R, Q, F_C, F_P\rangle$ be a model based on this frame such that for all $w'' \in W$, $F_P(P_1^1, w'') = Q(w'')$. In particular, for all $\overline{w}$ such that $wR\overline{w}$ then $F_P(P_1^1, \overline{w}) = Q(\overline{w})$, hence $M, w \Vdash \Box\forall v_1 P_1^1(v_1)$. However, clearly $M, w \not\Vdash \forall v_1 \Box P_1^1(v_1)$, since $\pi \notin F_P(P_1^1, w')$.

Hughes and Cresswell's axiomatisation covers then only the case of increasing domain semantics. Note that this limitation is included in any proof system for quantified modal logic which validates the classical axioms and which includes the necessitation rule. Any attempt to define quantified standard modal logics whilst preserving classical principles for quantifiers restricts the semantic expressivity to the case of increasing domains. Furthermore, for the particular quantified modal logics $QK\partial$ given by $\mathbf{B} \in \partial$, the restriction is even stronger since the increasing domain condition together with symmetric property of the accessibility relation makes the semantics collapse into constant domain semantics.

In addition to the above limitation, the underlying notions of truth and validity in Hughes & Cresswell's system are radically different from those given in propositional modal logic. To guarantee the validity of the classical axioms, Hughes and Cresswell have adopted a semantic approach known as the *truth values gaps* system[5]. This consists of expressing the notion of truth using three values (true, false and undefined). In particular, given an atomic formula $P(t_1, \ldots, t_n)$, an increasing domain model M, a variable assignment s and a possible world w, $M, w \Vdash P(t_1, \ldots, t_n)[s]$ if and only if for each t_i, $T_{F_C,s}(t_i) \in Q(w)$, and $\langle T_{F_C,s}(t_1), \ldots, T_{F_C,s}(t_n)\rangle \in F_P(P, n)$. If for some t_i, $T_{F_C,s}(t_i) \notin Q(w)$ then $P(t_1, \ldots, t_n)$ is said to be *undefined*. All the other satisfiability conditions are also appropriately modified with respect to this three-valued notion. In general, a formula α is said to be *undefined* at a possible world w if at least one of the terms that appears in α does not exist in the local domain associated with w. The notion of validity in a model is also redefined stating that a formula α is valid in a model M if for every possible world w, α is true wherever it is defined. This solves the problem of invalidity described in Example 4.1. In fact, in each possible world w where $\forall v\alpha(v)$ is true and the term t designates an object which does not exist in w, the formula $\forall v\alpha(v) \to \alpha(t)$ is undefined (and therefore no longer false), whereas at the possible worlds where t designates an object which locally exists, the truth of $\forall v\alpha(v)$ implies the truth of $\alpha(t)$. Hence, $\forall v\alpha(v) \to \alpha(t)$ is valid since it is true at any possible world where its truth value is "defined". The use of an underlying three valued system allows terms (both constants and variables) to refer to objects outside of the domain of a given world and predicates to be locally interpreted, without invalidating the classical first-order principles.

[5]This terminology was introduced by Garson in [Gar84].

The approach of truth values gaps thus solves the problem of "incompatibility" between the world-relative quantifiers and the classical first-order axioms. However, first-order axiomatisations of modal axiomatic systems still suffer from the increasing domain restriction given by the derivability of the CBF. Gabbay [Gab76] has suggested a different axiomatisation based on a weaker form of the necessitation rule, which allows the derivation of the classical axioms, but which does not prove the Converse Barcan Formula. This also allows the underlying semantics to be expressed in terms of frames with arbitrary domains. The disadvantages caused by allowing the generality of arbitrary domains together with the first-order principles is that the standard satisfiability conditions are significantly modified, as is the notion of validity in a model. (See [Gab76] Chapter 5 for a detailed presentation of this approach). The resulting quantified modal semantics is no longer a direct extension of the standard propositional possible world semantics.

Prefixed Tableaux

To conclude this brief introduction about existing quantified modal logics based on varying domains, it is worth mentioning Fitting's tableaux systems. Fitting [Fit83] has developed a quantified tableaux system for a wide class of modal logics which is sound and complete with respect to the increasing domain semantics. The underlying semantics differs slightly from that described by Hughes and Cresswell, in that the idea of the truth value gaps is only applied to the notion of validity. A formula α is valid in a model, if for every possible world where the interpretations of all the terms contained in α exist, α is true. The prefixed extensions of these quantified tableaux systems, which are described in [Fit83, Fit93] are particular interesting. They include three different sets of elimination rules for quantifiers which, together with the standard prefixed tableaux for propositional modal logic, capture the three cases of constant, increasing and arbitrary domain semantics. In the case of arbitrary domains, two different elimination rules for the universal quantifiers are suggested which are claimed to correspond respectively to the case of arbitrary non-empty local domains and arbitrary possibly-empty local domains, but which do not allow the derivations of first-order axioms. However, no detail is given about the underlying semantics, and hence no details are given regarding soundness or completeness. Nevertheless, prefixed tableaux systems are relevant here because they provide proof theoretic evidence that in defining a quantified extension of propositional modal system for arbitrary varying domain semantics it is necessary to withdraw certain classical axioms regarding quantifiers.

4.2 A Formal Presentation of Arbitrary Domain Semantics

This section describes the arbitrary domain semantics and the associated sound and complete axiomatisation defined by Kripke in one of his first papers [Kri63b], to which the semantics of quantified MLDSs given in Chapter 5 is proved to correspond under certain restrictions (see Chapter 6 for a correspondence theorem). In [Kri63b], Kripke presents a straightforward and simple way of extending propositional possible world semantics. He considers a quantified modal language, denoted here by $\mathcal{L}_K$ and referred to as a Kripke quantified modal language, whose set of terms is given only by an infinite set of variables (i.e. no constant symbols or function symbols are included in the language). First-order frames are defined by associating with each possible world a local domain – the domain of objects that exist in that world.

Definition 4.12 {Kripke first-order frame} Let $\langle W, R\rangle$ be a Kripke frame as given in Definition 1.2. A *Kripke first-order frame* is a tuple $\langle W, R, Q\rangle$, where Q is a mapping which assigns to each member w of W a (possibly empty) set, called the *local domain of* w. The union of each of the local domains is called the *global domain* and is denoted by D (i.e. $D = \bigcup_{w\in W} Q(w)$). D must be non-empty. □

In other words, the mapping Q assigns local domains to possible worlds arbitrarily, allowing the possibility that some of them are empty[6].

A *Kripke quantified model* based on the first-order frame $\mathcal{F}$ is a binary function ϕ which maps each n-ary predicate symbol of $\mathcal{L}_K$ and each possible world of $\mathcal{F}$ into a set of n-ary tuples of elements of the global domain of $\mathcal{F}$. This is formally described in the following two definitions.

Definition 4.13 {Kripke rigid interpretation} Let $\mathcal{L}_K$ be a quantified modal language and let $\langle W, R, Q\rangle$ be a Kripke first-order frame. A *Kripke rigid interpretation* based on $\langle W, R, Q\rangle$ is a mapping ϕ such that

- $\phi(P_i^n, w) \subseteq D^n$ for each n-ary predicate symbol P_i^n and possible world $w \in W$. □

[6] Note that at least one of the local domains must be non-empty to guarantee a non-empty global domain.

Definition 4.14 {Kripke first-order model} A *Kripke first-order model* M is a tuple $\langle W, R, Q, \phi\rangle$, where $\langle W, R, Q\rangle$ is a Kripke first-order frame, and ϕ is Kripke rigid interpretation based on $\langle W, R, Q\rangle$. □

Kripke's interpretation of predicate symbols differs from the rigid interpretation given in Definition 4.3 (and in most existing varying domain semantics) in that it is defined over the global domain of a first-order frame and not over the local domains. Variables are also assigned to objects of the global domain. The satisfiability condition of an atomic formula is given straightforwardly as in classical first-order logics, whereas for the quantifiers a world-relative interpretation is adopted.

Definition 4.15 {Kripke satisfiability conditions} Let $\mathcal{L}_K$ be a Kripke quantified modal language, let $M = \langle W, R, Q, \phi\rangle$ be a Kripke first-order model and let s be an assignment[7]. Given two wffs α and β of $\mathcal{L}_K$ and a possible world w of M, the satisfiability relation $\Vdash$ is uniquely defined as follows:

1. $M, w \Vdash P_i^n(t_1, \ldots, t_n)[s]$ iff $\langle s(t_1), \ldots, s(t_n)\rangle \in \phi(P_i^n, w)$.
2. $M, w \Vdash \neg\alpha[s]$ iff not $M, w \Vdash \alpha[s]$
3. $M, w \Vdash \alpha \wedge \beta[s]$ iff $M, w \Vdash \alpha[s]$ and $M, w \Vdash \beta[s]$
4. $M, w \Vdash \alpha \vee \beta[s]$ iff $M, w \Vdash \alpha[s]$ or $M, w \Vdash \beta[s]$
5. $M, w \Vdash \alpha \rightarrow \beta[s]$ iff $M, w \Vdash \beta[s]$ or not $M, w \Vdash \alpha[s]$
6. $M, w \Vdash \Box\alpha[s]$ iff, for all $w' \in W$, if wRw' then $M, w' \Vdash \alpha[s]$
7. $M, w \Vdash \Diamond\alpha[s]$ iff there exists a $w' \in W$ such that wRw' and $M, w' \Vdash \alpha[s]$
8. $M, w \Vdash \forall u\alpha[s]$ iff $M, w \Vdash \alpha[s[^u_a]]$ for every object $a \in Q(w)$.
9. $M, w \Vdash \exists u\alpha[s]$ iff $M, w \Vdash \alpha[s[^u_a]]$ for some object $a \in Q(w)$.

□

The notion of validity in a model is identical to that given in Definition 4.8, according to which a formula is valid in a Kripke first-order model if and only

[7]Note that in the present context, since $\mathcal{L}_K$ contains no constant or function symbols, it is not necessary to distinguish between variable and term assignments.

if for every variable assignment s, the formula is true under s at every possible world. The definition of semantic entailment is also identical to that given in Chapter 1 (see Definition 1.9) but adapted to quantified extensions. Note that in [Kri63b], Kripke consider only quantified extensions of the modal logics $K\partial_K$ given by $\partial_K \subseteq \{\mathbf{T}, \mathbf{4}, \mathbf{B}\}$. These are denoted in the following definition and subsequently by $QK\partial_K$.

Definition 4.16 {Kripke semantic entailment} Let $\mathbf{F}_{K\partial_K}$ be the class of Kripke first-order frames associated with the modal logic $QK\partial_K$ given by some $\partial_K \subseteq \{\mathbf{T}, \mathbf{4}, \mathbf{B}\}$. Let $\langle S, U\rangle$ be a theory written in the quantified modal language $\mathcal{L}_K$ and let α be a modal formula of $\mathcal{L}_K$. Then α is *entailed* from $\langle S, U\rangle$, written $S \models_{QK\partial_K} U \Rightarrow \alpha$, if for every frame $F_{K\partial} \in \mathbf{F}_{K\partial}$ and for every Kripke first-order model M based on $F_{K\partial}$ in which all members of S are valid, α is true under the same variable assignments and at every possible world $w \in M$ at which all members of U are true. □

It is easy to see that a consequence of the above semantic definitions is that formulae can be evaluated as true at a possible world w of a model even though some of their variables are instantiated with objects which do not exist at that world. For instance, consider a Kripke first-order model $M = \langle W, R, Q, \phi\rangle$ where $W = \{w, w^{'}\}$, $R = \{\}$, $Q(w) = \{a\}$ $Q(w^{'}) = \{b\}$ and the rigid interpretation is such that $\phi(P, w) = \{b\}$. Let s be a variable assignment such that $s(v) = b$. Then $M, w \Vdash P(v)$. Analogously, a formula can be false in a possible world w even when some of its variables are assigned to objects which do not exist in w.

Consequently, first-order axioms with free variable occurrences are not necessary valid. For example, it is easy to see, using the same argument adopted in Example 4.1, that instances of the universal quantification axiom [UQA1] can be falsified in a Kripke first-order model. Kripke argues that this is not a serious limitation. He points out that the notion of validity in a first-order model implies that any first-order axiom which includes free variables is valid if and only if its universal closure[8] is valid. Therefore, since the closures of standard first-order schemas are valid in a Kripke first-order model, it could be said that Kripke semantics validates the standard first-order principles.

Axiomatic systems have been suggested [Kri63b] for each of the $QK\partial_K$, which are sound and complete with respect to the Kripke arbitrary domain semantics defined in this section. A formal description is given below (starting from the

[8] The universal closure, or simply *closure* of a formula α is a formula without free variables obtained by binding each free variable in α by prefixed universal quantifiers.

simple case of $QK\{\}_K$). In Chapter 6 a correspondence will be proved between these Kripke systems and the quantified MLDSs. The correspondence results will confirm that quantified extensions of MLDSs, although syntactically more expressive than the traditional axiomatic approach to modal logic, preserve the principles of Kripke's arbitrary domain semantics.

Definition 4.17 {Kripke axiomatisation of quantified modal logic K} Let $\mathcal{L}_K$ be a quantified modal language, let α and β be two wffs of $\mathcal{L}_K$, and let v be a variable of $\mathcal{L}_K$. The *Kripke axiomatic system* for the quantified extension of the modal logic K, written $QK_{K(Ax)}$, is given by the closure of the axiom schemas [A1]-[A16] that appear in Definition 1.10 (see Chapter 1), the closure of the axiom [K] of Definition 1.10, the rules [MP] and [Nec] and the closure of the following axiom schemas

Axioms about $\forall$ and $\exists$

$$\alpha \rightarrow \forall v \alpha \qquad [A17]$$
$$\forall v(\alpha \rightarrow \beta) \rightarrow (\forall v \alpha \rightarrow \forall v \beta) \qquad [A18]$$
$$\forall u(\forall v \alpha(v) \rightarrow \alpha(u)) \qquad [A19]$$

where in [A17] the variable v is not free in α, and [A19] is Kripke's notation for $\forall u(\forall v \alpha \rightarrow \alpha(v/u))$. □

The axiomatisations of the quantified modal logics $QK\partial_K$ are given by simply including in the above system $QK_{K(Ax)}$ the appropriate subsets of the *closure* of the characteristic modal axiom schemas **T**, **4** and **B**, written in the quantified modal language $\mathcal{L}_K$. In the rest of this thesis the symbol $QK\partial_{K(Ax)}$ will refer to these Kripke axiomatic systems of the modal logics $QK\partial_K$.

The definitions of derivation and of the derivability relation for each of the systems $QK\partial_{K(Ax)}$ are identical to those given in Chapter 1 and in Section 4.1.1, but where global and local assumptions are sentences (i.e. do not contain free variables). Thus, $\langle S, U\rangle$ is said to be a *closed* theory if both S and U are sets of sentences. In particular, the following notation will be used extensively in Chapter 6.

Notation 4.4 Let $QK\partial_{K(Ax)}$ be a Kripke axiomatic system, let $\langle S, U\rangle$ be a closed theory written in the language $\mathcal{L}_K$ and let α be a sentence of $\mathcal{L}_K$. α is *Kripke-derivable* from $\langle S, U\rangle$, written $S \vdash_{QK\partial_{K(Ax)}} U \Rightarrow \alpha$ if there exists a derivation (see Definition 1.11 in Chapter 1) from the theory $\langle S, U\rangle$ in the axiomatic system $QK\partial_{K(Ax)}$. $\triangleleft$

Each axiomatic system $QK\partial_{K(Ax)}$ is sound and complete with respect to the arbitrary domain semantics given in this section. In other words, the following statement holds for an arbitrary closed theory $\langle S, U\rangle$ and an arbitrary sentence α written in the language $\mathcal{L}_K$.

$$S \vdash_{QK\partial_{K(Ax)}} U \Rightarrow \alpha \text{ iff } S \models_{QK\partial_K} U \Rightarrow \alpha \tag{4.4}$$

In Definition 4.17 and in each axiomatic system $QK\partial_{K(Ax)}$, the [Nec] rule is applicable only to closed formulae, and not to formulae with free variables as in the axiomatisation described in the previous section. This means that although the underlying classical axioms are equivalent, the Converse Barcan Formula is no longer derivable. Furthermore, in the case of modal logics based on symmetric accessibility relations the Barcan Formula is not derivable (see [Kri63b] for a discussion). This is completely compatible with the model-theoretic property of arbitrary domains of the underlying Kripke first-order models[9].

However, if the decreasing condition (i.e. $Q(w) \supseteq Q(w')$, whenever wRw') is added to Kripke first-order frames the associated models validate the closure of the Barcan Formula. In this case the associated semantic entailment relation is denoted here by $\models_{QK\partial_K+[\mathrm{dec}]}$ and the corresponding sound and complete axiomatisations of the quantified modal logics $QK\partial_K$ are given by $QK\partial_{K(Ax)} + [BF]$[10]. On the other hand, if the increasing domain condition (i.e. $Q(w) \subseteq Q(w')$, whenever wRw') is added to Kripke first-order frames the associated models validate the closure of the Converse Barcan Formula. In this case the associated semantic entailment relation is denoted here by $\models_{QK\partial_K+[\mathrm{inc}]}$ and the corresponding sound and complete axiomatisations of the quantified modal logics $QK\partial_K$ are given by $QK\partial_{K(Ax)} + [CBF]$. The following two statements formalise the equivalence between each of the derivability relations $\vdash_{QK\partial_{K(Ax)}+[BF]}$ and $\vdash_{QK\partial_{K(Ax)}+[CBF]}$ and their respective semantic entailments. Given a theory $\langle S, U\rangle$ and a sentence α in the language $\mathcal{L}_K$

$$S \vdash_{QK\partial_{K(Ax)+[BF]}} U \Rightarrow \alpha \text{ iff } S \models_{QK\partial_K+[\mathrm{dec}]} U \Rightarrow \alpha \tag{4.5}$$

$$S \vdash_{QK\partial_{K(Ax)+[CBF]}} U \Rightarrow \alpha \text{ iff } S \models_{QK\partial_K+[\mathrm{inc}]} U \Rightarrow \alpha \tag{4.6}$$

[9] Kripke shows in [Kri63b] that both Barcan Formula and its Converse are not valid by constructing first-order countermodels.

[10] In this section, [BF] and [CBF] denote the closures of the Barcan Formula and of the Converse Barcan Formula, respectively.

The above two properties can also be expressed with respect to the basic and most general axiomatic system $QK\partial_{K(Ax)}$, as shown below.

$$S \cup [BF] \vdash_{QK\partial_{K(Ax)}} U \Rightarrow \alpha \ \text{ iff } \ S \models_{QK\partial_K+[\text{dec}]} U \Rightarrow \alpha \tag{4.7}$$

$$S \cup [CBF] \vdash_{QK\partial_{K(Ax)}} U \Rightarrow \alpha \ \text{ iff } \ S \models_{QK\partial_K+[\text{inc}]} U \Rightarrow \alpha \tag{4.8}$$

The equivalences (4.4), (4.7) and (4.8) show that for each quantified modal logic $QK\partial_K$ the basic and most general semantics is that given by the class of Kripke arbitrary domain models, and the most general axiomatic system is $QK\partial_{K(Ax)}$. The addition of the increasing domain condition or of the decreasing domain condition generates more specific classes of models (and semantics) for which stronger axioms need to be included in $QK\partial_{K(Ax)}$ (i.e. the [CBF] and [BF] respectively).

In Chapter 5, quantified modal labelled deductive systems, (written $QK\delta$-MLDS) will be defined for the whole family of $QK\partial$ predicate modal logics which include the $QK\partial_K$ as special cases. A labelled natural deduction system is given for the case of arbitrary domains and two additional rules, called *Visa Rules*, are defined so that when added either separately or together to the system capture respectively the cases of either increasing or decreasing or fixed domains. In this way, the uniform feature of Kripke axiomatic systems will also be reflected by the $QK\delta$-MDLS natural deduction systems. Similarly, a model-theoretic semantics will be described in Chapter 5 which will encompass the Kripke semantic notions and principles described in this section. Finally, in Chapter 6 it will be proved that the most general Kripke axiomatic systems $QK\partial_{K(Ax)}$ is subsumed by the corresponding $QK\delta$-MDLS.

4.3 Extending Propositional MLDS

The formalisation of quantified modal logics as Labelled Deductive Systems is obtained by appropriately extending the propositional MLDSs given in Section 1.2 of Chapter 1. All the characteristics of the propositional MLDSs described in Chapter 1 are preserved by the corresponding quantified extension. The main difference is that the underlying modal language $\mathcal{L}_M$ is here the quantified modal language given in Definition 4.1 (including function symbols of non-zero arity). A quantified modal labelled deductive language is therefore defined as follows.

Definition 4.18 {Quantified MLDL} A *quantified modal labelled deductive language* is given by the ordered pair $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$, where $\mathcal{L}_L$ is the labelling language given in Definition 1.16 and $\mathcal{L}_M$ is the quantified modal language given in Definition 4.1. □

For proof-theoretic purposes, both the quantified modal language $\mathcal{L}_M$ and the labelling language $\mathcal{L}_L$ need to be extended with additional sets of terms, which will be used only in derivations. Specifically, the extension $Ext(\mathcal{L}_L, \mathcal{L}_M)$ of the language $\mathcal{L}_M$ needs to include two sets of constant symbols (which are somewhat analogous to Skolem constants) to formulate ∃-Elimination and ∀-Introduction derivation rules. These constants need to be indexed with ground terms of the extended labelling language $Func(\mathcal{L}_L, \mathcal{L}_M)$. $Func(\mathcal{L}_L, \mathcal{L}_M)$ itself needs to be defined in terms of $Ext(\mathcal{L}_L, \mathcal{L}_M)$[11].

Definition 4.19 {Semi-extended labelling language, $Func(\mathcal{L}_L, \mathcal{L}_M)$} Let $\mathcal{L}_L$ be a labelling language and let $Ext(\mathcal{L}_L, \mathcal{L}_M)$ be the extended quantified modal language defined below. Let $\alpha_1,\ldots,\alpha_n,\ldots$ be the ordered set of all wffs of $Ext(\mathcal{L}_L, \mathcal{L}_M)$. The first-order language $Func(\mathcal{L}_L, \mathcal{L}_M)$ is defined as the language $\mathcal{L}_L$ extended with the function symbol *succ* and the two sets of function symbols

$$\{f_{\alpha_1}, \ldots, f_{\alpha_n}, \ldots\} \qquad \text{and} \qquad \{box_{\alpha_1}, \ldots, box_{\alpha_n}, \ldots\}$$

□

Definition 4.20 {Extended quantified modal language, $Ext(\mathcal{L}_L, \mathcal{L}_M)$} Let $\mathcal{L}_M$ be a quantified modal language, let $\alpha_1, \ldots, \alpha_n, \ldots$ be the ordered set of wffs of $\mathcal{L}_M$, let $v_1, \ldots, v_n, \ldots$ be the ordered set of variables of $\mathcal{L}_M$ and let $\lambda_1, \ldots, \lambda_n, \ldots$ be the ordered set of ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ defined above. The *extended quantified modal language* $Ext(\mathcal{L}_L, \mathcal{L}_M)$ is defined as the language $\mathcal{L}_M$ extended with the following two sets of constant symbols

- $\{c_{\alpha_k}^{\langle \exists v_i, \lambda_j \rangle} \mid \text{for all } i, j, k \geq 1\}$

[11] Definitions 4.19 and 4.20 can be restated in a different way, replacing their interdependency with a constructive presentation. This consists of defining different *depths* of extensions for each of the two language $\mathcal{L}_L$ and $\mathcal{L}_M$ in the following way. First define a semi-extended labelling language of depth 0 and an extended quantified modal language of depth 0 given by the labelling language $\mathcal{L}_L$ and by the quantified modal language $\mathcal{L}_M$ respectively. Then define for each $n \geq 1$ a semi-extended labelling language of depth n in terms of the extended quantified modal language of depth $n-1$, and an extended quantified modal language of depth n in terms of a semi-extended labelling language of depth $n-1$. Finally, the semi-extended labelling language of Definition 4.19 will be the union over n of all the semi-extended labelling language of depth n, and the extended quantified modal language of Definition 4.20 will be the union over n of all the extended quantified modal language of depth n.

- $\{c_{\alpha_k}^{\langle \forall v_i, \lambda_j \rangle} \mid \text{for all } i, j, k \geq 1\}$

$\square$

In other words, in Definition 4.19, the function symbols f_{α_i} and box_{α_i} are indexed with formulae of the extended quantified modal language $Ext(\mathcal{L}_L, \mathcal{L}_M)$ which can also include constant symbols of the form $c_{\alpha_k}^{\langle \exists v_i, \lambda_j \rangle}$ and $c_{\alpha_k}^{\langle \forall v_i, \lambda_j \rangle}$. These are given for each variable v_i and each wff formula α_k of $\mathcal{L}_M$ and for each ground term λ_j of the semi-extended labelling language $Func(\mathcal{L}_L, \mathcal{L}_M)$. The set of labels of a quantified modal labelled deductive language is given by the set of ground terms of the semi-extended labelling language, whereas the set of terms of the logical language is given by the terms of the extended quantified modal language.

As shown in Part I, terms of the labelling language $\mathcal{L}_L$ constructed by using the function symbols f_{α_i} and box_{α_i} are essential in the natural deduction rules for the $\diamond$ and $\square$ modal operators, where they denote respectively particular and arbitrary possible worlds (see discussion in Section 1.2, Chapter 1). The constant symbols introduced in the extended quantified modal language play an analogous role for the $\exists$ and $\forall$ operators. For each possible world λ and wff α, $c_{\alpha}^{\langle \exists v_i, \lambda \rangle}$ names a *particular* object specifically associated with λ and α. Such constants will be used whenever Kripke quantified semantic notions of the form "there exists an object for which the property α holds..." need to be formalised. In contrast, constant symbols of the form $c_{\alpha}^{\langle \forall v_i, \lambda \rangle}$ can be seen as referring to any *arbitrary* object specifically associated with the formula α and the possible world λ. These constant symbols will be used whenever Kripke semantic notions of the form "for all the objects for which the property α holds..." need to be expressed.

The introduction of the constant symbols $c_{\alpha_k}^{\langle \exists v_i, \lambda_j \rangle}$ is comparable to Fitting's definition of *parameters* [Fit83]. In Fitting's prefixed tableaux system [Fit83], disjoint sets of parameters are added to the underlying quantified modal language, one for each possible world, where they are assumed to exist. The association with possible worlds is syntactically captured by the use of a prefix (index) for each parameter. In the extended quantified modal language $Ext(\mathcal{L}_L, \mathcal{L}_M)$, different symbols are also added for each label (i.e. possible world), but, as described below, the assumption of their existency is given by expressions called *declarative terms*.

Declarative terms allow the expression of information about the existence or the non-existence of objects in possible worlds. A declarative term is a pair of the form *"label : term"* or *"label $\not:$ term"* denoting that an object *exists* or *does not exist* at a particular possible world. The former is called a *positive* declarative term while the latter is called a *negative* declarative term. In a certain sense, the symbol ":" between the two components can be regarded as a sort of "*Exists*" predicate (similar to the E predicate described in the previous section), and the symbol "$\not:$" as the negated form (i.e. $\neg Exists$). This interpretation will be reflected in the semantics given in Chapter 6.

Definition 4.21 {Declarative term} Given a quantified modal labelled deductive language $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$, a *declarative term* is a pair $\lambda : \tau$ or $\lambda \not: \tau$ where λ is a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and τ is a ground term of $Ext(\mathcal{L}_L, \mathcal{L}_M)$. □

To distinguish a declarative term from its "opposite in sign", the notion of the *conjugate* of a declarative term is also introduced.

Definition 4.22 {Conjugate of a declarative term} Let Θ be a declarative term. The *conjugate* of Θ, written $\overline{\Theta}$, is defined as

- $\lambda \not: \tau$ if $\Theta = \lambda : \tau$
- $\lambda : \tau$ if $\Theta = \lambda \not: \tau$

□

Declarative terms of the form $\lambda : c_\alpha^{\langle \exists v, \lambda \rangle}$ denote that a *particular* object specifically associated with a formula α exists at the possible world λ. (The label of a declarative term facilitates a syntactic formalisation of world-relative interpretations of existential quantifications.) Analogously, declarative terms of the form $\lambda : c_\alpha^{\langle \forall v, \lambda \rangle}$ denote that any *arbitrary* object specifically associated with the formula α exists at the possible world λ. These types of declarative terms allow a syntactic formalisation of world-relative interpretations of universal quantifications. Other examples of declarative terms are $f_\alpha(W_0) \not: F_1^1(v)$, $f_\alpha(W_0) \not: F_1^0$, $box_\alpha(W_3) : F_2^0$ and $W_1 \not: F_n^0$.

Declarative units are defined as in Chapter 1 (see Definition 1.19) but with respect to the quantified MLDL.

Definition 4.23 {Declarative units of a quantified MLDL} Given a quantified MLDL $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$, a declarative unit is a pair $\lambda : \alpha$ where λ is a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and α is a wff of $Ext(\mathcal{L}_L, \mathcal{L}_M)$. □

R-literals, declarative units and declarative terms are the basic syntactic entities of a quantified modal labelled deductive language. Theories can therefore be defined so to include information about the relation between possible worlds (i.e. the component $\mathcal{D}$ of a configuration), information about formulae holding in particular possible worlds (i.e. the component $\mathcal{F}$ of a configuration) and information about objects existing in particular possible worlds. To capture this third type of information the definition of a configuration given in Chapter 1 for propositional MDLS (see Definition 1.23) is extended with two extra components, as shown in the following definition.

Definition 4.24 {Configuration in a quantified MLDS} Given a quantified MLDL, a *configuration* is a tuple

$$\langle \mathcal{D}, \mathcal{F}, \mathcal{T}^+, \mathcal{T}^- \rangle$$

where:
- $\mathcal{D}$ is a diagram;
- $\mathcal{F}$ is a function from the set of ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ to the set $\mathcal{P}(\text{wff}(Ext(\mathcal{L}_L, \mathcal{L}_M)))$ of sets of wffs of $Ext(\mathcal{L}_L, \mathcal{L}_M)$;
- $\mathcal{T}^+$,$\mathcal{T}^-$ are two functions from the set of ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ to the set $\mathcal{P}(\text{term}(Ext(\mathcal{L}_L, \mathcal{L}_M)))$ of sets of ground terms of $Ext(\mathcal{L}_L, \mathcal{L}_M)$;

□

The two functions $\mathcal{T}^+$ and $\mathcal{T}^-$ assign an empty set to the labels for which there is no information about the existence of objects, and an non-empty set of ground terms to the rest. Specifically, $\mathcal{T}^+$ captures the positive declarative terms included in a configuration, whereas $\mathcal{T}^-$ captures the negative declarative terms included in a configuration. This is formally expressed in the following notation of membership between declarative terms and configurations.

Notation 4.5 Given a configuration $\mathcal{C} = \langle \mathcal{D}, \mathcal{F}, \mathcal{T}^+, \mathcal{T}^- \rangle$, a declarative term Θ of the form $\lambda : \tau$ is said to be *member* of $\mathcal{C}$ if $\tau \in \mathcal{T}^+(\lambda)$, whereas a declarative terms of the form $\lambda \not: \tau$ is said to be *member* of $\mathcal{C}$ if $\tau \in \mathcal{T}^-$. ◁

In Section 1.2.3 of Chapter 1 it has been pointed out that configurations can be infinite and conditions have been given both on the diagram and on the function $\mathcal{F}$ under which this can occur. With the extension of the notion of a configuration, these are additional circumstances under which a configuration can be infinite. Namely a configuration can be infinite also when for some label λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$ either the value of $\mathcal{T}^+$ or the value of $\mathcal{T}^-$ is an infinite set of ground terms of $Ext(\mathcal{L}_L, \mathcal{L}_M)$ (i.e. the configuration contains an infinite number of declarative terms referring to λ), or when either $\mathcal{T}^+(\lambda) \neq \{\}$ or $\mathcal{T}^-(\lambda) \neq \{\}$ for an infinite number of labels λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$.

Finally the notions of a labelling algebra and of a quantified modal labelled deductive system are identical to those given in Chapter 1 (see Definitions 1.24 and 1.25 respectively), with the difference that the set $\mathcal{R}$ of inference rules is extended so as to include rules for quantifiers and declarative terms. In the rest of this thesis, quantified extensions of all the propositional $K\delta$-MLDSs considered in Part I will be described, and the following notation will be adopted to indicate each particular quantified MLDS considered.

Notation 4.6 Let $\mathcal{A}_{K\delta}$ be a particular labelling algebra. The term $QK\delta$-MLDS denotes the associated quantified MLDS. $\triangleleft$

4.4 Discussion

In this chapter, the notion of varying domain semantics has been introduced and some examples of existing quantified modal systems have been described. From this brief summary, it is clear that world-relative interpretation of quantifiers in world-relative domains is not entirely compatible with the classical first-order principles for quantified variables. Attempts to preserve such principles within a varying domain semantics lead necessarily to either a less general semantics (e.g. the increasing domain condition is automatically implied), or some satisfiability conditions of the underlying modal logic which no longer mirror those given in the propositional case. To paraphrase Garson [Gar84], there is little reason for preserving classical rules in formulating systems with world-relative domains and rigid designators, and the principles of free logics are more appropriate. This is reinforced in [Kri63b] where Kripke defines a semantics for arbitrary domains which does not validate the standard principles of first-order logic, but which validates equivalent forms given by their respective universal closures. The latter are in fact also valid in free logic, whenever the underlying language contains only variables as terms.

Furthermore, Kripke has shown [Kri63b] that an equivalent axiomatisation for varying domain semantics could be also obtained by introducing a monadic predicate E, for *existence*, which syntactically formalises the semantic characteristic of arbitrary local domains[12]. The first-order axioms for quantifiers would need to be replaced by axiom schemas of the form $\forall v\alpha(v) \wedge E(u) \rightarrow \alpha(u)$ and $\forall v E(v)$. The first schema is a quantifier principle of *free logic*. (See [Gar84] for a brief review

[12]The interpretation of the monadic predicate E in each possible world w is equal to the local domain $Q(w)$.

of free logic.) The natural deduction systems of quantified MLDSs described in the next chapter will include rules for quantifiers similar to Kripke's suggestion of having an E predicate, and Garson's suggestion of using an underlying free logic instead of classical first-order logic. In fact the definition of declarative terms allows explicit reference to ground terms existing in particular possible worlds. Declarative terms play an important role in the natural deduction rules for quantifiers.

4.5 Summary

In this chapter, several forms of quantified possible world semantics have been described and an indication has been given of what the main issues are in attempting to give quantified extensions to propositional modal logics. Particular attention has been given to Kripke arbitrary domain semantics and its related sound and complete axiomatisation. Finally, the basic notion and syntax of a quantified MLDS has been introduced.

Chapter 5

Derivability and Semantic Entailment in Quantified MLDS

This chapter describes quantified extensions of both the proof theory and the semantics of the family of propositional MLDSs given in Part I. A natural deduction system will be defined which includes, in addition to the three sets of rules presented in Chapter 2 (i.e. classical rules, modal rules and structural rules), introduction and elimination rules for the existential and the universal quantifiers. Such rules are general enough to be applied in situations where different (possibly empty) arbitrary domains are associated with different possible worlds. The proof system is based on a notion of derivability similar to that given in Chapter 2 for propositional $K\delta$-MLDSs. In the second part of this chapter a model-theoretic semantics is provided which extends the semantics of propositional MLDSs given in Chapter 2, using the same methodology of first-order translations. This allows the proofs of soundness, completeness and correspondence to be built upon the respective proofs for the propositional case already given in Chapter 3.

5.1 A Natural Deduction System

The proof theory of propositional MLDSs described in Chapter 2 is a generalisation of standard modal systems in that the syntactic consequence relation (as well as each inference rule) is defined between structures (or configurations) of local modal theories. Reasoning steps can occur either within each individual local theory (i.e. can use any of the "classical" rules), or between different theories which are in relation with each other (using any of the "modal" rules) or within a diagram and its interaction with the logical information (using any of the "structural" rules).

