Game Theory Tutorials

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1 Tutorial – 1

1.1 Example 1

The manager of a multinational company and the union of workers are preparing to sit down at the bargaining table to work out the details of a new contract for the workers. Each side has developed certain proposals for the contents of the new contract. Let us call union proposals "Proposal 1", "Proposal 2" and "Proposal 3", and manager's proposals "Contract A", "Contract B" and "Contract C". Both parties are aware of the financial aspects of each proposal–contract combination. The reward matrix is:

These values are the contract gains that the Union would secure and also the cost the company would have to bear.

Is there a clear–cut contract combination agreeable to both parties, or will they find it necessary to submit to arbitration in order to arrive at some sort of compromise?

1.1.1 Solution

First we find the union's optimal strategy. The minimum payoff for each strategy (which is the minimum payoff in each row) is shown in the **row**

Proposal	Contract		
	A	B	C
1	8.5	7.0	7.5
2	12.0	9.5	9.0
3	9.0	11.0	8.0

Table 1: The reward matrix.

min column of table 2. The maximum of these minimum payoffs is 9 in the second row. Consequently the union would select strategy 2 as its optimal one. In a similar way we find the manager's optimal strategy. The maximum

Proposal	Contract			
	A	B	C	Row min
1	8.5	7.0	7.5	7.0
2	12.0	9.5	9.0	9.0
3	9.0	11.0	8.0	8.0

Table 2: Minimum for each row.

pay out for each contract (which is the maximum pay out in each column) is shown in the **col max** column of table 3. The minimum of these maximum pay outs is 9 in the third column. Consequently the manager would select strategy C as its optimal one.

Proposal	Contract			
	A	B	C	Row min
1	8.5	7.0	7.5	7.0
2	12.0	9.5	9.0	9.0
3	9.0	11.0	8.0	8.0
Col. max	12.0	11.0	9.0	–

Table 3: Maximum for each column.

We see that both of the players in this game will select a strategy that has the same value. The *min* value in row 2 is also the *max* value in column C, so the solution is an equilibrium or saddle point.

$$\min\{col.max.\} = \min\{12, 11, 9\} = 9 = \max\{row.min.\} = max\{7.5, 9, 8\}.$$

1.2 Example 2

Consider the same problem as in Example 1, but with the following reward matrix:

- Is there an equilibrium point?
- Find the mixed strategies for the union and the manager.
- Formulate the LP problem to determine the optimum strategy for the union and the optimum strategy of the manager.

Proposal	Contract		
	A	B	C
1	9.5	12.0	7.0
2	7.0	8.5	6.5
3	6.0	9.0	10.0

Table 4: The reward matrix.

1.2.1 Solution

Similarly as in previous example we determine the row min and the col max for matrix 4:

Proposal	Contract			
	A	B	C	row min
1	9.5	12.0	7.0	7.0
2	7.0	8.5	6.5	6.5
3	6.0	9.0	10.0	6.0
col. max	9.5	12.0	10.0	–

Table 5: Optimum for each row and column.

- It is easy to see from table 5 that

  $$\max\{row.min.\} = 7 \neq \min\{col.max.\} = 9.5.$$

  Therefore there is no equilibrium point.

- Let us assign the following unknown probabilities $u_i, i = 1, 2, 3$ for each strategy, where u_i is the probability that the union chooses $i - th$ strategy. Similarly for the manager we assign $m_k, k = A, B, C$.

  If $u_1, u_2, u_3 \geq 0$ and $u_1 + u_2 + u_3 = 1$ then the strategy (u_1, u_2, u_3) is a randomised or mixed strategy for the union.

  If $m_A, m_B, m_C \geq 0$ and $m_A+m_B+m_C = 1$ then the strategy (m_A, m_B, m_C) is a randomised or mixed strategy for the manager.

