Algorithms for Optimal Decisions
Tutorial 7
Answers

Exercise 1

1. *An electric company needs to determine its pricing structure according to demand during peak and off–peak times. The prices and demands are as follows:*

	price()	*demand (KWh)*
peak–time	p_1	$(80 - 0.7p_1)$
off peak	p_2	$(50 - p_2)$

The company must have sufficient capacity to meet demand at all times. The capacity must not be less than $5kWh$ *at any time. It costs 12 to maintain each* kWh *of capacity at each period. The company wishes to maximize daily revenues less operating costs. Formulate this problem.*

2. *A mobile telephone company produces brands* A *and* B*. If the charge per unit is* p_1 *for brand* A *and* p_2 *for brand* B *it can sell* q_1 *of* A *and* q_2 *of* B*, where*

$$\begin{aligned} q_1 &= 5500 - 10p_1 + p_2 \\ q_2 &= 3700 - 9p_1 + 0.8p_2. \end{aligned}$$

The manufacturing requirements are:

	labor(min)	*chips*
Brand A	*20*	*3*
Brand B	*30*	*1*

At present, 1000 *hours of labor and* 5000 *chips are available. The company wants to maximize its revenue. Formulate the problem.*

Solution :

1. Let z be the capacity maintained. Then

$$\begin{aligned}
\max_{p_1,p_2} \quad & (80-0.7p_1)p_1 + (50-p_2)p_2 - 12z \\
s.t. \quad & (80-0.7p_1) \leq z, \\
& (50-p_2) \leq z, \\
& z \geq 5, \\
& p_1, p_2 \geq 0.
\end{aligned}$$

2.

$$\begin{aligned}
\max_{p_1,p_2} \quad & (5500-10p_1+p_2)p_1 + (3700-9p_2+0.8p_1)p_2 \\
s.t. \quad & 20(5500-10p_1+p_2) + 30(3700-9p_2+0.8p_1) \leq 6000, \\
s.t. \quad & 3(5500-10p_1+p_2) + (3700-9p_2+0.8p_1) \leq 5000, \\
& p_1, p_2 \geq 0.
\end{aligned}$$

Exercise 2

1. *Consider the problem:*

$$\begin{aligned}
\min_{x} \quad & x_1^4 + 2x_1^2 + 2x_1x_2 + 4x_2^2 \\
s.t. \quad & 2x_1 + x_2 = 10, \\
& x_1 + 2x_2 \geq 10, \\
& x_1, x_2 \geq 0. \qquad (1)
\end{aligned}$$

Write the KKT conditions for this problem.

If the SUMT algorithm were to be applied directly to this problem what would be the unconstrained function $l(x,\eta)$ *to be minimized at each iteration?*

2. *Consider the Frank-Wolfe algorithm for solving the nonlinear program:*

$$\min_x f(x), Ax \leq b,$$

using the following linear–programming (LP) subproblem at every iterate x_k, $Ax_k \leq b$:

$$\begin{aligned} \min_x \quad & \nabla f(x_k)^t x \\ & Ax \leq b. \end{aligned}$$

Let x_{LP} denote the solution of this LP. Suggest an amendment to the above subproblem that will ensure the following descent condition:

$$\nabla f(x_k)^t (x_{QP} - x_k) \leq -c(x_{QP} - x_k)^t (x_{LP} - x_k),$$

where x_{QP} denotes the solution of the amended subproblem and c a nonnegative constant. Establish this condition for the suggested amendment.

Solution :

1. KKT Conditions:

$$\begin{aligned} 4x_1^3 + 4x_1 + 2x_2 - 2\mu_1 - \mu_2 - \mu_3 &= 0 \\ 2x_1 + 8x_2 - \mu_1 - 2\mu_2 - \mu_4 &= 0 \\ 2x_1 + x_2 - 10 &= 0 \\ x_1 + 2x_2 - 10 &\geq 0 \\ x_1, x_2 &\geq 0 \\ \mu_2(x_1 + 2x_2 - 10) &= 0 \\ \mu_3 x_1 = 0, \quad \mu_4 x_2 = 0, \quad \mu_2, \mu_3, \mu_4 &\geq 0. \end{aligned}$$

