Game Theory
Tutorial 3
Answers

Exercise 1 (Duality Theory) *Find the dual problem of the following L.P. problem:*

$$\begin{array}{llrl} \max & x_0 = & 3x_1 + 2x_2 & \\ s.t. & & 5x_1 + 2x_2 & \leq 10 \\ & & 4x_1 + 6x_2 & \leq 24 \\ & & x_1 + x_2 & \geq 1 \\ & & x_1 + 3x_2 & = 9 \\ & & x_1 & \geq 0. \end{array} \tag{1}$$

Solution : We are going to use rules (1),(2) and (3) from your notes to find the dual of (1).

$$\begin{array}{c|cc l} \min y_0 & x_1 \geq 0 & x_2 \text{ free} & \\ \hline y_1 \geq 0 & 5 & 2 & \leq 10 \\ y_2 \geq 0 & 4 & 6 & \leq 24 \\ y_3? & 1 & 1 & \geq 1 \\ y_4? & 1 & 3 & = 9 \\ & \geq 3 & ?2 & \end{array}$$

- Since the *3rd* primal constraint is $\geq$ inequality, then *3rd* dual variable y_3 must satisfy $y_3 \leq 0$;

- Since the *4th* primal constraint is an equality constraint then *4th* dual variable y_4 must be free – unrestricted in sign;
- Since *2nd* primal variable x_2 is free then *2nd* dual constraint will be an equality.

The new table becomes:

$\min y_0$	$x_1 \geq 0$	x_2 free	
$y_1 \geq 0$	5	1	≤ 10
$y_2 \geq 0$	4	6	≤ 24
$y_3 \leq 0$	1	1	≥ 1
$y_4$free	1	3	$= 9$
	≥ 3	$=2$	

Hence, the dual problem of (1) is:

$$\begin{array}{lrl} \min \quad y_0 = & 10y_1 + 24y_2 + y_3 + 9y_4 & \\ s.t. & 5y_1 + 4y_2 + y_3 + y_4 & \geq 3 \\ & 2y_1 + 6y_2 + y_3 + 3y_4 & = 2 \\ & y_1, y_2 \geq 0 & y_3 \leq 0. \end{array} \tag{2}$$

Exercise 2 (Free Variables) *Solve the following problem:*

$$\begin{array}{lrl} \min \quad x_0 = & x_1 + 2x_2 - x_3 & \\ s.t. & x_1 - x_2 + x_3 & \leq 1 \\ & x_1 + x_2 - 2x_3 & \leq 4 \\ & x_1 & \geq 0. \end{array} \tag{3}$$

Solution : As x_2 and x_3 are free variables we introduce the following:

$$\begin{array}{ll} x_2 & = y_2 - w_2, \quad y_2, w_2 \geq 0, \\ x_3 & = y_3 - w_3, \quad y_3, w_3 \geq 0. \end{array} \tag{4}$$

After adding two slack variables s_1 and s_2 problem (3) becomes:

$$\begin{array}{lllr} \min & x_0 = & x_1 + 2(y_2 - w_2) - (y_3 - w_3) & \\ s.t. & & x_1 - (y_2 - w_2) + (y_3 - w_3) + s_1 & = 1, \\ & & x_1 + (y_2 - w_2) - 2(y_3 - w_3) + s_2 & = 4, \\ & & x_1 & \geq 0, \\ & & x_1, y_2, w_2, y_3, w_3, s_1, s_2 & \geq 0. \end{array} \quad (5)$$

Solution: $x^* = (0, -6, -5)$.

Exercise 3 (Game Theory) *Consider the following reward matrix:*

Player I	*Player II*		
	1	*2*	*3*
1	*17*	*23*	*48*
2	*17*	*3*	*51*
3	*3*	*17*	*-2*

Which strategy should each of the two players choose? One answer must be obtained by applying the concept of dominated strategies to rule out a succession of inferior strategies until only one choice remains.

Solution : At the initial table (reward matrix) there are no dominated strategies for player II. However, for Player I, strategy 3 is dominated by 1 because the latter has larger payoffs regardless of what player II does. Eliminating strategy 3 from further consideration the following reward matrix is obtained:

Player I	Player II		
	1	2	3
1	17	23	48
2	17	3	51

Player II now has a dominated strategy which is 3. It is dominated by both strategies 1 and 2 because they always have smaller losses. Eliminating this strategy we obtain the following reward matrix:

Player I	Player II	
	1	2
1	17	23
2	17	3

Now strategy 2 for player I becomes dominated by strategy 1 for player I. Eliminating the dominated strategy the following table is obtained:

Player I	Player II	
	1	2
1	17	23

Strategy 2 for player II is dominated by 1, as $17 \leq 23$. Consequently, both players should choose strategy 1.

Exercise 4 (Game Theory) *The manager of a multinational company and the union of workers are preparing to sit down at the bargaining table to work out the details of a new contract for the workers. Each side has developed certain proposals for the contents of the new contract. Let us call union proposals "Proposal 1", "Proposal 2" and "Proposal 3", and manager's proposals "Contract A", "Contract B" and "Contract C". Both parties are aware of the financial aspects of each proposal–contract combination. The reward matrix is:*

Proposal	*Contract*		
	A	*B*	*C*
1	*9.5*	*12.0*	*7.0*
2	*7.0*	*8.5*	*6.5*
3	*6.0*	*9.0*	*10.0*

- *Is there an equilibrium point?*
- *Find the mixed strategies for the union and the manager.*
- *Formulate the LP problem to determine the optimum strategy for the union and the optimum strategy of the manager.*

Solution :

- Union strategy – $(u_1, u_2, u_3) = (0.615, 0, 0.385)$;
- Manager's strategy $(m_A, m_B, m_C) = (0.462, 0, 0.538)$;
- Value of the game – $v = 8.154$.

Exercise 5 (Game Theory) *Consider the previous example, but with the following reward matrix:*

Proposal	*Contract*		
	A	*B*	*C*
1	*8.5*	*7.0*	*7.5*
2	*12.0*	*9.5*	*9.0*
3	*9.0*	*11.0*	*8.0*

- *Is there an equilibrium point?*
- *Find the strategies which are dominated by other strategies, and reduce the size of the reward matrix.*
- *Formulate the LP problem to determine the optimum strategy for the union and the optimum strategy of the manager.*

Solution : It can be seen that strategy 2 dominates strategy 1. For strategies 2 or 3 neither dominates the other. Depending on the strategy selected by the manager either of the two strategies can result in higher payoff.

Proposal	Contract		
	A	B	C
2	12.0	9.5	9.0
3	9.0	11.0	8.0

Similarly, since the Manager is seeking to choose the lowest pay outs possible, we see that strategy A always has a higher loss to the company then strategy C, regardless of the Union's action. So, the Manager would never choose strategy A, and it may be removed from the reward matrix:

Proposal	Contract	
	B	C
2	9.5	9.0
3	11.0	8.0

- The Union's mixed strategy (u_2, u_3);
- The Manager's mixed strategy (m_B, m_C).
- The manager will choose the strategy that makes Union's expected reward as small as possible:

$$\min\{9.5u_2 + 11u_3, 9u_2 + 8u_3\}.$$

- The union will choose the strategy that makes expected reward as large as possible:

$$\max\min\{9.5u_2 + 11u_3, 9u_2 + 8u_3\}.$$

- Similarly the manager will choose mixed strategy (m_B, m_C) to solve the following problem:

$$\min\max\{9.5m_B + 9m_C, 11m_B + 8m_C\}.$$