  If the union chooses the mixed strategy (u_1, u_2, u_3) then their expected reward against each of the manager's strategies are:

Manager chooses	Union's reward
A	$9.5u_1 + 7u_2 + 6u_3$
B	$12u_1 + 8.5u_2 + 9u_3$
C	$7u_1 + 6.5u_2 + 10u_3$

By the basic assumption the manager will choose a strategy that makes union's expected reward equal to

$$\min\{9.5u_1 + 7u_2 + 6u_3, 12u_1 + 8.5u_2 + 9u_3, 7u_1 + 6u_2 + 10u_3\}, \quad (1)$$

and at the same time the union should choose the strategy (u_1, u_2, u_3) to make (1) as large as possible:

$$\max\{\min\{9.5u_1+7u_2+6u_3, 12u_1+8.5u_2+9u_3, 7u_1+6u_2+10u_3\}\}. \quad (2)$$

Therefore the union's strategy is the solution of the following L.P. :

$$\begin{array}{rll} \max & v & \\ s.t. & v & -(9.5u_1 + 7u_2 + 6u_3) \leq 0 \\ & v & -(12u_1 + 8.5u_2 + 9u_3) \leq 0 \\ & v & -(7u_1 + 6u_2 + 10u_3) \leq 0 \\ & & u_1 + u_2 + u_3 = 1 \\ & & u_1, u_2, u_3 \geq 0. \end{array} \quad (3)$$

- Similarly – the manager's strategy will be determined by the solution of the following L.P. problem:

$$\begin{array}{rll} \min & w & \\ s.t. & w & -9.5m_A - 12m_B - 7m_C \geq 0 \\ & w & -7m_A - 8.5m_B - 6.5m_C \geq 0 \\ & w & -6m_A - 9m_B - 10m_C \geq 0 \\ & & m_A + m_B + m_C = 1 \\ & & m_A, m_B, m_C \geq 0. \end{array} \quad (4)$$

2 Tutorial –2

2.1 Example 1 – Minimax Problem

Three linear functions y_1, y_2 and y_3 are defined as follows:

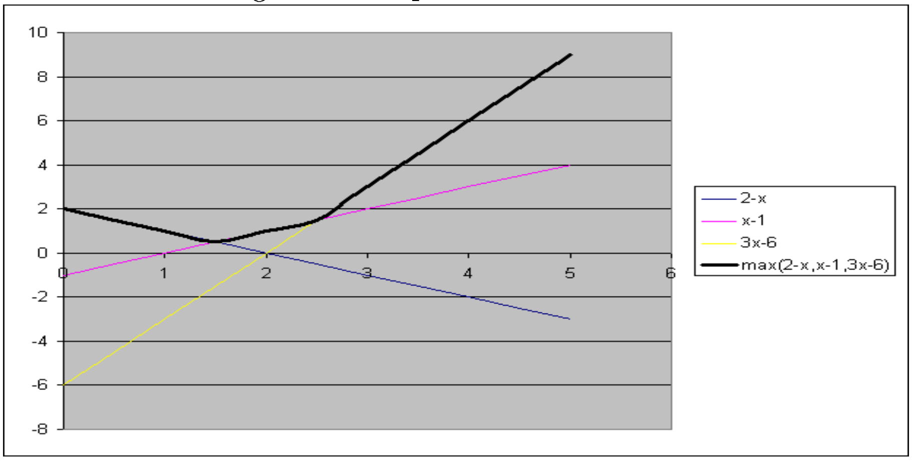
$$\begin{aligned} y_1 &= 2 - x_1, \\ y_2 &= x_1 - 1, \\ y_3 &= 2x_1 - 6. \end{aligned}$$

Find

$$\min_{x} \max_{i=1,2,3} \{y_i\}. \tag{5}$$

2.1.1 Solution

Figure 1: Graph of **max** function.



A way to solve problem (5) is to introduce an auxiliary variable, say x_0 and then to solve the equivalent problem:

$$\begin{aligned} \mathit{min} \quad & x_0 \\ \mathit{s.t.} \quad & 2 - x_1 \leq x_0, \\ & x_1 - 1 \leq x_0, \\ & 2x_1 - 6 \leq x_0. \end{aligned}$$

The solution to this problem, as can be seen from figure 1 is

$$x_1^* = 1.5, \quad x_0^* = 0.5.$$

What would the graph of the following function:

$$\max_x \min_i y_i$$

look like and what is the solution to that problem?

2.2 Example 2 – Minimax Problem Again

Find x_1, x_2 satisfying

$$\begin{aligned} x_1 + x_2 &\leq 2, \\ x_1, x_2 &\geq 0, \end{aligned} \tag{6}$$

and having the maximum of

$$\begin{aligned} 3x_1 - &\quad x_2 \\ -x_1 + &\quad x_2 \end{aligned} \tag{7}$$

as small as possible.