However, as shown in the previous chapter, configurations of a quantified

MLDS include an additional type of information, the declarative terms, which express the existence or non-existence of objects in individual possible worlds. Such information is not only important for its potential declarative use in the formalisation of particular domains but also for its logical relevance in reasoning processes based on world-relative quantifications. In a quantified modal framework such as that considered in this thesis, empty universes of discourse can be associated with particular possible worlds (or in MLDS's terminology with particular local modal theories), and therefore sound deductions on world-relative quantifiers can only be made if information about the existence of objects is provided. Elimination and introduction rules for quantifiers are therefore defined between "universal" or "existential" declarative units[1], and declarative terms. This rules extend the set of classical rules defined in Part I.

Declarative terms can be either given as initial assumptions or can be inferred during deduction processes. Two special rules are defined between declarative terms and arbitrary declarative units which provide consistency checking on the declarative terms with respect to the rest of the logical information contained in a configuration. As shown by the definitions given below, these rules are somewhat analogous to some of the rules defined in Part I for R-literals, and therefore are classified as structural rules.

Remark 5.1 For the rest of this thesis, a quantified MLDL=$\langle \mathcal{L}_L, \mathcal{L}_M \rangle$ is assumed. Meta-variables λ, λ_1, λ', ... etc. are terms of the semi-extended labelling language $Func(\mathcal{L}_L, \mathcal{L}_M)$ given in Definition 4.19 (see Chapter 4), whereas all meta-variables τ, τ', τ_1, ..., etc. are in general terms, unless otherwise specified, of the extended quantified modal language $Ext(\mathcal{L}_L, \mathcal{L}_M)$. All the wffs referred to are wffs of $Ext(\mathcal{L}_L, \mathcal{L}_M)$. As explained previously, given a ground term τ, a variable v and a formula α of $Ext(\mathcal{L}_L, \mathcal{L}_M)$, $\alpha(v/\tau)$ denotes the wff that is like α except that all free occurrences (if any) of the variable v are substituted by τ.

In what follows, inference rules for quantifiers and for declarative terms will be formally stated, in the same way as the inference rules for modal labelled deductive systems given in Chapter 2 (see Definition 2.1). Specifically, four natural deduction systems will be described for quantified MLDSs which correspond respectively to arbitrary, increasing, decreasing and constant domains. The following notation will be adopted to indicate each of these extensions.

Notation 5.1 Let $\iota \in \{\mathbf{a}, \mathbf{i}, \mathbf{d}, \mathbf{c}\}$ and let $\delta \subseteq \{(\mathbf{T}), (\mathbf{4}), (\mathbf{D}), (\mathbf{B}), (\mathbf{5}), (\mathbf{Irr})\}$ (see Notations 1.4 and 1.5). Then $Q_\iota K\delta$-MLDS=$\langle\langle \mathcal{L}_L, \mathcal{L}_M \rangle, \mathcal{A}_{K\delta}, \mathcal{R}_\iota \rangle$ signifies the

[1]These are declarative units of the form $\lambda : \forall v \alpha$ and $\lambda : \exists v \alpha$, i.e. where the main connectives in the wffs are quantifiers.

quantified MLDS associated with the labelling algebra $\mathcal{A}_{K\delta}$ (see Chapter 1 and 2) and the set of inference rules $\mathcal{R}_\iota$. $\mathcal{R}_\mathbf{a}$ is the set of rules (defined below) for arbitrary domains, $\mathcal{R}_\mathbf{i}$ denotes the set of rules for increasing domains, $\mathcal{R}_\mathbf{d}$ denotes the set of rules for decreasing domains and $\mathcal{R}_\mathbf{c}$ denotes the set of rules for constant domains. $\triangleleft$

Each of these systems (i.e. each $\mathcal{R}_\iota$) includes all the classical, modal and structural rules given in Part I, but with the meta-variables of the rules referring to the quantified MLDL.

Furthermore, for any $Q_\iota K\delta$-MLDS the notions of a natural deduction proof and of the derivability relation are identical to those given in Definition 2.2 and Definition 2.3 respectively (see Chapter 2), but with the set of inference rules $\mathcal{R}$ replaced with the particular set $\mathcal{R}_\iota$ and the notation $\vdash_{K\delta-\text{MLDS}}$ replaced with $\vdash_{Q_\iota K\delta-\text{MLDS}}$. However, where the choice of $K\delta$ is arbitrary and where the value of ι can be understood from the context, the derivability symbol $\vdash_{Q_\iota K\delta-\text{MLDS}}$ will often be written $\vdash_{\text{MLDS}}$.

Remark 5.2 Notations 2.1, 2.2 and 2.3, and Definitions 2.4, 2.5, 2.6 and 2.7 are assumed to be extended in the obvious way to cover declarative terms as well as declarative units and R-literals.

5.1.1 MLDS for Arbitrary Domains

Quantified extensions of $K\delta$-MLDSs for arbitrary domains are defined in this section. These are given by adding to the set $\mathcal{R}$ described in Chapter 2, inference rules for universal and existential quantifiers and special rules for declarative terms. These rules will also be included in every other $Q_\iota K\delta$-MLDS with $\iota \neq \mathbf{a}$, so making $Q_\mathbf{a}K\delta$-MLDSs the most general quantified modal labelled deductive system. For the other $Q_\iota K\delta$-MLDS, three extra rules will be defined, called *Visa Rules*, whose respective addition to the set $\mathcal{R}_\mathbf{a}$ will provide more specific quantified modal labelled deductive systems.

Additional Classical Rules

In this section *elimination* and *introduction* rules for quantifiers are given. As in Chapter 2 a graphical representation of each rule is also given.

Definition 5.1 $\{\forall$-Elimination, $\mathcal{I}_{\forall E}\}$ For all configurations $\mathcal{C}$, ground terms τ, labels λ, wffs α and variables v, $\mathcal{C}/\mathcal{C}+[\lambda:\alpha(v/\tau)]$ is a member of the inference rule $\forall$–Elimination (sometimes written $\mathcal{I}_{\forall-E}$) if

- $\lambda:\forall v\alpha \in \mathcal{C}$
- $\lambda:\tau \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle\lambda:\forall v\alpha\ ,\ \lambda:\tau\rangle}{\mathcal{C}'\langle\lambda:\alpha(v/\tau)\rangle}\ \mathcal{I}_{\forall E}$$

□

The above rule differs from the standard classical elimination rule of the universal quantifier (see for example [Sun83]) in that it is applied, within the scope of the local actual world under consideration, only if the object in question exists at that world. This reflects the world-relative interpretation of the universal quantifier discussed in the previous chapter which is also assumed in the quantified MLDS model-theory (see the semantics in the second part of this chapter). This rule does not facilitate the derivation of any formula of the form $\forall v\alpha \rightarrow \alpha$, since only ground substitutions of the variable v that are known to exist at the local world under consideration are allowed in the derived formula $\alpha(v/\tau)$. This is a first example of the fact that not all the quantified MLDS extensions considered in this thesis validate (in any local modal theory) the classical first-order axioms. However, as also pointed out by Kripke [Kri63b], this is not a real limitation – an example is given below which indicates that universal closures of first-order axioms are still derivable.

On the other hand, the above rule is analogous to the universal quantification rule of free logic (see [Gar84] for a brief survey[2]). In fact, in Garson's description [Gar84] of a free logic proof system a primitive predicate E (for "*Exists*") is introduced and the following rule for universal elimination is given, where P is a predicate symbol of arity 1.

$$\frac{\forall x P(x)}{E(t) \rightarrow P(t)}$$

The similarity with $\mathcal{I}_{\forall E}$ is immediate. Similar comparisons can also be drawn for the other inference rules for quantifiers.

[2]Or [Ben84] for a wide and detailed survey.

Definition 5.2 $\{\forall$-Introduction, $\mathcal{I}_{\forall I}\}$ For all configurations $\mathcal{C}$, labels λ, wffs α and variables v, $\mathcal{C}/\mathcal{C} + [\forall v \alpha]$ is a member of the inference rule $\forall$–Introduction (sometimes written $\mathcal{I}_{\forall - I}$) if

- $\mathcal{C} + [\lambda : c_\alpha^{\langle \forall v, \lambda \rangle}] \vdash_{\mathrm{MLDS}} \lambda : \alpha(v/c_\alpha^{\langle \forall v, \lambda \rangle})$

This is sometimes represented graphically as

$$\cfrac{\begin{array}{c}\mathcal{C}\langle [\lambda : c_\alpha^{\langle \forall v, \lambda \rangle}] \rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle \lambda : \alpha(v/c_\alpha^{\langle \forall v, \lambda \rangle}) \rangle\end{array}}{\mathcal{C}'\langle \lambda : \forall v \alpha \rangle}\ \mathcal{I}_{\forall I}$$

□

In the above rule, the temporary assumption $[\lambda : c_\alpha^{\langle \forall v, \lambda \rangle}]$ should be read as "given an arbitrary object which exists at the possible world λ". In standard classical logic this assumption is often formalised using any arbitrary name (i.e. variable) with a "side condition" which states that this name does not occur in any of the assumptions on which the sub-derivation depends [Sun83]. In $\mathcal{I}_{\forall - I}$ this condition is embedded into the syntactic name of constant symbol that appears in the assumption. In fact, the superscript index "$\forall v$" and the subscript index "α" of this constant symbol, together with some syntactic consistency constraints about its use in configurations proved in Chapter 6, guarantees a unique syntactic association of the constant $c_\alpha^{\langle \forall v, \lambda \rangle}$ with the particular formula α derived in the sub-proof. Moreover, since the rule can be applied in different local theories of a given configuration, it is also important for soundness reasons to guarantee uniqueness also with respect to each particular point (label) of the structure. This is provided by the use of the second superscript λ.

The $\mathcal{I}_{\forall I}$ is similar to the introduction rule for the universal quantifier of free logic. This rule allows the derivation of a universally quantified formula only if the latter can be proved from the assumption that an arbitrary object exists. A formalisation is given below, adopting Garson's notation [Gar84], where A is a wff representing any set of initial assumptions and t is an arbitrary term which does not appear in $A \rightarrow \forall x P(x)$[3].

$$\frac{A \rightarrow (E(t) \rightarrow P(t))}{A \rightarrow \forall x P(x)}$$

[3]The analogy with the $\mathcal{I}_{\forall I}$ rule is clearer if the symbol $\rightarrow$ is thought of a effectively signifying derivability.

As mentioned above, the two rules of universal quantification, i.e. $\mathcal{I}_{\forall E}$ and $\mathcal{I}_{\forall I}$, are also comparable with Kripke axiomatic rules described in Section 4.2 of Chapter 4. Although Kripke does not explicitly adopt free logic's axiom schemas, the universal closure condition of his axiomatization is also valid in free logic, and also provable using the MLDS rules given above. The following example shows, for instance, that the formula $\forall u_1((\forall v_1 P_1^1(v_1)) \rightarrow P_1^1(u_1))$ is provable from an initial empty configuration. This is an instance of Kripke's axiom schema [A19] given in Definition 4.17.

Example 5.1 Consider a configuration $\mathcal{C} = \langle \{\}, \mathcal{F}_{\{\}}, \mathcal{T}^+_{\{\}}, \mathcal{T}^-\{\}\rangle$, where for each ground term λ of the $Func(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{F}_{\{\}}(\lambda) = \{\}$, $\mathcal{T}^+_{\{\}}(\lambda) = \{\}$ and $\mathcal{T}^-_{\{\}}(\lambda) = \{\}$. Then

$$\mathcal{C} \vdash_{\text{MLDS}} W_0 : \forall u_1((\forall v_1 P_1^1(v_1)) \rightarrow P_1^1(u_1))$$

$$\dfrac{\mathcal{C}}{\dfrac{\mathcal{C}\langle [W_0 : c^{\langle \forall u_1, W_0\rangle}_{(\forall v_1 P_1^1(v_1)) \rightarrow P_1^1(u_1)}]\rangle \quad \text{(assumption)}}{\dfrac{\mathcal{C}\langle [W_0 : \forall v_1 P_1^1(v_1)]\rangle \quad \text{(assumption)}}{\dfrac{\tilde{\mathcal{C}}\langle W_0 : P(c^{\langle \forall u_1, W_0\rangle}_{(\forall v_1 P_1^1(v_1)) \rightarrow P_1^1(u_1)})\rangle \quad (\forall\text{-E})}{\dfrac{\tilde{\mathcal{C}}_1\langle W_0 : \forall v_1 P_1^1(v_1) \rightarrow P_1^1(c^{\langle \forall u_1, W_0\rangle}_{(\forall v_1 P_1^1(v_1)) \rightarrow P_1^1(u_1)})\rangle \quad (\rightarrow\text{-I})}{\mathcal{C}'\langle W_0 : \forall u_1((\forall v_1 P_1^1(v_1)) \rightarrow P_1^1(u_1))\rangle \quad (\forall\text{-I})}}}}}$$

•

Definition 5.3 $\{\exists$–Elimination, $\mathcal{I}_{\exists - E}\}$ For all configurations $\mathcal{C}$, labels λ, variables v and wffs α, $\mathcal{C}/\mathcal{C} + [\lambda : \alpha(v/c_\alpha^{\langle \exists v, \lambda\rangle})] + [\lambda : c_\alpha^{\langle \exists v, \lambda\rangle}]$ is a member of the inference rule $\exists$–Elimination (sometimes written $\mathcal{I}_{\exists E}$) if

- $\lambda : \exists v \alpha \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle [\lambda : \exists v \alpha]\rangle}{\mathcal{C}'\langle \lambda : \alpha(v/c_\alpha^{\langle \exists v, \lambda\rangle}) \ , \ \lambda : c_\alpha^{\langle \exists v, \lambda\rangle}\rangle} \ \mathcal{I}_{\exists E}$$

□

$\mathcal{I}_{\exists E}$ can be seen as a "skolemisation" of the existential quantifier local to the possible world λ. The constant symbol $c_\alpha^{\langle \exists v, \lambda \rangle}$ denotes a particular object uniquely associated with the formula α which exists at the possible world λ as stated by the inferred declarative term $\lambda : c_\alpha^{\langle \exists v, \lambda \rangle}$. This rule differs from the standard existential elimination rule of classical first-order logic [Sun83] in the following ways. (i) It infers two different types of unit of information whereas in the classical case only a single formula is inferred (the wff in the inferred declarative unit). (ii) In $\mathcal{I}_{\exists E}$ the instantiation of the existential variable is not discharged after the application of the rule but it expands the configuration to which it is applied with a new declarative term. (iii) the standard classical side condition (see [Sun83]) is embedded in the syntactic name of the constant $c_\alpha^{\langle \exists v, \lambda \rangle}$. Its superscript index "$\exists v$" and subscript index "α" (together with some syntactic consistency constraints about its use in starting configurations proved in Chapter 6) guarantees a unique syntactic association of the constant $c_\alpha^{\langle \exists v, \lambda \rangle}$ with the particular formula α where it is introduced. (iv) Finally, the instantiation of the existential variable is world-relative, because of the second superscript λ.

The following definition completes the description of the set of classical rules of the $Q_{\mathbf{a}}K\delta$-MLDS.

Definition 5.4 {$\exists$–Introduction, $\mathcal{I}_{\exists - I}$} For all configurations $\mathcal{C}$, ground terms τ, labels λ, wffs α and variables v, $\mathcal{C}/\mathcal{C} + [\lambda : \exists v \alpha]$ is a member of the inference rule $\exists$–Introduction (sometimes written $\mathcal{I}_{\exists I}$) if

- $\lambda : \alpha(v/\tau) \in \mathcal{C}$
- $\lambda : \tau \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda : \alpha(v/\tau), \lambda : \tau \rangle}{\mathcal{C}'\langle \lambda : \exists v \alpha \rangle} \; \mathcal{I}_{\exists I}$$

□

The above rule also has more aspects in common with free logic than with classical logic. The (local) introduction of the existential quantifier depends on the existence of an object in the domain of quantification. (This is expressed by the declarative term $\lambda : \tau$.) In free logic the above rule is captured by the axiom of *existential generalisation* described by Garson in [Gar84] which is given below (where P is a arbitrary unary predicate and E is the existence predicate that defines the domain of existential quantification)

$$(P(t) \wedge E(t)) \rightarrow \exists x P(x)$$

To summarise, the set of rules for quantifiers described above expresses a world-relative interpretation of both the existential and the universal quantifiers which makes the underlying local reasoning closer to free logic than to the "pure" classical first-order logic. Consequently, the standard first-order axiom schemas which include wffs with free variables are not always derivable within each local theory of a given configuration and therefore within a quantified MLDS. On the other hand, the universal closures of these schemas can be derived using the above rules, as exemplified in Example 5.1.

This is because the universal closure of any first-order axiom schema is valid both in the semantics of free logic and in the semantics of classical first-order logic where, moreover, it is also semantically equivalent to the corresponding not universally closed first-order schema (see discussion in Chapter 4). This observation provides the basis for the correspondence proof given in Chapter 6 between the Kripke axiomatic system $QK\partial_{K(Ax)}$ and the natural deduction system of $Q_{\mathbf{a}}K\delta$-MLDS.

Example derivations. Two example derivations are given below which together illustrate the classical equivalence $\forall v\alpha \equiv \neg\exists\neg\alpha$. These help illustrate how the above rules for existential and universal quantifiers work. Only graphical representations of the derivations are given.

Example 5.2 $\{\mathcal{C}\langle W_0 : \forall v_1 P_1^1(v_1)\rangle \vdash_{\text{MLDS}} W_0 : \neg\exists v_1 \neg P_1^1(v_1)\}$

Let $\mathcal{C} = \langle\{\}, \mathcal{F}, \mathcal{T}^+_{\{\}}, \mathcal{T}^-_{\{\}}\rangle$ where $\mathcal{F}(W_0) = \{\forall v_1 P_1^1(v_1)\}$ and $\mathcal{F}(\lambda) = \{\}$ for any $\lambda \neq W_0$, and for any λ, $\mathcal{T}^+_{\{\}}(\lambda) = \{\}$ and $\mathcal{T}^-_{\{\}}(\lambda) = \{\}$. Then

$$\mathcal{C} \vdash_{\text{MLDS}} W_0 : \neg\exists v_1 \neg P_1^1(v_1)$$

$$\mathcal{C}\langle W_0 : \forall v_1 P_1^1(v_1)\rangle$$

$$\mathcal{C}\langle [W_0 : \exists v_1 \neg P_1^1(v_1)]\rangle \qquad \text{(assumption)}$$

$$\tilde{\mathcal{C}}_1\langle W_0 : c_{\neg P_1^1(v_1)}^{\langle \exists v_1, W_0\rangle}, W_0 : \neg P_1^1(c_{\neg P_1^1(v_1)}^{\langle \exists v_1, W_0\rangle})\rangle \qquad (\exists\text{–E})$$

$$\tilde{\mathcal{C}}_2\langle W_0 : P_1^1(c_{\neg P_1^1(v_1)}^{\langle \exists v_1, W_0\rangle})\rangle \qquad (\forall\text{–E})$$

$$\tilde{\mathcal{C}}_3\langle W_0 : \bot\rangle \qquad (\wedge\text{–I})$$

$$\mathcal{C}'\langle W_0 : \neg\exists v_1 \neg P_1^1(v_1)\rangle \qquad (\neg\text{–I})$$

•

Example 5.3 $\{\mathcal{C}\langle W_0 : \neg\exists v_1 \neg P_1^1(v_1)\rangle \vdash_{\text{MLDS}} W_0 : \forall v_1 P_1^1(v_1)\}$

Let $\mathcal{C} = \langle \{\}, \mathcal{F}, \mathcal{T}_{\{\}}^+, \mathcal{T}_{\{\}}^-\rangle$, where $\mathcal{F}(W_0) = \{\neg\exists v_1 \neg P_1^1(v_1)\}$ and $\mathcal{F}(\lambda) = \{\}$ for any $\lambda \neq W_0$, and for any λ, $\mathcal{T}_{\{\}}^+ = \{\}$ and $\mathcal{T}_{\{\}}^- = \{\}$. Then

$$\mathcal{C} \vdash_{\text{MLDS}} W_0 : \forall v_1 P_1^1(v_1)$$

$$\mathcal{C}\langle W_0 : \neg\exists v_1 \neg P_1^1(v_1)\rangle$$

$$\mathcal{C}\langle [W_0 : c_{P_1^1(v_1)}^{\langle \forall v_1, W_0\rangle}]\rangle \qquad \text{(assumption)}$$

$$\mathcal{C}\langle [W_0 : \neg P_1^1(c_{P_1^1(v_1)}^{\langle \forall v_1, W_0\rangle})]\rangle \qquad \text{(assumption)}$$

$$\tilde{\mathcal{C}}_1\langle W_0 : \neg\exists v_1 \neg P_1^1(v_1), W_0 : \exists v_1 \neg P_1^1(v_1)\rangle \qquad (\exists\text{–I})$$

$$\tilde{\mathcal{C}}_2\langle W_0 : \bot\rangle \qquad (\wedge\text{–I})$$

$$\tilde{\mathcal{C}}_3\langle W_0 : \neg\neg P_1^1(c_{P_1^1(v_1)}^{\langle \forall v_1, W_0\rangle})\rangle \qquad (\neg\text{–I})$$

$$\tilde{\mathcal{C}}_4\langle W_0 : P_1^1(c_{P_1^1(v_1)}^{\langle \forall v_1, W_0\rangle})\rangle \qquad (\neg\text{–E})$$

$$\mathcal{C}'\langle W_0 : \forall v_1 P_1^1(v_1)\rangle \qquad (\forall\text{–I})$$

•

Additional Structural Rules

In the four rules above, declarative terms are only used in their positive form. However, the syntax of a quantified MLDS also includes negative declarative terms which can be either stated as assumptions of an initial configuration or inferred within a derivation process. It is important to be able to detect contradictory information in a configuration given by the inclusion of a declarative term and its conjugate. This ability is provided by the $\mathcal{I}_{\perp_t}$ rule described below, which, like the $\mathcal{I}_{\perp}$ rule described in Chapter 2 (see Definition 2.21), reflects the classical principle that any formula (declarative unit) can be derived from a contradiction.

Definition 5.5 $\{\perp_t$–Introduction, $\mathcal{I}_{\perp_t}\}$ For all configurations $\mathcal{C}$, declarative terms Θ and declarative units $\lambda : \alpha$, $\mathcal{C}/\mathcal{C} + [\lambda : \alpha]$ is a member of the inference rule $\perp_t$–Introduction (sometimes written $\mathcal{I}_{\perp_t}$) if

- $\Theta \in \mathcal{C}$
- $\overline{\Theta} \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda : \Theta, \lambda : \overline{\Theta} \rangle}{\mathcal{C}'\langle \lambda' : \alpha \rangle}\ \mathcal{I}_{\perp_t}$$

□

The following rule allows the deduction of declarative terms on the basis of the logical information which is already present in a given configuration. Specifically, it formalises a form of interaction between declarative terms and declarative units which facilitates the inference of a declarative term whenever a logical inconsistency (i.e. inconsistency between declarative units) would arise if the conjugate of the declarative term were included in the configuration.

Definition 5.6 $\{T$–Introduction, $\mathcal{I}_{T_i}\}$ For all configurations $\mathcal{C}$ and declarative terms Θ, $\mathcal{C}/\mathcal{C} + [\Theta]$ is a member of the inference rule T–Introduction (sometimes written $\mathcal{I}_{T_i}$) if for some label λ'

- $\mathcal{C} + [\overline{\Theta}] \vdash_{\mathrm{MLDS}} \lambda' : \perp$

This is sometimes represented graphically as

$$\cfrac{\begin{array}{c}\mathcal{C}\langle[\overline{\Theta}]\rangle\\ \vdots\\ \tilde{\mathcal{C}}\langle\lambda' : \bot\rangle\end{array}}{\mathcal{C}'\langle\Theta\rangle}\ \mathcal{I}_{T_i}$$

□

An example of when and how this rule is used is as follows. Let $\mathcal{C}$ be the configuration $\langle\{\}, \mathcal{F}, \mathcal{T}^+_{\{\}}, \mathcal{T}^-_{\{\}}\rangle$, where $\mathcal{F}(W_0) = \{\forall v_1 \neg P^1_1(v_1), P^1_1(F^0_1)\}$, for any label λ different from W_0 $\mathcal{F}(\lambda) = \{\}$, and for any λ $\mathcal{T}^+_{\{\}}(\lambda) = \{\}$ and $\mathcal{T}^-_{\{\}}(\lambda) = \{\}$. Then $\mathcal{C} \vdash_{\mathrm{MLDS}} W_0 \not\downarrow F^0_1$ as shown in the following derivation.

$$\cfrac{\mathcal{C}\langle W_0 : \forall v_1 \neg P^1_1(v_1), W_0 : P^1_1(F^0_1)\rangle}{\cfrac{\cfrac{\cfrac{\mathcal{C}\langle[W_0 : F^0_1]\rangle \qquad\qquad \text{(assumption)}}{\tilde{\mathcal{C}}_1\langle W_0 : \neg P^1_1(F^0_1), W_0 : P^1_1(F^0_1)\rangle \qquad (\forall\text{–E})}}{\tilde{\mathcal{C}}_2\langle W_0 : \bot\rangle \qquad (\wedge\text{–I})}}{\mathcal{C}'\langle W_0 \not\downarrow F^0_1\rangle \qquad (T_i)}}$$

The above two rules ($\mathcal{I}_{\bot_t}$ and $\mathcal{I}_{T_i}$) can be seen as a sort of "interface" between the sets of declarative terms (i.e. $\mathcal{T}^+$ and $\mathcal{T}^-$) and the set of declarative units $\mathcal{F}$ of a configuration, through which information about *inconsistency* can be exchanged. There is an obvious analogy between these structural rules and the structural rules $\mathcal{I}_{\bot I}$ and $\mathcal{I}_{R-I}$ introduced in Chapter 2, which provide a similar "interface" for exchanging inconsistency between the set $\mathcal{F}$ of declarative units and the set $\mathcal{D}$ of R-literals.

To summarise this section, the quantified extension of any $K\delta$-MLDS for arbitrary domains is obtained by (i) restating the sets of classical, modal and structural rules defined in Chapter 2 with respect to the quantified MLDL, the quantified semi-extended $Func(\mathcal{L}_L, \mathcal{L}_M)$ and $Ext(\mathcal{L}_L, \mathcal{L}_M)$; (ii) expanding the set of classical rules with the rules $\mathcal{I}_{\forall I}$, $\mathcal{I}_{\forall E}$, $\mathcal{I}_{\exists I}$ and $\mathcal{I}_{\exists E}$, and (iii) expanding the set of structural rules with $\mathcal{I}_{\bot_t}$ and $\mathcal{I}_{T_i}$. The set of modal rules is left unchanged. This is formally expressed in the following definition.

Definition 5.7 {A $Q_{\mathbf{a}}K\delta$-MLDS} Let $\mathcal{A}_{K\delta}$ be a particular labelling algebra, let $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$ be a quantified MLDL and let $\mathcal{R}$ be the set of inference rules of the associated $K\delta$-MLDS. Then the quantified extension $Q_{\mathbf{a}}K\delta$-MLDS for arbitrary domains is given by the set of inference rules

$$\mathcal{R}_{\mathbf{a}} = \mathcal{R} \cup \{\mathcal{I}_{\forall E}, \mathcal{I}_{\forall I}, \mathcal{I}_{\exists E}, \mathcal{I}_{\exists I}\} \cup \{\mathcal{I}_{\perp_t}, \mathcal{I}_{T_i}\}$$

□

5.1.2 MLDS for Specific Varying Domains

This section shows how extra deduction rules can be added to the MLDS framework to capture increasing, decreasing or constant domain semantics. The resulting proof systems are still uniform in the sense that they can be used in conjunction with any labelling algebra. Three special rules are introduced, called *Visa Rules*[4], that allow "cross-reasoning" about objects existing in different possible worlds in relation with each other.

MLDS for Increasing Domains

As described in Chapter 4, quantified modal logics are based on increasing domains whenever they characterise a class of models whose quantified frame structures satisfy the increasing domain condition (i.e. for all possible worlds w, w', if wRw' then $Q(w) \subseteq Q(w')$). This means that every object that belongs to the local domain of a particular possible world of any of these structures, belongs also to the local domains of all the possible worlds accessible from it. The notion of existence of an object in a possible world is syntactically expressed by a declarative term. Therefore, to capture the increasing domain property, the following rule needs to be included in the $Q_{\mathbf{i}}K\delta$-MLDS, which encompasses the increasing domain condition as a form of "interaction" between declarative terms and R-literals.

Definition 5.8 {Increasing Visa Rule, $\mathcal{I}_{VR_{\mathrm{i}}}$} For all configurations $\mathcal{C}$, ground terms τ of $Ext(\mathcal{L}_L, \mathcal{L}_M)$, and labels λ_1 and λ_2 of $Func(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{C}/\mathcal{C} + [\lambda_2 : \tau]$ is a member of the *increasing domain visa rule* (sometimes written $\mathcal{I}_{VR_{\mathrm{i}}}$) if

- $\lambda_1 : \tau \in \mathcal{C}$
- $R(\lambda_1, \lambda_2) \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda_1 : \tau \ , \ R(\lambda_1, \lambda_2) \rangle}{\mathcal{C}'\langle \lambda_2 : \tau \rangle} \ \mathcal{I}_{VR_{\mathrm{i}}}$$

□

[4]The word "Visa" is borrowed from Gabbay's [Gab92]. It has the figurative meaning of allowing terms to "travel" from one possible world to another!

A quantified MLDS for increasing domains can now be defined as follows.

Definition 5.9 {A $Q_{\mathbf{i}}K\delta$-MLDS} Let $\mathcal{A}_{K\delta}$ be a particular labelling algebra, let $\langle \mathcal{L}_L, \mathcal{L}_M \rangle$ be a quantified MLDL and let $\mathcal{R}_{\mathbf{a}}$ be the set of inference rules of the associated $Q_{\mathbf{a}}K\delta$-MLDS. Then the quantified extension $Q_{\mathbf{i}}K\delta$-MLDS for increasing domains is given by the set of inference rules

$$\mathcal{R}_{\mathbf{i}} = \mathcal{R}_{\mathbf{a}} \cup \{\mathcal{I}_{VR_{\mathbf{i}}}\}$$

$\Box$

As discussed in Chapter 4, standard axiomatic systems for quantified modal logics on increasing domains accept the Converse Barcan Formula as a theorem. The following example derivation shows that this specific formula is also derivable in a $Q_{\mathbf{i}}K\delta$-MLDS system.

Example 5.4 {MLDS derivation of the CBF} Given a $Q_{\mathbf{i}}K\delta$-MLDS and a configuration $\mathcal{C}_{\{\}} = \langle \{\}, \mathcal{F}, \mathcal{T}^+_{\{\}}, \mathcal{T}^-_{\{\}} \rangle$, where for any label λ, $\mathcal{F}(\lambda) = \{\}$, $\mathcal{T}^+_{\{\}}(\lambda) = \{\}$ and $\mathcal{T}^-_{\{\}}(\lambda) = \{\}$, and any arbitrary formula α of $Ext(\mathcal{L}_L, \mathcal{L}_M)$, then

$$\mathcal{C}_{\{\}} \vdash_{\text{MLDS}} W_0 : \Box \forall v_1 \alpha \rightarrow \forall v_1 \Box \alpha$$

In the following derivation, α' is shorthand for the wff $\alpha(v_1/c^{\langle \forall v_1, W_0 \rangle}_{\Box\alpha})$.

$\underline{\mathcal{C}_{\{\}}\quad}$

$\quad\underline{\mathcal{C}_{\{\}}\langle [W_0 : \Box \forall v_1 \alpha] \rangle}$ (assumption)

$\quad\quad\underline{\mathcal{C}_{\{\}}\langle [W_0 : c^{\langle \forall v_1, W_0 \rangle}_{\Box\alpha}] \rangle}$ (assumption)

$\quad\quad\quad\underline{\mathcal{C}_{\{\}}\langle [R(W_0, box_{\alpha'}(W_0))] \rangle}$ (assumption)

$\quad\quad\quad\underline{\tilde{\mathcal{C}}\langle box_{\alpha'}(W_0) : \forall v_1 \alpha \rangle}$ ($\Box$-E)

$\quad\quad\quad\underline{\tilde{\mathcal{C}}_1\langle box_{\alpha'}(W_0) : c^{\langle \forall v_1, W_0 \rangle}_{\Box\alpha} \rangle}$ ($VR_{\mathbf{i}}$)

$\quad\quad\quad\underline{\tilde{\mathcal{C}}_2\langle box_{\alpha'}(W_0) : \alpha' \rangle}$ ($\forall$-E)

$\quad\quad\underline{\tilde{\mathcal{C}}_3\langle W_0 : \Box \alpha' \rangle}$ ($\Box$-I)

$\quad\underline{\tilde{\mathcal{C}}_4\langle W_0 : \forall v_1 \Box \alpha \rangle}$ ($\forall$-I)

$\mathcal{C}'\langle W_0 : \Box \forall v_1 \alpha \rightarrow \forall v_1 \Box \alpha \rangle$ ($\rightarrow$-I)

•

MLDS for Decreasing Domains

Quantified modal logics are based on decreasing domains whenever they characterise a class of models whose quantified frame structures satisfy the decreasing domain condition (i.e. for all possible worlds w, w', if wRw' then $Q(w') \subseteq Q(w)$). This means that all the objects that belong to the local domains of all possible worlds accessible from a given world belong also to the local domain of this world. To capture this decreasing domain property the following rule needs to be included to each $Q_{\mathbf{d}}K\delta$-MLDS.

Definition 5.10 {Decreasing Visa Rule, $\mathcal{I}_{VR_{\mathbf{d}}}$} For all configurations $\mathcal{C}$, ground s τ of $Ext(\mathcal{L}_L, \mathcal{L}_M)$, and labels λ_1 and λ_2 of $Func(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{C}/\mathcal{C} + [\lambda_1 : \tau]$ is a member of the *decreasing domain visa rule* (sometimes written $\mathcal{I}_{VR_{\mathbf{d}}}$) if

- $\lambda_2 : \tau \in \mathcal{C}$
- $R(\lambda_1, \lambda_2) \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda_2 : \tau\ ,\ R(\lambda_1, \lambda_2)\rangle}{\mathcal{C}'\langle \lambda_1 : \tau\rangle}\ \mathcal{I}_{VR_{\mathbf{d}}}$$

□

A quantified MLDS for decreasing domains is then defined as follows.

Definition 5.11 {A $Q_{\mathbf{d}}K\delta$-MLDS} Let $\mathcal{A}_{K\delta}$ be a particular labelling algebra, let $\langle \mathcal{L}_L, \mathcal{L}_M\rangle$ be a quantified MLDL and let $\mathcal{R}_{\mathbf{a}}$ be the set of inference rules of the associated $Q_{\mathbf{a}}K\delta$-MLDS. Then the quantified extension $Q_{\mathbf{d}}K\delta$-MLDS for decreasing domains is given by the set of inference rules

$$\mathcal{R}_{\mathbf{d}} = \mathcal{R}_{\mathbf{a}} \cup \{\mathcal{I}_{VR_{\mathbf{d}}}\}$$

□

The following example derivation illustrates how specific Barcan Formula can be derived in a $Q_{\mathbf{d}}K\delta$-MLDS system. Note that in any standard axiomatisation of quantified modal logic (see Chapter 4) this formula has to be given as part of the set of axiom schemas.

