Duality Theory

Berc Rustem

November 23, 2010

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Definition: Primal and Dual Problem

- **Primal Problem.**

$$\max \left\{ c^\top x \, : \, Ax \leq b, \, x \geq 0 \right\}, \tag{P}$$

where $c, x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$.

- **Dual Problem.**

$$\min \left\{ b^\top y \, : \, A^\top y \geq c, \, y \geq 0 \right\}, \tag{D}$$

where c, A, b as in (P) and $y \in \mathbb{R}^m$.

- **Definition is 'symmetric'.** The dual of (D) is (P).
  - can be shown with techniques to be discussed later today

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Weak Duality

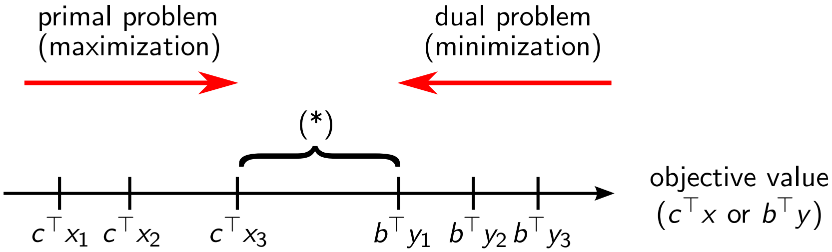
Theorem (Weak Duality). Assume that the problems

$$\max \left\{ c^\top x \,:\, Ax \leq b,\, x \geq 0 \right\} \tag{P}$$

and

$$\min \left\{ b^\top y \,:\, A^\top y \geq c,\, y \geq 0 \right\}. \tag{D}$$

are both feasible. Let $x \in \mathbb{R}^n$ be feasible for (P) and $y \in \mathbb{R}^m$ be feasible for (D). Then we have that $\boxed{\mathbf{c}^\top \mathbf{x} \leq \mathbf{b}^\top \mathbf{y}}$.

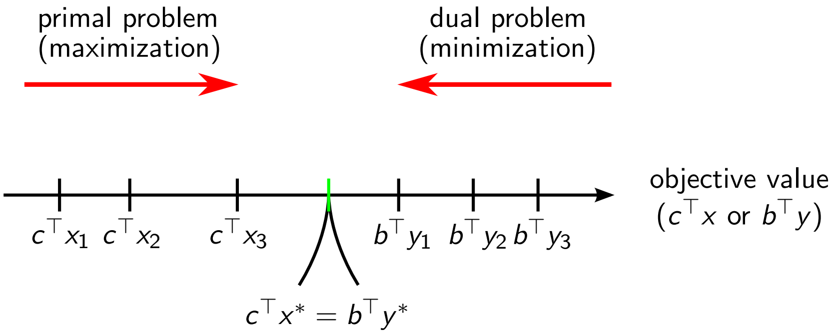


Strong Duality

Theorem (Strong Duality). Assume that the problems (P) and (D) are both feasible. Let B be optimal basis for (P), together with optimal basic solution (x_B^*, x_N^*). Then we have that:

(a) $y^* = (B^{-1})^\top c_B$ is an optimal solution for (D).

(b) $c^\top x^* = b^\top y^*$, that is, **the objective values coincide**.



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Complementary Slackness

Theorem (Complementary Slackness). Let $(x, s) \in \mathbb{R}^n \times \mathbb{R}^m$ be feasible for

$$\max \left\{ c^\top x \,:\, Ax + s = b,\ x \geq 0, s \geq 0 \right\} \qquad (\mathrm{P}')$$

and $(y, e) \in \mathbb{R}^m \times \mathbb{R}^n$ be feasible for

$$\min \left\{ b^\top y \,:\, A^\top y - e = c,\ y \geq 0, e \geq 0 \right\}. \qquad (\mathrm{D}')$$

Then x and y are optimal for (P) and (D), respectively, if and only if

$$\begin{aligned} s_i y_i &= 0 \quad \forall\, i = 1, \ldots, m \\ \text{and} \quad e_j x_j &= 0 \quad \forall\, j = 1, \ldots, n. \end{aligned}$$