2.2.1 Solution

The problem can be transformed into the following LP problem:

$$\begin{aligned} \min \quad & w \\ s.t. \quad & 3x_1 - x_2 \leq w \\ & -x_1 + 2x_2 \leq w \\ & x_1 + x_2 \leq 2, \\ & x_1, x_2 \geq 0. \end{aligned}$$

To use the simplex algorithm we need all variables to be ≥ 0. However, w could be any real value as it is free variable, so we define $w = x_3 - x_4$, where $x_3, x_4 \geq 0$. So we are left with the following problem to solve:

$$\begin{aligned} \min \quad & x_3 - x_4 \\ s.t. \quad & 3x_1 - x_2 - x_3 + x_4 \leq 0 \\ & -x_1 + 2x_2 - x_3 + x_4 \leq 0 \\ & x_1 + x_2 \leq 2, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Now, introduce slack variables x_5, x_6, x_7:

$$\begin{array}{ll} \min & x_0 = x_3 - x_4 \\ s.t. & 3x_1 - x_2 - x_3 + x_4 \quad +x_5 \qquad\qquad\quad = 0 \\ & -x_1 + 2x_2 - x_3 + x_4 \qquad\quad +x_6 \qquad = 0 \\ & x_1 + x_2 \qquad\qquad\qquad\qquad\qquad\qquad +x_7 = 2 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{array} \tag{8}$$

The solution is (**check!!!**) $(x_0, x_1, x_2) = (0, 0, 0)$.

2.3 Dual Theory

Given the primal L.P. problem:

$$\begin{array}{ll} \max\limits_{x} & c^t x \\ s.t. & Ax \leq b \\ & x \geq 0, \end{array} \tag{9}$$

and its dual pair:

$$\begin{array}{ll} \min\limits_{y} & b^t y \\ s.t. & A^t y \geq c \\ & y \geq 0, \end{array} \tag{10}$$

show that the dual of (10) is (9).

2.3.1 Solution

We start from the original dual problem (10):

$$\begin{array}{ll} \min\limits_{y} & b^t y \\ s.t. & A^t y \geq c \\ & y \geq 0, \end{array}$$

which is equivalent to :

$$\left.\begin{array}{rr} \max_y & (-b^t)y \\ s.t. & A^t y \geq c \\ & y \geq 0 \end{array}\right\} \rightarrow \left.\begin{array}{rr} \max_y & (-b^t)y \\ s.t. & -A^t y \leq -c \\ & y \geq 0 \end{array}\right\} \tag{11}$$

Thus problem (10) is equivalent to the problem (11). We will find the dual of (10) applying the rule onto the equivalent problem (11):

$$\left.\begin{array}{rr} \min_x & (-c^t)x \\ s.t. & (-A^t)^t x \geq -b \\ & x \geq 0 \end{array}\right\} \rightarrow \left.\begin{array}{rr} \min_x & (-c^t)x \\ s.t. & -Ax \geq -b \\ & x \geq 0 \end{array}\right\} \rightarrow \tag{12}$$

$$\rightarrow \left.\begin{array}{rr} \max_x & (c^t)x \\ s.t. & -Ax \geq -b \\ & x \geq 0 \end{array}\right\} \rightarrow \left.\begin{array}{rr} \max_y & c^t x \\ s.t. & Ax \leq b \\ & x \geq 0 \end{array}\right\} \rightarrow \tag{13}$$

Hence the dual of the dual problem (10) of the initial problem (9) is (13) which is equivalent to (9).

2.4 Duality Theory 2

Find the dual problem of the following L.P. problem:

$$\begin{array}{llrl} \max & x_0 = & 3x_1 + 2x_2 & \\ s.t. & & 5x_1 + 2x_2 & \leq 0 \\ & & 4x_1 + 6x_2 & \leq 24 \\ & & x_1 + x_2 & \geq 1 \\ & & x_1 + 3x_2 & = 1 \\ & & x_1 & \geq 0. \end{array} \tag{14}$$

2.4.1 Solution

We are going to use rules (1),(2) and (3) from your notes to find the dual of (14).

