Algorithms for Optimal Decisions
Tutorial 5
Answers

Exercise 1

Solve the following **QP** *using the active set method:*

$$
\begin{array}{rrcl}
\min\limits_{x} & f(x) & = & 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2 \\
s.t. & h_1(x) & = & x_1 + x_2 - 4 \le 0 \\
 & h_2(x) & = & -x_1 \le 0 \\
 & h_3(x) & = & -x_2 \le 0.
\end{array} \tag{1}
$$

Starting point : $x^{(0)} = (x_1^{(0)}, x_2^{(0)}) = (0, 0)$.

Solution : Problem (1) can be written in the following vector–matrix form:

$$
\begin{array}{rcl}
\min\limits_{x} f(x) & = & a^t x + \frac{1}{2} x^t Q x \\
s.t. & & H^t x \le h
\end{array} \tag{2}
$$

where $x^t = (x_1, x_2)$, $a^t = (-12, -10)$, and

$$
H^t = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad h = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}. \tag{3}
$$

1. $k = 0,\ I_0 = \{2, 3\}, x_0 = (0, 0)$
   - The starting point x_0 is feasible, since $h_j(x_0) \le 0, \quad j = 1, 2, 3$.

- Set $k = 0$, where k is the iteration counter. The set of active constraints at the point x_0 is $I_0 = \{2, 3\}$.
- The direction of movement $d_0 = \overline{x} - x_0 = \overline{x}$ will be found by solving the following equality constrained problem:

$$\begin{aligned} \min_x \quad f(x) &= 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2 \\ s.t. \quad h_2(x) &= -x_1 = 0 \\ h_3(x) &= -x_2 = 0. \end{aligned} \tag{4}$$

- It follows from (4) that $d_0 = 0$.
- Since $d_0 = 0$ we need to compute multipliers $\mu^{(1)} = (\mu_1^{(1)}, \mu_2^{(1)})$ for problem (4).
- The Lagrangian of (4) is:

$$L(x, \mu^{(1)}) = 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2 + \mu_1^{(1)}(-x_1) + \mu_2^{(1)}(-x_2). \tag{5}$$

- The optimality conditions for (4) are:

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 4x_1 + x_2 - 12 - \mu_1^{(1)} = 0 \\ \frac{\partial L}{\partial x_2} &= x_1 + 2x_2 - 10 - \mu_2^{(1)} = 0 \\ \frac{\partial L}{\partial \mu_1^{(1)}} &= -x_1 = 0 \\ \frac{\partial L}{\partial \mu_2^{(1)}} &= -x_2 = 0 \end{aligned}$$

  Solution to the above system is $(x_1, x_2, \mu_1^{(1)}, \mu_2^{(1)}) = (0, 0, -12, -10)$.
- Both of the Lagrange multipliers are negative. We choose the minimum of these, which is $-12 < 0$.
- Hence we can drop the constraint that corresponds to this multiplier, that is constraint number 2, i.e. $h_2(x) = -x_1 \leq 0$ from the active set I_0. Thus the new active set is $I_1 = \{3\}$.

2. $k = 1,\ I_1 = \{3\}, x_1 = (0, 0)$

   - Now we need to solve the following equality constrained quadratic problem:

$$\begin{aligned} \min_x \quad f(x) &= 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2 \\ s.t. \quad h_3(x) &= -x_2 = 0. \end{aligned} \tag{6}$$

- The Lagrangian of (6) is:

$$L(x, \mu^{(2)}) = 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2 + \mu^{(2)}(-x_2). \qquad (7)$$

- The optimality conditions for (6) are:

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 4x_1 + x_2 - 12 = 0 \\ \frac{\partial L}{\partial x_2} &= x_1 + 2x_2 - 10 - \mu^{(2)} = 0 \\ \frac{\partial L}{\partial \mu^{(2)}} &= -x_2 = 0 \end{aligned}$$

  Solution to the above system is $(x_1, x_2, \mu^{(2)}) = (3, 0, -7)$. In other words, $\overline{x} = (3, 0)$ and $\mu^{(2)} = -7$

- Thus $d_1 = \overline{x} - x_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \neq 0$

- Determine τ_1:
  - $j = 1$:$\langle \nabla h_1, d_1 \rangle = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3 > 0$
  - $j = 2$:$\langle \nabla h_2, d_1 \rangle = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = -3 < 0$ (the constraint $h_2(x) \leq 0$ is rejected)

  Therefore, $\tau_1 = \min\{1, \frac{4 - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}}{3}\} = \min\{1, 4/3\} = 1.$

- The Lagrange multiplier of problem (6) is negative ($\mu^{(2)} = -7$), so constraint $h_3(x)$ is dropped. Thus the new active set is $I_2 = \emptyset$.