Example 5.5 {MLDS derivation of the [BF]} Given a $Q_{\mathbf{d}}K\delta$-MLDS and a configuration $\mathcal{C}_{\{\}} = \langle \{\}, \mathcal{F}, \mathcal{T}^{+}_{\{\}}, \mathcal{T}^{-}_{\{\}} \rangle$ where for any label λ, $\mathcal{F}(\lambda) = \{\}$, $\mathcal{T}^{+}_{\{\}}(\lambda) = \{\}$ and $\mathcal{T}^{-}_{\{\}}(\lambda) = \{\}$ and any arbitrary formula α of $Ext(\mathcal{L}_L, \mathcal{L}_M)$, then

$$\mathcal{C}_{\{\}} \vdash_{\mathrm{MLDS}} W_0 : \forall v_1 \Box \alpha \rightarrow \Box \forall v_1 \alpha$$

$$\underline{\mathcal{C}_{\{\}}}$$

$$\underline{\mathcal{C}_{\{\}}\langle [W_0 : \forall v_1 \Box \alpha] \rangle} \qquad \text{(assumption)}$$

$$\underline{\mathcal{C}_{\{\}}\langle [R(W_0, box_{\forall v_1 \alpha}(W_0))] \rangle} \qquad \text{(assumption)}$$

$$\underline{\mathcal{C}_{\{\}}\langle [box_{\forall v_1 \alpha}(W_0) : c_{\alpha}^{\langle \forall v_1, box_{\forall v_1 \alpha}(W_0) \rangle}] \rangle} \qquad \text{(assumption)}$$

$$\underline{\tilde{\mathcal{C}}\langle W_0 : c_{\alpha}^{\langle \forall v_1, box_{\forall v_1 \alpha}(W_0) \rangle} \rangle} \qquad (VR_{\mathbf{d}})$$

$$\underline{\tilde{\mathcal{C}}_1\langle W_0 : \Box \alpha(v_1 / c_{\alpha}^{\langle \forall v_1, box_{\forall v_1 \alpha}(W_0) \rangle}) \rangle} \qquad (\forall\text{-E})$$

$$\underline{\tilde{\mathcal{C}}_2\langle box_{\forall v_1 \alpha}(W_0) : \alpha(v_1 / c_{\alpha}^{\langle \forall v_1, box_{\forall v_1 \alpha}(W_0) \rangle}) \rangle} \qquad (\Box\text{-E})$$

$$\underline{\tilde{\mathcal{C}}_3\langle box_{\forall v_1 \alpha}(W_0) : \forall v_1 \alpha \rangle} \qquad (\forall\text{-I})$$

$$\underline{\tilde{\mathcal{C}}_4\langle W_0 : \Box \forall v_1 \alpha \rangle} \qquad (\Box\text{-I})$$

$$\mathcal{C}'\langle W_0 : \forall v_1 \Box \alpha \rightarrow \Box \forall v_1 \alpha \rangle \qquad (\rightarrow\text{-I})$$

•

MLDS for Constant Domains

Quantified modal logics are based on constant domains whenever they characterise first-order frames where the local domains associated with each possible world are all identical to each other and to the global domain. Obviously this condition is much stronger than simply requiring that the frame structures satisfy both the increasing and the decreasing domain conditions, since this would only guarantee that only the local domains associated with possible worlds which are in relation with each other are identical. Therefore, to define a quantified MLDS for constant domains a third type of a visa rule needs to be defined and added to the basic quantified MLDS for arbitrary domains. Since, the individual conditions of increasing and decreasing domains are implied by the constant domain condition, this third rule subsumes both the increasing and the decreasing visa rules given in the previous two sections. A formal definition is given below.

Definition 5.12 {Constant Visa Rule, $\mathcal{I}_{VR_\mathbf{i}}$} For all configurations $\mathcal{C}$, ground terms τ of $Ext(\mathcal{L}_L, \mathcal{L}_M)$, and labels λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{C}/\mathcal{C} + [\lambda : \tau]$ is a member of the *constant domain visa rule* (sometimes written $\mathcal{I}_{VR_\mathbf{c}}$). This is sometimes represented graphically as

$$\frac{\mathcal{C}}{\mathcal{C}'\langle\lambda:\tau\rangle}\ \mathcal{I}_{VR_\mathbf{c}}$$

□

The quantified MLDS for constant domains is defined as follows:

Definition 5.13 {A $Q_\mathbf{c}K\delta$-MLDS} Let $\mathcal{A}_{K\delta}$ be a particular labelling algebra, let $\langle\mathcal{L}_L, \mathcal{L}_M\rangle$ be a quantified MLDL and let $\mathcal{R}_\mathbf{a}$ be the set of inference rules of the associated $Q_\mathbf{a}K\delta$-MLDS. Then the quantified extension $Q_\mathbf{c}K\delta$-MLDS for constant domains is given by the set of inference rules

$$\mathcal{R}_\mathbf{c} = \mathcal{R}_\mathbf{a} \cup \{\mathcal{I}_{VR_\mathbf{c}}\}$$

□

Of course, $\mathcal{R}_\mathbf{c}$ could also include the visa rule $\mathcal{I}_{VR_i}$ and $\mathcal{I}_{VR_d}$, but given the stronger rule $\mathcal{I}_{VR_c}$, these other rules are redundant.

It is obvious that in a constant domain semantics, where the global domain is equal to each local domain, formulae of the form $\forall v_1\alpha \rightarrow \alpha(\tau)$ are provable for any ground term τ. This is also the case in a quantified MLDS for constant domains, as illustrated in the following example.

Example 5.6 {MLDS derivation of [UQA1]} Given a $Q_\mathbf{c}K\delta$-MLDS and a configuration $\mathcal{C}_{\{\}} = \langle\{\}, \mathcal{F}, \mathcal{T}^+_{\{\}}, \mathcal{T}^-_{\{\}}\rangle$ where for any label λ, $\mathcal{F}(\lambda) = \{\}$, $\mathcal{T}^+_{\{\}}(\lambda) = \{\}$ and $\mathcal{T}^-_{\{\}}(\lambda) = \{\}$, and given any ground term τ and arbitrary formula α of $Ext(\mathcal{L}_L, \mathcal{L}_M)$, then

$$\mathcal{C}_{\{\}} \vdash_{\text{MLDS}} W_0 : \forall v_1\alpha \rightarrow \alpha(v_1/\tau)$$

$$\mathcal{C}_{\{\}}$$

$$\mathcal{C}\langle[W_0 : \forall v_1 \alpha] \qquad \text{(assumption)}$$

$$\tilde{\mathcal{C}}\langle W_0 : \tau\rangle \qquad (VR_{\mathbf{c}})$$

$$\tilde{\mathcal{C}}_1\langle W_0 : \alpha(v_1/\tau)\rangle \qquad (\forall\text{-E})$$

$$\mathcal{C}'\langle W_0 : \forall v_1 \alpha \rightarrow \alpha(v_1/\tau)\rangle \qquad (\rightarrow\text{-I})$$

•

It is worth pointing out that the introduction of the constant visa rule allows the natural deduction rules for quantifiers given in Section 5.1.1 to be preserved. The fact that any declarative term can now be inferred makes the specification of a declarative term in the antecedent of any of these rules somewhat redundant. In this case it would in fact be possible to replace these rules with a labelled formalisation of the corresponding standard classical natural deduction rules. However, the approach chosen here gives a "monotonic" way of defining natural deduction systems for different types of quantified modal logics based on the same overall set of rules for modal operators and quantifiers. The basic system is that of arbitrary domains where reasoning about quantification is world-relative, and any other systems related to more specific cases are obtained by appropriately strengthening the proof system with additional rules and therefore restricting the underlying class of models.

Once more, this shows the advantage of defining the quantified extension of propositional MLDSs in the style of free logic. It provides the most general way of dealing with quantifiers in the class of varying domain structures, in which constant domains can be seen as a specific subset.

5.2 A Semantics

In this second part of the chapter, a model-theoretic semantics for quantified MLDSs is defined following the same approach adopted in Chapter 2 for the propositional case. This consists of (i) providing a first-order translation method which transforms configurations into first-order theories and (ii) expressing the semantic entailment relation $\models_{\text{MLDS}}$ in terms of standard classical semantic entailment. In Part I it was sufficient to extend the labelling language with monadic predicates, one for each wff of the propositional modal language. Here, the translation method, as well as the target language, are slightly more complex, but still based on first-order logic. Wffs of the quantified modal language $Ext(\mathcal{L}_L, \mathcal{L}_M)$ are translated into *ground terms* of a reified first-order language, called the *extended labelling language* and written $Mon(\mathcal{L}_L, \mathcal{L}_M)$. Semantic notions of truth are expressed using a *Holds* predicate. As will be seen, this reification is necessary in order to deal with constants and variables within formulae which refer to objects in the world-relative domains.

The reified language $Mon(\mathcal{L}_L, \mathcal{L}_M)$is a three sorted first-order language which includes the first-order language $Func(\mathcal{L}_L, \mathcal{L}_M)$. The three sorts are respectively (i) the sort Ω of *worlds*, (ii) the sort Σ of *objects* and (iii) the sort Φ of *properties*. The set of ground terms and variables of Ω are given by the set of ground terms and variables of $Func(\mathcal{L}_L, \mathcal{L}_M)$. As for the sort Σ, the set of constant symbols is not only given by translations of the constant symbols of $Ext(\mathcal{L}_L, \mathcal{L}_M)$, but also by translations of the variables u_1, v_1, ... of $Ext(\mathcal{L}_L, \mathcal{L}_M)$. This allows the translation method to reify each wff occurring in a configuration into a *ground term*. The sort Φ has only constant symbols and functions. No variables are needed. However, as apparent from the axiomatisation given below, to capture a first-order reasoning in the reified theory, additional variables t, t_1, t_2, t_3, etc. are needed in an expanded version of the quantified modal language $Ext(\mathcal{L}_L, \mathcal{L}_M)$. These will be mainly used for universal and existential quantifications in the reified first-order theory over the sort Σ.

Definition 5.14 {Expanding $Ext(\mathcal{L}_L, \mathcal{L}_M)$} Given an extended quantified modal language $Ext(\mathcal{L}_L, \mathcal{L}_M)$, $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$ denotes the language $Ext(\mathcal{L}_L, \mathcal{L}_M)$ expanded with the following set of variables

$$\{t, t_1, t_2, t_3, \ldots\}$$

□

Note that no variable t_i appears in the wffs of declarative units included in configurations, since the declarative units are defined in the language $Ext(\mathcal{L}_L, \mathcal{L}_M)$. However, wffs of the form $[\alpha(v/t_i)]^*$ with occurrences of the variable t_i can be used in the reified theory. The definition of the reified language is given below.

Definition 5.15 {Extended labelling language, $Mon(\mathcal{L}_L, \mathcal{L}_M)$}

Let $Func(\mathcal{L}_L, \mathcal{L}_M)$ be a semi-extended labelling language and let $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$ be the expansion of the extended quantified modal language given in Definition 5.14.

Then $Mon(\mathcal{L}_L, \mathcal{L}_M)$ is a first-order language of three sorts, namely the sort of *worlds*, Ω, the sort of *objects* Σ and the sort of *properties* Φ. The sort Σ includes two sub-sorts Σ_g (the sort of *ground objects*) and Σ_v (the sort of *v-objects*). Σ_g and Σ_v are such that $\Sigma_v \cap \Sigma_g = \{\}$ and $(\Sigma_v \cup \Sigma_g) \subseteq \Sigma$. It is defined on the following alphabet.

- Constant Symbols:

$W_0, W_1, \ldots$	constant symbols of sort Ω of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$;
$\{\overbrace{c_{\alpha_k}^{\langle \exists v_i, \lambda_j \rangle}}^{*} \mid$ for each $i, j, k \geq 1\}$	constant symbols of sort Σ_g corresponding to each symbol $c_{\alpha_k}^{\langle \exists v_i, \lambda_j \rangle}$ of $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$;
$\{\overbrace{c_{\alpha_k}^{\langle \forall v_i, \lambda_j \rangle}}^{*} \mid$ for each $i, j, k \geq 1\}$	constant symbols of sort Σ_g corresponding to each symbol $c_{\alpha_k}^{\langle \forall v_i, \lambda_j \rangle}$ of $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$;
$\overset{*}{F}{}_1^0, \overset{*}{F}{}_2^0, \ldots$	constant symbols of sort Σ_g corresponding to each symbol F_i^0 of $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$;
$\overset{*}{v}_1, \overset{*}{u}_1, \overset{*}{v}_2, \overset{*}{u}_2, \ldots$	constant symbols of sort Σ_v corresponding to each variable symbol of $Ext(\mathcal{L}_L, \mathcal{L}_M)$
$\overset{*}{P}{}_1^0, \overset{*}{P}{}_2^0, \ldots$	constant symbols of sort Φ corresponding to each symbol P_i^0 of $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$;

- Variables:

$x, y, z, \ldots$	variables of sort Ω of the language $Func(\mathcal{L}_L, \mathcal{L}_M)$;
$t_1, t_2, \ldots$	variables of sort Σ_g of the language $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$;

- Function Symbols:

$succ$	function symbol of $Func(\mathcal{L}_L, \mathcal{L}_M)$ of sort $\Omega \longrightarrow \Omega$
$f_{\alpha_1}, f_{\alpha_2}, \ldots$	function symbols of $Func(\mathcal{L}_L, \mathcal{L}_M)$ of sort $\Omega \longrightarrow \Omega$
$box_{\alpha_1}, box_{\alpha_2}, \ldots$	function symbols of $Func(\mathcal{L}_L, \mathcal{L}_M)$ of sort $\Omega \longrightarrow \Omega$
$\overset{*}{F}{}_1^n, \overset{*}{F}{}_2^n, \ldots$	function symbols of arity n, for each $n \geq 1$, corresponding to each function symbol F_i^n of $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$, such that $\overset{*}{F}{}_i^n : (\Sigma_g)^n \longrightarrow \Sigma_g$; $\overset{*}{F}{}_i^n : (\Sigma^n - (\Sigma_g)^n) \longrightarrow (\Sigma - (\Sigma_g \cup \Sigma_v))$
$\overset{*}{P}{}_1^n, \overset{*}{P}{}_2^n, \ldots$	function symbols of arity n with $n \geq 1$, corresponding to each predicate P_i^n of $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$, such that $\overset{*}{P}{}_i^n : \Sigma^n \longrightarrow \Phi$

$\overset{*}{\forall}, \overset{*}{\exists}$	of sort $\Sigma_v \times \Phi \longrightarrow \Phi$ corresponding to the quantifiers $\forall$ and $\exists$ of $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$;
$\overset{*}{\neg}, \overset{*}{\Box}, \overset{*}{\Diamond}$	of sort $\Phi \longrightarrow \Phi$ corresponding to the operators $\neg$, $\Box$ and $\Diamond$ of $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$;
$\overset{*}{\wedge}, \overset{*}{\vee}, \overset{*}{\rightarrow}$	of sort $\Phi \times \Phi \longrightarrow \Phi$ corresponding to each operator $\wedge$, $\vee$ and $\rightarrow$ of $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$;

- Predicate Symbols:

$Holds$	of sort $\Phi \times \Omega$;
At	of sort $\Sigma_g \times \Omega$;
R	of sort $\Omega \times \Omega$ corresponding to the predicate R of $Func(\mathcal{L}_L, \mathcal{L}_M)$;

$\Box$

The definition of the two sub-sorts Σ_g and Σ_v is needed mainly because declarative terms allowed in a configuration are only defined on ground terms of $Ext(\mathcal{L}_L, \mathcal{L}_M)$. In the proof of completeness (see next chapter) this distinction will be crucial where the universe of discourse of the canonical model will be defined only on the sort Σ_g. For a similar reason, the interpretation of functions $\overset{*}{F}{}_i^n$ are appropriately restricted according to sorting of their argument. The stipulation that $\overset{*}{F}{}_i^n : (\Sigma_g)^n \longrightarrow \Sigma_g$ captures the meaning of "ground", whereas the stipulation that $\overset{*}{F}{}_i^n : (\Sigma^n - (\Sigma_g)^n) \longrightarrow (\Sigma - (\Sigma_g \cup \Sigma_v))$ captures the meaning of "non-ground".

The translation functor, denoted with $[\ ]^*$, is defined inductively from wffs and terms of the language $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$ (denoted with Λ) to terms of sort Σ or Φ of $Mon(\mathcal{L}_L, \mathcal{L}_M)$ (denoted with $[\Lambda]^*$) in Tables 5.1 and 5.2.

Λ	$[\Lambda]^*$
t_i	t_i
v_i, u_i	$\overset{*}{v}_i, \overset{*}{u}_i$
$c_{\alpha_k}^{\langle \forall v_i, \lambda_j \rangle}$	$\overbrace{c_{\alpha_k}^{\langle \forall v_i, \lambda_j \rangle}}^{*}$
$c_{\alpha_k}^{\langle \exists v_i, \lambda_j \rangle}$	$\overbrace{c_{\alpha_k}^{\langle \exists v_i, \lambda_j \rangle}}^{*}$
F_i^0	$\overset{*}{F}{}_i^0$
$F_i^n(\tau_1, \ldots, \tau_n)$	$\overset{*}{F}{}_i^n\ ([\tau_1]^*, \ldots, [\tau_n]^*)$

Table 5.1: Translation functor $[\ ]^*$ for terms of $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$

Note that the $[\ ]^*$ transformation of any wff or term of $Ext(\mathcal{L}_L, \mathcal{L}_M)$ which does not contain any of the variables t, t_1, t_2, ..., etc. is always a *ground term* of

Λ	$[\Lambda]^*$
P_i^0	$\overset{*}{P_i^0}$
$P_i^n(\tau_1, \ldots, \tau_n)$	$\overset{*}{P_i^n}\ ([\tau_1]^*, \ldots, [\tau_n]^*)$
$\neg\alpha$	$\overset{*}{\neg}\ ([\alpha]^*)$
$\alpha \wedge \beta$	$\overset{*}{\wedge}\ ([\alpha]^*, [\beta]^*)$
$\alpha \vee \beta$	$\overset{*}{\vee}\ ([\alpha]^*, [\beta]^*)$
$\alpha \rightarrow \beta$	$\overset{*}{\rightarrow}\ ([\alpha]^*, [\beta]^*)$
$\Box\alpha$	$\overset{*}{\Box}\ ([\alpha]^*)$
$\Diamond\alpha$	$\overset{*}{\Diamond}\ ([\alpha]^*)$
$\exists v\alpha$	$\overset{*}{\exists}\ (\overset{*}{v}, [\alpha]^*)$
$\forall v\alpha$	$\overset{*}{\forall}\ (\overset{*}{v}, [\alpha]^*)$

Table 5.2: Translation functor $[\ ]^*$ for wffs of $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$

$Mon(\mathcal{L}_L, \mathcal{L}_M)$. Wffs with free variable occurrences included in a configuration are reified into ground terms of sort Φ of $Mon(\mathcal{L}_L, \mathcal{L}_M)$. This is justified by the fact that the notion of truth of a quantified modal formula with free variable occurrences is generally defined (see Chapter 4) with respect to a given variable assignment. Under this assignment free variables behave semantically like constant symbols (i.e. their value is an object of the global domain of the semantic structure under consideration).

As in Part I of this thesis, the standard satisfiability condition of each type of wff α is captured by a first-order axiom schema which depends on the structure of the term $[\alpha]^*$. As before, a labelling algebra $\mathcal{A}$ is therefore extended with a set of axiom schemas, and the resulting three-sorted first-order theory is called an *extended algebra* and is denoted by $\mathcal{A}^+$. A definition of the extended algebra for a quantified MLDS based on arbitrary domains is given below. Subsequently, it is shown how this algebra is extended to capture specific classes of models based on increasing, decreasing and constant domains.

Definition 5.16 {Extended algebra $\mathcal{A}^+_{Q_{\mathrm{a}}K\delta}$ for arbitrary domains} Given an extended labelling language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and a labelling algebra $\mathcal{A}_{K\delta}$ written in $Func(\mathcal{L}_L, \mathcal{L}_M)$, the *extended algebra* $\mathcal{A}^+_{Q_{\mathrm{a}}K\delta}$ *for arbitrary domains* is a three-sorted first-order theory in $Mon(\mathcal{L}_L, \mathcal{L}_M)$ consisting of all instances of the following axiom schemas $(AxQ1)$–$(AxQ12)$ which are sentences[5], where α and β are wffs of $Ext(\mathcal{L}_L, \mathcal{L}_M)^+$, and λ is a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$.

[5]i.e. contain no free occurrences of the variables t, t_1, t_2, ...

$$\forall x(Holds([\alpha \wedge \beta]^*, x) \equiv (Holds([\alpha]^*, x) \wedge Holds([\beta]^*, x))) \tag{AxQ1}$$

$$\forall x(Holds([\neg\alpha]^*, x) \equiv \neg Holds([\alpha]^*, x)) \tag{AxQ2}$$

$$\forall x(Holds([\alpha \vee \beta]^*, x) \equiv (Holds([\alpha]^*, x) \vee Holds([\beta]^*, x))) \tag{AxQ3}$$

$$\forall x(Holds([\alpha \rightarrow \beta]^*, x) \equiv (Holds([\alpha]^*, x) \rightarrow Holds([\beta]^*, x))) \tag{AxQ4}$$

$$\forall x(Holds([\Diamond\alpha]^*, x) \rightarrow (R(x, f_\alpha(x)) \wedge Holds([\alpha]^*, f_\alpha(x)))) \tag{AxQ5}$$

$$\forall x(\exists y(R(x, y) \wedge Holds([\alpha]^*, y)) \rightarrow Holds([\Diamond\alpha]^*, x)) \tag{AxQ6}$$

$$\forall x((R(x, box_\alpha(x)) \rightarrow Holds([\alpha]^*, box_\alpha(x))) \rightarrow Holds([\Box\alpha]^*, x)) \tag{AxQ7}$$

$$\forall x(Holds([\Box\alpha]^*, x) \rightarrow (\forall y(R(x, y) \rightarrow Holds([\alpha]^*, y)))) \tag{AxQ8}$$

$$\begin{aligned} &Holds([\exists v\alpha]^*, \lambda) \rightarrow \\ &\qquad (At([c_\alpha^{\langle \exists v, \lambda \rangle}]^*, \lambda) \wedge Holds([\alpha(v/c_\alpha^{\langle \exists v, \lambda \rangle})]^*, \lambda)) \end{aligned} \tag{AxQ9}$$

$$\begin{aligned} &\forall x(\exists t(At(t, x) \wedge Holds([\alpha(v/t)]^*, x)) \rightarrow \\ &\qquad\qquad Holds([\exists v\alpha]^*, x)) \end{aligned} \tag{AxQ10}$$

$$\begin{aligned} &\forall x(Holds([\forall v\alpha]^*, x) \rightarrow \\ &\qquad (\forall t(At(t, x) \rightarrow Holds([\alpha(v/t)]^*, x)))) \end{aligned} \tag{AxQ11}$$

$$\begin{aligned} &(At([c_\alpha^{\langle \forall v, \lambda \rangle}]^*, \lambda) \rightarrow Holds([\alpha(v/c_\alpha^{\langle \forall v, \lambda \rangle})]^*, \lambda)) \rightarrow \\ &\qquad\qquad Holds([\forall v\alpha]^*, \lambda) \end{aligned} \tag{AxQ12}$$

$\Box$

It is easy to see that the first eight axiom schemas (i.e. (AxQ1)–AxQ8)) are equivalent to that given in the extended algebra of a propositional MLDS, with the difference that the monadic predicate symbols $[\alpha]^*$ have been reified.

Axioms (AxQ9)-(AxQ12) capture the Kripke world-relative interpretation of the quantifiers, where atomic formulae of the form $At(\tau, x)$ express the existence of the ground term τ at the possible world x. It can be shown that (AxQ9)-(AxQ10) together cover the Kripke semantic interpretation of a world-relative existential quantifier given in Definition 4.15. In fact the following statement can be proved from any extended algebra $\mathcal{A}^+_{Q_{\mathbf{a}}K\delta}$ and for any arbitrary α and λ.

$$Holds([\exists v\alpha]^*, \lambda) \equiv \exists t(At(t, \lambda) \wedge Holds([\alpha(v/t)]^*, \lambda))$$

Analogously, axiom schemas (AxQ11)-(AxQ12) together cover the Kripke semantic interpretation of a world-relative universal quantifier given in Definition 4.15. The following statement can be proved from any extended algebra $\mathcal{A}^+_{Q_{\mathbf{a}}K\delta}$ and for any arbitrary α and λ.

$$Holds([\forall v\alpha]^*, \lambda) \equiv \forall t(At(t, \lambda) \rightarrow Holds([\alpha(v/t)]^*, \lambda))$$

In Chapter 6 it will be proved that the class of classical models of the above extended algebra captures the class of first-order semantic structures defined by Kripke in [Kri63b] and described in Section 4.2 together with the notions of truth for each type of formula.

The following proposition shows that the standard interdefinability of the $\forall$ and $\exists$ quantifiers of the logical language $Ext(\mathcal{L}_L, \mathcal{L}_M)$ can be derived from the above extended algebra $\mathcal{A}^+_{Q_{\mathbf{a}}K\delta}$.

Proposition 5.1 Given the extended language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and an associated extended algebra $\mathcal{A}^+_{Q_{\mathbf{a}}K\delta}$, for any formula $\alpha \in Ext(\mathcal{L}_L, \mathcal{L}_M)$ and for any ground term $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$ the following equivalence is satisfied:

$$\mathcal{A}^+ \vdash_{FOL} Holds([\forall v\alpha]^*, \lambda) \equiv Holds([\neg\exists v\neg\alpha]^*, \lambda)$$

Proof:
It is sufficient to prove the following two results.

(a) $\mathcal{A}^+ \vdash_{FOL} Holds([\forall v\alpha]^*, \lambda) \rightarrow Holds([\neg\exists v\neg\alpha]^*, \lambda)$
(b) $\mathcal{A}^+ \vdash_{FOL} Holds([\neg\exists v\neg\alpha]^*, \lambda) \rightarrow Holds([\forall v\alpha]^*, \lambda)$

$Holds([\forall v \alpha]^*, \lambda)$	(i)
$\forall t(At(t, \lambda) \rightarrow Holds([\alpha(v/t)]^*, \lambda))$	(ii)
$\forall t(At(t, \lambda) \rightarrow \neg Holds([\neg\alpha(v/t)]^*, \lambda))$	(iii)
$\forall t \neg(At(t, \lambda) \wedge Holds([\neg\alpha(v/t)]^*, \lambda))$	(iv)
$\neg(At([c_\alpha^{\langle \exists v, \lambda \rangle}]^*, \lambda) \wedge Holds([\neg\alpha(v/c_\alpha^{\langle \exists v, \lambda \rangle}]^*, \lambda))$	(v)
$\neg Holds([\exists v \neg\alpha]^*, \lambda)$	(vi)
$Holds([\neg\exists v \neg\alpha]^*, \lambda)$	(vii)

Proof of (a)

(i) is a starting assumption, (ii) is derived from (i) and (AxQ11) by applying modus ponens, (iii) is obtained from (ii) and (AxQ2), (iv) is obtained by equivalence rewriting of (iii), (v) is derived from (iv) using classical rules for universal quantification, and (vi) is obtained from (v) and the contrapositive of (AxQ9) using modus ponens. Finally, (vii) is derived from (vi) and (AxQ2). Note that in the derivation of (v) the instantiation of t with the constant symbol $[c_\alpha^{\langle \exists v, \lambda \rangle}]^*$ makes the term $[\alpha(v/t)]^*$ equal to the term $[\alpha(v/c_\alpha^{\langle \exists v, \lambda \rangle})]^*$.

$Holds([\neg\exists v \neg\alpha]^*, \lambda)$	(i)
$\neg\exists t(At(t, \lambda) \wedge Holds([\neg\alpha(v/t)]^*, \lambda))$	(ii)
$\forall t(\neg At(t, \lambda) \vee \neg Holds([\alpha(v/t)]^*, \lambda))$	(iii)
$\forall t(At(t, \lambda) \rightarrow Holds([\alpha(v/t)]^*, \lambda))$	(iv)
$At([c_\alpha^{\langle \forall v, \lambda \rangle}]^*, \lambda) \rightarrow Holds([\alpha(v/c_\alpha^{\langle \forall v, \lambda \rangle}]^*, \lambda)$	(v)
$Holds([\forall v \alpha]^*, \lambda)$	(vi)

Proof of (b)

(i) is a starting assumption, (ii) is derived from (i), (AxQ2) and (AxQ10), (iii) is obtained from (ii) by equivalence rewriting, (iv) is obtained from (iii) and (AxQ2), (v) is derived from (iv) using classical rules for universal quantification and finally (vi) is derived from (v) and (AxQ12) by modus ponens. In the above deduction, the instantiation of t with the constant symbol $[c_\alpha^{\langle \forall v, \lambda \rangle}]^*$ in the derivation of (v), makes the term $[\alpha(v/t)]^*$ equal to the term $[\alpha(v/c_\alpha^{\langle \forall v, \lambda \rangle})]^*$. $\star$

Using the reified functor $[\ ]^*$, the definition of a first-order translation of a configuration can now be given. This associates configurations of a quantified MLDS with three-sorted first-order theories written in the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$, in the following way. Each declarative unit $\lambda : \alpha$ is translated into a ground $Holds$ literal of the form $Holds(\overset{*}{\alpha}, \lambda)$, each declarative term $\lambda : \tau$ is translated into a ground At literal of the form $At([\tau]^*, \lambda)$ and each R-literal is translated into itself.

Definition 5.17 {First-order translation in $Q_\iota K\delta$-MLDS}

Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F}, \mathcal{T}^+, \mathcal{T}^- \rangle$ be a configuration of a $Q_\iota K\delta$-MLDS. The first-order translation of $\mathcal{C}$, written $FOT(\mathcal{C})$, is the three-sorted theory written in $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and defined by the expression:

$$FOT(\mathcal{C}) = \mathcal{D} \cup DU \cup DT^+ \cup DT^-$$

where

- $DU = \{Holds(\overset{*}{\alpha}, \lambda) \mid \alpha \in \mathcal{F}(\lambda),\ \lambda \text{ is a ground term of } Func(\mathcal{L}_L, \mathcal{L}_M)\}$
- $DT^+ = \{At([\tau]^*, \lambda) \mid \tau \in \mathcal{T}^+(\lambda),\ \lambda \text{ is a ground term of } Func(\mathcal{L}_L, \mathcal{L}_M)\}$
- $DT^- = \{\neg At([\tau]^*, \lambda) \mid \tau \in \mathcal{T}^-(\lambda),\ \lambda \text{ is a ground term of } Func(\mathcal{L}_L, \mathcal{L}_M)\}$

□

It should be emphasised that since labels are all ground terms, and any free variable occurrence in the wffs of the declarative units is translated into a constant symbol of the reified language, then the three-sorted first-order translation of a configuration is a set of ground literals.

The definitions of a model, satisfiability and semantic entailment are similar to those given in Chapter 2 (see Definitions 2.27, 2.28 and 2.31). A semantic structure of a $Q_{\mathbf{a}}K\delta$-MLDS is simply a (sorted) classical first-order model of the associated extended labelling algebra $\mathcal{A}^+_{K\delta}$. A formal definition is given below.

Definition 5.18 {Semantic structure of a $Q_{\mathbf{a}}K\delta$-MLDS} Given a $Q_{\mathbf{a}}K\delta$-MLDS and the associated extended algebra $\mathcal{A}^+_{Q_{\mathbf{a}}K\delta}$, M is a *semantic structure based on arbitrary domains* of $Q_{\mathbf{a}}K\delta$-MLDS if M is a model of $\mathcal{A}^+_{Q_{\mathbf{a}}K\delta}$. □

Different classes of semantic structures based on arbitrary domains can be obtained by considering different underlying labelling algebrae (i.e. different values of δ).

This is because the axioms of a labelling algebra are also included in the extended algebra $\mathcal{A}^{+}_{Q_{\mathbf{a}}K\delta}$. Furthermore, as shown in the following sections, for each labelling algebra $\mathcal{A}_{K\delta}$, another three different types of semantic structure can be defined, which correspond to the cases of increasing, decreasing and constant domains. These are obtained in each case by extending $\mathcal{A}^{+}_{Q_{\mathbf{a}}K\delta}$ with an additional axiom using the At predicate. The following definitions hold for any type of semantic structure.

The notion of satisfiability of R-literals is identical to that given in Definition 2.29 (Chapter 2), since the same labelling language is shared between propositional and quantified MLDSs. However, the notions of satisfiability of a declarative unit and of a declarative term need to be stated with respect to the three-sorted extended labelling language.

Definition 5.19 {Satisfiability of a declarative unit} Let $\lambda : \alpha$ be a declarative unit. $\lambda : \alpha$ *is satisfiable* (with respect to $Q_{\mathbf{a}}K\delta$-MLDS) if there exists a semantic structure $\mathcal{M}_{Q_{\mathbf{a}}K\delta}$ such that $\mathcal{M}_{Q_{\mathbf{a}}K\delta} \Vdash_{FOL} Holds(\overset{*}{\alpha}, \lambda)$. In this case $\mathcal{M}_{Q_{\mathbf{a}}K\delta}$ is said to *satisfy* $\lambda : \alpha$, written as $\mathcal{M}_{Q_{\mathbf{a}}K\delta} \Vdash_{Q_{\mathbf{a}}K\delta-\text{MLDS}} \lambda : \alpha$. □

Definition 5.20 {Satisfiability of a declarative term} Let Θ be a declarative term. Θ *is satisfiable* (with respect to $Q_{\mathbf{a}}K\delta$-MLDS) if there exists a semantic structure $\mathcal{M}_{Q_{\mathbf{a}}K\delta}$ such that (i) if Θ is of the form $\lambda : \tau$ then $\mathcal{M}_{Q_{\mathbf{a}}K\delta} \Vdash_{FOL} At([\tau]^*, \lambda)$ and (ii) if Θ is of the form $\lambda \not: \tau$ then $\mathcal{M}_{Q_{\mathbf{a}}K\delta} \Vdash_{FOL} \neg At([\tau]^*, \lambda)$. In both case $\mathcal{M}_{Q_{\mathbf{a}}K\delta}$ is said to *satisfy* Θ, written as $\mathcal{M}_{Q_{\mathbf{a}}K\delta} \Vdash_{Q_{\mathbf{a}}K\delta-\text{MLDS}} \Theta$. □

Now the definition of satisfiability of a configuration can be given.

Definition 5.21 {Satisfiability of a configuration} Let $\mathcal{C}$ be a configuration. $\mathcal{C}$ is *satisfiable* (with respect to $Q_{\mathbf{a}}K\delta$-MLDS) if there exists a semantic structure $\mathcal{M}_{Q_{\mathbf{a}}K\delta}$ such that for each π, $\pi \in \mathcal{C}$, $\mathcal{M}_{Q_{\mathbf{a}}K\delta}$ satisfies π, (where π may be either a declarative unit or an R–literal or a declarative term). In this case $\mathcal{M}_{Q_{\mathbf{a}}K\delta}$ is said to *satisfy* $\mathcal{C}$, written $\mathcal{M}_{Q_{\mathbf{a}}K\delta} \Vdash_{Q_{\mathbf{a}}K\delta-\text{MLDS}} \mathcal{C}$. □

The notion of a semantic entailment for a quantified MLDS is similar to that given for propositional MLDSs.

Definition 5.22 {Semantic entailment in a quantified MLDS} Let $\mathcal{A}^{+}_{Q_{\mathbf{a}}K\delta}$ be the extended algebra of a MLDS $Q_{\mathbf{a}}K\delta$-MLDS. Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F}, \mathcal{T}^{+}, \mathcal{T}^{-} \rangle$ and

$\mathcal{C}_1 = \langle \mathcal{D}_1, \mathcal{F}_1, \mathcal{T}_1^+, \mathcal{T}_1^- \rangle$ be two configurations, and $FOT(\mathcal{C}) = \mathcal{D} \cup \mathcal{DU} \cup \mathcal{DT}^+ \cup \mathcal{DT}^-$ and $FOT(\mathcal{C}_1) = \mathcal{D}_1 \cup \mathcal{DU}_1 \cup \mathcal{DT}_1^+ \cup \mathcal{DT}_1^-$ their respective three-sorted first-order translations. The configuration $\mathcal{C}$ *semantically entails* $\mathcal{C}_1$, written $\mathcal{C} \models_{Q_{\mathbf{a}}K\delta-\text{MLDS}} \mathcal{C}_1$, iff

$$\mathcal{A}^+_{Q_{\mathbf{a}}K\delta} \cup FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}_1)$$

□

The above definitions of satisfiability and semantic entailment can equally be applied to any $Q_\iota K\delta$-MLDS as shown in the following sections.

