Exercise 1 Solve the following problem by using the active set method and taking $x^{(0)} = (x^{(0)}_1, x^{(0)}_2, x^{(0)}_3) = (0, 0, 1)$ as a starting point

\[
\min_x f(x) = x_1^2 + 2x_2^2 + 3x_3^2 \\
\text{s.t.} \quad x_1 + x_2 + x_3 - 1 \geq 0 \\
\quad x_1, x_2, x_3 \geq 0.
\]

Solution: First, we rewrite the problem, so we have constraints which are less or equal to zero:

\[
\min_x f(x) = x_1^2 + 2x_2^2 + 3x_3^2 \\
\text{s.t.} \quad 1 - x_1 - x_2 - x_3 \leq 0 \\
\quad -x_1, -x_2, -x_3 \leq 0.
\]

- The starting point $x^{(0)}$ is feasible, since $g_i(x^{(0)}) \leq 0, \quad i = 1, 2, 3, 4$.
- Set $k = 0$, where $k$ is the iteration counter. The set of active constraints at the point $x^{(0)}$ is $J_0 = \{1, 2, 3\}$.
- The direction of movement $d_0 = x - x^{(0)} = x$ will be found by solving the following equality constrained problem:

\[
\min_x f(x) = x_1^2 + 2x_2^2 + 3x_3^2 \\
\text{s.t.} \quad g_1(x) = 1 - x_1 - x_2 - x_3 = 0 \\
\quad g_2(x) = -x_1 = 0 \\
\quad g_3(x) = -x_2 = 0
\]
• It follows from (3) that $d_0 = 0$.

• Since $d_0 = 0$ we need to compute multipliers $\mu^{(1)} = (\mu_1^{(1)}, \mu_2^{(1)}, \mu_3^{(1)})$ for problem (3).

• The Lagrangian of (3) is:

$$L(x, \mu^{(1)}) = x_1^2 + 2x_2^2 + 3x_3^2 + \mu_1^{(1)}(1 - x_1 - x_2 - x_3) + \mu_2^{(1)}(-x_1) + \mu_3^{(1)}(-x_2).$$

(4)

• The optimality conditions for (3) are:

\[
\begin{align*}
\frac{\partial L}{\partial x_1} &= 2x_1 - \mu_1^{(1)} - \mu_2^{(1)} = 0 \\
\frac{\partial L}{\partial x_2} &= 4x_2 - \mu_1^{(1)} - \mu_3^{(1)} = 0 \\
\frac{\partial L}{\partial x_3} &= 6x_3 - \mu_1^{(1)} = 0 \\
\frac{\partial L}{\partial \mu_1^{(1)}} &= 1 - x_1 - x_2 - x_3 = 0 \\
\frac{\partial L}{\partial \mu_2^{(1)}} &= -x_1 = 0 \\
\frac{\partial L}{\partial \mu_3^{(1)}} &= -x_2 = 0
\end{align*}
\]

Solution to the above system is

$$(x_1, x_2, x_3, \mu_1^{(1)}, \mu_2^{(1)}, \mu_3^{(1)}) = (0, 0, 1, 6, -6, -6).$$

• Only one of the Lagrange multipliers are negative $\mu_2^{(1)}$.

• From step 3 of the algorithm (see your notes) we can drop the constraint $g_2(x) = -x_1 \leq 0$ from the active set $J_0$. Thus the new active set is $J_1 = \{1, 3\}$.