3. $k = 2,\ I_2 = \emptyset, x_2 = x_1 + \tau_1 d_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

- The direction d_2 is then the vector from point $x^{(2)} = (3, 0)$ to the solution of the following unconstrained quadratic problem:

$$\min_x \quad f(x) = 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2. \qquad (8)$$

- Optimality conditions of (8):

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= 4x_1 + x_2 - 12 = 0 \\ \frac{\partial L}{\partial x_2} &= x_1 + 2x_2 - 10 = 0 \end{aligned}$$

- Hence the point $\overline{x} = (2, 4)$ is the minimum of the latter unconstrained quadratic problem.
- The direction of movement is

$$d_2 = \overline{x} - x_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \neq 0. \tag{9}$$

- Determine the step length:
  - $j = 1$: $\langle \nabla h_1, d_2 \rangle = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = 3 > 0$
  - $j = 2$: $\langle \nabla h_2, d_2 \rangle = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = 1 > 0$
  - $j = 3$:$\langle \nabla h_3, d_2 \rangle = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = -4 < 0$ (the constraint $h_3(x) \leq 0$ is rejected)

  The value of the step length τ_1 is: $\tau_2 = \min\{1, \frac{4 - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}}{3}, \frac{0 - \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}}{1}\} = \min\{1, 1/3, 3\} = 1/3$ (it corresponds to $j = 1$).
- Therefore constraint $h_1(x) = 0$ is met and is added to the set of active constraints $I_3 = I_2 \cup \{1\}$.
- The new point is:

$$x_3 = x_2 + \tau_2 d_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{8}{3} \\ \frac{4}{3} \end{bmatrix} \tag{10}$$

4. $k = 3,\ I_3 = \{1\}, x_3 = \begin{bmatrix} \frac{8}{3} \\ \frac{4}{3} \end{bmatrix}$

- Now we need to solve the following equality constrained problem:

$$\begin{array}{rrcl} \min\limits_{x} & f(x) & = & 2x_1^2 + x_1 x_2 + x_2^2 - 12x_1 - 10x_2 \\ s.t. & h_1(x) & = & x_1 + x_2 - 4 = 0. \end{array} \tag{11}$$

- The Lagrangian of (8) is:

$$L(x, \mu^{(3)}) = 2x_1^2 + x_1 x_2 + x_2^2 - 12x_1 - 10x_2 + \mu^{(3)}(x_1 + x_2 - 4). \tag{12}$$

- The optimality conditions for (11) are:

$$\begin{aligned}\frac{\partial L}{\partial x_1} &= 4x_1 + x_2 - 12 + \mu^{(3)} = 0\\ \frac{\partial L}{\partial x_2} &= x_1 + 2x_2 - 10 + \mu^{(3)} = 0\\ \frac{\partial L}{\partial \mu^{(3)}} &= x_1 + x_2 - 4 = 0\end{aligned}$$

  Solution to the above system is $(x_1, x_2, \mu^{(4)}) = (\frac{3}{2}, \frac{5}{2}, \frac{7}{2})$.
- The direction d_3 is then the vector from point x_3 to point $\overline{x} = (\frac{3}{2}, \frac{5}{2})$:

$$d_3 = \overline{x} - x_3 = \begin{bmatrix} -\frac{7}{6} \\ \frac{7}{6} \end{bmatrix} \neq 0. \tag{13}$$

- Determine the step length:
  - $j = 2$: $\langle \nabla h_2, d_3 \rangle = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{7}{6} \\ \frac{7}{6} \end{bmatrix} = 7/6 > 0$
  - $j = 3$:$\langle \nabla h_3, d_3 \rangle = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{7}{6} \\ \frac{7}{6} \end{bmatrix} = -7/6 < 0$ (the constraint $h_3(x) \leq 0$ is rejected)

  The value of the step length τ_3 is: $\tau_3 = \min\{1, \frac{0 - \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{8}{3} \\ \frac{4}{3} \end{bmatrix}}{\frac{7}{6}}\} = \min\{1, 16/7\} = 1$.
- Since $\tau_3 = 1$ we compute the lagrange multiplier $\mu^{(4)}$. This is $\mu^{(4)} = \frac{7}{2} > 0$. Therefore $x_4 = x_3 + \tau_3 d_3 = \begin{bmatrix} \frac{8}{3} \\ \frac{4}{3} \end{bmatrix} + 1 \cdot \begin{bmatrix} \frac{-7}{6} \\ \frac{7}{6} \end{bmatrix} = \begin{bmatrix} \frac{9}{6} \\ \frac{15}{6} \end{bmatrix}$.
- Since $\mu^{(4)} = \frac{7}{2} > 0$, solution $x_4 = \begin{bmatrix} 1.5 \\ 2.5 \end{bmatrix}$ is the optimum of the original constrained **QP** problem (1) and the algorithm terminates.

Exercise 2 *Solve the following problem by using the active set method and taking*
$x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (0, 0, 1)$ *as a starting point*

$$\begin{aligned}\min_x f(x) &= x_1^2 + 2x_2^2 + 3x_3^2\\ s.t. \quad & x_1 + x_2 + x_3 - 1 \geq 0\\ & x_1, x_2, x_3 \geq 0.\end{aligned} \tag{14}$$

Solution :