SUMT function to be minimized is:

$$\begin{aligned} l(x, \eta) &= x_1^4 + 2x_1^2 + 2x_1x_2 + 4x_2^2 + \frac{1}{\eta}(2x_1 + x_2 - 10)^2 \\ &+ \eta\left(\frac{1}{x_1 + 2x_2 - 10} + \frac{1}{x_1} + \frac{1}{x_2}\right) \end{aligned}$$

2. Consider the following amended problem:

$$\begin{aligned} \min \quad & \nabla f(x_k)^t \cdot x + c(x - x_k)^t (x_{LP} - x_k) \\ s.t. \quad & Ax \leq b. \end{aligned} \tag{2}$$

The optimality condition:

$$\nabla f(x_k) + c(x_{LP} - x_k) + A^t \lambda = 0. \tag{3}$$

Multiplying the above equation with $(x_{QP} - x_k)$ we obtain:

$$(x_{QP}-x_k)^t \nabla f(x_k) + c(x_{QP}-x_k)^t (x_{LP}-x_k) + (x_{QP}-x_k)^t A^t \lambda = 0. \tag{4}$$

Next, we note that:

$$\begin{aligned} (x_{QP} - x_k)^t A^t \lambda &= x_{QP}^t A^t \lambda - x_k^t A^t \lambda \\ &\geq x_{QP}^t A^t \lambda - b^t \lambda \\ &= (A x_{QP} - b)^t \lambda = 0. \end{aligned} \tag{5}$$

Using this in (4) we have:

$$(x_{QP} - x_k)^t \nabla f(x_k) + c(x_{QP} - x_k)^t (x_{LP} - x_k) = -(x_{QP} - x_k)^t A^t \lambda \leq 0, \tag{6}$$

which yields:

$$\nabla f(x_k)^t (x_{QP} - x_k) \leq -c(x_{QP} - x_k)^t (x_{LP} - x_k). \tag{7}$$

Exercise 3 *We are considering investing in three stocks. The random variable S_i represents the annual return on 1 pound invested in stock i. We are given the expected values of these returns to be $E(S_1) = 0.15$, $E(S_2) = 0.21$ and $E(S_3) = 0.09$. The uncertainties are as follows:*

$$\begin{bmatrix} var S_1 & cov S_1 S_2 & cov S_1 S_3 \\ cov S_1 S_2 & var S_2 & cov S_2 S_3 \\ cov S_1 S_3 & cov S_2 S_3 & var S_3 \end{bmatrix} = \begin{bmatrix} 0.09 & 0.06 & -0.04 \\ 0.06 & 0.04 & 0.05 \\ -0.04 & 0.05 & 0.01 \end{bmatrix}. \tag{8}$$

We have 1000 pounds to invest and wish to have an expected return of at least 15%. *Formulate a quadratic programming problem to find the portfolio of minimum variance that attains an expected return of at least* 15%.

Solution : Let x_j be the number of invested in stock j. The annual return of the portfolio is:

$$\frac{x_1 S_1 + x_2 S_2 + x_3 S_3}{1000},$$

and the expected annual return on the portfolio is:

$$\frac{x_1 E(S_1) + x_2 E(S_2) + x_3 E(S_3)}{1000}.$$

To ensure that the portfolio has an expected return of at least 15% the following constraint has to be included:

$$0.15x_1 + 0.21x_2 + 0.09x_3 \geq 150.$$

Of course, the wealth constraint needs to be included and the amount invested must be nonnegative:

$$x_1 + x_2 + x_3 = 1000, x_1, x_2, x_3 \geq 0.$$

The objective is to minimize the variance of the portfolio:

$$var(x_1S_1 + x_2S_2 + x_3S_3) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^t \begin{bmatrix} varS_1 & covS_1S_2 & covS_1S_3 \\ covS_1S_2 & varS_2 & covS_2S_3 \\ covS_1S_3 & covS_2S_3 & varS_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{9}$$

Hence, we have the following QP problem:

$$\begin{array}{ll} \min\limits_{x} & var(x_1S_1 + x_2S_2 + x_3S_3) \\ s.t. & 0.15x_1 + 0.21x_2 + 0.09x_3 \geq 150, \\ & x_1 + x_2 + x_3 = 1000, \\ & x_1, x_2, x_3 \geq 0. \end{array}$$