GAME THEORY (GT)

Thus far, we have considered situations where a single DM chooses an optimal decision without reference to the effect his decision has on other DM's (and without reference to the effect the decision of others may have on him). A computer company, for example, must determine an advertising policy and pricing policy for its computers and each computer manufactures's decision will effect the revenues & profits of other computer manufacturers.

Noncooperative GT is useful for making decisions in cases where two or more DM's have conflicting interests. Mostly, we shall be concerned with two decision makers. However, GT extends to more than two DM's.

1. TWO PERSON ZERO SUM & CONSTANT SUM GAMES

Characteristics of 0-sum two person Games:

1. There are two players (called row player and column player)

2. The row player must choose 1 of m strategies. Simultaneously, the column player must choose 1 of n strategies.

3. If the row player chooses his/hers i^{th} strategy and the column player chooses his/her j^{th} strategy the row player receives a reward of a_{ij} and the column player looses an amount a_{ij}. Thus, we may think of the row player's reward of a_{ij} as coming from the column player.

This is a 2 person 0 – sum game : the matrix a_{ij} is the game's **reward matrix.**

Row Player's strategy	**Column player**'s strategy Col 1	Col 2	...	Col n
Row 1	a_{11}	a_{12}	...	a_{1n}
Row	a_{21}	a_{22}	...	a_{2n}
⋮	⋮	⋮		⋮
Row m	a_{m1}	a_{m2}	...	a_{mn}

Reward Matrix

Example 1:

1	2	3	– 1
2	1	– 2	0

Row player receives 2 units if the row player chooses the second strategy and the column player chooses the first strategy.

2 person 0-sum game: for any choice of strategies the sum of rewards to the players is zero. Every £ one player wins comes out of the other player's pocket. Thus the two players have totally conflicting interests - no cooperation can occur.

John-von Neumann and Oscar Morgenstern (Theory of Games and Economic Behaviour, J. Wiley 1943) developed the theory of 0-sum 2 person games and how they should be played.

Basic Assumption of 2 person 0-sum games:

Each player chooses a strategy that enables him to do the best he can given that the opponent knows the strategy he is following.

Row Player's Strategy	Column player's strategy Col 1	Col 2	Col 3	Row minimum
Row 1	4	4	10	4
Row 2	2	3	1	1
Row 3	6	5	7	5
Column maximum	6	5	10	

How should the Row Player (RP) play this game? If RP chooses R1, the assumption implies that the Column Player (CP) will choose C1 or C2 and hold the RP to a reward of 4 (the smallest number in row 1 of the game matrix). If RP chooses R2, CP will choose C3 and hold the RP's reward to 1 (the smallest-minimum in the second row). If RP chooses R3 then CP will allow him 5. Thus, the assumption $\Rightarrow$ RP should choose the row having the largest minimum. Since max $(4, 1, 5) = 5$, RP chooses R3. This ensures a win of at least max (row minimum) $= 5$.

If the CP chooses C1, the RP will choose R3 (to maximise earnings). If CP chooses C2 the RP will choose R3. If the CP chooses C3 the RP will choose R1 ($10 = \max(10, 1, 7)$). Thus the CP can hold his losses to min (column max) $= \min(6, 5, 10) = 5$ by choosing C2.

Thus, the RP can ensure at least 5 (win) and the CP can hold the RP's gains to at most 5. Thus, the only rational outcome of this game is for the RP to win 5. The RP cannot expect to win more because the CP (by choosing C2) can hold RP's win to 5.

The game matrix we have analysed satisfies the **SADDLE POINT CONDITION** property

$$\begin{pmatrix}\text{maximum}\\ \text{over all}\\ \text{rows}\end{pmatrix} (\text{ row minimum }) = \begin{pmatrix}\text{minimum}\\ \text{over all}\\ \text{columns}\end{pmatrix} (\text{ column maximum }) \tag{1}$$

Any 2 person 0-sum game (2p0sg) satisfying (1) is said to have a SADDLE POINT. If a 2p0sg has a saddle point the RP should choose any R strategy attaining the maximum on the LHS of (1) and a CP should choose a C strategy attaining the minimum on the RHS.

In the game considered a saddle point occurred at R3 and C2. If the game has a saddle point we call the common value to both sides of (1) the VALUE (v) of the game. In the above case $v = 5$.