5.2.1 Increasing Domains

As described in Chapter 4, quantified modal logics are based on increasing domains if they characterise a class of semantic structures whose frames satisfy the increasing domain condition (i.e. for every w, w', if wRw' then $Q(w) \subseteq Q(w')$). This semantic notion of increasing domains is formalised in the model-theoretic approach to quantified MLDS by adding to the extending labelling algebra $\mathcal{A}^+_{Q_{\mathbf{a}}K\delta}$ for arbitrary domains a particular axiom which expresses the increasing domain condition in terms of the At predicate. This is shown in the following definition.

Definition 5.23 {Extended algebra $\mathcal{A}_{Q_{\mathbf{i}}K\delta}$ for increasing domains} Given an extended labelling language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and a labelling algebra $\mathcal{A}_{K\delta}$ written in $Func(\mathcal{L}_L, \mathcal{L}_M)$, the *extended algebra $\mathcal{A}^+_{Q_{\mathbf{i}}K\delta}$ for increasing domains*, written $\mathcal{A}^+_{Q_{\mathbf{i}}K\delta}$, is given by adding the following axiom to the extended algebra $\mathcal{A}^+_{Q_{\mathbf{a}}K\delta}$

$$\forall x \forall y \forall t((At(t,x) \wedge R(x,y)) \rightarrow At(t,y)) \tag{AxI}$$

□

A semantic structure for quantified MLDS based on increasing domains is defined as follows.

Definition 5.24 {Semantic structure of a $Q_{\mathbf{i}}K\delta$-MLDS} Given a $Q_{\mathbf{i}}K\delta$-MLDS and the associated extended algebra $\mathcal{A}^+_{Q_{\mathbf{i}}K\delta}$, $M_{Q_{\mathbf{i}}K\delta}$ is a *semantic structure based on increasing domains* of $Q_{\mathbf{i}}K\delta$-MLDS if M is a model of $\mathcal{A}^+_{Q_{\mathbf{i}}K\delta}$. □

The definitions of satisfiability of declarative units, declarative terms and configurations given in Definitions 5.19, 5.20 and 5.21 hold for the case of increasing domains, but replacing the models $M_{Q_{\mathrm{a}}K\delta}$ with the models $M_{Q_{\mathrm{i}}K\delta}$. Analogously, semantic entailment between any pair of configurations $\mathcal{C}$ and $\mathcal{C}_1$, written $\mathcal{C} \models_{Q_{\mathrm{i}}K\delta-\mathrm{MLDS}} \mathcal{C}_1$, can be defined as

$$\mathcal{A}^{+}_{Q_{\mathrm{i}}K\delta} \cup FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}_1)$$

5.2.2 Decreasing Domains

Quantified MLDSs based on decreasing domains are also special cases of MLDSs for arbitrary domains. These are given by adding to the extended algebra $\mathcal{A}^{+}_{Q_{\mathrm{a}}K\delta}$ for arbitrary domains an axiom which expresses the increasing domain property (i.e. for every possible worlds w, w', if wRw' then $Q(w') \subseteq Q(w)$). The associated class of semantic structures is given by the class of first-oder models which validates $\mathcal{A}^{+}_{Q_{\mathrm{a}}K\delta}$ and this special axiom. This is shown in the following definitions.

Definition 5.25 {Extended algebra $\mathcal{A}_{Q_{\mathrm{d}}K\delta}$ for decreasing domains} Given an extended labelling language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and a labelling algebra $\mathcal{A}_{K\delta}$ written in $Func(\mathcal{L}_L, \mathcal{L}_M)$, the *extended algebra* $\mathcal{A}^{+}_{Q_{\mathrm{d}}K\delta}$ *for decreasing domains*, written $\mathcal{A}^{+}_{Q_{\mathrm{d}}K\delta}$, is given by adding the following axiom to the extended algebra $\mathcal{A}^{+}_{Q_{\mathrm{a}}K\delta}$

$$\forall x \forall y \forall t((At(t, y) \wedge R(x, y)) \rightarrow At(t, x)) \qquad \text{(AxD)}$$

□

A semantic structure for quantified MLDS based on decreasing domains is defined as follows.

Definition 5.26 {Semantic structure of a $Q_{\mathrm{d}}K\delta$-MLDS} Given a $Q_{\mathrm{d}}K\delta$-MLDS and the associated extended algerbra $\mathcal{A}^{+}_{Q_{\mathrm{d}}K\delta}$, $M_{Q_{\mathrm{d}}K\delta}$ is a *semantic structure base on decreasing domains* of $Q_{\mathrm{d}}K\delta$-MLDS if M is a model of $\mathcal{A}^{+}_{Q_{\mathrm{d}}K\delta}$. □

The definitions of satisfiability of declarative units, declarative terms and configurations given in Definitions 5.19, 5.20 and 5.21 are also applicable to this class of semantic structures but replacing the models $M_{Q_{\mathrm{a}}K\delta}$ with the models $M_{Q_{\mathrm{d}}K\delta}$. Semantic enatilment between any pair of configurations $\mathcal{C}$ and $\mathcal{C}_1$, written $\mathcal{C} \models_{Q_{\mathrm{d}}K\delta-\text{tiny MLDS}} \mathcal{C}_1$, can be defined as

$$\mathcal{A}^{+}_{Q_{\mathrm{d}}K\delta} \cup FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}_1)$$

5.2.3 Constant Domains

Quantified MLDSs based on constant domains are also special cases of MLDSs for arbitrary domains. These are given by adding to the extended algebra $\mathcal{A}^{+}_{Q_{\mathbf{a}}K\delta}$ for arbitrary domains an axiom which expresses the constant domain property (i.e. for every possible worlds w, w', $Q(w') = Q(w)$). The associated class of semantic structures is then given by the class of first-order models which validate $\mathcal{A}^{+}_{Q_{\mathbf{a}}K\delta}$ together with this special axiom. This is shown in the following definitions.

Definition 5.27 {Extended algebra $\mathcal{A}_{Q_{\mathbf{c}}K\delta}$ for constant domains} Given an extended labelling language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ and a labelling algebra $\mathcal{A}_{K\delta}$ written in $Func(\mathcal{L}_L, \mathcal{L}_M)$, the *extended algebra* $\mathcal{A}^{+}_{Q_{\mathbf{c}}K\delta}$ *for constant domains*, written $\mathcal{A}^{+}_{Q_{\mathbf{c}}K\delta}$, is given by adding the following axiom to the extended algebra $\mathcal{A}^{+}_{Q_{\mathbf{a}}K\delta}$

$$\forall x \forall t (At(t, x)) \qquad \text{(AxC)}$$

□

A semantic structure for quantified MLDS based on constant domains is defined as follows.

Definition 5.28 {Semantic structure of a $Q_{\mathbf{c}}K\delta$-MLDS} Given a $Q_{\mathbf{c}}K\delta$-MLDS and the associated extended algerbra $\mathcal{A}^{+}_{Q_{\mathbf{c}}K\delta}$, $M_{Q_{\mathbf{c}}K\delta}$ is a *semantic structure based on constant domains* of $Q_{\mathbf{c}}K\delta$-MLDS if M is a model of $\mathcal{A}^{+}_{Q_{\mathbf{c}}K\delta}$. □

The definitions of satisfiability of declarative units, declarative terms and configurations given in Definitions 5.19, 5.20 and 5.21 are also applicable to this class of semantic structures but replacing the models $M_{Q_{\mathbf{a}}K\delta}$ with the models $M_{Q_{\mathbf{c}}K\delta}$. Semantic entailment between any pair of configurations $\mathcal{C}$ and $\mathcal{C}_1$, written $\mathcal{C} \models_{Q_{\mathbf{c}}K\delta-\text{MLDS}} \mathcal{C}'$, is defined as

$$\mathcal{A}^{+}_{Q_{\mathbf{c}}K\delta} \cup FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}_1)$$

5.3 Discussion

In this chapter proof systems and semantics have been described for the wide family of quantified MLDSs $Q_{\iota}K\delta$-MLDS given by $\iota \in \{\mathbf{a}, \mathbf{i}, \mathbf{d}, \mathbf{c}\}$ and $\delta \subseteq \{(\mathbf{T}), (\mathbf{4}), (\mathbf{B}), (\mathbf{D}), (\mathbf{5}), (\mathbf{Irr})\}$.

The proof theoretic approach described in the first part of this chapter has the following features and advantages. (i) It generalises standard notions of proof theories for quantified modal logics such as the notion of a modal theory and the notion of a derivability relation (see the discussion section of Chapter 2). (ii) It is uniform for any δ with respect to the class of quantified modal logics $Q_\iota K\delta$. (iii) It allows explicit syntactic reference to the existence or non-existence of objects in individual possible worlds. (iv) It uses rules for quantifiers which are world-relative. (v) It solves the usual proof theoretic problem related to the "permutation" of quantifiers and modal operators, i.e. it does not necessitate the inclusion of the Barcan and Converse Barcan Formulae. (vi) It is in natural deduction style. Various proof systems for quantified modal logics have been described in the literature, but there is no deduction system in which all these six aspects co-exist.

The first main distinctive characteristic listed above, concerning reasoning about structures of local quantified modal theories shows that the underlying methodology of modal labelled deductive systems can be equally applied in both propositional and predicate settings. Moreover, the explicit representation of possible worlds and relationships between possible worlds facilitates a natural and monotonic first-order extension to the proof theory of propositional modal logics. The system is natural in the sense that rules for quantifiers are directly added to the underlying natural deduction system of propositional MLDS, allowing reasoning on arbitrary domain structures. The system is monotonic in the sense that the standard more specific cases of increasing, decreasing and constant domains are obtained by simply adding single "domain-related" rules (i.e. visa rules). This feature is mainly due to the use of declarative terms, which give an explicit syntactic formalisation of the notion of existence or non-existence of objects relative to particular "possible worlds".

A similar approach has been adopted by Fitting in his prefixed tableaux system for quantified modal logics, where the constant symbols of the language are indexed with prefixes denoting possible worlds. However, as already pointed out in Section 4.1.2 of Chapter 4, in the case of arbitrary domains Fitting does not give details about the underlying semantics and therefore about soundness and completeness. Another related approach can be found in a recent work on quantified modal logics by Artosi et al. [ABG96, JR89]. This describes a labelled tableux proof system for quantified modal logics which handles the cases of increasing, decreasing and constant domains via the introduction of "domain rules" that facilitate the inference of objects in different possible worlds. However, the underlying language does not include function symbols, and the system is not completely uniform since special rules need to be added to capture underlying modal logic based on symmetric accessibility relation. By the use of configurations quantified MLDSs incorporate more general notions of quantified modal theories and derivability.

An alternative approach to varying domain semantics has been suggested by Garson in [Gar84], using axiomatic proof systems. He suggests [Gar84] that the most natural way of extending propositional axiomatisations of modal logics to encompass world-relative quantifiers is by adding axiom schemas of free logics (expressed using an E ($Exists$) predicate) to an axiomatisation of a normal propositional modal logic. Kripke put forward the same suggestion in [Kri63b], but like Garson he has left the suggestion on a level of abstract speculation. The proof systems of quantified MLDSs described in this chapter substantiate this idea using a natural deduction style proof system instead of a Hilbert system. (The similarity of the elimination and introduction rules of the quantifiers with free logic axioms have been discussed throughout the first part of this chapter.) In the specific case of quantified MLDSs for arbitrary domains, it will be shown in Chapter 6 that this proof system subsumes the sound and complete axiomatisation described by Kripke in [Kri63b].

5.4 Summary

In this chapter, natural deduction systems have been defined for quantified MLDS which correspond to the notions of arbitrary, increasing, decreasing and constant domains. Corresponding definitions of semantic entailment have been given based on translations to sorted first-order classical theories.

Chapter 6

Soundness, Completeness and Correspondence for Quantified MLDSs

In the previous chapter proof systems and semantics have been defined for the family of quantified MLDSs $Q_\iota K\delta$-MLDS which correspond to the cases of arbitrary, increasing, decreasing and constant domains. For each $\iota \in \{\mathbf{a}, \mathbf{i}, \mathbf{d}, \mathbf{c}\}$, the associated notions of derivability $\vdash_{Q_\iota K\delta-\text{MLDS}}$ and semantic entailment $\models_{Q_\iota K\delta-MLDS}$ have been given in terms of the set of inference rules $\mathcal{R}_\iota$ and the extended algebra $\mathcal{A}^+_{Q_\iota K\delta}$ respectively. This chapter shows that for each $Q_\iota K\delta$-MLDS the related syntactic and semantic consequence relations are equivalent. The methodology adopted is the same as described in Part I. Most of the theorems, lemmas and propositions given here extend those given in Chapter 3 with additional cases corresponding to the quantifiers and to the existence of objects in possible worlds. The technical part of this thesis will conclude with a proof that Kripke's axiomatic system for arbitrary domain semantics, described in Chapter 4, is subsumed by the corresponding system $Q_{\mathbf{a}}K\delta$-MLDS.

6.1 Soundness

This section shows that for each type ι of quantified MLDS, the relation $\vdash_{Q_\iota K\delta-\text{MLDS}}$ implies the relation $\models_{Q_\iota K\delta-MLDS}$ (for a discussion about the adopted methodology see Section 3.1 in Chapter 3). By exploiting the soundness property of first-order logic, it is sufficient to show that for any pair of configurations $\mathcal{C}$ and $\mathcal{C}'$, if $\mathcal{C} \vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}'$ then $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$. Before going into details, the following notation is given which extends the notion of size of a member of an inference rule, given in Definition 3.1, to the classical rules for quantifiers, to the structural rules for declarative terms and to the visa rules.

Notation 6.1 In a quantified modal labelled deductive system, the category $\mathcal{I}^0$ of inference rules given in Notation 3.1 is extended so that

$$\mathcal{I}_{\forall E}, \mathcal{I}_{\exists I}, \mathcal{I}_{\exists E}, \mathcal{I}_{\perp_t}, \mathcal{I}_{VR_\mathbf{i}}, \mathcal{I}_{VR_\mathbf{d}}, \mathcal{I}_{VR_\mathbf{c}} \in \mathcal{I}^0$$

The category $\mathcal{I}^+$ is also extended so that

$$\mathcal{I}_{T_i}, \mathcal{I}_{\forall I} \subseteq \mathcal{I}^+$$

$\lhd$

Hence the $\forall$-Introduction and T-Introduction rules have size equal to $1+l_1$, where in both cases l_1 is the smallest of the sizes of all derivations that can be used as a condition of the rule in question. All the other additional rules of quantified MLDSs have size equal to 1 (i.e. they do not require subderivations as conditions).

The following lemma proves the soundness of each deduction $\mathcal{C} \vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}'$ with respect to the associated extended algebra $\mathcal{A}^+_{Q_\iota K\delta}$ and the first-order derivability $\vdash_{FOL}$. The proof is identical to that of Lemma 3.1, but augmented with additional cases to show the soundness of the rules for existential and universal quantifiers, declarative terms and visa rules. In fact, the axiom schemas (AxQ1)-(AxQ8) of an extended algebra $\mathcal{A}^+_{Q_\iota K\delta}$ are identical to the schemas (Ax1)-(Ax8) of the corresponding extended algebra $\mathcal{A}^+_{K\delta}$[1], except that the wffs are expressed in a quantified modal language and the monadic predicates $[\alpha]^*$ have been reified.

Lemma 6.1 {Soundness with respect to translations} Let $\iota \in \{\mathbf{a}, \mathbf{i}, \mathbf{d}, \mathbf{c}\}$, let $Q_\iota K\delta$-MLDS be the related quantified MLDS with the associated extended algebra $\mathcal{A}^+_{Q_\iota K\delta}$. Let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations and let $FOT(\mathcal{C})$ and $FOT(\mathcal{C}')$ be their respective first-order translations. If $\mathcal{C} \vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}'$ then $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$.

Proof:
The proof is analogous to the proof of Lemma 3.1 (see Chapter 3), i.e. it is by induction on the smallest size of derivations of the form $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, \overline{m}\rangle$, where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_n = \mathcal{C}'$. In what follows, $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle$ is a proof of this smallest size.

BASE CASE:
The proof is identical to the base case of the proof of Lemma 3.1.

[1]The monadic predicate $[\alpha]^*(\lambda)$ can be equivalently written as $Holds([\alpha]^*, \lambda)$ for any wff α of the propositional modal language and possible world λ.

INDUCTIVE STEP:
The proof is identical to the inductive step of the proof of Lemma 3.1, but augmented with additional cases to deal with the rules $\mathcal{I}_{\forall E}$, $\mathcal{I}_{\forall I}$, $\mathcal{I}_{\exists E}$, $\mathcal{I}_{\exists I}$, $\mathcal{I}_{\perp_t}$, $\mathcal{I}_{T_i}$, $\mathcal{I}_{VR_\mathrm{i}}$, $\mathcal{I}_{VR_\mathrm{d}}$ and $\mathcal{I}_{VR_\mathrm{c}}$ concerning quantifiers and declarative terms, and using first-order deductions in the reified logic to prove each case[2]. Hence only the additional cases are listed here.

Case 16: $\forall$-**Elimination** ($\mathcal{I}_{\forall E}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\forall E}$. Then there exists a declarative unit of the form $\lambda : \forall v\alpha \in \mathcal{C}_{n-1}$, a declarative term of the form $\lambda : \tau \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \alpha(v/\tau)]$. Therefore, $\{Holds([\forall v\alpha]^*, \lambda), At([\tau]^*, \lambda)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{Holds([\alpha(v/\tau)]^*, \lambda)\}$. Since $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} Holds([\alpha(v/\tau)]^*, \lambda)$. This is proved applying axiom schema $(AxQ11)$, as shown in the following outline derivation:

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{A}^+\\ \vdots \\ (AxQ11)\end{array} \qquad \begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots \\ Holds([\forall v\alpha]^*, \lambda)\end{array}}{\forall t(At(t,\lambda) \rightarrow Holds([\alpha(v/t)]^*, \lambda))}}{At([\tau]^*, \lambda) \rightarrow Holds([\alpha(v/\tau)]^*, \lambda)} \qquad \begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots \\ At([\tau]^*, \lambda)\end{array}}{Holds([\alpha(v/\tau)]^*, \lambda)}
$$

Case 17: $\forall$-**Introduction** ($\mathcal{I}_{\forall I}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\forall I}$. Then there exists a declarative term of the form $\lambda : c_\alpha^{\langle \forall v, \lambda\rangle}$ and a declarative unit of the form $\lambda : \alpha(v/c_\alpha^{\langle \forall v, \lambda\rangle})$ such that $\mathcal{C}_{n-1} + [\lambda : c_\alpha^{\langle \forall v, \lambda\rangle}] \vdash_{\mathrm{MLDS}} \lambda : \alpha(v/c_\alpha^{\langle \forall v, \lambda\rangle})$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \forall v\alpha]$. Therefore $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{Holds([\forall v\alpha]^*, \lambda)\}$. Since $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that

$$\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} Holds([\forall v\alpha]^*, \lambda)$$

[2] For example, in Case 1 (concerning $\wedge$-Elimination) the relevant deduction is now:

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{A}^+\\ \vdots \\ Holds([\alpha \wedge \beta]^*, \lambda) \rightarrow (Holds([\alpha]^*, \lambda) \wedge Holds([\beta]^*, \lambda))\end{array} \qquad \begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots \\ Holds([\alpha \wedge \beta]^*, \lambda)\end{array}}{Holds([\alpha]^*, \lambda) \wedge Holds([\beta]^*, \lambda)}}{Holds([\alpha]^*, \lambda)}
$$

Let $\langle\{\mathcal{C}_{n-1} + [\lambda : c_\alpha^{\langle\forall v,\lambda\rangle}], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle$, with $\lambda : \alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle}) \in \tilde{\mathcal{C}}$, be a derivation of the smallest size of $\lambda : \alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})$ from $\mathcal{C}_{n-1} + [\lambda : c_\alpha^{\langle\forall v,\lambda\rangle}]$. Since, $0 \leq l(\langle\{\mathcal{C}_{n-1} + [\lambda : c_\alpha^{\langle\forall v,\lambda\rangle}], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle) < L$, then by the inductive hypothesis,

$$\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \cup \{At([c_\alpha^{\langle\forall v,\lambda\rangle}]^*, \lambda)\} \vdash_{FOL} FOT(\tilde{\mathcal{C}})$$

and in particular

$$\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \cup \{At([c_\alpha^{\langle\forall v,\lambda\rangle}]^*, \lambda)\} \vdash_{FOL} Holds([\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})]^*, \lambda)$$

By the Deduction Theorem of first-order logic, $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} At([c_\alpha^{\langle\forall v,\lambda\rangle}]^*, \lambda) \rightarrow Holds([\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})]^*, \lambda)$. So by axiom schema (AxQ12), $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} Holds([\forall v\alpha]^*, \lambda)$ as shown in the following outline derivation:

$$\dfrac{\begin{array}{cc} \mathcal{A}^+ & \mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \\ \vdots & \vdots \\ (\text{AxQ12}) & At([c_\alpha^{\langle\forall v,\lambda\rangle}]^*, \lambda) \rightarrow Holds([\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})]^*, \lambda) \end{array}}{Holds([\forall v\alpha]^*, \lambda)}$$

Case 18: $\exists$-**Elimination** ($\mathcal{I}_{\exists E}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\exists E}$. Then there exists a declarative unit of the form $\lambda : \exists v\alpha \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \alpha(v/c_\alpha^{\langle\exists v,\lambda\rangle})] + [\lambda : c_\alpha^{\langle\exists v,\lambda\rangle}]$. Therefore,

$$\begin{gathered} \{Holds([\exists v\alpha]^*, \lambda)\} \subseteq FOT(\mathcal{C}_{n-1}) \text{ and} \\ FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{Holds([\alpha(v/c_\alpha^{\langle\exists v,\lambda\rangle})]^*, \lambda), At([c_\alpha^{\langle\exists v,\lambda\rangle}]^*, \lambda)\} \end{gathered}$$

Since $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that

$$\begin{gathered} \mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} Holds([\alpha(v/c_\alpha^{\langle\exists v,\lambda\rangle})]^*, \lambda) \text{ and} \\ \mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} At([c_\alpha^{\langle\exists v,\lambda\rangle}]^*, \lambda) \end{gathered}$$

These are proved applying axiom schema (AxQ9), as shown in the following outline derivation:

$$\dfrac{\begin{array}{cc} \mathcal{A}^+ & FOT(\mathcal{C}_{n-1}) \\ \vdots & \vdots \\ (AxQ9) & Holds([\exists v\alpha]^*, \lambda) \end{array}}{At([c_\alpha^{\langle\exists v,\lambda\rangle}]^*, \lambda) \wedge Holds([\alpha(v/c_\alpha^{\langle\exists v,\lambda\rangle})]^*, \lambda))}$$

Case 19: $\exists$-**Introduction** ($\mathcal{I}_{\exists I}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\exists I}$. Then there exists a declarative unit of the form $\lambda:\alpha(v/\tau) \in \mathcal{C}_{n-1}$, a declarative term $\lambda : \tau \in \mathcal{C}_{n-1}$, where τ is a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \exists v\alpha]$. Therefore, $\{Holds([\alpha(v/\tau)]^*, \lambda), At([\tau]^*, \lambda)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{Holds([\exists v\alpha]^*, \lambda)\}$ Since $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} Holds([\exists v\alpha]^*, \lambda)$. This is proved applying axiom schema (AxQ10), as shown in the following outline derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots \\ At([\tau]^*,\lambda)\end{array} \qquad \begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots\\ Holds([\alpha(v/\tau)]^*,\lambda)\end{array}}{At([\tau]^*,\lambda)\wedge Holds([\alpha(v/\tau)]^*,\lambda))}}{\exists t At(t,\lambda)\wedge Holds([\alpha(v/t)]^*,\lambda))} \qquad \begin{array}{c}\mathcal{A}^+_{Q_\iota K\delta}\\ \vdots\\ (\mathrm{AxQ10})\end{array}}{Holds([\exists v\alpha]^*,\lambda)}$$

Case 20: T-**Introduction** ($\mathcal{I}_{T_i}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{T_i}$. Then there exists a declarative term Θ such that $\mathcal{C}_{n-1} + [\overline{\Theta}] \vdash_{\text{MLDS}} \lambda' : \bot$ for some label λ', and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\Theta]$. Θ can be either of the form $\lambda : \tau$ or of the form $\lambda \not: \tau$, for some ground term τ. Only the first case is considered since the argument for the second case is analogous. So, $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{At([\tau]^*, \lambda)\}$. Since $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} At([\tau]^*, \lambda)$. Let $\langle\{\mathcal{C}_{n-1} + [\overline{\Theta}], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle$, be a derivation of the smallest size of $\lambda' : \bot$ from $\mathcal{C}_{n-1} + [\overline{\Theta}]$. Since, $0 \leq l(\langle\{\mathcal{C}_{n-1} + [\overline{\Theta}], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle) < L$, then by the inductive hypothesis, $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \cup \{\neg At([\tau]^*, \lambda)\} \vdash_{FOL} FOT(\tilde{\mathcal{C}})$ and in particular $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \cup \{\neg At([\tau]^*, \lambda)\} \vdash_{FOL} \bot$. By the Deduction Theorem of first-order logic, $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} \neg At([\tau]^*, \lambda) \rightarrow \bot$. Since $\neg At([\tau]^*, \lambda) \rightarrow \bot$ is equivalent to $\neg\neg At([\tau]^*, \lambda)$, which is also equivalent to $At([\tau]^*, \lambda)$, $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} At([\tau]^*, \lambda)$.

Case 21: $\bot_t$-**Introduction** $\mathcal{I}_{\bot_t}$

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\bot_t}$. Then there exists a declarative term $\lambda : \tau$, for some ground term τ, such that $\{\lambda : \tau, \lambda \not: \tau\} \subseteq \mathcal{C}_{n-1}$ and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \alpha]$ where $\lambda : \alpha$ is any declarative unit. Therefore $\{At([\tau]^*, \lambda), \neg At([\tau]^*, \lambda)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{Holds([\alpha]^*, \lambda)\}$. Since $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to prove that $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} Holds([\alpha]^*, \lambda)$.

Since $\{At([\tau]^*, \lambda), \neg At([\tau]^*, \lambda)\} \subseteq FOT(\mathcal{C}_{n-1})$, $FOT(\mathcal{C}_{n-1}) \vdash_{FOL} \bot$. Hence, $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} Holds([\alpha]^*, \lambda)$.

Case 21: **Increasing Visa Rule** $\mathcal{I}_{VR_{\mathrm{i}}}$

This case is necessary only for the family of quantified MLDSs $Q_{\mathbf{i}}K\delta$-MLDS for increasing domains. Then $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{VR_{\mathrm{i}}}$. Then there exists a declarative term of the form $\lambda : \tau$ and an R-literal $R(\lambda, \lambda')$ such that $\{\lambda : \tau, R(\lambda, \lambda')\} \subseteq \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda' : \tau]$. Therefore, $\{At([\tau]^*, \lambda), R(\lambda, \lambda')\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{At([\tau]^*, \lambda')\}$ Since $\mathcal{A}^+_{Q_{\mathrm{i}}K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+_{Q_{\mathrm{i}}K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} At([\tau]^*, \lambda')$. This is proved applying axiom schema (AxI), as shown in the following outline derivation:

$$\dfrac{\dfrac{\begin{matrix} FOT(\mathcal{C}_{n-1}) \\ \vdots \\ At([\tau]^*, \lambda) \end{matrix} \qquad \begin{matrix} FOT(\mathcal{C}_{n-1}) \\ \vdots \\ R(\lambda, \lambda') \end{matrix}}{At([\tau]^*, \lambda) \wedge R(\lambda, \lambda')} \qquad \begin{matrix} \mathcal{A}^+_{Q_{\mathrm{i}}K\delta} \\ \vdots \\ (\mathrm{AxI}) \end{matrix}}{At([\tau]^*, \lambda')}$$

Case 22: **Decreasing Visa Rule** $\mathcal{I}_{VR_{\mathrm{d}}}$

This case is necessary only for the family of quantified MLDSs $Q_{\mathbf{d}}K\delta$-MLDS for decreasing domains. Then $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{VR_{\mathrm{d}}}$. Then there exists a declarative term of the form $\lambda' : \tau$ and an R-literal $R(\lambda, \lambda')$ such that $\{\lambda' : \tau, R(\lambda, \lambda')\} \subseteq \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \tau]$. Therefore, $\{At([\tau]^*, \lambda'), R(\lambda, \lambda')\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{At([\tau]^*, \lambda)\}$ Since $\mathcal{A}^+_{Q_{\mathrm{d}}K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+_{Q_{\mathrm{d}}K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} At([\tau]^*, \lambda)$. This is proved applying axiom schema (AxD), as shown in the following outline derivation:

$$\dfrac{\dfrac{\begin{matrix} FOT(\mathcal{C}_{n-1}) \\ \vdots \\ At([\tau]^*, \lambda') \end{matrix} \qquad \begin{matrix} FOT(\mathcal{C}_{n-1}) \\ \vdots \\ R(\lambda, \lambda') \end{matrix}}{At([\tau]^*, \lambda') \wedge R(\lambda, \lambda')} \qquad \begin{matrix} \mathcal{A}^+_{Q_{\mathrm{d}}K\delta} \\ \vdots \\ (\mathrm{AxD}) \end{matrix}}{At([\tau]^*, \lambda)}$$

Case 22: **Constant Visa Rule** $\mathcal{I}_{VR_{\mathrm{c}}}$

This case holds only for the family of quantified MLDSs $Q_{\mathbf{c}}K\delta$-MLDS for constant domains. Then $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{VR_{\mathrm{c}}}$. Then there exists a declarative term of the form $\lambda : \tau$ such that $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \tau]$. Therefore, $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{At([\tau]^*, \lambda)\}$. Since $\mathcal{A}^+_{Q_c K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+_{Q_c K\delta} \cup FOT(\mathcal{C}_{n-1}) \vdash_{FOL} At([\tau]^*, \lambda)$. This is given by the monotonicity property of $\vdash_{FOL}$ and the fact that by axiom (AxC) $\mathcal{A}^+_{Q_c K\delta} \vdash_{FOL} At([\tau]^*, \lambda)$. ⋆

Theorem 6.1 {Soundness for quantified MLDSs} Let $QK_\iota\delta$-MLDS be a quantified MLDS and let $\mathcal{C}$, $\mathcal{C}'$ be two configurations of $Q_\iota K\delta$-MLDS. If $\mathcal{C} \vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}'$ then $\mathcal{C} \models_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}'$.

Proof:
By hypothesis

$$\mathcal{C} \vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}'$$

Hence by Lemma 6.1

$$\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}) \vdash_{FOL} FOT(\mathcal{C}')$$

where $\mathcal{A}^+_{Q_\iota K\delta}$ is the extended algebra associated with $Q_\iota K\delta$-MLDS, and $FOT(\mathcal{C})$, $FOT(\mathcal{C}')$ are the respective first-order translations of $\mathcal{C}$ and $\mathcal{C}'$. By soundness of first-order logic,

$$\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}) \models_{FOL} FOT(\mathcal{C}')$$

Hence by the definition of semantic entailment $\models_{Q_\iota K\delta-\text{MLDS}}$ given in Chapter 5

$$\mathcal{C} \models_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}'$$

⋆

6.2 Completeness

This section shows that the derivability relations $\vdash_{Q_\iota K\delta-\text{MLDS}}$ for every $\iota \in \{\mathbf{a}, \mathbf{i}, \mathbf{d}, \mathbf{c}\}$ are complete with respect to the semantic entailments $\models_{Q_\iota K\delta-\text{MLDS}}$. The proof is based on the Henkin style methodology described in Chapter 3, Part I. In particular, the construction of maximal consistent configurations will be identical to that given for propositional MLDSs. This is an interesting technical feature of the whole MLDS approach – i.e. explicit information about possible worlds, relations between possible worlds and existence of objects in possible worlds facilitates a construction of maximal consistent configurations where information can be added without any particular order whenever this can be done consistently.

This differs from the standard quantified modal logics, where Henkin completeness proofs are normally more complex than those used in the proposition modal logics (see discussion later in this chapter).

Most of the properties of the derivability relation $\vdash_{\mathrm{MLDS}}$ proved in Section 3.2.1, such as reflexivity, transitivity and monotonicity, and properties related to R-literals and modal declarative units[3] also hold for any $Q_\iota K\delta$-MLDS, with all wffs now considered to be wffs of the quantified modal language $\mathcal{L}_M$. In particular, the proofs of Propositions 3.3, 3.4, 3.5, 3.6 and 3.7, and Lemma 3.2 are identical in the propositional and predicate case, and so these results do not need to be restated.

As in the propositional case, a configuration of a $Q_\iota K\delta$-MLDS is defined to be inconsistent whenever it is possible to derive a declarative unit of the form $\lambda:\bot$ for some label λ, and is defined to be consistent whenever it is not inconsistent.

In the axiomatisation of an extended algebra $\mathcal{A}^+_{Q_\iota K\delta}$ the function symbols f_α and box_α, and the constant symbols $c_\alpha^{\langle\forall v,\lambda\rangle}$ and $c_\alpha^{\langle\exists v,\lambda\rangle}$ are strictly related to the interpretation of the modal operators and of the quantifiers respectively (see e.g. (AxQ5), (AxQ7), (AxQ9) and (AxQ12)). It has been shown in Chapter 3 (see Proposition 3.3) that to preserve consistency the use of labels of the form $f_\alpha(\lambda)$ and $box_\alpha(\lambda)$ in a configuration has to respect certain constraints. This is also the case for configurations of a $Q_\iota K\delta$-MLDS and moreover, similar constraints have also to be imposed on the "skolem constants" $c_\alpha^{\langle\forall v,\lambda\rangle}$ and $c_\alpha^{\langle\exists v,\lambda\rangle}$, as shown by the following proposition.

Proposition 6.1 {Constraints on consistent configurations}

Let $\mathcal{C} = \langle \mathcal{D}, \mathcal{F}, \mathcal{T}^+, \mathcal{T}^- \rangle$ be a consistent configuration of a $Q_\iota K\delta$-MLDS. Let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$, let α be a wff of $Ext(\mathcal{L}_L, \mathcal{L}_M)$ and let v be a variable of $Ext(\mathcal{L}_L, \mathcal{L}_M)$. Then the following two statements hold:

1. If $\exists v\alpha \in \mathcal{F}(\lambda)$ then $c_\alpha^{\langle\exists v,\lambda\rangle} \notin \mathcal{T}^-(\lambda)$ and $\neg\alpha(v/c_\alpha^{\langle\exists v,\lambda\rangle}) \notin \mathcal{F}(\lambda)$
2. If $\neg\forall v\alpha \in \mathcal{F}(\lambda)$ then $c_\alpha^{\langle\forall v,\lambda\rangle} \notin \mathcal{T}^-(\lambda)$ and $\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle}) \notin \mathcal{F}(\lambda)$

Proof:

[3]These are declarative units of the form $\lambda:\Box\alpha$ or $\lambda:\Diamond\alpha$, where the main operator of the wff part is a modal operator.

The proof is by contradiction.