$\min y_0$	$x_1 \geq 0$	x_2 free	
$y_1 \geq 0$	5	2	≤ 0
$y_2 \geq 0$	4	6	≤ 24
$y_3?$	1	1	≥ 1
$y_4?$	1	3	$= 1$
	≥ 3	?2	

- Since the *3rd* primal constraint is $\geq$ inequality, then *3rd* dual variable y_3 must satisfy $y_3 \leq 0$;

- Since the *4th* primal constraint is an equality constraint then *4th* dual variable y_4 must be free – unrestricted in sign;
- Since *2nd* primal variable x_2 is free then *2nd* dual constraint will be an equality.

The new table becomes:

$\min y_0$	$x_1 \geq 0$	x_2 free	
$y_1 \geq 0$	5	2	≤ 0
$y_2 \geq 0$	4	6	≤ 24
$y_3 \leq 0$	1	1	≥ 1
$y_4$free	1	3	$= 1$
	≥ 3	$=2$	

Hence, the dual problem of (14) is:

$$\begin{array}{lll} \min \quad y_0 = & 0y_1 + 24y_2 + y_3 + y_4 & \\ s.t. & 5y_1 + 4y_2 + y_3 + y_4 & \geq 3 \\ & 2y_1 + 6y_2 + y_3 + 3y_4 & = 2 \\ & y_1, y_2 \geq 0 & y_3 \leq 0 \end{array} \tag{15}$$

3 Tutorial – 3

3.1 Duality Theory

Find the dual problem of the following L.P. problem:

$$\begin{array}{llrl} \max & x_0 = & 3x_1 + 2x_2 & \\ s.t. & & 5x_1 + 2x_2 & \leq 10 \\ & & 4x_1 + 6x_2 & \leq 24 \\ & & x_1 + x_2 & \geq 1 \\ & & x_1 + 3x_2 & = 9 \\ & & x_1 & \geq 0. \end{array} \tag{16}$$

3.1.1 Solution

We are going to use rules (1),(2) and (3) from your notes to find the dual of (16).

$\min y_0$	$x_1 \geq 0$	x_2 free	
$y_1 \geq 0$	5	2	≤ 10
$y_2 \geq 0$	4	6	≤ 24
$y_3?$	1	1	≥ 1
$y_4?$	1	3	$= 9$
	≥ 3	$?2$	

- Since the *3rd* primal constraint is $\geq$ inequality, then *3rd* dual variable y_3 must satisfy $y_3 \leq 0$;
- Since the *4th* primal constraint is an equality constraint then *4th* dual variable y_4 must be free – unrestricted in sign;
- Since *2nd* primal variable x_2 is free then *2nd* dual constraint will be an equality.

The new table becomes:
Hence, the dual problem of (16) is:

$$\begin{array}{llrl} \min & y_0 = & 10y_1 + 24y_2 + y_3 + 9y_4 & \\ s.t. & & 5y_1 + 4y_2 + y_3 + y_4 & \geq 3 \\ & & 2y_1 + 6y_2 + y_3 + 3y_4 & = 2 \\ & & y_1, y_2 \geq 0 & y_3 \leq 0. \end{array} \tag{17}$$

$\min y_0$	$x_1 \geq 0$	x_2 free	
$y_1 \geq 0$	5	1	≤ 10
$y_2 \geq 0$	4	6	≤ 24
$y_3 \leq 0$	1	1	≥ 1
$y_4$free	1	3	$= 9$
	≥ 3	$=2$	

3.2 Free Variables

Solve the following problem:

$$\begin{array}{lrl} \min \quad x_0 = & x_1 + 2x_2 - x_3 & \\ s.t. & x_1 - x_2 + x_3 & \leq 1 \\ & x_1 + x_2 - 2x_3 & \leq 4 \\ & x_1 & \geq 0. \end{array} \tag{18}$$

3.2.1 Solution

As x_2 and x_3 are free variables we introduce the following:

$$\begin{array}{ll} x_2 & = y_2 - w_2, \quad y_2, w_2 \geq 0, \\ x_3 & = y_3 - w_3, \quad y_3, w_3 \geq 0. \end{array} \tag{19}$$

After adding two slack variables s_1 and s_2 problem (18) becomes:

$$\begin{array}{lrl} \min \quad x_0 = & x_1 + 2(y_2 - w_2) - (y_3 - w_3) & \\ s.t. & x_1 - (y_2 - w_2) + (y_3 - w_3) + s_1 & = 1, \\ & x_1 + (y_2 - w_2) - 2(y_3 - w_3) + s_2 & = 4, \\ & x_1 & \geq 0, \\ & x_1, y_2, w_2, y_3, w_3, s_1, s_2 & \geq 0. \end{array} \tag{20}$$

Solution: $x^* = (0, -6, -5)$.

