Game Theory
Tutorial 1
Answers

Exercise 1 *The manager of a multinational company and the union of workers are preparing to sit down at the bargaining table to work out the details of a new contract for the workers. Each side has developed certain proposals for the contents of the new contract. Let us call union proposals "Proposal 1", "Proposal 2" and "Proposal 3", and manager's proposals "Contract A", "Contract B" and "Contract C". Both parties are aware of the financial aspects of each proposal–contract combination. The reward matrix is:*

Proposal	*Contract*		
	A	*B*	*C*
1	*8.5*	*7.0*	*7.5*
2	*12.0*	*9.5*	*9.0*
3	*9.0*	*11.0*	*8.0*

Reward matrix

These values are the contract gains that the Union would secure and also the cost the company would have to bear.

Is there a clear–cut contract combination agreeable to both parties, or will they find it necessary to submit to arbitration in order to arrive at some sort of compromise?

Solution : First we find the union's optimal strategy. The minimum payoff for each strategy (which is the minimum payoff in each row) is shown in the **row min** column of the following table. The maximum of these minimum payoffs is 9 in the second row. Consequently the union would select strategy 2 as its optimal one.

Proposal	Contract			Row min
	A	B	C	
1	8.5	7.0	7.5	7.0
2	12.0	9.5	9.0	9.0
3	9.0	11.0	8.0	8.0

Minimum for each row.

In a similar way we find the manager's optimal strategy. The maximum pay out for each contract (which is the maximum pay out in each column) is shown in the **col max** column of the following table. The minimum of these maximum pay outs is 9 in the third column. Consequently the manager would select strategy C as its optimal one.

Proposal	Contract			Row min
	A	B	C	
1	8.5	7.0	7.5	7.0
2	12.0	9.5	9.0	9.0
3	9.0	11.0	8.0	8.0
Col. max	12.0	11.0	9.0	–

Maximum for each column.

We see that both of the players in this game will select a strategy that has the same value. The *min* value in row 2 is also the *max* value in column C, so the solution is an equilibrium or saddle point.

$$\min\{col.max.\} = \min\{12, 11, 9\} = 9 = \max\{row.min.\} = max\{7.5, 9, 8\}.$$

Exercise 2 *Consider the same problem as in Example 1, but with the following reward matrix:*

Proposal	*Contract*		
	A	*B*	*C*
1	*9.5*	*12.0*	*7.0*
2	*7.0*	*8.5*	*6.5*
3	*6.0*	*9.0*	*10.0*

Reward matrix

- *Is there an equilibrium point?*
- *Find the mixed strategies for the union and the manager.*
- *Formulate the LP problem to determine the optimum strategy for the union and the optimum strategy of the manager.*

Solution : Similarly as in previous example we determine the row min and the col max for the reward matrix:

Proposal	Contract			row min
	A	B	C	
1	9.5	12.0	7.0	7.0
2	7.0	8.5	6.5	6.5
3	6.0	9.0	10.0	6.0
col. max	9.5	12.0	10.0	–

Optimum for each row and column.

- It is easy to see from table 5 that

  $$\max\{row.min.\} = 7 \neq \min\{col.max.\} = 9.5.$$

  Therefore there is no equilibrium point.

- Let us assign the following unknown probabilities $u_i, i = 1, 2, 3$ for each strategy, where u_i is the probability that the union chooses $i - th$ strategy. Similarly for the manager we assign $m_k, k = A, B, C$.

  If $u_1, u_2, u_3 \geq 0$ and $u_1 + u_2 + u_3 = 1$ then the strategy (u_1, u_2, u_3) is a randomised or mixed strategy for the union.

  If $m_A, m_B, m_C \geq 0$ and $m_A+m_B+m_C = 1$ then the strategy (m_A, m_B, m_C) is a randomised or mixed strategy for the manager.

  If the union chooses the mixed strategy (u_1, u_2, u_3) then their expected reward against each of the manager's strategies are:

Manager chooses	Union's reward
A	$9.5u_1 + 7u_2 + 6u_3$
B	$12u_1 + 8.5u_2 + 9u_3$
C	$7u_1 + 6.5u_2 + 10u_3$

By the basic assumption the manager will choose a strategy that makes union's expected reward equal to

$$\min\{9.5u_1 + 7u_2 + 6u_3, 12u_1 + 8.5u_2 + 9u_3, 7u_1 + 6u_2 + 10u_3\}, \quad (1)$$

and at the same time the union should choose the strategy (u_1, u_2, u_3) to make (1) as large as possible:

$$\max\{\min\{9.5u_1+7u_2+6u_3, 12u_1+8.5u_2+9u_3, 7u_1+6u_2+10u_3\}\}. \quad (2)$$

Therefore the union's strategy is the solution of the following L.P. :

$$\begin{array}{rll} \max & v & \\ s.t. & v & -(9.5u_1 + 7u_2 + 6u_3) \leq 0 \\ & v & -(12u_1 + 8.5u_2 + 9u_3) \leq 0 \\ & v & -(7u_1 + 6u_2 + 10u_3) \leq 0 \\ & & u_1 + u_2 + u_3 = 1 \\ & & u_1, u_2, u_3 \geq 0. \end{array} \quad (3)$$

- Similarly – the manager's strategy will be determined by the solution of the following L.P. problem:

$$\begin{array}{rll} \min & w & \\ s.t. & w & -9.5m_A - 12m_B - 7m_C \geq 0 \\ & w & -7m_A - 8.5m_B - 6.5m_C \geq 0 \\ & w & -6m_A - 9m_B - 10m_C \geq 0 \\ & & m_A + m_B + m_C = 1 \\ & & m_A, m_B, m_C \geq 0. \end{array} \quad (4)$$