Exercise 1

Solve the following QP using the active set method:

\[
\begin{align*}
\min_x & \quad f(x) = 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2 \\
\text{s.t.} & \quad h_1(x) = x_1 + x_2 - 4 \leq 0 \\
& \quad h_2(x) = -x_1 \leq 0 \\
& \quad h_3(x) = -x_2 \leq 0.
\end{align*}
\]

Starting point : \(x^{(0)} = (x_1^{(0)}, x_2^{(0)}) = (0, 0)\).

Solution : Problem (1) can be written in the following vector–matrix form:

\[
\begin{align*}
\min_x & \quad f(x) = a^T x + \frac{1}{2} x^T Q x \\
\text{s.t.} & \quad H^T x \leq h
\end{align*}
\]

where \(x^T = (x_1, x_2)\), \(a^T = (-12, -10)\), and

\[
H^T = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad h = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}.
\]

1. \(k = 0, \ I_0 = \{2, 3\}, x_0 = (0, 0)\)
   - The starting point \(x_0\) is feasible, since \(h_j(x_0) \leq 0, \ j = 1, 2, 3\).
• Set $k = 0$, where $k$ is the iteration counter. The set of active constraints at the point $x_0$ is $I_0 = \{2, 3\}$.

• The direction of movement $d_0 = \bar{x} - x_0 = \bar{x}$ will be found by solving the following equality constrained problem:

$$
\begin{align*}
\min_x & \quad f(x) = 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2 \\
s.t. & \quad h_2(x) = -x_1 = 0 \\
& \quad h_3(x) = -x_2 = 0.
\end{align*}
$$

(4)

It follows from (4) that $d_0 = 0$.

• Since $d_0 = 0$ we need to compute multipliers $\mu^{(1)} = (\mu_1^{(1)}, \mu_2^{(1)})$ for problem (4).

• The Lagrangian of (4) is:

$$
L(x, \mu^{(1)}) = 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2 + \mu_1^{(1)}(-x_1) + \mu_2^{(1)}(-x_2).
$$

(5)

• The optimality conditions for (4) are:

$$
\begin{align*}
\frac{\partial L}{\partial x_1} & = 4x_1 + x_2 - 12 - \mu_1^{(1)} = 0 \\
\frac{\partial L}{\partial x_2} & = x_1 + 2x_2 - 10 - \mu_2^{(1)} = 0 \\
\frac{\partial L}{\partial \mu_1^{(1)}} & = -x_1 = 0 \\
\frac{\partial L}{\partial \mu_2^{(1)}} & = -x_2 = 0
\end{align*}
$$

Solution to the above system is $(x_1, x_2, \mu_1^{(1)}, \mu_2^{(1)}) = (0, 0, -12, -10)$.

• Both of the Lagrange multipliers are negative. We choose the minimum of these, which is $-12 < 0$.

• Hence we can drop the constraint that corresponds to this multiplier, that is constraint number 2, i.e. $h_2(x) = -x_1 \leq 0$ from the active set $I_0$. Thus the new active set is $I_1 = \{3\}$.

2. $k = 1$, $I_1 = \{3\}$, $x_1 = (0, 0)$

• Now we need to solve the following equality constrained quadratic problem:

$$
\begin{align*}
\min_x & \quad f(x) = 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2 \\
s.t. & \quad h_3(x) = -x_2 = 0.
\end{align*}
$$

(6)
• The Lagrangian of (6) is:
\[ L(x, \mu^{(2)}) = 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2 + \mu^{(2)}(-x_2). \] (7)

• The optimality conditions for (6) are:
\[
\begin{align*}
\frac{\partial L}{\partial x_1} &= 4x_1 + x_2 - 12 = 0 \\
\frac{\partial L}{\partial x_2} &= x_1 + 2x_2 - 10 - \mu^{(2)} = 0 \\
\frac{\partial L}{\partial \mu^{(2)}} &= -x_2 = 0
\end{align*}
\]
Solution to the above system is \((x_1, x_2, \mu^{(2)}) = (3, 0, -7)\). In other words, \(\bar{x} = (3, 0)\) and \(\mu^{(2)} = -7\).

• Thus \(d_1 = \bar{x} - x_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \neq 0\)

• Determine \(\tau_1\):
\[
\begin{align*}
- j &= 1: \langle \nabla h_1, d_1 \rangle = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3 > 0 \\
- j &= 2: \langle \nabla h_2, d_1 \rangle = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = -3 < 0 \quad \text{(the constraint } h_2(x) \leq 0 \text{ is rejected)}
\end{align*}
\]

Therefore, \(\tau_1 = \min\{1, \frac{4 - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}}{3}\} = \min\{1, \frac{4}{3}\} = 1\).

• The Lagrange multiplier of problem (6) is negative \(\mu^{(2)} = -7\), so constraint \(h_3(x)\) is dropped. Thus the new active set is \(I_2 = \emptyset\).