• Now we need to solve the following equality constrained quadratic problem:

\[
\begin{align*}
\min_x f(x) &= x_1^2 + 2x_2^2 + 3x_3^2 \\
\text{s.t. } g_1(x) &= 1 - x_1 - x_2 - x_3 = 0 \\
g_3(x) &= -x_2 = 0
\end{align*}
\]

(5)
The Lagrangian of (5) is:

$$L(x, \mu^{(2)}) = x_1^2 + 2x_2^2 + 3x_3^2 + \mu^{(2)}_1 (1 - x_1 - x_2 - x_3) + \mu^{(2)}_2 (-x_2).$$  (6)

The optimality conditions for (5) are:

$$\frac{\partial L}{\partial x_1} = 2x_1 - \mu^{(2)}_1 = 0$$
$$\frac{\partial L}{\partial x_2} = 4x_2 - \mu^{(2)}_1 - \mu^{(2)}_2 = 0$$
$$\frac{\partial L}{\partial x_3} = 6x_3 - \mu^{(2)}_1 = 0$$
$$\frac{\partial L}{\partial \mu^{(2)}_1} = 1 - x_1 - x_2 - x_3 = 0$$
$$\frac{\partial L}{\partial \mu^{(2)}_2} = -x_2 = 0.$$  (7)

Solution to the above system is

$$(x_1, x_2, x_3, \mu^{(2)}_1, \mu^{(2)}_2) = \left(\frac{3}{4}, 0, \frac{1}{4}, \frac{3}{2}, -\frac{3}{2}\right).$$

One of the Lagrange multipliers of problem (5) is negative, so constraint $g_3(x)$ is dropped.

The direction $d_1$ is then the vector from point $x^{(1)} = \left(\frac{3}{4}, 0, \frac{1}{4}\right)$ to the solution of the following constrained quadratic problem:

$$\min_x \quad f(x) = x_1^2 + 2x_2^2 + 3x_3^2$$
$$\text{s.t.} \quad 1 - x_1 - x_2 - x_3 = 0.$$  (8)
• Optimality conditions of (8):
\[
\begin{align*}
\frac{\partial L}{\partial x_1} &= 2x_1 - \mu_1^{(3)} = 0 \\
\frac{\partial L}{\partial x_2} &= 4x_2 - \mu_1^{(3)} = 0 \\
\frac{\partial L}{\partial x_3} &= 6x_3 - \mu_1^{(3)} = 0 \\
\frac{\partial L}{\partial \mu_1^{(2)}} &= 1 - x_1 - x_2 - x_3 = 0.
\end{align*}
\]
(9)

• The point \((x^*_1, x^*_2, x^*_3, \mu_1^{(1)}) = (\frac{6}{11}, \frac{3}{11}, \frac{2}{11}, \frac{3}{11})\).

• New point is feasible, so we can take that point as a new point. That means that \(\tau = 1\). Also the Lagrange multiplier is positive, so point \(x^* = (\frac{6}{11}, \frac{3}{11}, \frac{2}{11})\) is the solution to our problem.

Exercise 2: Solve the following problem using the interior point method:
\[
\begin{align*}
\min_x \quad f(x) &= x_1 + x_2 \\
\text{s.t.} \quad g_1(x) &= -x_2^2 + x_2 \geq 0 \\
&\quad g_2(x) = x_1 \geq 0.
\end{align*}
\]
(10)

Solution: We shall use the logarithmic barrier function to solve the problem (10). Thus problem (10) is approximated by a sequence of unconstrained problems:
\[
\min_x f(x) - \eta_k \sum_{i=1}^{2} \log(g_i(x)),
\]
(11)
where the values of the parameter \(\eta_k\) decrease and approach zero. We are going to solve a number of problems \(\eta_k\) decrease and approach zero. We are going to solve a number of problems (11) for a decreasing sequence of values of the barrier parameter \(\eta_k\), such that
\[
\lim_{k \to \infty} \eta_k = 0.
\]