1. Suppose that $\exists v\alpha \in \mathcal{F}(\lambda)$ and that $c_\alpha^{\langle \exists v,\lambda\rangle} \in \mathcal{T}^-(\lambda)$. Then $\mathcal{C} \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda:\bot$, as shown in the following derivation. Hence $\mathcal{C}$ is inconsistent which is in contradiction with the original hypothesis.

$$\begin{array}{ll}
\mathcal{C}\langle \lambda:\exists v\alpha, \lambda \not< c_\alpha^{\langle \exists v,\lambda\rangle}\rangle & (\exists\text{-E}) \\ \hline
\mathcal{C}'\langle \lambda:\ c_\alpha^{\langle \exists v,\lambda\rangle}, \lambda:\alpha(v/c_\alpha^{\langle \exists v,\lambda\rangle}), \lambda \not< c_\alpha^{\langle \exists v,\lambda\rangle}\rangle & (\bot_t\text{-I}) \\ \hline
\mathcal{C}_1'\langle \lambda:\bot\rangle &
\end{array}$$

Now suppose that $\exists v\alpha \in \mathcal{F}(\lambda)$ and that $\neg\alpha(v/c_\alpha^{\langle \exists v,\lambda\rangle}) \in \mathcal{F}(\lambda)$. Then $\mathcal{C} \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda : \bot$ as shown in the following derivation. Hence $\mathcal{C}$ is an inconsistent configuration which is in contradiction with the original hypothesis.

$$\begin{array}{ll}
\mathcal{C}\langle \lambda:\exists v\alpha, \lambda:\neg\alpha(v/c_\alpha^{\langle \exists v,\lambda\rangle})\rangle & (\exists\text{-E}) \\ \hline
\mathcal{C}'\langle \lambda:c_\alpha^{\langle \exists v,\lambda\rangle}, \lambda:\alpha(v/c_\alpha^{\langle \exists v,\lambda\rangle}), \lambda:\neg\alpha(v/c_\alpha^{\langle \exists v,\lambda\rangle})\rangle & (\wedge\text{-I}) \\ \hline
\mathcal{C}_1'\langle \lambda:\bot\rangle &
\end{array}$$

2. Suppose that $\neg\forall v\alpha \in \mathcal{F}(\lambda)$ and that $c_\alpha^{\langle \forall v,\lambda\rangle} \in \mathcal{T}^-(\lambda)$. Then $\mathcal{C} \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda:\bot$, as shown in the following derivation. Hence $\mathcal{C}$ is inconsistent which is in contradiction with the original hypothesis.

$$\begin{array}{ll}
\mathcal{C}\langle \lambda:\neg\forall v\alpha, \lambda \not< c_\alpha^{\langle \forall v,\lambda\rangle}\rangle & \\ \hline
\quad \mathcal{C}\langle \lambda \not< c_\alpha^{\langle \forall v,\lambda\rangle}, [\lambda:c_\alpha^{\langle \forall v,\lambda\rangle}]\rangle & (\text{new assumption}) \\ \hline
\quad \tilde{\mathcal{C}}_1\langle \lambda:\alpha(v/c_\alpha^{\langle \forall v,\lambda\rangle})\rangle & (\bot_t\text{-I}) \\ \hline
\mathcal{C}'\langle \lambda:\forall v\alpha\rangle & (\forall\text{-I}) \\ \hline
\mathcal{C}_1'\langle \lambda:\bot\rangle & (\wedge\text{-I})
\end{array}$$

Now suppose that $\neg\forall v\alpha \in \mathcal{F}(\lambda)$ and that $\alpha(v/c_\alpha^{\langle \forall v,\lambda\rangle}) \in \mathcal{F}(\lambda)$. Then $\mathcal{C} \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda : \bot$, as shown in the following derivation. Hence $\mathcal{C}$ is inconsistent which is in contradiction with the original hypothesis.

$$\begin{array}{ll}
\mathcal{C}\langle \lambda:\neg\forall v\alpha, \lambda:\alpha(v/c_\alpha^{\langle \forall v,\lambda\rangle})\rangle & \\ \hline
\quad \mathcal{C} + [\lambda:c_\alpha^{\langle \forall v,\lambda\rangle}] & (\text{new assumption}) \\ \hline
\quad \tilde{\mathcal{C}}\langle \lambda:\alpha(v/c_\alpha^{\langle \forall v,\lambda\rangle})\rangle & (\mathcal{C}\text{-R}) \\ \hline
\mathcal{C}'\langle \lambda:\neg\forall v\alpha, \lambda:\forall v\alpha\rangle & (\forall\text{-I}) \\ \hline
\mathcal{C}_1'\langle \lambda:\bot\rangle & (\wedge\text{-I})
\end{array}$$

⋆

The following theorem characterises the relation of derivability between two configurations $\mathcal{C}$ and $\mathcal{C}'$ in terms of relation of derivability between $\mathcal{C}$ and each element contained in $\mathcal{C}'$. This result simply extends Theorem 3.2 (Chapter 3) to the declarative terms contained in $\mathcal{C}'$.

Theorem 6.2 {Characterisation of derivability} Let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations of an arbitrary $Q_\iota K\delta$-MLDS such that the configuration $\mathcal{C}' - \mathcal{C}$ is finite. $\mathcal{C} \vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}'$ if and only if for each $\pi \in \mathcal{C}' - \mathcal{C}$, $\mathcal{C} \vdash_{Q_\iota K\delta-\text{MLDS}} \pi$, where π is either a declarative unit or an R–literal or a declarative term.

Proof:
The proof is identical to the proof of Theorem 3.2 where for any i, $1 \leq i \leq n$, π_i may be a declarative unit, R-literal or a declarative term[4]. $\star$

The following two properties of a consistent configuration also extend corresponding properties shown in Chapter 3 (i.e. Proposition 3.7 and Theorem 3.3) to the case of declarative terms.

Proposition 6.2 {Consistency of sub-configurations} Let $\mathcal{C}$ be a consistent configuration of a $Q_\iota K\delta$-MLDS and let π be either a declarative unit or an R–literal or a declarative term. If $\mathcal{C} + [\pi]$ is a consistent configuration then for any configuration $\mathcal{C}'$, $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{C}' + [\pi]$ is consistent too.

Proof:
The proof is identical to the proof of Proposition 3.7 (see Chapter 3). $\star$

Theorem 6.3 {Finiteness} Let $\mathcal{C}$ be a configuration of a $Q_\iota K\delta$-MLDS. Let π be either a declarative unit or an R–literal or a declarative term. If $\mathcal{C} \vdash_{Q_\iota K\delta-\text{MLDS}} \pi$ then there exists a configuration $\mathcal{C}'$ such that $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{C}' \vdash_{Q_\iota K\delta-\text{MLDS}} \pi$ and $\mathcal{C}'$ is finite.

Proof:
The proof is analogous to the proof of Theorem 3.3 (see Chapter 3), i.e. it is by induction on the smallest size of derivations of the form $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, \overline{m}\rangle$, where $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_n$ is the configuration such that (by hypothesis) $\mathcal{C} \vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}_n$ and $\pi \in \mathcal{C}_n$. In what follows $\langle\{\mathcal{C}_0, \ldots, \mathcal{C}_n\}, m\rangle$ is a proof of the smallest size.

[4] Note that Lemma 3.2 is trivially extended to any inference rule $\mathcal{I} \in \mathcal{R}_\iota$.

BASE CASE:
The proof is identical to the proof of the base case of Theorem 3.3 augmented with the following argument to show that it also holds when π is a declarative term. There are two cases to consider. (i) If π is a declarative term of the form $\lambda:\tau$, then let $\mathcal{C}' = \langle \mathcal{D}', \mathcal{F}', \mathcal{T}'^{+}, \mathcal{T}'^{-}\rangle$, where $\mathcal{D}' = \{\}$ and for each ground term λ' of $Func(\mathcal{L}_L, \mathcal{L}_M)$ $\mathcal{F}'(\lambda') = \{\}$, $\mathcal{T}'^{-}(\lambda') = \{\}$, and if $\lambda' = \lambda$, $\mathcal{T}'^{+}(\lambda) = \{\tau\}$ otherwise $\mathcal{T}'^{+}(\lambda') = \{\}$. (ii) If π is a declarative term of the form $\lambda \not: \tau$, then let $\mathcal{C}' = \langle \mathcal{D}', \mathcal{F}', \mathcal{T}'^{+}, \mathcal{T}'^{-}\rangle$, where $\mathcal{D}' = \{\}$ and for each ground term λ' of $Func(\mathcal{L}_L, \mathcal{L}_M)$ $\mathcal{F}'(\lambda') = \{\}$, $\mathcal{T}'^{+}(\lambda') = \{\}$, and if $\lambda' = \lambda$, $\mathcal{T}'^{-}(\lambda) = \{\tau\}$ otherwise $\mathcal{T}'^{-}(\lambda') = \{\}$. In both cases (i) and (ii), $\mathcal{C}' \subseteq \mathcal{C}_0$, $\mathcal{C}'$ is finite and $\mathcal{C}' \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \pi$.

INDUCTIVE STEP:
The proof is identical to the inductive step of Theorem 3.3, but augmented with additional cases to deal with the rules $\mathcal{I}_{\forall E}$, $\mathcal{I}_{\forall I}$, $\mathcal{I}_{\exists E}$, $\mathcal{I}_{\exists I}$, $\mathcal{I}_{\perp_t}$, $\mathcal{I}_{T_\mathrm{i}}$, $\mathcal{I}_{VR_\mathrm{i}}$, $\mathcal{I}_{VR_\mathrm{d}}$ and $\mathcal{I}_{VR_\mathrm{c}}$ concerning quantifiers and declarative terms. Hence only the additional cases are listed here.

Case 16: $\forall$-**Elimination** ($\mathcal{I}_{\forall E}$)

In this case $m(0) = \mathcal{I}_{\forall - E}$. Then there exist a declarative term of the form $\lambda:\tau$ and two declarative units of the form $\lambda:\forall v\alpha$ and $\lambda:\alpha(v/\tau)$ such that $\lambda:\tau \in \mathcal{C}_0$, $\lambda:\forall v\alpha \in \mathcal{C}_0$ and $\mathcal{C}_1 = \mathcal{C}_0 + [\lambda:\alpha(v/\tau)]$. Then $[\,]_0 = [\lambda:\alpha(v/\tau)]$. By the inductive hypothesis, $\mathcal{C}_1' \subseteq \mathcal{C}_1$ and $\mathcal{C}_1'$ is finite. By assumption $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$, then $[\lambda:\alpha(v/\tau)] \in \mathcal{C}_1'$. Since it is not necessarily the case that $\lambda:\forall v\alpha \in \mathcal{C}_1'$ and that $\lambda:\tau \in \mathcal{C}_1'$, let $\mathcal{C}'$ be the configuration $(\mathcal{C}_1' - [\lambda:\alpha(v/\tau)]) + [\lambda:\forall v\alpha] + [\lambda:\tau]$. Then $\mathcal{C}'$ is a finite configuration and $\langle\{\mathcal{C}', \mathcal{C}' + [\lambda:\alpha(v/\tau)], \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{\forall E}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence $\mathcal{C}' \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \mathcal{C}_1'$.

Case 17: $\forall$-**Introduction** ($\mathcal{I}_{\forall I}$)

In this case $m(0) = \mathcal{I}_{\forall I}$. Then there exist a declarative unit of the form $\lambda:c_\alpha^{\langle\forall v,\lambda\rangle}$ and two declarative terms of the form $\lambda:\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})$ and $\lambda:\forall v\alpha$ such that $\mathcal{C}_0 + [\lambda:c_\alpha^{\langle\forall v,\lambda\rangle}] \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda:\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})$, and $\mathcal{C}_1 = \mathcal{C}_0 + [\lambda:\forall v\alpha]$, where $[\,]_0 = \{\lambda:\forall v\alpha\}$. By the inductive hypothesis $\mathcal{C}_1' \subseteq \mathcal{C}_1$ and $\mathcal{C}_1'$ is finite. Since by assumption $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$, then $[\lambda:\forall v\alpha] \in \mathcal{C}_1'$. Let $\langle\{\mathcal{C}_0 + [\lambda:c_\alpha^{\langle\forall v,\lambda\rangle}], \ldots, \tilde{\mathcal{C}}_n\}, \tilde{m}\rangle$ (with $\lambda:\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle}) \in \tilde{\mathcal{C}}_n$) be a proof of the smallest size of $\lambda:\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})$ from $\mathcal{C}_0 + [\lambda:c_\alpha^{\langle\forall v,\lambda\rangle}]$. Since $0 \le l(\langle\{\mathcal{C}_0 + [\lambda:c_\alpha^{\langle\forall v,\lambda\rangle}], \ldots, \tilde{\mathcal{C}}_n\}, \tilde{m}\rangle) < L$, by the inductive hypothesis, there exists a finite configuration $\mathcal{C}_0^* \subseteq \mathcal{C}_0 + [\lambda:c_\alpha^{\langle\forall v,\lambda\rangle}]$ such that $\mathcal{C}_0^* \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda:\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})$. Let $\mathcal{C}'$ be the configuration $(\mathcal{C}_1' - [\lambda:\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})]) \cup (\mathcal{C}_0^* - [\lambda:c_\alpha^{\langle\forall v,\lambda\rangle}])$. Then $\mathcal{C}'$ is a finite configuration and $\mathcal{C}' \subseteq \mathcal{C}_0$. Since $\mathcal{C}_0^* \subseteq \mathcal{C}' + [\lambda:c_\alpha^{\langle\forall v,\lambda\rangle}]$, by monotonicity $\mathcal{C}' + [\lambda:c_\alpha^{\langle\forall v,\lambda\rangle}] \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda:\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})$. So $\mathcal{C}'/\mathcal{C}' + [\lambda:\forall v\alpha] \in \mathcal{I}_{\forall I}$. Therefore $\langle\{\mathcal{C}', \mathcal{C}' + [\lambda:\forall v\alpha], \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{\forall I}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence

$\mathcal{C}' \vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}_1'$.

Case 18: $\exists$-**Elimination** ($\mathcal{I}_{\exists E}$)

In this case $m(0) = \mathcal{I}_{\exists-E}$. Then there exist a declarative term of the form $\lambda\!:\!c_\alpha^{\langle\exists v,\lambda\rangle}$ and two declarative units of the form $\lambda\!:\!\exists v\alpha$ and $\lambda\!:\!\alpha(v/c_\alpha^{\langle\exists v,\lambda\rangle})$ such that $\lambda\!:\!\exists v\alpha \in \mathcal{C}_0$, $[\;]^0 = \{\lambda\!:\!c_\alpha^{\langle\exists v,\lambda\rangle}, \lambda\!:\!\alpha(v/c_\alpha^{\langle\exists v,\lambda\rangle})\}$ and $\mathcal{C}_1 = \mathcal{C}_0 + [\;]_0$. By the inductive hypothesis and the assumption that $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$, then $[\;]_0 \subseteq \mathcal{C}_1'$. Then let $\mathcal{C}'$ be the configuration $(\mathcal{C}_1' - [\;]_0) + [\lambda\!:\!\exists v\alpha]$. $\mathcal{C}'$ is a finite configuration such that $\mathcal{C}' \subseteq \mathcal{C}_0$ and $\langle\{\mathcal{C}', \mathcal{C}' + [\;]_0, \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{\exists E}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence $\mathcal{C}' \vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}_1'$.

Case 19: $\exists$-**Introduction** ($\mathcal{I}_{\exists I}$)

In this case $m(0) = \mathcal{I}_{\exists-I}$. Then there exist a declarative term of the form $\lambda : \tau$ and two declarative units of the form $\lambda : \alpha(v/\tau)$ and $\lambda : \exists v\alpha$ such that $[\lambda : \tau, \lambda : \alpha(v/\tau)] \subseteq \mathcal{C}_0$, $[\;]_0 = \{\lambda : \exists v\alpha\}$, and $\mathcal{C}_1 = \mathcal{C}_0 + [\;]_0$. By the inductive hypothesis and the assumption that $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$, $[\;]_0 \subseteq \mathcal{C}_1'$. Then let $\mathcal{C}'$ be the configuration $(\mathcal{C}_1' - [\;]_0) + [\lambda\!:\!\tau] + [\lambda\!:\!\alpha(v/\tau)]$. $\mathcal{C}'$ is a finite configuration such that $\mathcal{C}' \subseteq \mathcal{C}_0$ and $\langle\{\mathcal{C}', \mathcal{C}' + [\;]_0, \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{\exists I}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence $\mathcal{C}' \vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}_1'$.

Case 20: $\perp_t$-**Introduction** ($\mathcal{I}_{\perp_t-I}$)

In this case $m(0) = \mathcal{I}_{\perp_t-I}$. Then there exist declarative term Θ and a declarative unit of the form $\lambda\!:\!\alpha$ such that $\Theta \in \mathcal{C}_0$, $\overline{\Theta} \in \mathcal{C}_0$, $[\;]_0 = [\lambda\!:\!\alpha]$ and $\mathcal{C}_1 = \mathcal{C}_0 + [\;]_0$. By the inductive hypothesis and the assumption that $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$, $[\;]_0 \subseteq \mathcal{C}_1'$. Then let $\mathcal{C}'$ be the configuration $(\mathcal{C}_1' - [\;]_0) + [\Theta, \overline{\Theta}]$. Then $\mathcal{C}'$ is a finite configuration, $\mathcal{C}' \subseteq \mathcal{C}_0$ and $\langle\{\mathcal{C}', \mathcal{C}' + [\;]_0, \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{\perp_t I}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence $\mathcal{C}' \vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}_1'$.

Case 21: T-**Introduction** ($\mathcal{I}_{T-I}$)

In this case $m(0) = \mathcal{I}_{T-I}$. Then there exist a declarative term Θ and a declarative unit of the form $\lambda : \perp$ such that $\mathcal{C}_0 + [\overline{\Theta}] \vdash_{Q_\iota K\delta-\text{MLDS}} \lambda : \perp$, $[\;]_0 = [\Theta]$ and $\mathcal{C}_1 = \mathcal{C}_0 + [\;]_0$. By the inductive hypothesis and the assumption $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$, $[\;]_0 \subseteq \mathcal{C}_1'$. Let $\langle\{\mathcal{C}_0 + [\overline{\Theta}], \ldots, \tilde{\mathcal{C}}_n\}, \tilde{m}\rangle$ (with $\lambda : \perp \in \tilde{\mathcal{C}}_n$) be a proof of the smallest size of $\lambda : \perp$ from $\mathcal{C}_0 + [\overline{\Theta}]$. Since, $0 \leq l(\langle\{\mathcal{C}_0 + [\overline{\Theta}], \ldots, \tilde{\mathcal{C}}_n\}, \tilde{m}\rangle) < L$, by the inductive hypothesis there exists a finite configuration $\mathcal{C}_0^* \subseteq \mathcal{C}_0 + [\overline{\Theta}]$ such that $\mathcal{C}_0^* \vdash_{Q_\iota K\delta-\text{MLDS}} \lambda\!:\!\perp$. Let $\mathcal{C}'$ be the configuration $(\mathcal{C}_1' - [\;]_0) \cup (\mathcal{C}_0^* - [\overline{\Theta}])$. Then $\mathcal{C}'$ is a finite configuration and $\mathcal{C}' \subseteq \mathcal{C}_0$. By the monotonicity property of $\vdash_{Q_\iota K\delta-\text{MLDS}}$,

$\mathcal{C}' + [\overline{\Theta}] \vdash_{Q_\iota K\delta-\text{MLDS}} \lambda : \bot$. Therefore $\langle\{\mathcal{C}', \mathcal{C}' + [\ ]_0, \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{\Theta I}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence $\mathcal{C}' \vdash_S \mathcal{C}_1'$.

Case 22: **Increasing Visa Rule** ($\mathcal{I}_{VR_\mathrm{i}}$)

This case is necessary only for the family of quantified MLDSs $Q_\mathbf{i}K\delta$-MLDS for increasing domains. $m(0) = \mathcal{I}_{VR_\mathrm{i}}$, then there exist a declarative term of the form $\lambda:\tau$ and an R-literal $R(\lambda, \lambda')$ such that $\lambda:\tau \in \mathcal{C}_0$, $R(\lambda, \lambda') \in \mathcal{C}_0$, $[\ ]_0 = [\lambda':\tau]$ and $\mathcal{C}_1 = \mathcal{C}_0 + [\ ]_0$. By the inductive hypothesis and the assumption that $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$, $[\ ]_0 \subseteq \mathcal{C}_1'$. Then let $\mathcal{C}'$ be the configuration $(\mathcal{C}_1' - [\ ]_0) + [\lambda:\tau] + [R(\lambda, \lambda')]$. Then $\mathcal{C}'$ is a finite configuration, $\mathcal{C}' \subseteq \mathcal{C}_0$ and $\langle\{\mathcal{C}', \mathcal{C}' + [\ ]_0, \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{VR_\mathrm{i}}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence $\mathcal{C}' \vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}_1'$.

Case 23: **Decreasing Visa Rule** ($\mathcal{I}_{VR_\mathrm{d}}$)

This case is necessary only for the family of quantified MLDSs $Q_\mathbf{d}K\delta$-MLDS for decreasing domains. $m(0) = \mathcal{I}_{VR_\mathrm{d}}$, then there exist a declarative term of the form $\lambda:\tau$ and an R-literal $R(\lambda', \lambda)$ such that $\lambda:\tau \in \mathcal{C}_0$, $R(\lambda', \lambda) \in \mathcal{C}_0$, $[\ ]_0 = [\lambda:\tau]$ and $\mathcal{C}_1 = \mathcal{C}_0 + [\ ]_0$. By the inductive hypothesis and the assumption that $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$, $[\ ]_0 \subseteq \mathcal{C}_1'$. Then let $\mathcal{C}'$ be the configuration $(\mathcal{C}_1' - [\ ]_0) + [\lambda:\tau] + [R(\lambda', \lambda)]$. Then $\mathcal{C}'$ is a finite configuration, $\mathcal{C}' \subseteq \mathcal{C}_0$ and $\langle\{\mathcal{C}', \mathcal{C}' + [\ ]_0, \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{VR_\mathrm{d}}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence $\mathcal{C}' \vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}_1'$.

Case 24: **Constant Visa Rule** ($\mathcal{I}_{VR_\mathrm{c}}$)

This case is necessary only for the family of quantified MLDSs $Q_\mathbf{c}K\delta$-MLDS for constant domains. $m(0) = \mathcal{I}_{VR_\mathrm{c}}$, Then there exist a declarative term of the form $\lambda:\tau$ such that $[\ ]_0 = [\lambda:\tau]$ and $\mathcal{C}_1 = \mathcal{C}_0 + [\ ]_0$. By the inductive hypothesis and the assumption that $\mathcal{C}_1' \not\subseteq \mathcal{C}_0$, $[\ ]_0 \subseteq \mathcal{C}_1'$. Then let $\mathcal{C}'$ be the configuration $(\mathcal{C}_1' - [\ ]_0)$. Then $\mathcal{C}'$ is a finite configuration, $\mathcal{C}' \subseteq \mathcal{C}_0$ and $\langle\{\mathcal{C}', \mathcal{C}' + [\ ]_0, \mathcal{C}_1'\}, \overline{m}\rangle$ is a proof, where $\overline{m}(0) = \mathcal{I}_{VR_\mathrm{d}}$ and $\overline{m}(1) = \mathcal{I}_{C-R}$. Hence $\mathcal{C}' \vdash_{Q_\mathbf{c}K\delta-\text{MLDS}} \mathcal{C}_1'$. ⋆

As a consequence of the above theorem, the *compactness* property for any quantified MLDS $Q_\iota K\delta$-MLDS can be stated.

Corollary 6.1 {Compactness for quantified MLDS} Let $\mathcal{C}$ be a configuration of an arbitrary $Q_\iota K\delta$-MLDS. If for any finite configuration $\mathcal{C}'$, $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{C}'$ is consistent then $\mathcal{C}$ is consistent.

Proof:
The proof is identical to the proof of Corollary 3.1 given in Chapter 3. ⋆

6.2.1 Completeness Proof

Some useful preliminary properties of quantified MLDSs have been proved in the previous section. In this section the completeness theorem is proved for any $Q_\iota K\delta$-MLDS. The methodology is analogous to that in Section 3.2.2 of Chapter 3. The general statement of the theorem is given below, followed by some preliminary propositions and lemmas needed in the main proof, which is given at the end of this section

Theorem 6.4 {Completeness for quantified MLDSs} Let $Q_\iota K\delta$-MLDS be an arbitrary propositional MLDS. Let $\mathcal{C}$ and $\mathcal{C}'$ be two configurations of $Q_\iota K\delta$-MLDS such that the configuration difference $\mathcal{C}' - \mathcal{C}$ is finite.

$$\text{If } \mathcal{C} \models_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}' \text{ then } \mathcal{C} \vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}'$$

The Henkin methodology described in Chapter 3 for the completeness of propositional MLDSs is also applied here. To recap the main steps, the proof is by contradiction and, in particular, relies on the fact that $\mathcal{C} \nvdash_{\text{MLDS}} \mathcal{C}'$ implies that there exists a $\pi \in \mathcal{C}' - \mathcal{C}$ (where π is either a declarative unit or an R–literal or a declarative term) such that $\mathcal{C} \nvdash_{Q_\iota K\delta-\text{MLDS}} \pi$. (This has already been proved by Theorem 6.2.) So, $\mathcal{C} + [\neg\pi]$ is a consistent configuration (see Proposition 6.15 given below). This implies by a *Model Existence Lemma* that the configuration $\mathcal{C}+[\neg\pi]$ is satisfiable. Therefore, there exists a semantic structure $\mathcal{M}_{Q_\iota K\delta}$ of $Q_\iota K\delta$-MLDS which satisfies $\mathcal{C}$ and that also satisfies $\neg\pi$. Then, as shown below, this semantic structure $\mathcal{M}_{Q_\iota K\delta}$ does not satisfy π. Since $\pi \in \mathcal{C}'$, by the definition of satisfiability of a configuration, $\mathcal{M}_{Q_\iota K\delta}$ does not satisfy $\mathcal{C}'$ too. Hence, $\mathcal{A}^+_{Q_\iota K\delta} \cup FOT(\mathcal{C}) \not\models_{FOL} FOT(\mathcal{C}')$, which implies by definition of semantic entailment that $\mathcal{C} \not\models_{Q_\iota K\delta} \mathcal{C}'$. As in the propositional case, the proof of the Model Existence Lemma is based on two main steps: (i) prove that for any consistent configuration $\mathcal{C}$ it is possible to construct a maximal consistent configuration $\mathcal{C}_{mcc}$ which includes it, and (ii) prove that it is possible to construct a semantic structure $\mathcal{M}_{Q_\iota K\delta}(\mathcal{C}_{mcc})$ which satisfies $\mathcal{C}_{mcc}$ and therefore which satisfies $\mathcal{C}$.

Definition 6.1 {Maximal Consistent Configuration} Let $Q_\iota K\delta$-MLDS be a quantified MLDS. $\mathcal{C}_{\text{mcc}}$ is a *maximal consistent configuration relative to $Q_\iota K\delta$-MLDS*, if it is a consistent configuration and if for any $\pi \notin \mathcal{C}_{\text{mcc}}$ (where π is either a declarative unit or an R–literal or a declarative term), the configuration $\mathcal{C}_{\text{mcc}} + [\pi]$ is not consistent. □

To construct a maximal consistent configuration for a quantified MLDS it is assumed that the set of all declarative units, R-literals and declarative terms of $Q_\iota K\delta$-MLDS is ordered so that it is possible to speak about the 1st, 2nd, 3th, ..., nth, ..., element π of $Q_\iota K\delta$-MLDS[5]. Then, given a configuration $\mathcal{C}$ of a $Q_\iota K\delta$-MLDS, the construction of a maximal consistent configuration $\mathcal{C}_{mcc}$ relative to $Q_\iota K\delta$-MLDS which contains $\mathcal{C}$ is identical to that described in Definition 3.6, but with the sequence $\pi_1, \pi_2, \ldots, \pi_n, \ldots$ denoting the ordered sequence of all declarative units, R-literals and declarative terms of $Q_\iota K\delta$-MLDS[6].

The properties proved in Propositions 3.9-3.20 for maximal consistent configurations relative to propositional MLDSs also hold for maximal consistent configurations relative to quantified MLDSs, with the wffs of their declarative units referring in this case to a quantified modal language $\mathcal{L}_M$. These properties are listed in the following general proposition.

Proposition 6.3 $\{$Basic properties of $\mathcal{C}_{mcc}\}$ Let $\mathcal{C}_{mcc}$ be a maximal consistent configuration of $Q_\iota K\delta$-MLDS. Then for any wffs α and β of $Ext(\mathcal{L}_L, \mathcal{L}_M)$, and any labels λ and λ' of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and any R-literal Δ of $Func(\mathcal{L}_L, \mathcal{L}_M)$, the following statments hold:

1. $\lambda : \alpha$ and $\lambda : \neg\alpha$ are not both in $\mathcal{C}_{mcc}$.
2. Δ and $\neg\Delta$ are not both in $\mathcal{C}_{mcc}$.
3. Either $\lambda : \alpha \in \mathcal{C}_{mcc}$ or $\lambda : \neg\alpha \in \mathcal{C}_{mcc}$.
4. Either $\Delta \in \mathcal{C}_{mcc}$ or $\neg\Delta \in \mathcal{C}_{mcc}$.
5. $\lambda : \alpha \wedge \beta \in \mathcal{C}_{mcc}$ if and only if $\lambda : \alpha \in \mathcal{C}_{mcc}$ and $\lambda : \beta \in \mathcal{C}_{mcc}$.
6. $\lambda : \alpha \vee \beta \in \mathcal{C}_{mcc}$ if and only if $\lambda : \alpha \in \mathcal{C}_{mcc}$ or $\lambda : \beta \in \mathcal{C}_{mcc}$.
7. $\lambda : \alpha \rightarrow \beta \in \mathcal{C}_{mcc}$ if and only if $\lambda : \alpha \in \mathcal{C}_{mcc}$ implies $\lambda : \beta \in \mathcal{C}_{mcc}$.
8. If $\lambda : \Diamond\alpha \in \mathcal{C}_{mcc}$ then $f_\alpha(\lambda) : \alpha \in \mathcal{C}_{mcc}$ and $R(\lambda, f_\alpha(\lambda)) \in \mathcal{C}_{mcc}$.
9. If $R(\lambda, \lambda') \in \mathcal{C}_{mcc}$ and $\lambda' : \alpha \in \mathcal{C}_{mcc}$ then $\lambda : \Diamond\alpha \in \mathcal{C}_{mcc}$.
10. If $\lambda : \Box\alpha \in \mathcal{C}_{mcc}$ and $R(\lambda, \lambda') \in \mathcal{C}_{mcc}$ then $\lambda' : \alpha \in \mathcal{C}_{mcc}$.

[5] Once again, this assumption is justified because both the quantified modal language $Ext(\mathcal{L}_L, \mathcal{L}_M)$ and the semi-extended labelling language $Func(\mathcal{L}_L, \mathcal{L}_M)$ are composed of countably infinite sets of symbols.

[6] The proof that the configuration resulting from this construction is maximal and consistent is identical to that given in Proposition 3.8.

11. If $\neg R(\lambda, box_\alpha(\lambda)) \in \mathcal{C}_{mcc}$ or $box_\alpha(\lambda) : \alpha \in \mathcal{C}_{mcc}$ then $\lambda : \Box\alpha \in \mathcal{C}_{mcc}$.

Proof:
The proofs of the above statements are identical to the proofs of Propositions 3.9–3.20 respectively. $\star$

Various properties of a maximal consistent configuration related to the $Q_\iota K\delta$-MLDS for a particular value of δ are also proved, in the same way as for propositional MLDSs. These properties are stated in the following proposition.

Proposition 6.4 {Properties of $\mathcal{C}_{mcc}$} Let $\delta \subseteq \{(\mathbf{T}), (\mathbf{4}), (\mathbf{D}), (\mathbf{B}), (\mathbf{5}), (\mathbf{Irr})\}$ and let $Q_\iota K\delta$-MLDS be the associated quantified MLDS. Let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $Q_\iota K\delta$-MLDS. Then the following statments hold:

1. If $(\mathbf{T}) \in \delta$ then for any ground term $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $R(\lambda, \lambda) \in \mathcal{C}_{mcc}$.
2. If $(\mathbf{Irr}) \in \delta$ then for any ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$, $\neg R(\lambda, \lambda) \in \mathcal{C}_{mcc}$.
3. If $(\mathbf{4}) \in \delta$ then for any ground terms λ, λ' and λ'' of $Func(\mathcal{L}_L, \mathcal{L}_M)$, if $R(\lambda, \lambda') \in \mathcal{C}_{mcc}$ and $R(\lambda', \lambda'') \in \mathcal{C}_{mcc}$ then $R(\lambda, \lambda'') \in \mathcal{C}_{mcc}$.
4. If $(\mathbf{B}) \in \delta$ then for any ground terms λ and λ' of $Func(\mathcal{L}_L, \mathcal{L}_M)$, if $R(\lambda, \lambda') \in \mathcal{C}_{mcc}$ then $R(\lambda', \lambda) \in \mathcal{C}_{mcc}$.
5. If $(\mathbf{5}) \in \delta$ then for any ground terms λ, λ' and λ'' of $Func(\mathcal{L}_L, \mathcal{L}_M)$, if $R(\lambda, \lambda') \in \mathcal{C}_{mcc}$ and $R(\lambda, \lambda'') \in \mathcal{C}_{mcc}$ then $R(\lambda', \lambda'') \in \mathcal{C}_{mcc}$.
6. If $(\mathbf{D}) \in \delta$ then for any ground term $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_M)$, $R(\lambda, succ(\lambda)) \in \mathcal{C}_{mcc}$.

Proof:
The proofs of the above statements are identical to the proofs of Propositions 3.21–3.26 respectively. $\star$

In addition to the above properties inherited from the propositional case, properties of maximal consistent configurations relative to declarative terms and quantified declarative units are given below.

Proposition 6.5 {Consistency on declarative terms} Let $Q_\iota K\delta$-MLDS be a quantified MLDS and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $Q_\iota K\delta$-MLDS. Then for any declarative term Θ, Θ and $\overline{\Theta}$ are not both in $\mathcal{C}_{mcc}$.

Proof:
The proof follows directly from the definition of the $\perp_t$-Introduction rule. $\star$

Proposition 6.6 {Maximality on declarative terms} Let $Q_\iota K\delta$-MDLS be a quantified MLDS and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $Q_\iota K\delta$-MLDS. Then for any declarative term Θ, either $\Theta \in \mathcal{C}_{mcc}$ or $\overline{\Theta} \in \mathcal{C}_{mcc}$.

Proof:
The proof is by contradiction. Suppose that $\Theta \notin \mathcal{C}_{mcc}$ and that $\overline{\Theta} \notin \mathcal{C}_{mcc}$. By definition of a maximal consistent configuration, $\mathcal{C}_{mcc} + [\Theta]$ and $\mathcal{C}_{mcc} + [\overline{\Theta}]$ are inconsistent configurations. Then $\mathcal{C}_{mcc} + [\Theta] \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda' : \perp$ and $\mathcal{C}_{mcc} + [\overline{\Theta}] \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda'' : \perp$, for some ground terms λ and λ'' of $Func(\mathcal{L}_L, \mathcal{L}_M)$. By Theorem 6.3 there exist two finite configurations $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{C}'' \subseteq \mathcal{C}$ such that $\mathcal{C}' + [\Theta] \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda' : \perp$ and $\mathcal{C}'' + [\overline{\Theta}] \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda'' : \perp$. Let $\overline{\mathcal{C}} = \mathcal{C}' \cup \mathcal{C}''$. By monotonicity $\overline{\mathcal{C}} + [\overline{\Theta}] \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda'' : \perp$ which implies that $\overline{\mathcal{C}} \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \overline{\mathcal{C}} + [\Theta]$. Since, also by monotonicity, $\overline{\mathcal{C}} + [\Theta] \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda' : \perp$, then by transitivity $\overline{\mathcal{C}} \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda' : \perp$, which means that $\overline{\mathcal{C}}$ is inconsistent. Hence $\mathcal{C}_{mcc}$ is inconsistent which is in contradiction with the original hypothesis. $\star$

Proposition 6.7 {Property for $\exists$-formulae (i)} Let $Q_\iota K\delta$-MLDS be a quantified MLDS, let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $Q_\iota K\delta$-MLDS, let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$, let α be a wff of $Ext(\mathcal{L}_L, \mathcal{L}_M)$ and let τ be a ground term of $Ext(\mathcal{L}_L, \mathcal{L}_M)$. If $\lambda : \alpha(v/\tau) \in \mathcal{C}_{mcc}$ and $\lambda : \tau \in \mathcal{C}_{mcc}$ then $\lambda : \exists v\alpha \in \mathcal{C}_{mcc}$.