An easy way of identifying saddle points is to observe that the reward for a saddle point must be the smallest number in its row and the largest number in its column. Like the centre point of a horse's saddle, a saddle point for a 2p0sg is a local minimum in one direction (looking across the row) and local maximum in another direction (down the column).

A saddle point can also be seen as an **EQUILIBRIUM POINT** in that neither player can benefit from a unilateral change from the optimal strategy (of Row 3 to either R1 or 2) since his reward would decrease. If the column player changed from the optimal strategy (of C2 to C1 or 3) RP's reward would increase. $\Rightarrow$ A saddle point is stable in that neither player has an incentive to deviate from it.

Many 2p0sg's do not have saddle points.

Example 2:

$$\begin{bmatrix} -1 & +1 \\ +1 & -1 \end{bmatrix}$$

$$\text{Max (row min)} = -1 < \text{min (col. max)} = +1$$

TWO PERSON CONSTANT SUM GAMES (2PCSG)

Two players can still be in total conflict.

Definition: A 2pcsg is a two player game in which, for any choice of both players' strategies, the RP's reward and the CP's reward add up to a constant c.

Note: 2p0sg is a 2pcsg with $c = 0$.

2pcsg maintains the total conflict between RP and CP. A unit increase in RP's reward $\Rightarrow$ a unit decrease in CP's reward.

The optimal strategies and value of a 2pcsg can be found by the same methods used to solve a 2p0sg.

Example 3:

TV. 8 − 9 pm slot. Two channels competing for an audience of 10 million. Each channel (N1 & N2) must simultaneously announce their programme. Possible choices for N1 & N2 and the number of N1 viewers (millions) for each choice are:

N1	N2 Western	Soap	Comedy	Row minimum
W	3 · 5	1 · 5	6 · 0	1 · 5
S	4 · 5	5 · 8	5 · 0	4 · 5
C	3 · 8	1 · 4	7 · 0	1 · 4
Column max.	4 · 5	5 · 8	7 · 0	

Value of the game for N1? ∃ a saddle point?

If both N1 and N2 show W then N1 gets 3 · 5m viewers, N2 gets (10 − 3.5 =) 6 · 5m.

2pcsg with c = 10m.

Looking at the row minima: choosing a soap, N1 can be sure of at least max (1 · 5, 4 · 5, 1 · 4) = 4 · 5m viewers. Looking at column maxima, choosing a western, N2 can hold N1 at most min (4 · 5, 5 · 8, 7 · 0) = 4 · 5m viewers. Since

$$\textbf{max (row minimum)} = \textbf{(col. maximum)} = 4 \cdot 5$$

(1) is satisfied. Thus, N1 choosing a soap and N2 choosing a western yield a saddle point. Neither side will do better if it unilaterally changes strategy (check this). Thus, the value of the game to N1 = 4 · 5m. viewers and the value of the game to N2 = 10 · 0 − 4 · 5 = 5 · 5m The optimal strategy for N1 is soap and N2 is western.

2. 2P0SG − RANDOMISED STRATEGIES

NOT ALL 2P0SG HAVE SADDLE POINTS. We discuss how one can find the value and optimal strategies for a 2p0sg that does not have a saddle point.

Example 4: ODDS and EVENS

2 Players: Odd (O) and Even (E) simultaneously choose the number of fingers (1 or 2) to put out. If the sum of the fingers put out by both players is odd, O wins £1 from E. If the sum of the fingers is even, E wins £1 from O. Row player: Odd; Column player : E. Reward matrix:

R. Player: O	C. Player: E 1 Finger	2 Fingers	row min
1 Finger	− 1	+ 1	− 1
2 Fingers	+ 1	− 1	− 1
Col. max	+ 1	+ 1	

This is a 0 − sum game: the amount gained by one player = - the amount lost by the other.
Since max (row min) = 1 and min (col. max) + 1, (1) is not satisfied ⇒ no saddle point.

O can be sure of a reward of − 1 (at least) and E can hold O to a reward of at most + 1. Thus, it is unclear how to determine the value of the game and the optimal strategies. For any choice of strategies by both players, there is a player who can benefit by unilaterally changing his/her strategy. E.G. if both players put out 1 finger, then O can increase O's reward from -1 to $+1$ by changing from 1 to 2 fingers. Thus, no choice of strategies by both players is stable.