3. \(k = 2\), \(I_2 = \emptyset\), \(x_2 = x_1 + \tau_1 d_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}\)

• The direction \(d_2\) is then the vector from point \(x^{(2)} = (3, 0)\) to the solution of the following unconstrained quadratic problem:
\[ \min_x f(x) = 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2. \] (8)

• Optimality conditions of (8):
\[
\begin{align*}
\frac{\partial L}{\partial x_1} &= 4x_1 + x_2 - 12 = 0 \\
\frac{\partial L}{\partial x_2} &= x_1 + 2x_2 - 10 = 0
\end{align*}
\]
Hence the point $\bar{x} = (2, 4)$ is the minimum of the latter unconstrained quadratic problem.

The direction of movement is

$$d_2 = \bar{x} - x_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \neq 0.$$  \hspace{1cm} (9)

Determine the step length:

- $j = 1$: $\langle \nabla h_1, d_2 \rangle = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = 3 > 0$
- $j = 2$: $\langle \nabla h_2, d_2 \rangle = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = 1 > 0$
- $j = 3$: $\langle \nabla h_3, d_2 \rangle = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = -4 < 0$ (the constraint $h_3(x) \leq 0$ is rejected)

The value of the step length $\tau_1$ is: $\tau_2 = \min\{1, \frac{3}{4}, \frac{0 - [-1 \ 0]}{1} 3\} = \min\{1, \frac{1}{3}, 3\} = \frac{1}{3}$ (it corresponds to $j = 1$).

Therefore constraint $h_1(x) = 0$ is met and is added to the set of active constraints $I_3 = I_2 \cup \{1\}$.

The new point is:

$$x_3 = x_2 + \tau_2 d_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{8}{3} \\ \frac{4}{3} \end{bmatrix}.$$  \hspace{1cm} (10)

4. $k = 3$, $I_3 = \{1\}$, $x_3 = \begin{bmatrix} \frac{8}{3} \\ \frac{4}{3} \end{bmatrix}$

Now we need to solve the following equality constrained problem:

$$\begin{align*}
\min_x & \quad f(x) = 2x_1^2 + x_1 x_2 + x_2^2 - 12x_1 - 10x_2 \\
\text{s.t.} & \quad h_1(x) = x_1 + x_2 - 4 = 0.
\end{align*}$$  \hspace{1cm} (11)

The Lagrangian of (8) is:

$$L(x, \mu^{(3)}) = 2x_1^2 + x_1 x_2 + x_2^2 - 12x_1 - 10x_2 + \mu^{(3)}(x_1 + x_2 - 4).$$  \hspace{1cm} (12)
• The optimality conditions for (11) are:
\[
\frac{\partial L}{\partial x_1} = 4x_1 + x_2 - 12 + \mu^{(3)} = 0 \\
\frac{\partial L}{\partial x_2} = x_1 + 2x_2 - 10 + \mu^{(3)} = 0 \\
\frac{\partial L}{\partial \mu^{(3)}} = x_1 + x_2 - 4 = 0
\]
Solution to the above system is \((x_1, x_2, \mu^{(4)}) = (\frac{3}{2}, \frac{5}{2}, \frac{7}{2})\).

• The direction \(d_3\) is then the vector from point \(x_3\) to point \(x\) = \((\frac{3}{2}, \frac{5}{2})\):
\[
d_3 = x - x_3 = \begin{bmatrix} -\frac{7}{6} \\ \frac{7}{6} \end{bmatrix} \neq 0. \quad (13)
\]

• Determine the step length:
\[
- j = 2: \langle \nabla h_2, d_3 \rangle = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{7}{6} \\ \frac{7}{6} \end{bmatrix} = 7/6 > 0
- j = 3: \langle \nabla h_3, d_3 \rangle = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{7}{6} \\ \frac{7}{6} \end{bmatrix} = -7/6 < 0 \text{ (the constraint } h_3(x) \leq 0 \text{ is rejected)}
\]
The value of the step length \(\tau_3\) is: \(\tau_3 = \min\{1, \frac{16}{7}\} = 1\).

• Since \(\tau_3 = 1\) we compute the lagrange multiplier \(\mu^{(4)}\). This is \(\mu^{(4)} = \frac{7}{2} > 0\). Therefore \(x_4 = x_3 + \tau_3 d_3 = \begin{bmatrix} \frac{8}{3} \\ \frac{8}{3} \end{bmatrix} + 1 \cdot \begin{bmatrix} -\frac{7}{6} \\ \frac{7}{6} \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}\).

• Since \(\mu^{(4)} = \frac{7}{2} > 0\), solution \(x_4 = \begin{bmatrix} 1.5 \\ 2.5 \end{bmatrix}\) is the optimum of the original constrained QP problem (1) and the algorithm terminates.

**Exercise 2** Solve the following problem by using the active set method and taking 
\(x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (0, 0, 1)\) as a starting point
\[
\min_x f(x) = x_1^2 + 2x_2^2 + 3x_3^2 \\
\text{s.t.} \quad x_1 + x_2 + x_3 - 1 \geq 0 \\
\quad x_1, x_2, x_3 \geq 0. \quad (14)
\]

Solution: 