Explanation. Conditions are satisified if and only if:

(A1) i^{th} slack in (P) $> 0 \Rightarrow i^{\text{th}}$ variable in (D) $= 0$

(A2) i^{th} slack in (P) $= 0 \Leftarrow i^{\text{th}}$ variable in (D) > 0

(B1) j^{th} slack in (D) $> 0 \Rightarrow j^{\text{th}}$ variable in (P) $= 0$

(B2) j^{th} slack in (D) $= 0 \Leftarrow j^{\text{th}}$ variable in (P) > 0

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Indirect Way

Idea. Bring problem to form of (P) or (D) and apply duality definition.

'Algorithm'.

1. **Bring LP to the form of either (P) or (D).**
   - Replace variables $x_i \in \mathbb{R}$ with $(x_i^+ - x_i^-)$ where $x_i^+, x_i^- \geq 0$.
   - Replace equality constraints with two inequality constraints.
   - Change constraint direction $(\leq, \geq)$ by multiplication with (-1) if necessary.
   - Change direction of objective function by multiplication with (-1) if necessary.
2. **Obtain dual according to definition.**
   - If LP is in the form of (P), its dual is (D).
   - If LP is in the form of (D), its dual is (P).
3. **Simplify dual problem.** (Optional)
   - Replace variable pairs $y_i, y_j \geq 0$, $i \neq j$, that occur in all functions as $\alpha y_i - \alpha y_j$ by one variable $y_k \in \mathbb{R}$.
   - Replace matching inequality constraints by equality constraints.

Indirect Way: Example

Obtain the dual of

$$\max_{x_1, x_2} \; 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &\leq 1, \end{aligned}$$

where $x_1 \geq 0$ and $x_2 \in \mathbb{R}$.

Indirect Way: Example

$$\max_{x_1, x_2} \; 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &\leq 1, \end{aligned}$$

where $x_1 \geq 0$ and $x_2 \in \mathbb{R}$.

'Algorithm'.

1. **Bring LP to the form of either (P) or (D).**
   - Replace variables $x_i \in \mathbb{R}$ with $(x_i^+ - x_i^-)$ where $x_i^+, x_i^- \geq 0$.

Indirect Way: Example

$$\max_{x_1, x_2} \; 2x_1 + x_2^+ - x_2^-$$

subject to

$$\begin{aligned} x_1 + x_2^+ - x_2^- &= 2 \\ 2x_1 - x_2^+ + x_2^- &\geq 3 \\ x_1 - x_2^+ + x_2^- &\leq 1, \end{aligned}$$

where $x_1, x_2^+, x_2^- \geq 0$.

'Algorithm'.

1. **Bring LP to the form of either (P) or (D).**
   - Replace equality constraints with two inequality constraints.

Indirect Way: Example

$$\max_{x_1, x_2} \; 2x_1 + x_2^+ - x_2^-$$

subject to

$$\begin{aligned} x_1 + x_2^+ - x_2^- &\leq 2 \\ x_1 + x_2^+ - x_2^- &\geq 2 \\ 2x_1 - x_2^+ + x_2^- &\geq 3 \\ x_1 - x_2^+ + x_2^- &\leq 1, \end{aligned}$$

where $x_1, x_2^+, x_2^- \geq 0$.

'Algorithm'.

1. **Bring LP to the form of either (P) or (D).**
   - Change constraint direction $(\leq, \geq)$ by multiplication with (-1) if necessary.

Indirect Way: Example

$$\max_{x_1, x_2} \; 2x_1 + x_2^+ - x_2^-$$

subject to

$$\begin{aligned} x_1 + x_2^+ - x_2^- &\leq 2 \\ -x_1 - x_2^+ + x_2^- &\leq -2 \\ -2x_1 + x_2^+ - x_2^- &\leq -3 \\ x_1 - x_2^+ + x_2^- &\leq 1, \end{aligned}$$

where $x_1, x_2^+, x_2^- \geq 0$.

'Algorithm'.

1. **Bring LP to the form of either (P) or (D).**
   - Change direction of objective function by multiplication with (-1) if necessary.