RANDOMISED OR MIXED STRATEGIES

Until now we have assumed that each time a player plays a game, the player will choose the same strategy. Why not allow each player to select a probability of playing each strategy?

Example 5:

x_1 = probability that O puts out 1 fingers
x_2 = probability that O puts out 2 fingers
y_1 = probability that E puts out 1 finger
y_2 = probability that E puts out 2 fingers

If $x_1 \geq 0, x_2 \geq 0$ and $x_1 + x_2 = 1$, (x_1, x_2) is a randomized, or mixed, strategy for O. e.g. $(\frac{1}{2}, \frac{1}{2})$ could be realised by O if O tossed a coin before each play of the game and put out 1 finger for heads and 2 fingers for tails. Similarly if $y_1, y_2 \geq 0$ and $y_1 + y_2 = 1$, (y_1, y_2) is a mixed strategy for E.

Any mixed strategy $(x_1, x_2, ..., x_m)$ for R player is a PURE STRATEGY if any of the $x_i = 1$. Similarly, any mixed strategy $(y_1, ..., y_n)$ for C player is a pure strategy if for any, i, $y_i = 1$. A pure strategy is a special case of a mixed strategy in which a player always chooses the same action.

Example 6:

In the example on page 2, the game had a value of 5 (saddle point). R's optimal strategy could be represented as the pure strategy $(0, 0, 1)$ and C's strategy was the pure strategy $(0, 1, 0)$.

We continue to assume that both players play a 2p0sg in accordance of the basic assumption on page 1. In the context of randomised strategies the assumption (from the point of view of the odd) may be stated as follows: Odd should choose x_1 and x_2 to maximise O's expected reward under the assumption that Even knows the values of x_1, on a particular play of the game E is not assumed to know O's actual strategy choice until the game is played.

Each player knows that his/her opponent will choose the probabilities

$(x_1, x_2, ...)$ and $(y_1, y_2 ...)$

to maximise their own expected reward. The choice of the strategy itself is done according to the probabilities that have been computed.

3. <u>FURTHER LINEAR PROGRAMMING: DUALITY</u> & <u>COMPLEMENTARY</u> <u>SLACKNESS</u>

3.1 DUALITY

Consider the LP (called the **<u>PRIMAL</u>** problem)

$$\text{(P)} \qquad \max \left\{ c^T x \;\middle|\; A x \leq b ; x \geq 0 \right\}; \quad c, x \in \mathbb{R}^n, \ A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m$$

The **<u>DUAL</u>** of this problem is defined as

$$\text{(D)} \qquad \min \left\{ b^T y \;\middle|\; A^T y \geq c, \ y \geq 0 \right\} \quad y \in \mathbb{R}^m$$

NOTE: The **<u>DUAL</u> of (D) is (P)**. Hence the **DUAL of the DUAL is the PRIMAL**.

Example 7:

Furniture company (FC) manufactures desks (D), tables (T) and chairs (CH). The manufacture of each type of furniture requires wood (W) and two types of labour: finishing (FI) and carpentry (CA). Resources needed for each D, T, CH

Resources	D	T	CH
W	8 ft	6 ft	1 ft
FI	4 hrs	2 hrs	1 · 5 hrs
CA	2 hrs	1 · 5 hrs	0 · 5 hrs

Available: 48 ft wood. 20 F1 hours and 8 CA hours. Demand for T,D & CH unlimited. FC wishes to maximise revenue.

$x_1 = $ number of D, $x_2 = $ number of T, $x_3 = $ number of CH

A Desk sells £60, a Table for £30 , a Chair for £20.