First, we find the optimality conditions of the unconstrained problem (11) where the value of the barrier parameter is fixed:
\[
\begin{align*}
\frac{\partial}{\partial x_1} (x_1 + x_2 - \eta_k (\log(-x_2^2 + x_2) + \log(x_1))) &= 0 \\
\frac{\partial}{\partial x_2} (x_1 + x_2 - \eta_k (\log(-x_2^2 + x_2) + \log(x_1))) &= 0 \quad \Rightarrow
\end{align*}
\]
(12)
\[
\begin{align*}
1 - \eta_k \frac{1}{-x_2^2 + x_2} \cdot (-2x_1) - \frac{\eta_k}{x_1} &= 0 \\
1 - \eta_k \frac{1}{-x_2^2 + x_2} &= 0
\end{align*}
\]
(13)
Solving (13) we have:
\[-\frac{\eta_k}{-x_1^2 + x_2} = -1,\]  
and
\[1 - (-2x_1) - \frac{\eta_k}{x_1} = 0 \Rightarrow 1 + 2x_1 - \frac{\eta_k}{x_1} = 0 \Rightarrow 2x_1^2 + x_1 - \eta_k = 0.\]  
The solution of (15) is given by the formula:
\[x_1 = \frac{-1 \pm \sqrt{1 + 8\eta_k}}{4}.\]  
Since \(x_1\) must be positive, only the root
\[x_1 = \frac{-1 + \sqrt{1 + 8\eta_k}}{4}\]  
is of interest. Substituting (17) into (14) yields:
\[\frac{-\eta_k}{-\left(\frac{-1 + \sqrt{1 + 8\eta_k}}{4}\right)^2 + x_2} = -1 \Rightarrow \ldots \Rightarrow \Rightarrow x_2 = \frac{\left(-1 + \sqrt{1 + 8\eta_k}\right)^2}{16} + \eta_k.\]  
Formulae (17) and (18) give the optimum of the unconstrained problem (10) where the value of the barrier parameter \(\eta_k\) is fixed. For example, if \(\eta_k = 1\) then the point
\[(x_1^{(1)}, x_2^{(1)}) = \left(\frac{-1 + \sqrt{1 + 8}}{4}, \frac{\left(-1 + \sqrt{1 + 8}\right)^2}{16} + 1\right) = (0.5, 1.25)\]  
is the optimum solution of the following unconstrained problem:
\[\min_x f(x) - 1 \cdot \sum_{i=1}^{2} \log(g_i(x)).\]  
Now, if \(\eta_k\) is fixed to a smaller value, say \(\eta_k = \frac{1}{2}\) then the point
\[(x_1^{(2)}, x_2^{(2)}) = \left(\frac{-1 + \sqrt{1 + 8\frac{1}{2}}}{4}, \frac{\left(-1 + \sqrt{1 + 8\frac{1}{2}}\right)^2}{16} + 1\right) = (0.309, 0.595)\]  
is the optimum solution of the following unconstrained problem:
\[\min_x f(x) - \frac{1}{2} \cdot \sum_{i=1}^{2} \log(g_i(x)).\]  
The following table shows the computed value of the points \((x_1^{(k)}, x_2^{(k)})\) for different values of \(\eta_k\).
In the limit (i.e. \( \lim_{k \to \infty} \eta_k = 0 \)) the minimizing points \((x_1^{(k)}, x_2^{(k)})\) approach the solution \((x_1^*, x_2^*) = (0, 0)\) of the original constrained problem (10).

In this problem there is only one unconstrained local minimum for each value of \(\eta_k\). The problem happens to have the unique solution. It turns out that in problems with many local optima there is a sequence of local unconstrained minima converging to each set of constrained local minima. This is illustrated in the next example.

<table>
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<tr>
<th>(k)</th>
<th>(\eta_k)</th>
<th>(x_1^{(k)})</th>
<th>(x_2^{(k)})</th>
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<td>1</td>
<td>(\eta_1 = 1)</td>
<td>0.5</td>
<td>1.25</td>
</tr>
<tr>
<td>2</td>
<td>(\eta_2 = \frac{1}{2})</td>
<td>0.309</td>
<td>0.595</td>
</tr>
<tr>
<td>3</td>
<td>(\eta_2 = \frac{1}{4})</td>
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<td>0.283</td>
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<tr>
<td>4</td>
<td>(\eta_2 = \frac{1}{10})</td>
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<td>0.107</td>
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<td>↓</td>
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<td>↓</td>
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<tr>
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