3.3 Game Theory – Example 1

Consider the following reward matrix:

Which strategy should each of the two players choose? One answer must be obtained by applying the concept of dominated strategies to rule out a succession of inferior strategies until only one choice remains.

Player I	Player II		
	1	2	3
1	17	23	48
2	17	3	51
3	3	17	-2

Table 6: The reward matrix 1.

3.3.1 Solution

At the initial table (reward matrix) there are no dominated strategies for player II. However, for Player I, strategy 3 is dominated by 1 because the latter has larger payoffs regardless of what player II does. Eliminating strategy 3 from further consideration the following reward matrix is obtained:

Player I	Player II		
	1	2	3
1	17	23	48
2	17	3	51

Table 7: The reward matrix 2.

Player II now has a dominated strategy which is 3. It is dominated by both strategies 1 and 2 because they always have smaller losses. Eliminating this strategy we obtain the following reward matrix:

Player I	Player II	
	1	2
1	17	23
2	17	3

Table 8: The reward matrix 3.

Now strategy 2 for player I becomes dominated by strategy 1 for player I. Eliminating the dominated strategy the following table is obtained:

Strategy 2 for player II is dominated by 1, as $17 \leq 23$. Consequently, both players should choose strategy 1.

Player I	Player II	
	1	2
1	17	23

Table 9: The reward matrix 4.

3.4 Game Theory – Example 2

The manager of a multinational company and the union of workers are preparing to sit down at the bargaining table to work out the details of a new contract for the workers. Each side has developed certain proposals for the contents of the new contract. Let us call union proposals "Proposal 1", "Proposal 2" and "Proposal 3", and manager's proposals "Contract A", "Contract B" and "Contract C". Both parties are aware of the financial aspects of each proposal–contract combination. The reward matrix is:

Proposal	Contract		
	A	B	C
1	9.5	12.0	7.0
2	7.0	8.5	6.5
3	6.0	9.0	10.0

Table 10: The reward matrix.

- Is there an equilibrium point?
- Find the mixed strategies for the union and the manager.
- Formulate the LP problem to determine the optimum strategy for the union and the optimum strategy of the manager.

3.4.1 Solution

- Union strategy – $(u_1, u_2, u_3) = (0.615, 0, 0.385)$;
- Manager's strategy $(m_A, m_B, m_C) = (0.462, 0, 0.538)$;
- Value of the game – $v = 8.154$.

Proposal	Contract		
	A	B	C
1	8.5	7.0	7.5
2	12.0	9.5	9.0
3	9.0	11.0	8.0

Table 11: The reward matrix.

3.5 Game Theory – Example 3

Consider the previous example, but with the following reward matrix:

- Is there an equilibrium point?
- Find the strategies which are dominated by other strategies, and reduce the size of the reward matrix.
- Formulate the LP problem to determine the optimum strategy for the union and the optimum strategy of the manager.

3.5.1 Solution

It can be seen that strategy 2 dominates strategy 1. For strategies 2 or 3 neither dominates the other. Depending on the strategy selected by the manager either of the two strategies can result in higher payoff.

Proposal	Contract		
	A	B	C
2	12.0	9.5	9.0
3	9.0	11.0	8.0

Table 12: The reward matrix.

Similarly, since the Manager is seeking to choose the lowest pay outs possible, we see that strategy A always has a higher loss to the company then strategy C, regardless of the Union's action. So, the Manager would never choose strategy A, and it may be removed from the reward matrix:

- The Union's mixed strategy (u_2, u_3);
- The Manager's mixed strategy (m_B, m_C).

Proposal	Contract	
	B	C
2	9.5	9.0
3	11.0	8.0

Table 13: The reward matrix.

- The manager will choose the strategy that makes Union's expected reward as small as possible:

$$\min\{9.5u_2 + 11u_3, 9u_2 + 8u_3\}.$$

- The union will choose the strategy that makes expected reward as large as possible:

$$\max\min\{9.5u_2 + 11u_3, 9u_2 + 8u_3\}.$$

- Similarly the manager will choose mixed strategy (m_B, m_C) to solve the following problem:

$$\min\max\{9.5m_B + 9m_C, 11m_B + 8m_C\}.$$