The PRIMAL problem:

$$\max \quad 60x_1 + 30x_2 + 20x_3$$

$$\begin{aligned} \text{S.T.} \quad & 8x_1 + 6\ x_2 + \quad x_3 \le 48 \text{ (W. Constraint)} \\ & 4x_1 + 2\ x_2 + 1 \cdot 5x_3 \le 20 \text{ (FI. constraint)} \\ & 2x_1 + 1 \cdot 5x_2 + \quad \cdot 5x_3 \le 8 \text{ (CA constraint)} \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

The DUAL problem:

$$\min \quad 48y_1 + 20y_2 + 8y_3$$

$$\begin{aligned} \text{S.T.} \quad & 8y_1 + 4y_2 + 2y_3 \ge 60 \text{ (Desk constraint)} \\ & 6y_1 + 2y_2 + 1 \cdot 5y_3 \ge 30 \text{ (Table constraint)} \\ & y_1 + 1 \cdot 5y_2 + 0 \cdot 5y_3 \ge 20 \text{ (Chair constraint)} \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

The first constraint corresponds to x_1 (DESKS) because each number in the first dual constraint comes from the x_1 (DESK) column of the primal. Similarly the second dual constraint is associated with tables and the third with chairs. Also, y_1 is associated with wood y_2 with finishing hours and y_3 with carpentry hours.

Resource/Product

Resources	D	T	CH	resources Available
W	8ft	6ft	1ft	48 ft
FI	4	2	1 · 5	20 hrs
CA	2	1 · 5	· 5	8 hrs
Price (selling)	£60	£30	£20	

<u>To interpret the Dual</u>: You are an entrepreneur who wants to purchase all of FC's resources. Then you must determine the price you are willing to pay for a unit of each of FC's resources. Define

y_1: price paid for 1ft of wood
y_2: price paid for 1 finishing hour
y_3: price paid for 1 carpentry hour

Now we show that the resource price y_1, y_2, y_3 should be determined by solving the DUAL above. The total price you must pay for all of these resources is $48y_1 + 20y_2 + 8y_3$. You wish to minimise the cost of your purchase:

$$\min 48y_1 + 20y_2 + 8y_3$$

In setting resource prices what constraints do you face? You must set the resource prices high enough to induce FC to sell you its resources:

You must offer at least £60 for a combination of the resources that includes 8ft of W, 4 of FI hrs, 2 of CA hrs since FC could, if it wished, use these to produce a desk & sell it for £60. Thus,

$$8y_1 + 4y_2 + 2y_3 \geq 60$$

Similar reasoning shows that you must pay at least £30 for the resources used to produce a table (6 ft of W, 2 FI hrs, 1.5 CA hrs). This means that y_1, y_2, y_3 must satisfy

$$6y_1 + 2y_2 + 1.5y_3 \geq 30$$

Similarly, the third (chair) constraint (dual)

$$y_1 + 1 \cdot 5y_2 + 0 \cdot 5y_3 \geq 20$$

states that you must pay at least £20 (price of a chair) for the resources needed to produce a chair (1ft. W., $1 \cdot 5$ FI hrs., $0 \cdot 5$ CA hrs). The sign restrictions $y_1, y_2\ y_3 \geq 0$ must also hold. The i^{th} dual variable thus corresponds, in a natural way, to the i^{th} primal constraint.

DUAL THEOREM (DT) 1

P: $$\boxed{\max \left\{ c^T x \,\middle|\, A x \leq b, x \geq 0 \right\}; \quad A \in \mathbb{R}^{m \times n},\ x, c \in \mathbb{R}^n}$$

D: $$\boxed{\min \left\{ b^T y \,\middle|\, A^T y \geq c, y \geq 0 \right\}; \quad y, b \in \mathbb{R}^m}$$

DT states that **P and D have equal optimal objective function values**.

Lemma 1 Let **x, y be feasible** w.r.t. the primal and the dual problems respectively i.e.

$$\boxed{x:\ Ax \leq b,\ x \geq 0 \quad \text{and} \quad y:\ A^T y \geq c,\ y \geq 0}$$

Then

$$\boxed{\mathbf{c^T x \leq b^T y}}$$

Proof

Since $y \geq 0$, multiplying $A x \leq b$ by y^T yields

$$y^T Ax \leq y^T b$$

Since $x \geq 0$ multiplying $A^T y \geq c$ by x^T yields

$$x^T A^T y \geq x^T c$$

Since $y^T A x = x^T A^T y$, we have

$$x^T c \leq x^T A^T y = y^T Ax \leq y^T b. \qquad \square$$

Example 8:

If a feasible solution to either the primal or the dual is known, it can be used to obtain a bound on the optimal value of the objective function of the other problem.