Indirect Way: Example

$$\max_{x_1, x_2} \; 2x_1 + x_2^+ - x_2^-$$

subject to

$$\begin{aligned} x_1 + x_2^+ - x_2^- &\le 2 \\ -x_1 - x_2^+ + x_2^- &\le -2 \\ -2x_1 + x_2^+ - x_2^- &\le -3 \\ x_1 - x_2^+ + x_2^- &\le 1, \end{aligned}$$

where $x_1, x_2^+, x_2^- \ge 0$.

'Algorithm'.

1. **Obtain dual according to definition.**
   - If LP is in the form of (P), its dual is (D).

Indirect Way: Example

Primal Problem:

$$\max_{x} \; c^\top x \quad \text{with} \quad c = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \text{ and } x = \begin{pmatrix} x_1 \\ x_2^+ \\ x_2^- \end{pmatrix}$$

subject to

$$Ax \leq b \quad \text{with} \quad A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ -2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ -2 \\ -3 \\ 1 \end{pmatrix}$$

$$x \geq 0.$$

'Algorithm'.

1. **Obtain dual according to definition.**
   - If LP is in the form of (P), its dual is (D).

Indirect Way: Example

Dual Problem:

$$\min_y \; b^\top y \quad \text{with} \quad b = \begin{pmatrix} 2 \\ -2 \\ -3 \\ 1 \end{pmatrix} \text{ and } y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

subject to

$$A^\top y \geq c \quad \text{with} \quad A^\top = \begin{pmatrix} 1 & -1 & -2 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \text{ and } c = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$y \geq 0.$$

'Algorithm'.

1. **Obtain dual according to definition.**
   - If LP is in the form of (P), its dual is (D).

Indirect Way: Example

Dual Problem:

$$\min_{y_1,y_2,y_3,y_4} \; 2y_1 - 2y_2 - 3y_3 + y_4$$

subject to

$$\begin{aligned} y_1 - y_2 - 2y_3 + y_4 &\geq 2 \\ y_1 - y_2 + y_3 - y_4 &\geq 1 \\ -y_1 + y_2 - y_3 + y_4 &\geq -1, \end{aligned}$$

where $y_1, \ldots, y_4 \geq 0$.

'Algorithm'.

1. **Simplify dual problem.** (Optional)
   - Replace variable pairs $y_i, y_j \geq 0$, $i \neq j$, that occur in all functions as $\alpha y_i - \alpha y_j$ by one variable $y_k \in \mathbb{R}$.

Indirect Way: Example

$$\min_{y_1', y_3, y_4} \quad 2y_1' - 3y_3 + y_4$$

subject to

$$\begin{aligned} y_1' - 2y_3 + y_4 &\geq 2 \\ y_1' + y_3 - y_4 &\geq 1 \\ -y_1' - y_3 + y_4 &\geq -1, \end{aligned}$$

where $y_1' \in \mathbb{R}$ and $y_3, y_4 \geq 0$.

'Algorithm'.

1. **Simplify dual problem.** (Optional)
   - Replace matching inequality constraints by equality constraints.

Indirect Way: Example

The simplified dual problem is:

$$\min_{y_1', y_3, y_4} \; 2y_1' - 3y_3 + y_4$$

subject to

$$\begin{aligned} y_1' - 2y_3 + y_4 &\geq 2 \\ y_1' + y_3 - y_4 &= 1, \end{aligned}$$

where $y_1' \in \mathbb{R}$ and $y_3, y_4 \geq 0$.

Direct Way

Idea. Apply duality directly without detour via (P) or (D).

'Algorithm'.