Proof:
Proof is by contradiction. Suppose that $\lambda : \exists v\alpha \notin \mathcal{C}_{mcc}$. By statement (3) of Proposition 6.3 $\lambda : \neg\exists v\alpha \in \mathcal{C}_{mcc}$. Then $\mathcal{C}_{mcc} \vdash_{Q_\iota K\delta-\mathrm{MLDS}} \lambda : \perp$ as shown in the following derivation. Hence $\mathcal{C}_{mcc}$ is inconsistent which is in contradiction with the original hypothesis.

$$\begin{array}{ll}
\mathcal{C}_{mcc}\langle \lambda : \alpha(v/\tau), \lambda : \tau, \neg\exists v\alpha \rangle & (\exists\text{-I}) \\
\hline
\mathcal{C}'\langle \lambda : \exists v\alpha, \lambda : \neg\exists v\alpha \rangle & (\wedge\text{-I}) \\
\hline
\mathcal{C}_1'\langle \lambda : \perp \rangle & (\wedge\text{-I})
\end{array}$$

$\star$

Proposition 6.8 {Property for $\exists$-formulae (ii)} Let $Q_\iota K\delta$-MLDS be a quantified MLDS, let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $Q_\iota K\delta$-MLDS, let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$, let α be a wff of $Ext(\mathcal{L}_L, \mathcal{L}_M)$ and let v be a variable of $Ext(\mathcal{L}_L, \mathcal{L}_M)$. If $\lambda : \exists v\alpha \in \mathcal{C}_{mcc}$ then $\lambda : c_\alpha^{\langle \exists v, \lambda\rangle} \in \mathcal{C}_{mcc}$ and $\lambda : \alpha(v/c_\alpha^{\langle \exists v, \lambda\rangle}) \in \mathcal{C}_{mcc}$.

Proof:
By hypothesis and Proposition 6.1 $\lambda \not: c_\alpha^{\langle \exists v, \lambda\rangle} \notin \mathcal{C}_{mcc}$ and $\lambda : \neg\alpha(v/c_\alpha^{\langle \exists v, \lambda\rangle}) \notin \mathcal{C}_{mcc}$. Hence, by Proposition 6.6 and statement (3) of Proposition 6.3, $\lambda : c_\alpha^{\langle \exists v, \lambda\rangle} \in \mathcal{C}_{mcc}$ and $\lambda : \alpha(v/c_\alpha^{\langle \exists v, \lambda\rangle}) \in \mathcal{C}_{mcc}$. ⋆

Proposition 6.9 {Property for $\forall$-formulae (ii)} Let $Q_\iota K\delta$-MLDS be a quantified MLDS, let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $Q_\iota K\delta$-MLDS, let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$, let α be a wff of $Ext(\mathcal{L}_L, \mathcal{L}_M)$ and let τ be a ground term of $Ext(\mathcal{L}_L, \mathcal{L}_M)$. If $\lambda : \forall v\alpha \in \mathcal{C}_{mcc}$ and $\lambda : \tau \in \mathcal{C}_{mcc}$ then $\lambda : \alpha(v/\tau) \in \mathcal{C}_{mcc}$.

Proof:
The proof is by contradiction. Suppose that $\lambda : \alpha(v/\tau) \notin \mathcal{C}_{mcc}$. Then by statement (3) of Proposition 6.3 $\lambda : \neg\alpha(v/\tau) \in \mathcal{C}_{mcc}$. Then $\mathcal{C}_{mc} \vdash_{Q_\iota K\delta-\text{MLDS}} \lambda : \bot$ as shown in the following derivation. Hence $\mathcal{C}_{mcc}$ is inconsistent which is in contradiction with the original hypothesis.

$$\cfrac{\cfrac{\mathcal{C}_{mcc}\langle \lambda : \forall v\alpha, \lambda : \tau, \lambda : \neg\alpha(v/\tau)\rangle}{\mathcal{C}'\langle \lambda : \alpha(v/\tau), \lambda : \neg\alpha(v/\tau)\rangle}\ (\forall\text{-E})}{\mathcal{C}'_1\langle \lambda : \bot\rangle}\ (\wedge\text{-I})$$

⋆

Proposition 6.10 {Property for $\exists$-formulae (ii)} Let $Q_\iota K\delta$-MLDS be a quantified MLDS, let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $Q_\iota K\delta$-MLDS, let λ be a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$, let α be a wff of $Ext(\mathcal{L}_L, \mathcal{L}_M)$ and let v be a variable of $Ext(\mathcal{L}_L, \mathcal{L}_M)$. If $\lambda \not: c_\alpha^{\langle \forall v, \lambda\rangle} \in \mathcal{C}_{mcc}$ or $\lambda : \alpha(v/c_\alpha^{\langle \forall v, \lambda\rangle}) \in \mathcal{C}_{mcc}$ then $\lambda : \forall v\alpha \in \mathcal{C}_{mcc}$.

Proof:
Proof is by contradiction. Suppose that $\lambda : \forall v\alpha \notin \mathcal{C}_{mcc}$. Then by statment (3) of Proposition 6.3 $\lambda : \neg\forall v\alpha \in \mathcal{C}_{mcc}$. Then by Proposition 6.1 $\lambda : \alpha(v/c_\alpha^{\langle \forall v, \lambda\rangle}) \notin \mathcal{C}_{mcc}$ and $\lambda \not: c_\alpha^{\langle \forall v, \lambda\rangle} \notin \mathcal{C}_{mcc}$. This is in contradiction with the original hypothesis. ⋆

The properties described above (Propositions 6.3-6.10) hold for any maximal consistent configuration relative to any quantified $Q_\iota K\delta$-MLDS for any ι. However, in the construction of a maximal consistent configuration, different sets of declarative terms are added for different values of ι (i.e. depending on whether the quantified MLDS under consideration is for constant, increasing or decreasing domains). The following three extra propositions cover these three cases.

Proposition 6.11 {Property for increasing domains} Let $Q_{\mathbf{i}}K\delta$-MLDS be an arbitrary MLDS for increasing domains and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $Q_{\mathbf{i}}K\delta$-MLDS. Let λ and λ' be two ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let τ be a ground term of $Ext(\mathcal{L}_L, \mathcal{L}_M)$. If $\lambda:\tau \in \mathcal{C}_{mcc}$ and $R(\lambda, \lambda') \in \mathcal{C}_{mcc}$ then $\lambda':\tau \in \mathcal{C}_{mcc}$.

Proof:
The proof is by contradiction. Suppose that $\lambda':\tau \notin \mathcal{C}_{mcc}$. Then by statement (4) of Proposition 6.3, $\lambda' \not: \tau \in \mathcal{C}_{mcc}$, which implies that $\mathcal{C}_{mcc} \vdash_{Q_{\mathbf{i}}K\delta-\text{MLDS}} \lambda':\bot$ as shown in the following derivation. Hence $\mathcal{C}_{mcc}$ is inconsistent which is in contradiction with the original hypothesis.

$$\cfrac{\cfrac{\mathcal{C}_{mcc}\langle \lambda:\tau, R(\lambda,\lambda'), \lambda' \not: \tau\rangle}{\mathcal{C}'\langle \lambda':\tau, \lambda' \not: \tau\rangle}\ (VR_i)}{\mathcal{C}'_1\langle \lambda':\bot\rangle}\ (\bot_t\text{-I})$$

⋆

Proposition 6.12 {Property for decreasing domains} Let $Q_{\mathbf{d}}K\delta$-MLDS be an arbitrary MLDS for decreasing domains and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $Q_{\mathbf{d}}K\delta$-MLDS. Let λ and λ' be two ground terms of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and let τ be a ground term of $Ext(\mathcal{L}_L, \mathcal{L}_M)$. If $\lambda':\tau \in \mathcal{C}_{mcc}$ and $R(\lambda, \lambda') \in \mathcal{C}_{mcc}$ then $\lambda:\tau \in \mathcal{C}_{mcc}$.

Proof:
The proof is by contradiction. Suppose that $\lambda:\tau \notin \mathcal{C}_{mcc}$. Then by statement (4) of Proposition 6.3, $\lambda \not: \tau \in \mathcal{C}_{mcc}$, which implies that $\mathcal{C}_{mcc} \vdash_{Q_{\mathbf{d}}K\delta-\text{MLDS}} \lambda:\bot$ as shown in the following derivation. Hence $\mathcal{C}_{mcc}$ is inconsistent which is in contradiction with the original hypothesis.

$$\cfrac{\cfrac{\mathcal{C}_{mcc}\langle \lambda':\tau, R(\lambda,\lambda'), \lambda \not: \tau\rangle}{\mathcal{C}'\langle \lambda:\tau, \lambda \not: \tau\rangle}\ (VR_d)}{\mathcal{C}'_1\langle \lambda:\bot\rangle}\ (\bot_t\text{-I})$$

⋆

Proposition 6.13 {Property for constant domains} Let $Q_c K\delta$-MLDS be an arbitrary MLDS for constant domains and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $Q_c K\delta$-MLDS. Then for all labels λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$ and ground terms τ of $Ext(\mathcal{L}_L, \mathcal{L}_M)$, $\lambda : \tau \in \mathcal{C}_{mcc}$.

Proof:
The proof is by contradiction. Suppose that for some ground term λ' and ground term τ', $\lambda' : \tau' \notin \mathcal{C}_{mcc}$. Then by the statement (4) of Proposition 6.3 $\lambda' \not\!/ \tau' \in \mathcal{C}_{mcc}$, which implies that $\mathcal{C}_{mcc} \vdash_{Q_c K\delta-\text{MLDS}} \lambda' : \bot$ as shown in the following derivation. Hence $\mathcal{C}_{mcc}$ is inconsistent which is in contradiction with the original hypothesis.

$$\dfrac{\dfrac{\mathcal{C}_{mcc}\langle \lambda' \not\!/ \tau' \rangle}{\mathcal{C}'\langle \lambda' \not\!/ \tau', \lambda : \tau' \rangle}\,(VR_c)}{\mathcal{C}_1'\langle \lambda' : \bot \rangle}\,(\bot_t\text{-I})$$

⋆

To summarise the discussion so far, given a consistent configuration $\mathcal{C}$, (i) a method has been given of enlarging $\mathcal{C}$ into a maximal consistent configuration $\mathcal{C}_{mcc}$, and (ii) properties of $\mathcal{C}_{mcc}$ have been shown with respect to the declarative units, the R-literals and the declarative terms contained in it. In what follows, it is proved that $\mathcal{C}_{mcc}$ is also satisfiable – i.e. it is possible to construct a semantic structure relative to the $Q_\iota K\delta$-MLDS under consideration which satisfies $\mathcal{C}_{mcc}$.

Definition 6.2 {Maximal Interpretation, $\mathcal{M}(\mathcal{C}_{mcc})$} Let $Q_\iota K\delta$-MLDS be a quantified MLDS, let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $Q_\iota K\delta$-MLDS and let $FOT(\mathcal{C}_{mcc})$ be its first-order translation. Let $\mathcal{U} = \Sigma^H \cup \Omega^H \cup \Phi^H$ be the (sorted) Herbrand universe of $Mon(\mathcal{L}_L, \mathcal{L}_M)$, so that $\Sigma^H_g \cap \Sigma^H_v = \{\}$. $I(\mathcal{C}_{mcc})$ is the interpretation function on the language $Mon(\mathcal{L}_L, \mathcal{L}_M)$ defined as follows.

- All ground terms (of any sort) are interpreted as themselves.
- For the binary predicate R
  $\| R \|_{I(\mathcal{C}_{mcc})} = \{\langle \lambda_i, \lambda_j \rangle \mid R(\lambda_i, \lambda_j) \in FOT(\mathcal{C}_{mcc})\}$[7].
- For the binary predicate $Holds$
  $\| Holds \|_{I(\mathcal{C}_{mcc})} = \{\langle [\alpha]^*, \lambda \rangle \mid Holds([\alpha]^*, \lambda) \in FOT(\mathcal{C}_{mcc})\}$[8].

[7] Recall that $FOT(\mathcal{C}_{mcc})$ contains only ground R-literals.

[8] Note that $\| Holds \|_{I(\mathcal{C}_{mcc})}$ is a relation over $\Phi^H \times \Omega^H$.

- For the binary predicate At

  $\| At \|_{I(\mathcal{C}_{\mathrm{mcc}})} = \{\langle [\tau]^*, \lambda\rangle \mid At([\tau]^*, \lambda) \in FOT(\mathcal{C}_{mcc})\}$[9].

$\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}) = \langle \mathcal{U}, I(\mathcal{C}_{\mathrm{mcc}})\rangle$ is called the *canonical interpretation.* □

Remark 6.1 Given a variable assignment V associated with the universe of discourse $\mathcal{U}$ of Definition 6.2, V maps any variable t of sort Σ_g into an object of the Herbrand universe Σ_g^H, and any variable x of sort Ω into an object of the Herbrand universe Ω^H.

In what follows it is shown that, given a $Q_\iota K\delta$-MLDS and maximal consistent configuration $\mathcal{C}_{mcc}$ relative to it, the maximal interpretation $\mathcal{M}(\mathcal{C}_{mcc})$ is a semantic structure of $Q_\iota K\delta$-MLDS. This is done in the following steps. (i) Prove that $\mathcal{M}(\mathcal{C}_{mcc})$ satisfies all the axioms of the labelling algebra $\mathcal{A}_{K\delta}$, (ii) prove that $\mathcal{M}(\mathcal{C}_{mcc})$ satisfies all the axiom schemas (AxQ1)-(AxQ12), (iii) prove that for each $\iota \neq \mathbf{a}$, $\mathcal{M}(\mathcal{C}_{mcc})$ satisfies the associated axiom (i.e. (AxI) or (AxD) or (AxC)) of the particular extended algebra $\mathcal{A}^+_{Q_\iota K\delta}$.

The proof of step (i) is identical to that given in the propositional case. In fact for each value of δ (i.e. $\delta \subseteq \{(\mathbf{T}), (\mathbf{4}), (\mathbf{B}), (\mathbf{5}), (\mathbf{D}), (\mathbf{Irr})\}$), the labelling algebra of the associated quantified MLDS is given by the same set of axioms as the labelling algebra of the associated propositional MLDS. Besides, a maximal interpretation of a quantified MLDS is within its sort Ω identical to a maximal interpretation relative to the corresponding propositional MLDS.

Proposition 6.14 {Satisfiability of δ} Let $Q_\iota K\delta$-MLDS be a quantified MLDS, let $\mathcal{M}(\mathcal{C}_{mcc})$ be a maximal consistent configuration relative to $Q_\iota K\delta$-MLDS and let $FOT(\mathcal{C}_{mcc})$ be its first-order translation. Then the following statements hold.

1. If $(\mathbf{T}) \in \mathcal{A}_{Q_\iota K\delta}$ then $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} (\mathbf{T})$.
2. If $(\mathbf{Irr}) \in \mathcal{A}_{Q_\iota K\delta}$ then $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} (\mathbf{Irr})$.
3. If $(\mathbf{4}) \in \mathcal{A}_{Q_\iota K\delta}$ then $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} (\mathbf{4})$.
4. If $(\mathbf{B}) \in \mathcal{A}_{Q_\iota K\delta}$ then $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} (\mathbf{B})$.
5. If $(\mathbf{D}) \in \mathcal{A}_{Q_\iota K\delta}$ then $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} (\mathbf{D})$.

[9] Note that $\| At \|_{I(\mathcal{C}_{\mathrm{mcc}})}$ is a relation over $\Sigma g^H \times \Omega^H$ since τ can only be a ground term of $Mon(\mathcal{L}_L, \mathcal{L}_M)$.

6. If $(\mathbf{5}) \in \mathcal{A}_{Q_\iota K\delta}$ then $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} (\mathbf{5})$.

Proof:
The proof of each of the above statements is identical to the proof of Propositions 3.27–3.32 respectively. ⋆

The following theorem proves steps (ii) and (iii) described above.

Theorem 6.5 {Model existence for $\mathcal{A}^+_{Q_\iota K\delta} - \mathcal{A}_{Q_\iota K\delta}$ } Let $\mathcal{A}^+_{Q_\iota K\delta}$ be the extended algebra associated with a $Q_\iota K\delta$-MLDS, let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $Q_\iota K\delta$-MLDS and let $FOT(\mathcal{C}_{mcc})$ be its first-order translation. Then $\mathcal{M}(\mathcal{C}_{mcc})$ is a model of $\mathcal{A}^+_{Q_\iota K\delta} - \mathcal{A}_{Q_\iota K\delta}$.

Proof:
The proof is by cases on each of the axiom schemas (AxQ1)–(AxQ12), and on the additional axiom schemas (AxI), (AxD), (AxC) for the systems $Q_{\mathbf{i}}K\delta$-MLDS, $Q_{\mathbf{d}}K\delta$-MLDS and $Q_{\mathbf{c}}K\delta$-MLDS respectively. In what follows S, V and U are arbitrary variable assignments (which respect the sorty of the logic).

Cases 1-8 (i.e. **(AxQ1)–(AxQ8)**) are exactly analogous to Cases 1-8 of Theorem 3.5 (but with the monadic predicates $[\alpha]^*$, etc. reified as terms inside the $Holds$ predicate).

Case 9: **(AxQ9)** $Holds([\exists v\alpha]^*, \lambda) \rightarrow (At([c_\alpha^{\langle\exists v,\lambda\rangle}]^*, \lambda) \wedge Holds([\alpha(v/c_\alpha^{\langle\exists v,\lambda\rangle})]^*, \lambda))$.

It is sufficient to prove that

if $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} Holds([\exists v\alpha]^*, \lambda)$
then $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} At([c_\alpha^{\langle\exists v,\lambda\rangle}]^*, \lambda)$ and
$\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} Holds([\alpha(v/c_\alpha^{\langle\exists v,\lambda\rangle})]^*, \lambda)$.

By assumption $Holds([\exists v\alpha]^*, \lambda) \in FOT(\mathcal{C}_{mcc})$. This implies $\lambda : \exists v\alpha \in \mathcal{C}_{mcc}$. By Proposition 6.8, $\lambda : c_\alpha^{\langle\exists v,\lambda\rangle} \in \mathcal{C}_{mcc}$ and $\lambda : \alpha(v/c_\alpha^{\langle\exists v,\lambda\rangle}) \in \mathcal{C}_{mcc}$. Then, $At([c_\alpha^{\langle\exists v,\lambda\rangle}]^*, \lambda) \in FOT(\mathcal{C}_{mcc})$ and $Holds([\alpha(v/c_\alpha^{\langle\exists v,\lambda\rangle})]^*, \lambda) \in FOT(\mathcal{C}_{mcc})$. Hence

$\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} At([c_\alpha^{\langle\exists v,\lambda\rangle}]^*, \lambda)$ and
$\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} Holds([\alpha(v/c_\alpha^{\langle\exists v,\lambda\rangle})]^*, \lambda)))$.

Case 10: **(AxQ10)** $\forall x(\exists t(At(t, x) \wedge Holds([\alpha(v/t)]^*, x)) \rightarrow Holds([\exists v\alpha]^*, x))$.

By definition, $S(t) = [\tau]^*$ for some $[\tau]^* \in \Sigma_g^H$. It is sufficient to prove that

if $\mathcal{M}(\mathcal{C}_{mcc}) \| \!\!-_{FOL} At(S(t), V(x))$ and
$\mathcal{M}(\mathcal{C}_{mcc}) \| \!\!-_{FOL} Holds([\alpha(v/\tau)]^*, V(x)))$
then $\mathcal{M}(\mathcal{C}_{mcc}) \| \!\!-_{FOL} Holds([\exists v\alpha]^*, V(x)))$.

By assumption $At(S(t), V(x)) \in FOT(\mathcal{C}_{mcc})$ and $Holds([\alpha(v/\tau)]^*, V(x))) \in FOT(\mathcal{C}_{mcc})$, which implies that $V(x) : \tau \in \mathcal{C}_{mcc}$ and $V(x) : \alpha(v/\tau) \in \mathcal{C}_{mcc}$. Then by Proposition 6.7, $V(x) : \exists v\alpha \in \mathcal{C}_{mcc}$, which implies that $Holds([\exists v\alpha]^*, V(x)) \in FOT(\mathcal{C}_{mcc})$. Hence $\mathcal{M}(\mathcal{C}_{mcc}) \| \!\!-_{FOL} Holds([\exists v\alpha]^*, V(x)))$.

Case 11: **(AxQ11)** $\forall x(Holds([\forall v\alpha]^*, x) \to \forall t(At(t, x) \to Holds([\alpha(v/t)]^*, x))$.

Consider the equivalent formula

$$\forall x \forall t(Holds([\forall v\alpha]^*, x) \wedge At(t, x)) \to Holds([\alpha(v/t)]^*, x))$$

By definition, $S(t) = [\tau]^*$ for some $[\tau]^* \in \Sigma_g^H$. It is sufficient to prove that

if $\mathcal{M}(\mathcal{C}_{mcc}) \| \!\!-_{FOL} Holds([\forall v\alpha]^*, V(x))$
and $\mathcal{M}(\mathcal{C}_{mcc}) \| \!\!-_{FOL} At(S(t), V(x))$ then
$\mathcal{M}(\mathcal{C}_{mcc}) \| \!\!-_{FOL} Holds([\alpha(v/\tau)]^*, V(x)))$.

By assumption, $At(S(t), V(x)) \in FOT(\mathcal{C}_{mcc})$ and $Holds([\forall v\alpha]^*, V(x)) \in FOT(\mathcal{C}_{mcc})$, which implies that $V(x) : \tau \in \mathcal{C}_{mcc}$ and $V(x) : \forall v\alpha \in \mathcal{C}_{mcc}$. By Proposition 6.9, $V(x) : \alpha(v/\tau) \in \mathcal{C}_{mcc}$ which implies that $Holds([\alpha(v/\tau)]^*, V(x)) \in FOT(\mathcal{C}_{mcc})$. Hence, $\mathcal{M}(\mathcal{C}_{mcc}) \| \!\!-_{FOL} Holds([\alpha(v/\tau)]^*, V(x))$.

Case 12: **(AxQ12)**
$(At([c_\alpha^{\langle\forall v,\lambda\rangle}]^*, \lambda) \to Holds([\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})]^*, \lambda)) \to Holds([\forall v\alpha]^*, \lambda)$.

It is sufficient to prove that

if $\mathcal{M}(\mathcal{C}_{mcc}) \| \!\!-_{FOL} \neg At([c_\alpha^{\langle\forall v,\lambda\rangle}]^*, \lambda)$ or
$\mathcal{M}(\mathcal{C}_{mcc}) \| \!\!-_{FOL} Holds([\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})]^*, \lambda)$ then
$\mathcal{M}(\mathcal{C}_{mcc}) \| \!\!-_{FOL} Holds([\forall v\alpha]^*, \lambda)$

By assumption

either $\neg At([c_\alpha^{\langle\forall v,\lambda\rangle}]^*, \lambda) \in FOT(\mathcal{C}_{mcc})$ or
$Holds([\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})]^*, \lambda) \in FOT(\mathcal{C}_{mcc})$

This implies that either $\lambda \not: c_\alpha^{\langle\forall v,\lambda\rangle} \in \mathcal{C}_{mcc}$ or $\lambda : \alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle}) \in \mathcal{C}_{mcc}$. By Proposition 6.10, $\lambda : \forall v\alpha \in \mathcal{C}_{mcc}$ which implies that $Holds([\forall v\alpha]^*, \lambda) \in FOT(\mathcal{C}_{mcc})$. Hence $\mathcal{M}(\mathcal{C}_{mcc}) \| \!\!-_{FOL} Holds([\forall v\alpha]^*, \lambda)$.

Case 13: **(AxI)** $\forall x \forall y \forall t((At(t,x) \wedge R(x,y)) \rightarrow At(t,y))$.

This case is necessary only for the extended algebra $\mathcal{A}^{+}_{Q_{\mathbf{i}}K\delta}$ associated with the family of quantified MLDSs $Q_{\mathbf{i}}K\delta$-MLDS for increasing domains. By definition, $S(t) = [\tau]^*$ for some $[\tau]^* \in \Sigma^h_g$. It is sufficient to show that if $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} At(S(t), V(x))$ and $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} R(V(x), U(y))$ then $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} At(S(t), U(y))$. By assumption $At(S(t), V(x)) \in FOT(\mathcal{C}_{mcc})$ and $R(V(x), U(y)) \in FOT(\mathcal{C}_{mcc})$, which implies that $V(x) : \tau \in \mathcal{C}_{mcc}$ and $R(V(x), U(y)) \in \mathcal{C}_{mcc}$. By Proposition 6.11, $U(y) : \tau \in \mathcal{C}_{mcc}$, which implies that $At(S(t), U(y)) \in FOT(\mathcal{C}_{mcc})$. Hence $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} At(S(t), U(y))$.

Case 14: **(AxD)** $\forall x \forall y \forall t((At(t,y) \wedge R(x,y)) \rightarrow At(t,x))$.

This case is necessary only for the extended algebra $\mathcal{A}^{+}_{Q_{\mathbf{d}}K\delta}$ associated with the family of quantified MLDSs $Q_{\mathbf{d}}K\delta$-MLDS for decreasing domains. By definition, $S(t) = [\tau]^*$ for some $[\tau]^* \in \Sigma^h_g$. It is sufficient to show that if $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} At(S(t), U(y))$ and $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} R(V(x), U(y))$ then $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} At(S(t), V(x))$. By assumption $At(S(t), U(y)) \in FOT(\mathcal{C}_{mcc})$ and $R(V(x), U(y)) \in FOT(\mathcal{C}_{mcc})$, which implies that $U(y) : \tau \in \mathcal{C}_{mcc}$ and $R(V(x), U(y)) \in \mathcal{C}_{mcc}$. By Proposition 6.12, $V(x) : \tau \in \mathcal{C}_{mcc}$, which implies that $At(S(t), V(x)) \in FOT(\mathcal{C}_{mcc})$. Hence $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} At(S(t), V(x))$.

Case 15: **(AxC)** $\forall x \forall t At(t,y)$.

This case is necessary only for the extended algebra $\mathcal{A}^{+}_{Q_{\mathbf{c}}K\delta}$ associated with the family of quantified MLDSs $Q_{\mathbf{c}}K\delta$-MLDS for constant domains. It is sufficient to show that $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} \forall t At(t, V(x))$. By Proposition 6.13, $V(x) : \tau \in \mathcal{C}_{mcc}$ for all $\tau \in \Sigma_g$. Therefore, $At([\tau]^*, V(x)) \in FOT(\mathcal{C}_{mcc})$ for all $[\tau]^* \in \Sigma^H_g$. Hence $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} \forall t At(t, V(x))$. $\star$

To summarise the above results, it has been shown that for any quantified MLDS $Q_\iota K\delta$-MLDS, given a consistent configuration $\mathcal{C}$ of $Q_\iota K\delta$-MLDS it is possible to construct a maximal consistent configuration $\mathcal{C}_{mcc}$ which includes $\mathcal{C}$, and a canonical interpretation $\mathcal{M}(\mathcal{C}_{mcc})$ which is a semantic structure of $Q_\iota K\delta$-MLDS (i.e. a model of $\mathcal{A}^{+}_{Q_\iota K\delta}$). The following Model Existence Lemma shows that this model also satisfies the maximal consistent configuration $\mathcal{C}_{mcc}$ and therefore satifies the configuration $\mathcal{C}$.

Lemma 6.2 {Model Existence Lemma} Let $Q_\iota K\delta$-MLDS be a quantified MLDS and let $\mathcal{C}_{mcc}$ be a maximal consistent configuration relative to $Q_\iota K\delta$-MLDS. Then for any π (where π is a declarative unit, R-literal or declarative term) of $Q_\iota K\delta$-MLDS, $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{Q_\iota K\delta-\text{MLDS}} \pi$ if $\pi \in \mathcal{C}_{mcc}$ and $\mathcal{M}(\mathcal{C}_{mcc}) \not\Vdash_{Q_\iota K\delta-\text{MLDS}} \pi$ if $\pi \notin \mathcal{C}_{mcc}$.

Proof:
There are three cases to consider:

1. π is a declarative unit.

   The proof is identical to the proof of (1) of Lemma 3.3 (see Chapter 3), but with the monadic predicates $[\alpha]^*$, etc. reified as terms inside the *Holds* predicate.

2. π is an R-literal.

   The proof is identical to the proof of (2) of Lemma 3.3.

3. π is a declarative term.

   Then π can be either of the form $\lambda : \tau$ or of the form $\lambda \not: \tau$, where τ is a ground term of $Ext(\mathcal{L}_L, \mathcal{L}_M)$ and λ is a ground term of $Func(\mathcal{L}_L, \mathcal{L}_M)$. Only the first case is considered, since the argument for the second case is analogous. Suppose that π is of the form $\lambda : \tau$. If $\lambda : \tau \in \mathcal{C}_{mcc}$ then $At([\tau]^*, \lambda) \in FOT(\mathcal{C}_{mcc})$, which implies that $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} At([\tau]^*, \lambda)$. Hence $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{Q_\iota K\delta-\mathrm{MLDS}} \pi$. If $\lambda : \tau \notin \mathcal{C}_{mcc}$ by statement (4) of Proposition 6.3 $\lambda \not: \tau \in \mathcal{C}_{mcc}$ which implies that $\neg At([\tau]^*, \lambda) \in FOT(\mathcal{C}_{mcc})$ and therefore, $\mathcal{M}(\mathcal{C}_{mcc}) \Vdash_{FOL} \neg At([\tau]^*, \lambda)$. Hence $\mathcal{M}(\mathcal{C}_{mcc}) \not\Vdash_{Q_\iota K\delta-\mathrm{MLDS}} \pi$.

$\star$

Corollary 6.2 Let $Q_\iota K\delta$-MLDS be a quantified MLDS and let $\mathcal{C}$ be a consistent configuration of $Q_\iota K\delta$-MLDS. Then $\mathcal{M}(\mathcal{C}_{mcc})$ satisfies $\mathcal{C}$.

Proof:
The proof follows trivially from Lemma 6.2 and from the fact that $\mathcal{C} \subseteq \mathcal{C}_{mcc}$. $\star$

Proposition 6.15 Let $Q_\iota K\delta$-MLDS be a quantified MLDS and let $\mathcal{C}$ be a configuration of $Q_\iota K\delta$-MLDS. Let π be a declarative unit, R-literal or declarative term such that $\pi \notin \mathcal{C}$. If $\mathcal{C} \nvdash_{Q_\iota K\delta-\mathrm{MLDS}} \pi$ then $\mathcal{C}+[\neg\pi]$ is a consistent configuration.

Proof:
There are three cases to consider: (i) π is a declarative unit, (ii) π is an R-literal and (iii) π is a declarative term.

(i) The proof is identical to the proof of (i) of Proposition 3.33 (see Chapter 3).

(ii) The proof is identical to the proof of (ii) of Proposition 3.33.

(iii) Suppose that π is a declarative term. The contrapositive of the proposition statement is proved. Assume that $\mathcal{C} + [\overline{\pi}]$ is not consistent. Then $\mathcal{C} + [\overline{\pi}] \vdash_{Q_\iota K\delta-\text{MLDS}} \lambda : \bot$ for some ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$. Then the pair $\mathcal{C}/\mathcal{C} + [\pi] \in \mathcal{I}_{\bot_t}$ and $\langle\{\mathcal{C}, \mathcal{C} + [\pi]\}, m\rangle$ is a proof, where $m(0) = \mathcal{I}_{\bot_t}$. Hence $\mathcal{C} \vdash_{Q_\iota K\delta-\text{MLDS}} \pi$.

⋆

Finally, the completeness theorem given at the beginning of this section can now be proved. Note that the following proof uses the "$MCC(-)$" notation introduced in Definition 3.6.

Proof of Completeness (Theorem 6.4)

The proof is by contrapositive. Assume that $\mathcal{C} \not\vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}'$. Then by Theorem 6.2, there exists a $\pi \in \mathcal{C}' - \mathcal{C}$, where π is a declarative unit, R-literal or declarative term, such that $\mathcal{C} \not\vdash_{Q_\iota K\delta-\text{MLDS}} \pi$. Then by Proposition 6.15, $\mathcal{C} + [\neg\pi]$ is a consistent configuration. By Corollary 6.2, $\mathcal{M}_m = \mathcal{M}(MCC(\mathcal{C} + [\neg\pi]))$ satisfies the configuration $\mathcal{C} + [\neg\pi]$. So $\mathcal{M}_m \Vdash_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}$ and $\mathcal{M}_m \Vdash_{Q_\iota K\delta-\text{MLDS}} \neg\pi$. Now there are two cases to consider.

1. π is a declarative unit or an R-literal. Then $\mathcal{M}_m \not\Vdash_{Q_\iota K\delta-\text{MLDS}} \pi$, hence $\mathcal{C} \not\models_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}'$ (the proof is identical to cases (1) and (2) in the proof of Theorem 3.4 given in Chapter 3 for propositional MLDSs, but with the monadic predicates $[\alpha]^*$ reified as terms of the predicate $Holds$).

2. π is a declarative term. π can be either of the form $\lambda : \tau$ or of the form $\lambda \not: \tau$. Only the first case is considered since the argument for the second case is analogous. Suppose that π is of the form $\lambda : \tau$. Then $\mathcal{M}_m \Vdash_{FOL} \neg At([\tau]^*, \lambda)$ which implies that $\mathcal{M}_m \not\Vdash_{FOL} At([\tau]^*, \lambda)$. Therefore, $\mathcal{M}_m \not\Vdash_{Q_\iota K\delta-\text{MLDS}} \pi$, hence $\mathcal{C} \not\models_{Q_\iota K\delta-\text{MLDS}} \mathcal{C}'$.

⋆

6.3 Correspondence

In this section a correspondence theorem is proved which shows that, whenever the initial configuration contains no local assumptions except possibly at a single point, and for the type of underlying modal language considered by Kripke, the model-theoretic semantics defined in Chapter 5 for quantified MLDSs is equivalent to that given by Kripke in [Kri63b] and formalised in Section 4.2. Throughout this section, the quantified modal language $\mathcal{L}_M$ of a quantified MLDS is restricted to the Kripke language $\mathcal{L}_K$ where the set of terms is given only by a set of variables.

The extended quantified modal language is therefore the language $Ext(\mathcal{L}_L, \mathcal{L}_K)$, the semi-extended labelling language is $Func(\mathcal{L}_L, \mathcal{L}_K)$ and the extended labelling language is $Mon(\mathcal{L}_L, \mathcal{L}_K)$.

First, and as in the propositional case in Chapter 3, a "weak correspondence" result is given for the family of $Q_{\mathbf{a}}K$-MLDS (i.e. modal logic K based on arbitrary domains). This states that any declarative unit of the form $\lambda : \alpha$ (for any label λ), where α is a sentence of $\mathcal{L}_K$, is derivable from an empty initial configuration if and only if the sentence α is provable in the Kripke axiomatic system $QK_{K(Ax)}$ (see Definition 4.17). This result is then extended to the modal logics T, $S4$ and $S5$ to show that such a correspondence also holds with respect to particular classes of frames. (Recall that Kripke only considered the logics K, T, $S4$ and $S5$.) A "strong correspondence" theorem is then proved which states that an equivalence between the Kripke semantic entailment and $Q_{\mathbf{a}}K\delta$-MLDS's semantic entailment exists whenever initial configurations are singleton configurations (i.e. whenever the local modal theory differs only at one single actual world W_0). This more general result shows that quantified extensions of a MLDS are generalisations of standard quantified modal logics.