In the FC problem $x_1 = x_2 = x_3 = 1$ is feasible. This has an objective function value of

$$60(1) + 30(1) + 20(1) = 110$$

Lemma 1 $\Rightarrow$ Any dual feasible solution must satisfy

$$48y_1 + 20y_2 + 8y_3 \geq 110$$

Lemma 2

Let $\hat{x}$ be a feasible solution to the primal i.e.

$$\hat{x} \in \mathbb{R}^n : A\,x \leq b,\ x \geq 0$$

and $\hat{y}$ be feasible solution to the dual i.e.

$$\hat{y} \in \mathbb{R}^m : A^T y \geq c,\ y \geq 0$$

If $c^T \hat{x} = \hat{y}^T b$, then $\hat{x}$ **is optimal for (P) and** $\hat{y}$ **is optimal for (D)**.

Proof

Lemma 1 $\Rightarrow$ for any feasible point x

$$c^T x \leq b^T \hat{y}$$

Thus, any primal feasible x must yield an objective function value $c^T x$ that does not exceed $b^T \hat{y}$. Since $\hat{x}$ is primal feasible and has objective function value $c^T \hat{x} = b^T \hat{y}$, $\hat{x}$ corresponds to the largest value $c^T x$ can take. Hence $\hat{x}$ is optimal. Similarly Lemma 1 $\Rightarrow$

$$c^T \hat{x} \leq y^T b$$

and for any dual feasible y the objective function $y^T b$ exceeds $c^T \hat{x}$. Since $\hat{y}$ is dual feasible and objective value $\hat{y}^T b = c^T \hat{x}$, $\hat{y}$ corresponds the smallest value $b^T y$ can take. Hence $\hat{y}$ is an optimal solution for the dual. ☐

Example 9:

For the FC problem $\hat{x} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$; $\hat{y} = \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} \Rightarrow c^T x = 280 = b^T y.$

Lemma 3: If (P) is unbounded then (D) is infeasible.

Lemma 4: If (D) is unbounded then (P) is infeasible.

DUAL THEOREM 2

Given the primal problem

$$\max \left\{ c^T x \;\middle|\; A\,x \leq b,\ x \geq 0 \right\}$$

and the dual

$$\min \left\{ b^T y \;\middle|\; A^T y \geq c,\ y \geq 0 \right\}$$

Suppose B is an optimal basis for the primal. Recall:

$$A\,x = b \quad \Rightarrow \quad B\,x_B + N\,x_N = b.$$

$$x_0 = c^T x \quad \Rightarrow \quad x_0 = c_B^T x_B + c_N^T x_N$$

$$\mathbf{x_B + B^{-1} N\, x_N = B^{-1} b \qquad (\Rightarrow x_B = B^{-1} b\,;\, x_N = 0)}$$

$$x_0 + (c_B^T B^{-1} N - c_N^T)\, x_N = c_B^T B^{-1} b \ \ (= y^T b)$$

Then $\mathbf{y = B^{-T} c_B}$ (and $y^T = c_B^T B^{-1}$) **is an optimal solution to the dual and** $\mathbf{b^T y = c^T x}$**,** for the optimal values of x and y.

Proof

Plan:

1. We use the fact that B is an optimal basis for the primal, to show that $y = B^{-T} c_B$ is dual feasible. (For simplicity, we assume that all slacks of the primal problem are non-basic at the optimum solution. For the case in which a slack variable is basic, note that it can be shown that the corresponding element of $B^{-T} c_B$ is zero.)

2. Show that the optimal primal objective function value = the dual objective function value for $B^{-T} c_B$.

3. Having found a primal feasible solution from B and a dual feasible solution, $B^{-T} c_B$ that have equal objective values, we invoke Lemma 2 to conclude that $B^{-T} c_B$ is optimal for the dual and $b^T x = c^T y$. Thus:

1. Let B be an optimal basis and let $y = B^{-T} c_B$ ($y = [y_1, ..., y_m]^T$). Thus, y_i is the i^{th} element of $B^{-T} c_B$. We use the fact that B is primal optimal to show that $B^{-T} c_B$ is feasible for the dual. Since B is primal optimal, the coefficient of each variable in the reduced cost

$$(c_B^T \; B^{-1} \; N \; - \; c_N^T)$$

must be nonnegative:

$$(c_B^T \; B^{-1} \; N \; - \; c_N^T) \; \geq \; 0$$

using $y = B^{-T} c_B$

$$(y^T \; N \; - \; c_N^T) \; \geq \; 0 \qquad \text{or} \qquad N^T y \; \geq \; c_N.$$