1. For every primal constraint, create one dual variable.
   For every primal variable, create one dual constraint.
2. Transpose coefficient matrix A.
   Former right-hand sides b become new objective coefficients.
   Former objective coefficients c become new right-hand sides.
3. If primal is max problem: Dual is min problem.
   - If i^{th} primal constraint is $[\geq, =, \leq]$, i^{th} dual variable becomes $[y_i \leq 0,\ y_i \in \mathbb{R},\ y_i \geq 0]$, respectively.
   - If j^{th} primal variable is $[x_j \geq 0,\ x_j \in \mathbb{R},\ x_k \leq 0]$, j^{th} dual constraint becomes $[\geq, =, \leq]$, respectively.
4. If primal is min problem: Dual is max problem.
   - If i^{th} primal constraint is $[\geq, =, \leq]$, i^{th} dual variable becomes $[y_i \geq 0,\ y_i \in \mathbb{R},\ y_i \leq 0]$, respectively.
   - If j^{th} primal variable is $[x_j \geq 0,\ x_j \in \mathbb{R},\ x_k \leq 0]$, j^{th} dual constraint becomes $[\leq, =, \geq]$, respectively.

Direct Way: Example

Same example as before: obtain the dual of

$$\max_{x_1, x_2} \; 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &\leq 1, \end{aligned}$$

where $x_1 \geq 0$ and $x_2 \in \mathbb{R}$.

Direct Way: Example

$$\max_{x_1, x_2} \; 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &\leq 1, \end{aligned}$$

where $x_1 \geq 0$ and $x_2 \in \mathbb{R}$.

'Algorithm'.

1. For every primal constraint, create one dual variable.
   For every primal variable, create one dual constraint.
2. Transpose coefficient matrix A.
   Former right-hand sides b become new objective coefficients.
   Former objective coefficients c become new right-hand sides.

Direct Way: Example

$$\underset{y_1, y_2, y_3}{?} \quad 2y_1 + 3y_2 + y_3$$

subject to

$$\begin{aligned} y_1 + 2y_2 + y_3 &\; ?\; 2 && [x_1] \\ y_1 - y_2 - y_3 &\; ?\; 1, && [x_2] \end{aligned}$$

where the domain of y_1, y_2, y_3 is not yet defined.

'Algorithm'.

1. If primal is max problem: Dual is min problem.
   - If i^{th} primal constraint is $[\geq, =, \leq]$, i^{th} dual variable becomes $[y_i \leq 0,\ y_i \in \mathbb{R},\ y_i \geq 0]$, respectively.

Direct Way: Example

$$\min_{y_1,y_2,y_3} \; 2y_1 + 3y_2 + y_3$$

subject to

$$y_1 + 2y_2 + y_3 \;?\; 2$$
$$y_1 - y_2 - y_3 \;?\; 1,$$

where $y_1 \in \mathbb{R}$, $y_2 \leq 0$ and $y_3 \geq 0$.

'Algorithm'.

1. If primal is max problem: Dual is min problem.
   - If j^{th} primal variable is $[x_j \geq 0,\ x_j \in \mathbb{R},\ x_k \leq 0]$, j^{th} dual constraint becomes $[\geq, =, \leq]$, respectively.

Direct Way: Example

The resulting dual problem is:

$$\min_{y_1,y_2,y_3} \; 2y_1 + 3y_2 + y_3$$

subject to

$$\begin{aligned} y_1 + 2y_2 + y_3 &\geq 2 \\ y_1 - y_2 - y_3 &= 1, \end{aligned}$$

where $y_1 \in \mathbb{R}$, $y_2 \leq 0$ and $y_3 \geq 0$.

Equivalence of Indirect and Direct Way

Indirect way led us to the problem:

$$\min_{y_1', y_3, y_4} \; 2y_1' - 3y_3 + y_4$$

subject to

$$\begin{aligned} y_1' - 2y_3 + y_4 &\geq 2 \\ y_1' + y_3 - y_4 &= 1, \end{aligned}$$

where $y_1' \in \mathbb{R}$ and $y_3, y_4 \geq 0$.

Direct way led us to the problem:

$$\min_{y_1, y_2, y_3} \; 2y_1 + 3y_2 + y_3$$

subject to

$$\begin{aligned} y_1 + 2y_2 + y_3 &\geq \; 2 \\ y_1 - y_2 - y_3 &= 1, \end{aligned}$$

where $y_1 \in \mathbb{R}$, $y_2 \leq 0$ and $y_3 \geq 0$.