Lemma 6.3 {Simple Correspondence for a quantified modal logic K based on arbitrary domains}

Let $Q_{\mathbf{a}}K\delta$-MLDS be a quantified MLDS whose associated labelling algebra $\mathcal{A}_{Q_{\mathbf{a}}K} = \{\}$, and let $\langle QK_{Ax}, \vdash_{QK_{Ax}} \rangle$ be the Kripke axiomatic system given in Definition 4.17. Let $\mathcal{C}_{\{\}} = \langle \{\}, \mathcal{F}_{\{\}}, \mathcal{T}^+_{\{\}}, \mathcal{T}^-_{\{\}} \rangle$, where for any ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_K)$, $\mathcal{F}_{\{\}}(\lambda) = \{\}$, $\mathcal{T}^+_{\{\}}(\lambda) = \{\}$ and $\mathcal{T}^+_{\{\}}(\lambda) = \{\}$. Let α be a sentence of the language $\mathcal{L}_K$. Then

$$\vdash_{QK_{Ax}} \alpha$$
$$\text{if and only if}$$
$$\text{for all ground terms } \lambda \in Func(\mathcal{L}_L, \mathcal{L}_K),\ \mathcal{C}_{\{\}} \vdash_{Q_{\mathbf{a}}K\text{-MLDS}} \lambda : \alpha$$

Proof:
("If" half:) The proof is by showing that the contrapositive statement holds – i.e. given a sentence $\alpha \in \mathcal{L}_K$, if $\nvdash_{QK_{Ax}} \alpha$ then there exists a ground term $\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_K)$ such that $\mathcal{C}_{\{\}} \nvdash_{Q_{\mathbf{a}}K\text{-MLDS}} \lambda : \alpha$. Since both the two derivability relations $\vdash_{QK_{Ax}}$ and $\vdash_{Q_{\mathbf{a}}K\text{-MLDS}}$ are sound and complete with respect to their semantics[10] it is sufficient to prove that if $\not\models_{QK_{Ax}} \alpha$ then there exists a ground term

[10] $\vdash_{QK_{Ax}}$ is complete with respect to the Kripke semantics given in Section 4.2 (see statment 4.4) and $\vdash_{Q_{\mathbf{a}}K\text{-MLDS}}$ is complete with respect to the semantics defined in Chapter 5.

$\lambda \in Mon(\mathcal{L}_L, \mathcal{L}_K)$ such that $\mathcal{C}_{\{\}} \not\models_{Q_{\mathbf{a}}K-\text{MLDS}} \lambda : \alpha$. Suppose that $\not\models_{QK_{Ax}} \alpha$. Since $\mathcal{C}_{\{\}}$ is an empty configuration, it is sufficient to show that there exists a semantic structure of $Q_{\mathbf{a}}K$-MLDS (i.e. a model of $\mathcal{A}^{+}_{Q_{\mathbf{a}}K}$) which satisfies the declarative unit $\lambda : \neg\alpha$. By hypothesis and by the Kripke semantic definition of validity (see Section 4.2), there exists a Kripke model which satisfies the sentence $\neg\alpha$ (i.e. a countermodel of the sentence α). Let $M = \langle \widehat{W}, \widehat{R}, Q, \phi \rangle$ be such a model, with D as global domain, and let s be an arbitrary variable assignment[11]. Thus there exists some possible world $w \in \widehat{W}$ such that $M, w \Vdash \neg\alpha[s]$. Assume a canonical well-ordering on the sets $\widehat{W}$ and D[12]. Let $Sat = \{w_i \mid w_i \in \widehat{W} \text{ and } M, w_i \Vdash \neg\alpha[s]\}$ and let w' be the first element of Sat according to $\widehat{W}$'s canonical well-ordering relation. Then $M, w' \Vdash \neg\alpha[s]$. An interpretation I of $Mon(\mathcal{L}_L, \mathcal{L}_K)$ (which will subsequently be proved to be a model of $\mathcal{A}^{+}_{Q_{\mathbf{a}}K}$) is constructed below using the 4-sorted universe of discourse $U = \widehat{W} \cup \Sigma^H_v \cup D \cup \Phi^H$, where Σ^H_v and Φ^H are the Herbrand universes of sorts Σ_v and Φ respectively of the language $Mon(\mathcal{L}_L, \mathcal{L}_K)$. I maps terms of sort Ω to $\widehat{W}$, terms of sort Σ_g to D, terms of sort Σ_v to Σ^H_v, and terms of sort Φ to Φ^H. I is defined s follows:

- All terms of sort Σ_v and Φ are interpreted as themselves.
- For each constant symbol W_i, function symbols $succ$, f_β and box_β, $\| W_i \|_I$, $\| succ \|_I$, $\| f_\beta \|_I$ and $\| box_\beta \|_I$ are identical to the corresponding interpretations given in Lemma 3.4 (see Chapter 3), but with the notion of truth $M, \tilde{w} \Vdash \beta$ replaced by Kripke notion of truth $M, \tilde{w} \Vdash \beta[s]$ given in Definition 4.15.
- For each constant symbol of the form $[c^{\langle \exists v, \lambda \rangle}_\beta]^*$, $\| [c^{\langle \exists v, \lambda \rangle}_\beta]^* \|_I \in D$ and it is defined as follows:
  - If $Sat^{\lambda}_{\exists v \beta} = \{a \mid a \in Q(\| \lambda \|_I) \text{ and } M, \| \lambda \|_I \Vdash \beta[s[^v_a]]\}$ is a non–empty set, then $\| [c^{\langle \exists v, \lambda \rangle}_\beta]^* \|_I = \overline{a}$ where $\overline{a}$ is the first element of $Sat^{\lambda}_{\exists v \beta}$ with respect to D's canonical well–ordering.
  - Otherwise $\| [c^{\langle \exists v, \lambda \rangle}_\beta]^* \|_I = a_0$, where a_0 is the first element of D according to its assumed canonical well-ordering.
- For each constant symbol of the form $[c^{\langle \forall v, \lambda \rangle}_\beta]^*$, $\| [c^{\langle \forall v, \lambda \rangle}_\beta]^* \|_I \in D$ and it is defined as follows:
  - If $Q(\| \lambda \|_I) = \{\}$, then $\| [c^{\langle \forall v, \lambda \rangle}_\beta]^* \|_I = a_0$ where a_0 is the first element of D according to its assumed canonical well-ordering.

[11] Note that the variable assignment s is not particularily relevant for the truth of α since α is a sentence, but it allows the definition of truth of any other arbitrary formula of $\mathcal{L}_K$.

[12] Note that both W and D are non-empty sets and thus, according to Zermelo's *Well–Ordering Theorem*, it is possible to define a well-ordering relation on them.

- If $Q(\| \lambda \|_I) \neq \{\}$ and for all $a \in Q(\| \lambda \|_I)$, $M, \| \lambda \|_I \Vdash \beta[s[^v_a]]$, then $\| [c_\beta^{\langle \forall v, \lambda \rangle}]^* \|_I = \overline{a}$ where $\overline{a}$ is the first element of $Q(\| \lambda \|_I)$ with respect to D's canonical well–ordering.
- Otherwise $\| [c_\beta^{\langle \forall v, \lambda \rangle}]^* \|_I = \tilde{a}$, where $\tilde{a}$ is the first element of $Sat^\lambda_{\exists v \neg \beta}$ with respect to D's canonical well–ordering.

- For the binary predicate R

  $\| R \|_I = \widehat{R}$

- For the binary predicate At

  $\| At \|_I = \{\langle a, w \rangle \mid a \in Q(w) \text{ and } a = \| \tau \|_I \text{ for some ground term } \tau \text{ of } Ext(\mathcal{L}_L, \mathcal{L}_K)\}$

- For the binary predicate $Holds$

  $\| Holds \|_I = \{\langle [\beta]^*, w \rangle \mid M, w \Vdash \beta[s]\}$

It is shown now that $\langle U, I \rangle$ is a $Q_\mathbf{a}K$-MLDS semantic structure. This requires proving that $\langle U, I \rangle$ is a classical model of the extended algebra $\mathcal{A}^+_{Q_\mathbf{a}K}$. Observe that in a $Q_\mathbf{a}K$-MLDS, the labelling algebra $\mathcal{A}_{Q_\mathbf{a}K} = \{\}$. Therefore the extended algebra $\mathcal{A}^+_{Q_\mathbf{a}K}$ is given only by the set of axiom schemas (AxQ1)–(AxQ12). The proof is by cases on each of the axiom schemas (AxQ1)–(AxQ12). Let β, γ be two wffs of $\mathcal{L}_K$.

Cases 1–8: (AxQ1) – (AxQ8)

The proof is exactly analogous to Cases 1–8 ((**Ax1**)–(**Ax8**)) of the proof of Lemma 3.4 in Chapter 3, with the monadic predicates $[\alpha]^*$ reified as terms of the predicate *Holds*.

Case 9: (AxQ9) $Holds([\exists v \beta]^*, \lambda) \rightarrow (At([c_\beta^{\langle \exists v, \lambda \rangle}]^*, \lambda) \wedge Holds([\beta(v/c_\beta^{\langle \exists v, \lambda \rangle})]^*, \lambda))$

Let $w \in \widehat{W}$ such that $\| \lambda \|_I = w$. It is sufficient to prove that if $\langle [\exists v \beta]^*, w \rangle \in \| Holds \|_I$ then $\langle \| [c_\beta^{\langle \exists v, \lambda \rangle}]^* \|_I, w \rangle \in \| At \|_I$ and $\langle [\beta(v/c_\beta^{\langle \exists v, \lambda \rangle})]^*, w \rangle \in \| Holds \|_I$. By assumption and definition of $\| Holds \|_I$, $M, w \Vdash \exists v \beta[s]$, which implies that $M, w \Vdash \beta[s[^v_a]]$ for some $a \in Q(w)$. Therefore, the set $Sat^\lambda_{\exists v \beta} \neq \{\}$ and $\| [c_\beta^{\langle \exists v, \lambda \rangle}]^* \|_I = \overline{a}$, where $\overline{a}$ is the first element of $Sat^\lambda_{\exists v \beta} \neq \{\}$ according to D's canonical well-ordering. Then by definition of I, $\langle \| [c_\beta^{\langle \exists v, \lambda \rangle}]^* \|_I, w \rangle \in \| At \|_I$ and $\langle [\beta(v/c_\beta^{\langle \exists v, \lambda \rangle})]^*, w \rangle \in \| Holds \|_I$.

Case 10: (AxQ10) $\forall x(\exists t(At(t,x) \wedge Holds([\beta(v/t)]^*, x)) \rightarrow Holds([\exists v\beta]^*, x))$

Let $w \in \widehat{W}$. There are two cases to consider. (i) There is not $a \in \Sigma_g$ such that $\langle a, w\rangle \in \| At \|_I$. Then the axiom is trivially satisfied. (ii) there is a member $a \in \Sigma_g$ such that $\langle a, w\rangle \in \| At \|_I$. Then by definition, $a = \| \tau \|_I$ for some ground term $\tau \in Ext(\mathcal{L}_L, \mathcal{L}_K)$. Hence, in this case it is sufficient to show that if $\langle \ \| [\beta(v/\tau)]^* \|_I,\ w\rangle \in \| Holds \|_I$ then $\langle \ \| [\exists v\beta]^* \|_I,\ w\rangle \in \| Holds \|_I$. Suppose that $\langle \ \| [\beta(v/\tau]^* \|_I,\ w\rangle \in \| Holds \|_I$. Then $M, w \Vdash \beta[s[^v_a]]$, so that, since $a \in Q(w)$, $M, w \Vdash \exists v\beta[s]$. Hence $\langle \| [\exists v\beta]^* \|_I, w\rangle \in \| Holds \|_I$.

Case 11: (AxQ11) $\forall x(Holds([\forall v\beta]^*, x) \rightarrow (\forall t(At(t,x) \rightarrow Holds([\beta(v/t)]^*, x)))$

The following equivalent axiom is considered.

$$\forall x \forall t((Holds([\forall v\beta]^*, x) \wedge At(t,x)) \rightarrow Holds([\alpha(v/t)]^*, x))$$

It is sufficient to consider the following case. Let $w \in \widehat{W}$ and let $a \in \Sigma_g$, such that $\langle \| [\forall v\beta]^* \|_I, w\rangle \in \| Holds \|_I$ and $\langle a, w\rangle \in \| At \|_I$. Then by definition there exists a ground term $\tau \in Ext(\mathcal{L}_L, \mathcal{L}_K)$ such that $a = \| \tau \|_I$. By assumption, $M, w \Vdash \forall v\beta[s]$. This implies that for all objects $a' \in Q(w)$, $M, w \Vdash \beta[s[^v_{a'}]]$. Hence by definition of I, $\langle \ \| [\beta(v/\tau)]^* \|_I,\ w\rangle \in \| Holds \|_I$.

Case 12: (AxQ12) $(At([c_\beta^{\langle \forall v, \lambda\rangle}]^*, \lambda) \rightarrow Holds([\beta(v/c_\beta^{\langle \forall v, \lambda\rangle})]^*, \lambda)) \rightarrow Holds([\forall v\beta]^*, \lambda)$

Let w be an arbitrary element of $\widehat{W}$, such that $\| \lambda \|_I = w$. It is sufficient to prove that if the tuple $\langle \ \| [c_\beta^{\langle \forall v, \lambda\rangle}]^* \|_I,\ w\rangle \notin \| At \|_I$ or $\langle \ \| [\beta(v/c_\beta^{\langle \forall v, \lambda\rangle})]^* \|_I,\ w\rangle \in \| Holds \|_I$, then $\langle \| [\forall v\beta]^* \|_I, w\rangle \in \| Holds \|_I$. Suppose first, that the tuple $\langle \ \| [c_\beta^{\langle \forall v, \lambda\rangle}]^* \|_I,\ w\rangle \notin \| At \|_I$. This implies, by definition of $\| [c_\beta^{\langle \forall v, \lambda\rangle}]^* \|_I$, that $Q(w) = \{\}$. Therefore $M, w \Vdash \forall v\beta[s]$. Hence $\langle \ \| [\forall v\beta]^* \|_I,\ w\rangle \in \| Holds \|_I$. Suppose now that $\langle \ \| [\beta(v/c_\beta^{\langle \forall v, \lambda\rangle})]^* \|_I,\ w\rangle \in \| Holds \|_I$, then by definition of I, there are two sub-cases to consider: (i) $Q(w) = \{\}$. Then by Kripke semantic definition of satisfiability of the $\forall$ quantifier, $M, w \Vdash \forall v\beta[s]$. Hence by definition of I, $\langle \ \| [\forall v\beta]^* \|_I\ , w\rangle \in \| Holds \|_I$. (ii) For all $a \in Q(w)$, $M, w \Vdash \beta[s[^v_a]]$. Then again by Kripke semantic definition of satisfiability of the $\forall$ operator, $M, w \Vdash \forall v\beta[s]$. Hence by definition of I, $\langle \ \| [\forall v\beta]^* \|_I,\ w\rangle \in \| Holds \|_I$.

(End of cases)

Therefore the tuple $\langle U, I\rangle$ is a model of $\mathcal{A}^+_{Q_{\mathbf{a}}K}$. Thus, since by hypothesis $M, w' \Vdash \neg\alpha[s]$ and, by definition of I, $\| W_i \|_I = w'$, where W_i are constant symbols of $Mon(\mathcal{L}_L, \mathcal{L}_K)$, there exists a ground term λ, $\lambda = W_i$ for some $i \geq 0$, such that $\langle U, I\rangle \Vdash_{FOL} Holds([\neg\alpha]^*, \lambda)$. By (AxQ2) $\langle U, I\rangle \nVdash_{FOL} Holds([\alpha]^*, \lambda)$. Hence $\mathcal{C}_{\{\}} \not\models_{Q_{\mathbf{a}}K-\mathrm{MLDS}} \lambda : \alpha$.

(End of "If" half)

("Only if" half: i.e. if $\vdash_{QK_{Ax}} \alpha$ then $\mathcal{C}_{\{\}} \vdash_{Q_\iota K-\text{MLDS}} \lambda : \alpha$ for any ground term λ of $Mon(\mathcal{L}_L, \mathcal{L}_K)$.)

The proof is analogous to the "If" half of the proof of Lemma 3.4 (see Chapter 3) augmented with the additional arguments to cover axioms [A17]-[A19] that extend in QK_{Ax} the basic propositional axiomatisation of the modal logic K. Therefore, suppose that $\vdash_{QK_{K(Ax)}} \alpha$ and that $\alpha_1, \ldots, \alpha_m$, where $m \geq 1$ and $\alpha_m = \alpha$, is the shortest derivation of α with length (i.e. number of steps) $l \geq 1$[13]. The proof $\mathcal{C}_{\{\}} \vdash_{Q_{\mathbf{a}}K-\text{MLDS}} \lambda : \alpha$, for any arbitrary λ, is by induction on l.

BASE CASE:
The proof is analogous to the base case of the "If" half of Lemma 3.4 augmented with the following cases.

Case 16: [A17] Let $\mathcal{C}_0 = \mathcal{C}_{\{\}}$ and let λ be an arbitrary ground term of $Mon(\mathcal{L}_L, \mathcal{L}_K)$. Then the following derivation proves that given the universally closed axiom schema $\alpha \rightarrow \forall v\alpha$, where v is not free in α, $\mathcal{C}_0 \vdash_{Q_{\mathbf{a}}K-\text{MLDS}} \lambda : \alpha \rightarrow \forall v\alpha$. Only the case in which α is a sentence (i.e. the closure is equal to the schema itself) is considered. More general instantiations of the schema, given by prefixing $\alpha \rightarrow \forall v\alpha$ with an appropriate number of universal quantifiers, are proved by adding at the end of the following derivation a corresponding number of $\forall$-I rules (and using an appropriately "$\forall$–skolemised" assumption in line 2).

$$
\begin{array}{ll}
\mathcal{C}_0\langle\rangle & \\
\quad \mathcal{C}_0\langle[\lambda : \alpha]\rangle & \text{(assumption)} \\
\quad\quad \mathcal{C}_0\langle[\lambda : c_\alpha^{\langle\forall v,\lambda\rangle}]\rangle & \text{(assumption)} \\
\quad\quad \mathcal{C}_1\langle\lambda : \alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})\rangle & \text{(C–R)} \\
\quad \mathcal{C}_2\langle\lambda : \forall v\alpha\rangle & (\forall\ \text{I}) \\
\mathcal{C}'\langle\lambda : \alpha \rightarrow \forall v\alpha\rangle & (\rightarrow \text{I})
\end{array}
$$

(N.B. line 3 relies on the fact that $\alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle}) = \alpha$.)

Case 17: [A18] Let $\mathcal{C}_0 = \mathcal{C}_{\{\}}$ and let λ be an arbitrary ground term of $Mon(\mathcal{L}_L, \mathcal{L}_K)$. Then the following derivation proves that given the closed axiom schema $\forall v(\alpha \rightarrow \beta) \rightarrow (\forall v\alpha \rightarrow \forall v\beta)$, $\mathcal{C}_0 \vdash_{Q_{\mathbf{a}}K-\text{MLDS}} \forall v(\alpha \rightarrow \beta) \rightarrow (\forall v\alpha \rightarrow \forall v\beta)$. Again, more general instantiations of the schema (in which α and β contain free variables other than v), given by prefixing the schema with an appropriate

[13]Note that since there are no local assumptions the derivation is all global.

number of universal quantifiers, are proved by adding to the following derivation a corresponding number of $\forall$-I rules.

$$
\begin{array}{ll}
\mathcal{C}_0\langle\rangle & \\
\quad \mathcal{C}_0\langle[\lambda:\forall v(\alpha\rightarrow\beta)]\rangle & \text{(assumption)}\\
\quad\quad \mathcal{C}_0\langle[\lambda:\forall v\alpha]\rangle & \text{(assumption)}\\
\quad\quad\quad \mathcal{C}_0\langle[\lambda:c_\beta^{\langle\forall v,\lambda\rangle}]\rangle & \text{(assumption)}\\
\quad\quad\quad \tilde{\mathcal{C}}_1\langle\lambda:\alpha(v/c_\beta^{\langle\forall v,\lambda\rangle})\rightarrow\beta(v/c_\beta^{\langle\forall v,\lambda\rangle})\rangle & (\forall\text{-E})\\
\quad\quad\quad \tilde{\mathcal{C}}_2\langle\lambda:\alpha(v/c_\beta^{\langle\forall v,\lambda\rangle})\rangle & (\forall\text{-E})\\
\quad\quad\quad \tilde{\mathcal{C}}_3\langle\lambda:\beta(v/c_\beta^{\langle\forall v,\lambda\rangle})\rangle & (\rightarrow\text{-E})\\
\quad\quad \tilde{\mathcal{C}}_3\langle\lambda:\forall v\beta\rangle & (\forall\text{-I})\\
\quad \tilde{\mathcal{C}}_4\langle\lambda:\forall v\alpha\rightarrow\forall v\beta\rangle & (\rightarrow\text{-I})\\
\mathcal{C}'\langle\lambda:\forall v(\alpha\rightarrow\beta)\rightarrow(\forall v\alpha\rightarrow\forall v\beta)\rangle & (\rightarrow\text{ I})
\end{array}
$$

Case 18: [Ax19] Let $\mathcal{C}_0=\mathcal{C}_{\{\}}$ and let λ be an arbitrary ground term of $Mon(\mathcal{L}_L,\mathcal{L}_K)$. Then the following derivation proves that given the closed axiom schema $\forall u(\forall v\alpha\rightarrow\alpha(v/u))$, $\mathcal{C}_0\vdash_{Q_{\mathbf{a}}K-\text{MLDS}}\forall u(\forall v\alpha\rightarrow\alpha(v/u))$. Again, more general instantiations of the schema (in which α and β contain free variables other than v), given by prefixing the above formula with an appropriate number of universal quantifiers, are proved by adding to the following derivation a corresponding number of $\forall$-I rules. In what follows, $\gamma(u)$ denotes the subformula $\forall v\alpha\rightarrow\alpha(v/u)$.

$$
\begin{array}{ll}
\mathcal{C}_0\langle\rangle & \\
\quad \mathcal{C}_0\langle[\lambda:c_{\gamma(u)}^{\langle\forall u,\lambda\rangle}]\rangle & \text{(assumption)}\\
\quad\quad \mathcal{C}_0\langle[\lambda:\forall v\alpha]\rangle & \text{(assumption)}\\
\quad\quad \tilde{\mathcal{C}}\langle\lambda:\alpha(v/c_{\gamma(u)}^{\langle\forall u,\lambda\rangle})\rangle & (\forall\text{-E})\\
\quad \tilde{\mathcal{C}}_1\langle\lambda:\forall v\alpha\rightarrow\alpha(v/c_{\gamma(u)}^{\langle\forall u,\lambda\rangle})\rangle & (\rightarrow\text{-I})\\
\mathcal{C}'\langle\lambda:\forall u(\forall v\alpha\rightarrow\alpha(v/u))\rangle & (\forall\text{-I})
\end{array}
$$

INDUCTIVE STEP:
The proof is identical to the inductive step of the proof of Lemma 3.4, since the inference rules of the Kripke system QK_{Ax} are identical to that considered in the propositional modal axiomatic system K_{Ax} (see Definition 1.10), where wffs of the propositional $\mathcal{L}_M$ are replaced by sentences of the quantified modal language $Ext(\mathcal{L}_L,\mathcal{L}_K)$.

⋆

The following theorem generalises the above result to the family of quantified modal logics T, $S4$ and $S5$. Recall that these are the only logics considered by Kripke in [Kri63b].

Theorem 6.6 {Simple Correspondence for modal logic $QK\partial_K$}

Let $\partial_K \in \{T, S4, S5\}$, let $\langle QK\partial_{K(Ax)}, \vdash_{QK\partial_{K(Ax)}}\rangle$ be the corresponding Kripke axiomatic system obtained by adding the (closure of the) appropriate set of modal axiom schemas from the set $\{\mathbf{T}, \mathbf{4}, \mathbf{B}\}$ to QK_{Ax}, and let $QK\delta_K$-MLDS be the corresponding quantified MLDS whose labelling algebra is $\mathcal{A}_{K\delta_K}$. Let $\mathcal{C}_{\{\}} = \langle\{\}, \mathcal{F}_{\{\}}, \mathcal{T}^+_{\{\}}, \mathcal{T}^-_{\{\}}\rangle$, where for any ground term $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_K)$, $\mathcal{F}_{\{\}}(\lambda) = \{\}$, $\mathcal{T}^+_{\{\}}(\lambda) = \{\}$ and $\mathcal{T}^-_{\{\}}(\lambda) = \{\}$. Let α be a sentence of $\mathcal{L}_K$. Then

$$\vdash_{QK\partial_{K(Ax)}} \alpha$$
if and only if
for all ground terms $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_K)$, $\mathcal{C}_{\{\}} \vdash_{Q_{\mathbf{a}}K\delta_K-\text{MLDS}} \lambda:\alpha$

Proof:
The proof is identical to the proof of Theorem 3.6, but building on Lemma 6.3 instead of Lemma 3.4. ⋆

Theorem 6.6 shows how the definition of validity of a modal sentence given in Kripke arbitrary domain semantics can be reformulated in terms of the quantified MLDS semantics. To conclude the technical results of this thesis, and as in Chapter 3, the following two theorems generalise this result to the notion of semantic entailment from a modal theory, considering first a derivability from a set of global assumptions and second a derivability from both a set of global and a set of local assumptions.

Theorem 6.7 {Restricted Strong Correspondence for modal logic $QK\partial_K$}

Let $\partial_K \in \{T, S4, S5\}$, let $\langle QK\partial_{K(Ax)}, \vdash_{QK\partial_{K(Ax)}}\rangle$ be the corresponding Kripke axiomatic system, and let $QK\delta_K$-MLDS be the corresponding quantified MLDS whose labelling algebra is $\mathcal{A}_{K\delta_K}$. Let S be a set of sentences of $\mathcal{L}_K$. Let $\mathcal{C}_S = \langle\{\}, \mathcal{F}_S, \mathcal{T}^+_{\{\}}, \mathcal{T}^-_{\{\}}\rangle$, where for any ground term $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_K)$, $\mathcal{F}_S(\lambda) = S$, $\mathcal{T}^+_{\{\}}(\lambda) = \{\}$ and $\mathcal{T}^-_{\{\}}(\lambda) = \{\}$. Let α be a sentence of $\mathcal{L}_K$. Then

$$S \vdash_{QK\partial_{K(Ax)}} \{\} \Rightarrow \alpha$$
if and only if
for all ground terms $\lambda \in Func(\mathcal{L}_L, \mathcal{L}_K)$, $\mathcal{C}_S \models_{QK\delta_K-\text{MLDS}} \lambda:\alpha$

Proof:
The proof is identical to the proof of Theorem 3.7, but building on Theorem 6.6 instead of Theorem 3.6. ⋆

Theorem 6.8 {Strong Correspondence for modal logic $QK\partial_K$}

Let $\partial_K \in \{T, S4, S5\}$, let $\langle QK\partial_{K(Ax)}, \vdash_{QK\partial_{K(Ax)}}\rangle$ be the corresponding axiomatic system and let $QK\delta_K$-MLDS be the corresponding quantified MLDS whose labelling algebra is $\mathcal{A}_{K\delta_K}$. Let S and U be sets of sentences of $\mathcal{L}_K$. Let $\mathcal{C}_{SU} = \langle\{\}, \mathcal{F}_{SU}, \mathcal{T}^+_{\{\}}, \mathcal{T}^-_{\{\}}\rangle$, where for any ground term λ of $Func(\mathcal{L}_L, \mathcal{L}_K)$, $\mathcal{T}^+_{\{\}}(\lambda) = \{\}$, $\mathcal{T}^-_{\{\}}(\lambda) = \{\}$ and if $\lambda \neq W_0$, $\mathcal{F}_{SU}(\lambda) = S$, and $\mathcal{F}_{SU}(W_0) = S \cup U$. Let α be a sentence of $\mathcal{L}_K$. Then

$$\mathcal{C}_{SU} \models_{QK\delta_K-\text{MLDS}} W_0 : \alpha \quad \text{if and only if} \quad S \models_{QK\partial_{K(Ax)}} U \Rightarrow \alpha$$

Proof:
The proof is identical to the proof of Theorem 3.8, but building on Theorem 6.7 instead of Theorem 3.7. ⋆

6.4 Discussion

The soundness (Theorem 6.1), completeness (Theorem 6.4) and strong correspondence (Theorem 6.8) results that have been proved in this chapter justify the claims of Chapter 4 and 5 – that quantified extensions of MLDS can be defined with respect to arbitrary, increasing, decreasing and constant domains. It is perhaps worth emphasising a technical feature of the proof of completeness. The proof of Theorem 6.4 is a direct extension of the completeness proof for propositional MLDSs (Theorem 3.4), no additional structural complexity is needed. This is made possible by the explicit syntactic reference to possible worlds within the logic, and the use of Skolem-like constants for existential and universal quantifiers which are indexed with particular associated formulae. This simplicity is in contrast to the standard Henkin completeness proof for first-order logic, where extra notions such as "E-formulae" and "E-properties" of sets have to be introduced in order to deal with existentially quantified sentences (e.g. see [HC68]). This use of syntactic names for both worlds and objects also allows inference rules to be defined in a natural way, without requiring special side conditions.

6.5 Summary

The results of this chapter regarding quantified MLDSs mirror those given in Chapter 3 for the propositional case. Theorems 6.1 and 6.4 show that the natural deduction systems described in Chapter 5 for arbitrary, increasing, decreasing and constant domains are all sound and complete with respect to their associated model-theoretic semantics. Finally, Theorem 6.8 shows that any quantified modal theory as defined by Kripke in [Kri63b] can be translated into an equivalent quantified MLDS configuration. This translation preserves both derivability and semantic entailment as defined in terms of Kripke arbitrary varying domain semantics. Again, quantified MLDSs are more general than the quantified axiomatic systems described by Kripke in the sense that inverse translation cannot be applied to any arbitrary configuration.

Conclusions

In this thesis a new formalism has been developed for a wide family of propositional and predicate modal logics, which retains and extends the benefits of the two standard approaches (implicit and explicit) to modal logic. In the preface to this thesis, a number of claims were made regarding the contribution of this work. The following discussion summarises how each of these claims has be substantiated.

A proof theory based on natural deduction has been defined which is *uniform* with respect to a wide family of propositional and predicate modal logics, i.e. the same set of inference rules is applicable to each logic. As explained in Chapter 1, existing results in Correspondence Theory [vB84] have shown that various classes of modal logics can be identified by particular classes of frames, which are essentially classically definable mathematical structures. In other words, different modal logics can be generated by imposing different properties on the accessibility relations of their underlying structures. However, in existing implicit proof theoretical systems, semantic notions such as possible worlds and relations between them are embedded in the behaviour of the modal operators. Hence (with the exception of Hilbert systems), different deduction rules are required to mirror different corresponding semantic structures. Fitting's natural deduction system has been given as an example of such a "non-uniform" approach in Chapter 1. MLDSs are able to avoid this non-uniformity because they include syntactic mechanisms to express information about semantic structures directly, side by side with other logical information. Structural inference rules provide a disciplined interface between the two types of information.

The use of labelling algebrae, as defined in Chapter 1, allows the formalisation of new modal logics which none of the existing proof systems (including Hilbert systems) can deal with. An example of an *irreflexive* modal labelled deductive system is given in Chapter 1, and many other extensions can be defined, as discussed in Chapter 2, Section 2.2.1. The irreflexive system is of interest because, as shown by Correspondence Theory, it is not possible to identify the set of irreflexive frames by a characteristic modal axiom schema.

The notion of a configuration, in either propositional or predicate MLDSs, generalises the notion of a modal theory, since it can be used to describe domains containing not just one, but an explicitly specified structure of local actual worlds. Configurations are effectively structures of local modal theories. This is illustrated by the *strong correspondence* theorems of Chapters 3 and 6 (Theorems 3.8 and 6.8) which show that there is a one-way translation of standard modal theories into configurations, which preserves both derivability and semantic entailment.

Part II of this thesis has contributed to the theoretical study of predicate modal logic as follows. It has been shown that the MLDS approach facilitates a "conservative" extension of propositional MLDSs to first-order systems. In Chapter 5, inference rules for world-relative quantifiers are simply added to the underlying body of propositional modal rules. The resulting proof system is sound and complete with respect to an arbitrary varying domain semantics. Further simple additions of appropriate intuitive "visa" rules give expanded proof systems which are sound and complete with respect to increasing, decreasing and fixed domain semantics respectively. Moreover, each of these semantics is generated in a natural way by extending standard propositional possible world semantics with world-relative interpretations of quantifiers. Both the semantics and the proof theory allow the inclusion of function symbols of arbitrary arity in the language. As mentioned above, Theorem 6.8 shows that the arbitrary varying domain semantics given in Chapter 5 is equivalent to Kripke arbitrary varying domain semantics for the restricted class of domains and languages considered by Kripke [Kri63b].

The work in this thesis suggests several interesting avenues for future investigation – in terms of theory, applications and implementation.

As regards the theoretical study of classical quantified modal logic, it seems likely that the addition of the equality predicate to the quantified modal language would be unproblematical, and could lead to even greater system flexibility. In particular, it would be interesting to investigate the extent to which equality could be used to "simulate" systems based on non-rigid designators [Fit93], without requiring any additional syntactic constructions. Extensions to other types of logics also suggest themselves. Chapter 2 already includes a brief discussion on how MLDSs can be formulated with multiple modalities. It seems likely that non-classical modal logics (e.g. substructural modal logics) could also be formulated, by making appropriate adjustments to individual (non-modal) deduction rules and individual axiom schemas in the first-order formalisation of the semantics.

As regards applications, preliminary investigations are already underway on the use of MLDSs to formalise distributed and active database systems. It seems likely that structures of databases can be represented as configurations, where each constant label is associated with a local database, and physical connections between databases are described by R-literals. Within this setting, modal operators could be used to impose constraints on the "evolution" of local databases whenever "updates" are made at a particular node. This idea has also been discussed in [Ben95].

Finally, it seems possible that automated theorem provers for MLDS could be developed, either based on the labelled natural deduction systems described in Sections 2.1 and 5.1, or by applying classical theorem provers to the extended algebrae defined in Sections 2.2 and 5.2. Before exploring such possibilities, it would of course be beneficial to undertake an analysis of the complexity and decidability of the various MLDS systems described here, and, if possible, to prove normalisation properties for the natural deduction systems.

Associated Publication

Part I of this thesis is published in summarised form in [Rus96]: *Generalising Propositional Modal Logic Using Labelled Deductive Systems.*

Appendix A

Remaining Details of Proofs in Part I

A.1 Extra Cases in Proof of Lemma 3.1

In this section proofs omitted in Lemma 3.1 are given.