Thus, $B^{-T} c_B$ satisfies the n dual constraints:

$$A^T y = [\, B \,\vdots\, N \,]^T y = \begin{bmatrix} B^T \\ N^T \end{bmatrix} y = \begin{bmatrix} B^T y \\ N^T y \end{bmatrix} \begin{matrix} = \\ \geq \end{matrix} \begin{bmatrix} B^T B^{-T} c_B \\ c_N \end{bmatrix} \begin{matrix} = \\ = \end{matrix} \begin{bmatrix} c_B \\ c_N \end{bmatrix}$$

Since, B is an optimal basis for the primal, we also know that the coefficient of the i^{th} slack variable in

$$\mathbf{x_0 \; + (\; c_B^T \; B^{-1} \; N \; - \; c_N^T) \, x_N \; = \; c_B^T \, B^{-1} \, b}$$

is the $\mathbf{i^{th}}$ **element** ($\mathbf{y_i}$) of $y = B^{-T} c_B$. (The reason for this is that in the initial simplex tableau the coefficient of a slack variable is zero and the corresponding (i th) column of N is e_i -the column of the i^{th} slack which is null everywhere except in
the i^{th} row where it has a unit entry- the i^{th} element of $(c_B^T \; B^{-1} \; N \; - \; c_N^T)$ is in general given by

$$(c_B^T \; B^{-1} \; n_i \; - \; c_i)$$

where n_i is the i^{th} column of N. In the case of the i^{th} slack, the original objective function coefficient $c_i = 0$ and $n_i = e_i$. Thus, the coefficient of the i^{th} slack is given by $c_B^T \; B^{-1} \; e_i$, which is the i^{th} element of y.) Thus, for $i = 1, 2, ..., m$, $y_i \geq 0$. We have shown that $y = B^{-T} c_B$ satisfies all n constraints of the dual problem and that all elements of $B^{-T} c_B$ are nonnegative. Thus $y = B^{-T} c_B$ is dual feasible.

2. We now need to show that dual objective function value for $B^{-T} c_B$ = primal objective value for B. From $x_0 + (c_B^T \; B^{-1} \; N \; - \; c_N^T) \, x_N = c_B^T B^{-1} b$ we know that the primal objective value is $x_0 = c_B^T B^{-1} b$. But the dual objective value for the dual feasible solution $B^{-T} c_B$ is

$$b^T y \; = \; b^T \; B^{-T} c_B$$

which is the required result.

3. We now invoke Lemma 2 to establish the Dual Theorem. □

<u>Example</u> <u>10</u>: FC $\hat{x} = 2, 0, 8$ SLACKS: $s_1 = 24, s_2 = 0, s_3 = 0$

							<u>Basic</u> <u>Variable</u>
x_0	$+$	$5x_2$		$+ 10s_2 + 10s_3$	$=$	280	$x_0 = 280$
	$-$	$2x_2$	$+ s_1$	$+ 2s_2 - 8s_3$	$=$	24	$s_1 = 24$
	$-$	$2x_2 + x_3$		$+ 2s_2 - 4s_3$	$=$	8	$x_3 = 8$

$$x_1 \quad + 1 \cdot 25x_2 \qquad - \cdot 5s_2 + 1 \cdot 5s_3 = 2 \qquad x_1 = 2$$

We may also compute the dual optimal solution directly:

$$B^{-T}c_{BV} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & - \cdot 5 \\ -8 & -4 & 1 \cdot 5 \end{bmatrix} \begin{bmatrix} 0 \\ 20 \\ 60 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix}.$$

3.2 COMPLEMENTARY SLACKNESS

Relates the primal and dual optimal solutions. Let

P: $\max\left\{ c^T x \;\middle|\; A\,x \le b, x \ge 0 \right\} \quad x, c \in \mathbb{R}^n \quad A \in \mathbb{R}^{m\times n} \quad b \in \mathbb{R}^m$

Let $S \in \mathbb{R}^m$ be the slack variables for (P) i.e. $s = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{bmatrix}$.

D: $\min\left\{ b^T y \;\middle|\; A^T y \ge c,\ y \ge 0 \right\} \quad y \in \mathbb{R}^m$

Let $e \in \mathbb{R}^n$ be the dual slack variables, i.e. $e = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$.

THEOREM: COMPLEMENTARY SLACKNESS

Let x be feasible to P and y be feasible to D. Then **x is primal optimal and y is dual optimal iff**

$$s_i\, y_i = 0 \quad i = 1, 2, \ldots, m \tag{2}$$
$$e_j\, x_j = 0 \quad j = 1, 2, \ldots, n \tag{3}$$

From (2) and (3) it follows that

$$\mathbf{i^{th}\ primal\ slack > 0 \Rightarrow i^{th}\ dual = 0} \tag{4}$$

$$\mathbf{i^{th}\ dual > 0 \Rightarrow i^{th}\ primal\ slack = 0} \tag{5}$$

$$\mathbf{j^{th}\ dual\ slack > 0 \Rightarrow j^{th}\ primal = 0} \tag{6}$$

$$\mathbf{j^{th}\ primal > 0 \Rightarrow j^{th}\ dual\ slack = 0} \tag{7}$$

(4) and (6) $\Rightarrow$ if a constraint in either primal or dual is not satisfied as an equality (has either $s_i > 0$ or $e_j > 0$) then the corresponding variable of the other (or complementary) problem must equal zero. Hence complementary slackness.

Example 11:

FC $\quad x = \begin{bmatrix} x_1 = 2 \\ x_2 = 0 \\ x_3 = 8 \end{bmatrix} \quad s =$

$$s_1 = 48 - (8(2) + 6(0) + 1\,(8)) = 24$$

$$s_2 = 20 - (4(2) + 2(0) + 1 \cdot 5(8)) = 0$$

$$s_3 = 8 - (2(2) + 1 \cdot 5(0) + 0 \cdot 5(8)) = 0$$

$$e_1 = (8(0) + 4(10) + 2(10)) - 60 = 0$$

$$y = \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} \qquad e = \quad e_2 = (6(0) + 2(10) + 1\cdot 5(10)) - 30 = 5$$

$$e_3 = (1(0) + 1\cdot 5(10) + 0\cdot 5(10)) - 20 = 0$$

$$s_1\, y_1 = s_2\, y_2 = s_3\, y_3 = 0$$

$$e_1\, x_1 = e_2\, x_2 = e_3\, x_3 = 0$$

$s_1 > 0 \Rightarrow y_1 = 0$: A positive slack in the wood constraint $\Rightarrow$ wood must have zero price. Since slack in the wood constraint means that extra wood will not be used, an extra foot of wood would indeed be worthless.

$y_2 > 0 \Rightarrow s_2 = 0$: $y_2 > 0 \Rightarrow$ an extra finishing hour has some value. This can only occur if we are using all available finishing hours (i.e. $s_2 = 0$).

3.3 FINDING THE DUAL OF AN LP NOT IN NORMAL FORM

Example 12: Maximisation problem

$$\begin{aligned} \max\ x_0 &= 2x_1 + x_2 \\ \text{ST} \quad x_1 + x_2 &= 2 \\ 2x_1 - x_2 &\geq 3 \\ x_1 - x_2 &\leq 1 \\ x_1 \geq 0\ & x_2 \text{ unrestricted} \end{aligned}$$

To convert to NF:

(1) **Multiply each $\geq$ constraint by -1**. This converts $\geq$ into a $\leq$ constraint.

$$2x_1 - x_2 \geq 3 \quad \rightarrow \quad -2x_1 + x_2 \leq -3$$

(2) **Replace each $=$ constraint by two inequality constraints: a $\leq$ and a $\geq$ constraint. Then convert the $\geq$ constraint to a $\leq$ constraint using (1)**.

$$x_1 + x_2 = 2 \quad \rightarrow \quad \left\{ \begin{array}{l} x_1 + x_2 \geq 2 \rightarrow -x_1 - x_2 \leq -2 \\ x_1 + x_2 \leq 2 \end{array} \right\} \text{Two constraints}$$

(3) **Replace each unrestricted variable x_i by $x_i' - x_i''$ where $x_i', x_i'' \geq 0$**.

$$x_2 \rightarrow x_2' - x_2''$$

we have

$$\begin{aligned} \max x_0 &= 2x_1 + x_2' - x_2'' \\ \text{ST} \quad x_1 + x_2' - x_2'' &\leq 2 \\ -x_1 - x_2' + x_2'' &\leq -2 \\ -2x_1 + x_2' - x_2'' &\leq -3 \\ x_1 - x_2' + x_2'' &\leq 1 \\ x_1, x_2', x_2'' &\geq 0 \end{aligned}$$

We can now find the dual.