Case 3: $\wedge$–**Introduction** ($\mathcal{I}_{\wedge I}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\wedge I}$. Then there exist declarative units of the form $\lambda : \alpha \in \mathcal{C}_{n-1}$, and $\lambda : \beta \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \alpha \wedge \beta]$. Therefore $\{[\alpha]^*(\lambda), [\beta]^*(\lambda)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\alpha \wedge \beta]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha \wedge \beta]^*(\lambda)$. This is proved applying axiom schema $(Ax1)$, as shown in the following derivation:

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{A}^+\\ \vdots \\ \forall x(([\alpha]^*(x) \wedge [\beta]^*(x)) \rightarrow [\alpha \wedge \beta]^*(x))\end{array}}{([\alpha]^*(\lambda) \wedge [\beta]^*(\lambda)) \rightarrow [\alpha \wedge \beta]^*(\lambda)} \qquad \dfrac{\begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots \\ [\alpha]^*(\lambda)\end{array} \qquad \begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots \\ [\beta]^*(\lambda)\end{array}}{[\alpha]^*(\lambda) \wedge [\beta]^*(\lambda)}}{[\alpha \wedge \beta]^*(\lambda)}
$$

Case 4: $\vee$–**Introduction** ($\mathcal{I}_{\vee I}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\vee I}$. Then there exists a declarative unit of the form $\lambda : \alpha \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is either equal to $\mathcal{C}_{n-1} + [\lambda : \alpha \vee \beta]$ or to $\mathcal{C}_{n-1} + [\lambda : \beta \vee \alpha]$. Only the first case in considered since the argument for the second case is analogous. Then, $\{[\alpha]^*(\lambda)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\alpha \vee \beta]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that

$\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha \vee \beta]^*(\lambda)$. This is proved applying axiom schema $(Ax3)$, as shown in the following derivation:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{A}^+\\ \vdots \\ \forall x([\alpha]^*(x) \vee [\beta]^*(x) \to [\alpha \vee \beta]^*(x))\end{matrix}}{([\alpha]^*(\lambda) \vee [\beta]^*(\lambda)) \to [\alpha \vee \beta]^*(\lambda)} \qquad \dfrac{\begin{matrix}FOT(\mathcal{C}_{n-1})\\ \vdots \\ [\alpha]^*(\lambda)\end{matrix}}{[\alpha]^*(\lambda) \vee [\beta]^*(\lambda)}}{[\alpha \vee \beta]^*(\lambda)}$$

Case 5: $\to$–**Elimination** $(\mathcal{I}_{\to E})$

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\to E}$. Then there exist declarative units of the form $\lambda : \alpha \to \beta \in \mathcal{C}_{n-1}$, and $\lambda : \alpha \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \beta]$. Therefore $\{[\alpha \to \beta]^*(\lambda), [\alpha]^*(\lambda)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\beta]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\beta]^*(\lambda)$. This is proved applying axiom schema $(Ax4)$, as shown in the following derivation:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{A}^+\\ \vdots \\ [\alpha \to \beta]^*(\lambda) \to ([\alpha]^*(\lambda) \to [\beta]^*(\lambda))\end{matrix} \qquad \begin{matrix}FOT(\mathcal{C}_{n-1})\\ \vdots \\ [\alpha \to \beta]^*(\lambda)\end{matrix}}{[\alpha]^*(\lambda) \to [\beta]^*(\lambda)} \qquad \begin{matrix}FOT(\mathcal{C}_{n-1})\\ \vdots \\ \vdots \\ [\alpha]^*(\lambda)\end{matrix}}{[\beta]^*(\lambda)}$$

Case 6: $\neg$–**Elimination** $(\mathcal{I}_{\neg E})$

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\neg E}$. Then there exists a declarative unit of the form $\lambda : \neg\neg\alpha \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \alpha]$. Therefore $\{[\neg\neg\alpha]^*(\lambda)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda)$. This is proved applying axiom schema $(Ax2)$ as shown in the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}FOT(\mathcal{C}_{n-1})\\ \vdots \\ [\neg\neg\alpha]^*(\lambda)\end{matrix} \qquad \begin{matrix}\mathcal{A}^+\\ \vdots \\ [\neg\neg\alpha]^*(\lambda) \to \neg[\neg\alpha]^*(\lambda)\end{matrix}}{\neg[\neg\alpha]^*(\lambda)} \qquad \begin{matrix}\mathcal{A}^+\\ \vdots \\ \vdots \\ \neg[\neg\alpha]^*(\lambda) \to \neg\neg[\alpha]^*(\lambda)\end{matrix}}{\neg\neg[\alpha]^*(\lambda)}}{[\alpha]^*(\lambda)}$$

Case 7: $\Diamond$–**Introduction** ($\mathcal{I}_{\Diamond I}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\Diamond I}$. Then there exists a declarative unit of the form $\lambda_2 : \alpha \in \mathcal{C}_{n-1}$, an R-literal of the form $R(\lambda_1, \lambda_2) \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda_1 : \Diamond\alpha]$. Therefore $\{[\alpha]^*(\lambda_2), R(\lambda_1, \lambda_2)\} \subseteq FOT(\mathcal{C}_{n-1})$ and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\Diamond\alpha]^*(\lambda_1)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\Diamond\alpha]^*(\lambda_1)$. This is proved applying axiom schema ($Ax7$), as shown in the following derivations:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots\\ R(\lambda_1,\lambda_2)\end{array}\qquad \begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots\\ [\alpha]^*(\lambda_2)\end{array}}{R(\lambda_1,\lambda_2)\wedge[\alpha]^*(\lambda_2)}}{\exists y(R(\lambda_1,y)\wedge[\alpha]^*(y))}\qquad \begin{array}{c}\mathcal{A}^+\\ \vdots\\ \vdots\\ \vdots\\ (\mathrm{Ax7})\end{array}}{[\Diamond\alpha]^*(\lambda_1)}$$

Case 8: $\Box$–**Elimination** ($\mathcal{I}_{\Box E}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\Box E}$. Then there exists a declarative unit of the form $\lambda_1 : \Box\alpha \in \mathcal{C}_{n-1}$, an R-literal of the form $R(\lambda_1, \lambda_2) \in \mathcal{C}_{n-1}$, and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda_2 : \alpha]$. Therefore $\{[\Box\alpha]^*(\lambda_1), R(\lambda_1, \lambda_2)\} \subseteq FOT(\mathcal{C}_{n-1})$, and $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda_2)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda_2)$. This is proved applying axiom schema (Ax8), as shown in the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{A}^+\\ \vdots\\ (\mathrm{Ax8})\end{array}\qquad \begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots\\ [\Box\alpha]^*(\lambda_1)\end{array}}{\forall y(R(\lambda_1,y)\rightarrow[\alpha]^*(y))}}{R(\lambda_1,\lambda_2)\rightarrow[\alpha]^*(\lambda_2)}\qquad \begin{array}{c}FOT(\mathcal{C}_{n-1})\\ \vdots\\ \vdots\\ \vdots\\ R(\lambda_1,\lambda_2)\end{array}}{[\alpha]^*(\lambda_2)}$$

Case 12: $\rightarrow$–**Introduction** ($\mathcal{I}_{\rightarrow I}$)

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\rightarrow I}$. Then there exist declarative units of the form $\lambda : \alpha$, $\lambda : \beta$, $\lambda : \alpha \rightarrow \beta$ such that $\mathcal{C}_{n-1} + [\lambda : \alpha] \vdash_{\mathrm{MLDS}} \lambda : \beta$ and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \alpha \rightarrow \beta]$. Therefore $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\alpha \rightarrow \beta]^*(\lambda)\}$.

Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha \rightarrow \beta]^*(\lambda)$. Let $\langle\{\mathcal{C}_{n-1} + [\lambda : \alpha], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle$, with $\lambda : \beta \in \tilde{\mathcal{C}}$, be a proof of the smallest size of $\mathcal{C}_{n-1} + [\lambda : \alpha] \vdash_{\text{MLDS}} \lambda : \beta$, condition of $\mathcal{I}_{\rightarrow I}$. By hypothesis of the inductive step, $0 < l(\mathcal{C}_{n-1}/\mathcal{C}_n, \mathcal{I}_{\rightarrow I}) = 1 + l_1 \leq L$. Then $0 \leq l_1 = l(\langle\{\mathcal{C}_{n-1} + [\lambda : \alpha], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle) < L$. By inductive hypothesis, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\} \vdash_{FOL} FOT(\tilde{\mathcal{C}})$ and in particular $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\} \vdash_{FOL} [\beta]^*(\lambda)$. By the Deduction Theorem of first order logic, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda) \rightarrow [\beta]^*(\lambda)$. So by axiom schema $(Ax4)$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha \rightarrow \beta]^*(\lambda)$ as shown in the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{A}^+\\ \vdots\\ \forall x(([\alpha]^*(x) \rightarrow [\beta]^*(x)) \rightarrow [\alpha \rightarrow \beta]^*(x))\end{array}}{([\alpha]^*(\lambda) \rightarrow [\beta]^*(\lambda)) \rightarrow [\alpha \rightarrow \beta]^*(\lambda)} \qquad \dfrac{\begin{array}{c}\mathcal{A}^+ \cup FOT(\mathcal{C}_{n-1})\\ \vdots\\ \vdots\end{array}}{[\alpha]^*(\lambda) \rightarrow [\beta]^*(\lambda)}}{[\alpha \rightarrow \beta]^*(\lambda)}$$

Case 13: $\neg$–**Introduction** $(\mathcal{I}_{\neg I})$

In this case $\mathcal{C}_{n-1}/\mathcal{C}_n \in \mathcal{I}_{\neg I}$. Then there exists a declarative unit of the form $\lambda : \alpha$ and a term $\overline{\lambda}$ such that $\mathcal{C}_{n-1} + [\lambda : \alpha] \vdash_{\text{MLDS}} \overline{\lambda} : \bot$ and $\mathcal{C}_n$ is equal to $\mathcal{C}_{n-1} + [\lambda : \neg\alpha]$. Therefore $FOT(\mathcal{C}_n) = FOT(\mathcal{C}_{n-1}) \cup \{[\neg\alpha]^*(\lambda)\}$. Since, by reflexivity of $\vdash_{FOL}$, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} FOT(\mathcal{C}_{n-1})$, it remains to show that $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\neg\alpha]^*(\lambda)$. Let $\langle\{\mathcal{C}_{n-1} + [\lambda : \alpha], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle$, with $\overline{\lambda} : \bot \in \tilde{\mathcal{C}}$, be a proof of the smallest size of $\mathcal{C}_{n-1} + [\lambda : \alpha] \vdash_{\text{MLDS}} \overline{\lambda} : \bot$, condition of $\mathcal{I}_{\neg I}$. By hypothesis of the inductive step, $0 < l(\mathcal{C}_{n-1}/\mathcal{C}_n, \mathcal{I}_{\neg I}) = 1 + l_1 \leq L$. Then $0 \leq l_1 = l(\langle\{\mathcal{C}_{n-1} + [\lambda : \alpha], \ldots, \tilde{\mathcal{C}}\}, \tilde{m}\rangle) < L$. By inductive hypothesis, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\} \vdash_{FOL} FOT(\tilde{\mathcal{C}})$ and in particular $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \cup \{[\alpha]^*(\lambda)\} \vdash_{FOL} \bot$. By the Deduction Theorem of first order logic, $\mathcal{A}^+, FOT(\mathcal{C}_{n-1}) \vdash_{FOL} [\alpha]^*(\lambda) \rightarrow \bot$. So, since $[\alpha]^*(\lambda) \rightarrow \bot$ is equivalent to $\neg[\alpha]^*(\lambda)$, by axiom schema (Ax2), $\mathcal{A}^+, FOT(\mathcal{C}_n) \vdash_{FOL} [\neg\alpha]^*(\lambda)$ as shown in the following derivation:

$$\dfrac{\begin{array}{c}\mathcal{A}^+ \cup FOT(\mathcal{C}_{n-1})\\ \vdots\\ \neg[\alpha]^*(\lambda)\end{array} \qquad \begin{array}{c}\mathcal{A}^+\\ \vdots\\ \neg[\alpha]^*(\lambda) \rightarrow [\neg\alpha]^*(\lambda)\end{array}}{[\neg\alpha]^*(\lambda)}$$

A.2 Extra Cases in Proof of Theorem 3.5

In this section proofs omitted in Theorem 3.5 are given.

Case 2: **(Ax2):** $\forall x([\neg\alpha]^*(x) \equiv \neg[\alpha]^*(x))$

It is sufficient to prove that $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\neg\alpha]^*(V(x))$ if and only if $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} \neg[\alpha]^*(V(x))$.

("Only if" half:) Assume that $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\neg\alpha]^*(V(x))$. Then $[\neg\alpha]^*(V(x)) \in FOT(\mathcal{C}_{\mathrm{mcc}})$ which implies that $V(x) : \neg\alpha \in \mathcal{C}_{\mathrm{mcc}}$. Thus, by Proposition 3.11, $V(x) : \alpha \notin \mathcal{C}_{\mathrm{mcc}}$ and therefore $[\alpha]^*(V(x)) \notin FOT(\mathcal{C}_{\mathrm{mcc}})$. So $V(x) \notin \| [\alpha]^* \|_{I(\mathcal{C}_{\mathrm{mcc}})}$. Hence $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} \neg[\alpha]^*(V(x))$.

("If" half:) Assume that $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} \neg[\alpha]^*(V(x))$.
Then, $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \not\Vdash_{FOL} [\alpha]^*(V(x))$, which implies that $V(x) \notin \| [\alpha]^* \|_{I(\mathcal{C}_{\mathrm{mcc}})}$. Thus $[\alpha]^*(V(x)) \notin FOT(\mathcal{C}_{\mathrm{mcc}})$ and then $V(x) : \alpha \notin \mathcal{C}_{\mathrm{mcc}}$. By Proposition 3.11, $V(x) : \neg\alpha \in \mathcal{C}_{\mathrm{mcc}}$. So $[\neg\alpha]^*(V(x)) \in FOT(\mathcal{C}_{\mathrm{mcc}})$. Hence $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\neg\alpha]^*(V(x))$.

Case 3: **(Ax3):** $\forall x([\alpha \vee \beta]^*(x) \equiv ([\alpha]^*(x) \vee [\beta]^*(x))$.

It is sufficient to prove that $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\alpha \vee \beta]^*(V(x))$ if and only if $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\alpha]^*(V(x))$ or $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\beta]^*(V(x))$.

("Only If" half:) Assume that $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\alpha \vee \beta]^*(V(x))$.
Then, $[\alpha \vee \beta]^*(V(x)) \in FOT(\mathcal{C}_{\mathrm{mcc}})$, which implies that $V(x) : \alpha \vee \beta \in \mathcal{C}_{\mathrm{mcc}}$. By Proposition 3.14, $V(x) : \alpha \in \mathcal{C}_{\mathrm{mcc}}$ or $V(x) : \beta \in \mathcal{C}_{\mathrm{mcc}}$. So $[\alpha]^*(V(x)) \in FOT(\mathcal{C}_{\mathrm{mcc}})$ or $[\beta]^*(V(x)) \in FOT(\mathcal{C}_{\mathrm{mcc}})$. Hence $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\alpha]^*(V(x))$ or $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\beta]^*(V(x))$.

("If" half:)

Assume that $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\alpha]^*(V(x))$ or $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\beta]^*(V(x))$. Then either $[\alpha]^*(V(x)) \in FOT(\mathcal{C}_{\mathrm{mcc}})$ or $[\beta]^*(V(x)) \in FOT(\mathcal{C}_{\mathrm{mcc}})$. So either $V(x) : \alpha \in \mathcal{C}_{\mathrm{mcc}}$ or $V(x) : \beta \in \mathcal{C}_{\mathrm{mcc}}$. By Proposition 3.14, $V(x) : \alpha \vee \beta \in \mathcal{C}_{\mathrm{mcc}}$. So $[\alpha \vee \beta]^*(V(x)) \in FOT(\mathcal{C}_{\mathrm{mcc}})$. Hence $\mathcal{M}(\mathcal{C}_{\mathrm{mcc}}), V \Vdash_{FOL} [\alpha \vee \beta]^*(V(x))$.

Case 4: **(Ax4):** $\forall x([\alpha \rightarrow \beta]^*(x) \equiv ([\alpha]^*(x) \rightarrow [\beta]^*(x)))$

It is sufficient to prove that $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} [\alpha \rightarrow \beta]^*(V(x))$ if and only if,

$$\text{if } \mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} [\alpha]^*(V(x)) \text{ then } \mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} [\beta]^*(V(x))$$

("Only if" half:) Assume that $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} [\alpha \rightarrow \beta]^*(V(x))$ and that $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} [\alpha]^*(V(x))$. Then $[\alpha \rightarrow \beta]^*(V(x)) \in FOT(\mathcal{C}_{\text{mcc}})$ and $[\alpha]^*(V(x)) \in FOT(\mathcal{C}_{\text{mcc}})$. Then $V(x) : \alpha \rightarrow \beta \in \mathcal{C}_{\text{mcc}}$ and $V(x) : \alpha \in \mathcal{C}_{\text{mcc}}$. By Proposition 3.16, $V(x) : \beta \in \mathcal{C}_{\text{mcc}}$ and then $[\beta]^*(V(x)) \in FOT(\mathcal{C}_{\text{mcc}})$. Hence $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} [\beta]^*(V(x))$.

("If" half:) Assume that $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} \neg[\alpha]^*(V(x))$ or $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} [\beta]^*(V(x))$. By satisfiability of $(Ax2)$, $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} [\neg\alpha]^*(V(x))$. So, $[\neg\alpha]^*(V(x)) \in FOT(\mathcal{C}_{\text{mcc}})$ or $[\beta]^*(V(x)) \in FOT(\mathcal{C}_{\text{mcc}})$, which implies that $V(x) : \neg\alpha \in \mathcal{C}_{\text{mcc}}$ or $V(x) : \beta \in \mathcal{C}_{\text{mcc}}$. Therefore, by Proposition 3.15, $V(x) : \alpha \rightarrow \beta \in \mathcal{C}_{\text{mcc}}$ and $[\alpha \rightarrow \beta]^*(V(x)) \in FOT(\mathcal{C}_{\text{mcc}})$. Hence $\mathcal{M}(\mathcal{C}_{\text{mcc}}), V \Vdash_{FOL} [\alpha \rightarrow \beta]^*(V(x))$.

A.3 Collected Inference Rules for MLDS

A.3.1 Inference rules for propositional and predicate MLDS

Classical rules

$\wedge$–**Elimination,** $\mathcal{I}_{\wedge E}$

For all configurations $\mathcal{C}$, terms λ and wffs α and β, $\mathcal{C}/\mathcal{C}+[\lambda:\alpha]$ and $\mathcal{C}/\mathcal{C}+[\lambda:\beta]$ are members of the inference rule $\wedge$–Elimination (sometimes written $\mathcal{I}_{\wedge E}$) if

- $\lambda:\alpha \wedge \beta \in \mathcal{C}$

This is sometimes represented graphically as

$$\mathcal{I}_{\wedge E}\ \frac{\mathcal{C}\langle\lambda:\alpha\wedge\beta\rangle}{\mathcal{C}'\langle\lambda:\alpha\rangle} \qquad\qquad \frac{\mathcal{C}\langle\lambda:\alpha\wedge\beta\rangle}{\mathcal{C}'\langle\lambda:\beta\rangle}\ \mathcal{I}_{\wedge E}$$

$\wedge$–**Introduction,** $\mathcal{I}_{\wedge I}$

For all configurations $\mathcal{C}$, terms λ and wffs α and β, $\mathcal{C}/\mathcal{C}+[\lambda:\alpha\wedge\beta]$ is a member of the inference rule $\wedge$-Introduction (sometimes written $\mathcal{I}_{\wedge I}$) if

- $\lambda:\alpha \in \mathcal{C}$
- $\lambda:\beta \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle\lambda:\alpha,\lambda:\beta\rangle}{\mathcal{C}'\langle\lambda:\alpha\wedge\beta\rangle}\ \mathcal{I}_{\wedge I}$$

$\vee$–**Elimination,** $\mathcal{I}_{\vee E}$

For all configurations $\mathcal{C}$, terms λ and wffs γ, $\mathcal{C}/\mathcal{C}+[\lambda:\gamma]$ is a member of the inference rule $\vee$–Elimination (sometimes written $\mathcal{I}_{\vee E}$) if there exist wffs α and β such that

- $\lambda:\alpha\vee\beta \in \mathcal{C}$
- $\mathcal{C}+[\lambda:\alpha] \vdash_{\mathrm{MLDS}} \lambda:\gamma$
- $\mathcal{C}+[\lambda:\beta] \vdash_{\mathrm{MLDS}} \lambda:\gamma$

This is sometimes represented graphically as

$$\dfrac{\mathcal{C}\langle\lambda : \alpha \vee \beta\rangle \quad \begin{array}{c}\mathcal{C}\langle[\lambda : \alpha]\rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle\lambda : \gamma\rangle\end{array} \quad \begin{array}{c}\mathcal{C}\langle[\lambda : \beta]\rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle\lambda : \gamma\rangle\end{array}}{\mathcal{C}'\langle\lambda : \gamma\rangle}\ \mathcal{I}_{\vee E}$$

$\vee$–Introduction, $\mathcal{I}_{\vee I}$

For all configurations $\mathcal{C}$, terms λ and wffs α and β, $\mathcal{C}/\mathcal{C} + [\lambda : \alpha \vee \beta]$ and $\mathcal{C}/\mathcal{C} + [\lambda : \beta \vee \alpha]$ are members of the inference rule $\vee$–Introduction (sometimes written $\mathcal{I}_{\vee I}$) if

- $\lambda : \alpha \in \mathcal{C}$

This is sometimes represented graphically as

$$\mathcal{I}_{\vee I}\ \dfrac{\mathcal{C}\langle\lambda : \alpha\rangle}{\mathcal{C}'\langle\lambda : \alpha \vee \beta\rangle} \qquad\qquad \dfrac{\mathcal{C}\langle\lambda : \alpha\rangle}{\mathcal{C}'\langle\lambda : \beta \vee \alpha\rangle}\ \mathcal{I}_{\vee I}$$

$\rightarrow$–Elimination, $\mathcal{I}_{\rightarrow E}$

For all configurations $\mathcal{C}$, terms λ and wffs β, $\mathcal{C}/\mathcal{C} + [\lambda : \beta]$ is a member of the inference rule $\rightarrow$–Elimination (sometimes written $\mathcal{I}_{\rightarrow E}$) if for some wff α,

- $\lambda : \alpha \rightarrow \beta \in \mathcal{C}$
- $\lambda : \alpha \in \mathcal{C}$

This is sometimes represented graphically as

$$\dfrac{\mathcal{C}\langle\lambda : \alpha \rightarrow \beta, \lambda : \alpha\rangle}{\mathcal{C}'\langle\lambda : \beta\rangle}\ \mathcal{I}_{\rightarrow E}$$

$\rightarrow$–Introduction, $\mathcal{I}_{\rightarrow I}$

For all configurations $\mathcal{C}$, terms λ and wffs α and β, $\mathcal{C}/\mathcal{C} + [\alpha \rightarrow \beta]$ is a member of the inference rule $\rightarrow$–Introduction (sometimes written $\mathcal{I}_{\rightarrow I}$) if

- $\mathcal{C} + [\lambda : \alpha]\ \vdash_{\mathrm{MLDS}}\ \lambda : \beta$

This is sometimes represented graphically as

$$\frac{\begin{array}{c}\mathcal{C}\langle[\lambda:\alpha]\rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle\lambda:\beta\rangle\end{array}}{\mathcal{C}'\langle\lambda:\alpha\rightarrow\beta\rangle}\ \mathcal{I}_{\rightarrow I}$$

$\neg$–Elimination, $\mathcal{I}_{\neg E}$

For all configurations $\mathcal{C}$, terms λ and wffs α, $\mathcal{C}/\mathcal{C}+[\lambda:\alpha]$ is a member of the inference rule $\neg$–Elimination (sometimes written $\mathcal{I}_{\neg E}$) if

- $\lambda:\neg\neg\alpha \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle\lambda:\neg\neg\alpha\rangle}{\mathcal{C}'\langle\lambda:\alpha\rangle}\ \mathcal{I}_{\neg E}$$

$\neg$–Introduction, $\mathcal{I}_{\neg I}$

For all configurations $\mathcal{C}$, terms λ and wffs α, $\mathcal{C}/\mathcal{C}+[\lambda:\neg\alpha]$ is a member of the inference rule $\neg$–Introduction (sometimes written $\mathcal{I}_{\neg I}$) if for some term λ'

- $\mathcal{C}+[\lambda:\alpha]\ \vdash_{\mathrm{MLDS}}\ \lambda':\perp$

This is sometimes represented graphically as

$$\frac{\begin{array}{c}\mathcal{C}\langle[\lambda:\alpha]\rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle\lambda':\perp\rangle\end{array}}{\mathcal{C}'\langle\lambda:\neg\alpha\rangle}\ \mathcal{I}_{\neg I}$$

Modal rules

$\Diamond$–Elimination, $\mathcal{I}_{\Diamond E}$

For all configurations $\mathcal{C}$, terms λ and wffs α, $\mathcal{C}/\mathcal{C}+[f_\alpha(\lambda):\alpha]+[R(\lambda, f_\alpha(\lambda))]$ is a member of the inference rule $\Diamond$–Elimination (sometimes written $\mathcal{I}_{\Diamond E}$) if

- $\lambda : \Diamond\alpha \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda : \Diamond\alpha \rangle}{\mathcal{C}'\langle f_\alpha(\lambda) : \alpha, R(\lambda, f_\alpha(\lambda))\rangle}$$

$\Diamond$–Introduction, $\mathcal{I}_{\Diamond I}$

For all configurations $\mathcal{C}$, terms λ_1 and λ_2 and wffs α, $\mathcal{C}/\mathcal{C} + [\lambda_1 : \Diamond\alpha]$ is a member of the inference rule $\Diamond$–Introduction (sometimes written $\mathcal{I}_{\Diamond I}$) if

- $\lambda_2 : \alpha \in \mathcal{C}$
- $R(\lambda_1, \lambda_2) \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda_2 : \alpha, R(\lambda_1, \lambda_2)\rangle}{\mathcal{C}'\langle \lambda_1 : \Diamond\alpha \rangle}$$

$\Box$–Elimination, $\mathcal{I}_{\Box E}$

For all configurations $\mathcal{C}$, terms λ_1 and λ_2 and wffs α, $\mathcal{C}/\mathcal{C} + [\lambda_2 : \alpha]$ is a member of the inference rule $\Box$–Elimination (sometimes written $\mathcal{I}_{\Box E}$) if

- $\lambda_1 : \Box\alpha \in \mathcal{C}$
- $R(\lambda_1, \lambda_2) \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda_1 : \Box\alpha, R(\lambda_1, \lambda_2)\rangle}{\mathcal{C}'\langle \lambda_2 : \alpha \rangle}$$

$\Box$–Introduction, $\mathcal{I}_{\Box I}$

For all configurations $\mathcal{C}$, terms λ and wffs α, $\mathcal{C}/\mathcal{C} + [\lambda : \Box\alpha]$ is a member of the inference rule $\Box$–Introduction (sometimes written $\mathcal{I}_{\Box I}$) if

- $\mathcal{C} + [R(\lambda, box_\alpha(\lambda))] \vdash_{\text{MLDS}} box_\alpha(\lambda) : \alpha$

This is sometimes represented graphically as

$$\frac{\begin{array}{c}\mathcal{C}\langle[R(\lambda, box_\alpha(\lambda))]\rangle \\ \vdots \\ \tilde{\mathcal{C}}\langle box_\alpha(\lambda):\alpha\rangle\end{array}}{\mathcal{C}'\langle\lambda:\Box\alpha\rangle}$$

Structural rules

R–Assertion, $\mathcal{I}_{R-A}$

For all configurations $\mathcal{C} = \langle\mathcal{D}, \mathcal{F}\rangle$, and R-literals Δ, $\mathcal{C}/\mathcal{C} + [\Delta]$ is a member of the inference rule R–Assertion (sometimes written $\mathcal{I}_{R-A}$) if

- $\mathcal{D}, \mathcal{A} \vdash_{FOL} \Delta$

where $\mathcal{A}$ is a labelling algebra.

$\perp$–Introduction, $\mathcal{I}_{\perp I}$

For all configurations $\mathcal{C}$, any R–literals Δ, and any declarative unit $\lambda : \alpha$, $\mathcal{C}/\mathcal{C} + [\lambda:\alpha]$ is a member of the inference rule $\perp$–Introduction (sometimes written $\mathcal{I}_{\perp I}$) if

- $\Delta \in \mathcal{C}$
- $\overline{\Delta} \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle\Delta, \overline{\Delta}\rangle}{\mathcal{C}'\langle\lambda:\alpha\rangle}$$

R–Introduction, $\mathcal{I}_{R-I}$

For all configurations $\mathcal{C}$, and R-literals Δ, $\mathcal{C}/\mathcal{C} + [\Delta]$ is a member of the inference rule R–Introduction (sometimes written $\mathcal{I}_{R-I}$) if for some term λ'

- $\mathcal{C} + [\overline{\Delta}] \vdash_{\mathrm{MLDS}} \lambda' : \perp$

This is sometimes represented graphically as

$$\frac{\begin{array}{c}\mathcal{C}\langle\overline{\Delta}\rangle\\ \vdots\\ \tilde{\mathcal{C}}\langle\lambda' : \bot\rangle\end{array}}{\mathcal{C}'\langle\Delta\rangle}$$

$\mathcal{C}$–Reduction, $\mathcal{I}_{C-R}$

For all configurations $\mathcal{C}$ and $\mathcal{C}'$, $\mathcal{C}/\mathcal{C}'$ is a member of the inference rule $\mathcal{C}$–Reduction (sometimes written $\mathcal{I}_{C-R}$) if

- $\mathcal{C}' \subseteq \mathcal{C}$

A.3.2 Additional rules for predicate MLDS only

Additional classical rules

$\forall$-Elimination, $\mathcal{I}_{\forall E}$

For all configurations $\mathcal{C}$, ground terms τ, labels λ, wffs α and variables v, $\mathcal{C}/\mathcal{C} + [\lambda : \alpha(v/\tau)]$ is a member of the inference rule $\forall$–Elimination (sometimes written $\mathcal{I}_{\forall-E}$) if

- $\lambda : \forall v\alpha \in \mathcal{C}$
- $\lambda : \tau \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle\lambda : \forall v\alpha\ ,\ \lambda : \tau\rangle}{\mathcal{C}'\langle\lambda : \alpha(v/\tau)\rangle}\ \mathcal{I}_{\forall E}$$

$\forall$-Introduction, $\mathcal{I}_{\forall I}$

For all configurations $\mathcal{C}$, labels λ, wffs α and variables v, $\mathcal{C}/\mathcal{C} + [\forall v\alpha]$ is a member of the inference rule $\forall$–Introduction (sometimes written $\mathcal{I}_{\forall-I}$) if

- $\mathcal{C} + [\lambda : c_\alpha^{\langle\forall v,\lambda\rangle}] \vdash_{\mathrm{MLDS}} \lambda : \alpha(v/c_\alpha^{\langle\forall v,\lambda\rangle})$

This is sometimes represented graphically as

$$\dfrac{\begin{array}{c}\mathcal{C}\langle[\lambda : c_{\alpha}^{\langle\forall v,\lambda\rangle}]\rangle\\ \vdots \\ \tilde{\mathcal{C}}\langle\lambda : \alpha(v/c_{\alpha}^{\langle\forall v,\lambda\rangle})\rangle\end{array}}{\mathcal{C}'\langle\lambda : \forall v\alpha\rangle}\ \mathcal{I}_{\forall I}$$

∃–Elimination, $\mathcal{I}_{\exists-E}$ For all configurations $\mathcal{C}$, labels λ, variables v and wffs α, $\mathcal{C}/\mathcal{C} + [\lambda : \alpha(v/c_{\alpha}^{\langle\exists v,\lambda\rangle})] + [\lambda : c_{\alpha}^{\langle\exists v,\lambda\rangle}]$ is a member of the inference rule ∃–Elimination (sometimes written $\mathcal{I}_{\exists E}$) if

- $\lambda : \exists v\alpha \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle[\lambda : \exists v\alpha]\rangle}{\mathcal{C}'\langle\lambda : \alpha(v/c_{\alpha}^{\langle\exists v,\lambda\rangle})\ \ ,\ \ \lambda : c_{\alpha}^{\langle\exists v,\lambda\rangle}\rangle}\ \mathcal{I}_{\exists E}$$

∃–Introduction, $\mathcal{I}_{\exists-I}$ For all configurations $\mathcal{C}$, ground terms τ, labels λ, wffs α and variables v, $\mathcal{C}/\mathcal{C} + [\lambda : \exists v\alpha]$ is a member of the inference rule ∃–Introduction (sometimes written $\mathcal{I}_{\exists I}$) if

- $\lambda : \alpha(v/\tau) \in \mathcal{C}$
- $\lambda : \tau \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle\lambda : \alpha(v/\tau), \lambda : \tau\rangle}{\mathcal{C}'\langle\lambda : \exists v\alpha\rangle}\ \mathcal{I}_{\exists I}$$

Additional structural rules

$\perp_t$–Introduction, $\mathcal{I}_{\perp_t}$

For all configurations $\mathcal{C}$, declarative terms Θ and declarative units $\lambda : \alpha$, $\mathcal{C}/\mathcal{C}+[\lambda : \alpha]$ is a member of the inference rule $\perp_t$–Introduction (sometimes written $\mathcal{I}_{\perp_t}$) if

- $\Theta \in \mathcal{C}$
- $\overline{\Theta} \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda : \Theta, \lambda : \overline{\Theta} \rangle}{\mathcal{C}'\langle \lambda' : \alpha \rangle} \ \mathcal{I}_{\perp_t}$$

T–Introduction, $\mathcal{I}_{T_i}$

For all configurations $\mathcal{C}$ and declarative terms Θ, $\mathcal{C}/\mathcal{C} + [\Theta]$ is a member of the inference rule T–Introduction (sometimes written $\mathcal{I}_{T_i}$) if for some label λ'

- $\mathcal{C} + [\overline{\Theta}] \vdash_{\text{MLDS}} \lambda' : \perp$

This is sometimes represented graphically as

$$\begin{array}{c} \mathcal{C}\langle [\overline{\Theta}] \rangle \\ \vdots \\ \dfrac{\tilde{\mathcal{C}}\langle \lambda' : \perp \rangle}{\mathcal{C}'\langle \Theta \rangle} \ \mathcal{I}_{T_i} \end{array}$$

Extra Visa Rule for Increasing Domains

Increasing Visa Rule, $\mathcal{I}_{VR_{\text{i}}}$ For all configurations $\mathcal{C}$, ground terms τ of $Ext(\mathcal{L}_L, \mathcal{L}_M)$, and labels λ_1 and λ_2 of $Func(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{C}/\mathcal{C} + [\lambda_2 : \tau]$ is a member of the *increasing domain visa rule* (sometimes written $\mathcal{I}_{VR_{\text{i}}}$) if

- $\lambda_1 : \tau \in \mathcal{C}$
- $R(\lambda_1, \lambda_2) \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda_1 : \tau \ , \ R(\lambda_1, \lambda_2) \rangle}{\mathcal{C}'\langle \lambda_2 : \tau \rangle} \ \mathcal{I}_{VR_{\text{i}}}$$

Extra Visa Rule for Decreasing Domains

Decreasing Visa Rule, $\mathcal{I}_{VR_{\text{d}}}$ For all configurations $\mathcal{C}$, ground s τ of $Ext(\mathcal{L}_L, \mathcal{L}_M)$, and labels λ_1 and λ_2 of $Func(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{C}/\mathcal{C} + [\lambda_1 : \tau]$ is a member of the *decreasing domain visa rule* (sometimes written $\mathcal{I}_{VR_{\text{d}}}$) if

- $\lambda_2 : \tau \in \mathcal{C}$
- $R(\lambda_1, \lambda_2) \in \mathcal{C}$

This is sometimes represented graphically as

$$\frac{\mathcal{C}\langle \lambda_2 : \tau \ , \ R(\lambda_1, \lambda_2) \rangle}{\mathcal{C}' \langle \lambda_1 : \tau \rangle} \ \mathcal{I}_{VR_{\mathrm{d}}}$$

Extra Visa Rule for Constant Domains

Constant Visa Rule, $\mathcal{I}_{VR_{\mathrm{i}}}$

For all configurations $\mathcal{C}$, ground terms τ of $Ext(\mathcal{L}_L, \mathcal{L}_M)$, and labels λ of $Func(\mathcal{L}_L, \mathcal{L}_M)$, $\mathcal{C}/\mathcal{C} + [\lambda : \tau]$ is a member of the *constant domain visa rule* (sometimes written $\mathcal{I}_{VR_{\mathrm{c}}}$). This is sometimes represented graphically as

$$\frac{\mathcal{C}}{\mathcal{C}' \langle \lambda : \tau \rangle} \ \mathcal{I}_{VR_{\mathrm{c}}}$$

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Index

E

F

G