Rules for finding the Dual directly

(1) If the i^{th} primal constraint is a $\geq$ constraint the i^{th} dual variable must satisfy $y_i \leq 0$

(2) The i^{th} primal constraint $=$ constraint, the dual variable y_i is unrestricted in sign

(3) The i^{th} primal variable is unrestricted, the i^{th} dual constraint is an = constraint.

	max x_0	
min y_0	$x_1 \geq 0$	x_2^* unrs
	x_1	x_2

	y_1	1	1		$= 2^*$
	y_2	2	$-$	1	$\geq 3^*$
$y_3 \geq 0$	y_3	1	$-$	1	≤ 1
		≥ 2	1		

		max x_0		
min y_0		$x_1 \geq 0$	x_2 unrs	
		x_1	x_2	
y_1 unrs	y_1	1	1	$= 2$
$y_2 \leq 0$	y_2	2	-1	≥ 3
$y_3 \geq 0$	y_3	1	-1	≤ 1
		≥ 2	$= 1$	

DUAL PROBLEM:

min $y_0 = 2y_1 + 3y_2 + y_3$

ST $y_1 + 2y_2 + y_3 \geq 2$

$y_1 - y_2 - y_3 = 1$

y_1 unrs, $y_2 \leq 0, y_3 \geq 0$

Example 13: minimisation problem

min $y_0 = 2y_1 + 4y_2 + 6y_3$

ST

$$\begin{aligned} y_1 + 2y_2 + y_3 &\geq 2 \\ y_1 - y_3 &\geq 1 \\ y_2 + y_3 &= 1 \quad (=) \\ 2y_1 + y_2 &\leq 3 \quad (\leq) \\ y_1\text{(unrs)} \quad y_2, y_3 &\geq 0 \end{aligned}$$

(1) Change each $\leq$ constraint to $\geq$ by multiplying $\leq$ constraint by -1:

$$2y_1 + y_2 \leq 3 \rightarrow -2y_1 - y_2 \geq -3$$

(2) Replace each $=$ constraint by a $\geq$ constraint and a $\leq$ constraint. Then transform $\leq$ to a $\geq$ constraint:

$$y_2 + y_3 = 1 \quad \rightarrow \quad \left\{ \begin{array}{l} y_2 + y_3 \geq 1 \\ y_2 + y_3 \leq 1 \rightarrow -y_2 - y_3 \geq -1 \end{array} \right\} \text{Two constraint}$$

(3) Replace unrs variable y_i by $y_i' - y_i''$; $y_i', y_i'' \geq 0$.

min $y_0 = 2y_1' - 2y_1'' + 4y_2 + 6y_3$

ST

$$\begin{aligned} y_1' - y_1'' + 2y_2 + y_3 &\geq 2 \\ y_1' - y_1'' - y_3 &\geq 1 \\ y_2 + y_3 &\geq 1 \end{aligned}$$

$$\begin{aligned} & \quad\quad\quad\quad\quad - y_2 - y_3 \geq -1 \\ & -2y_1' + 2y_1'' - y_2 \quad \geq -3 \\ & y_1', \; y_1'', \; y_2, \; y_3 \geq 0 \end{aligned}$$

now find the dual.

Rules for finding the Dual (for min problem) directly

min y_0		max x_0				
		$x_1 \geq 0$	$x_2 \geq 0$	x_3 unrs	$x_4 \leq 0$	
		x_1	x_2	x_3	x_4	
y_1 unrs*	y_1	1	1	0	2	$= 2$
$y_2 \geq 0$	y_2	2	0	1	1	≤ 4
$y_3 \geq 0$	y_3	1	- 1	1	0	≤ 6
		≥ 2	≥ 1	$= 1^*$	$\leq 3^*$	

(1) If the i^{th} primal constraint is a $\leq$ constraint, the dual variable x_i must satisfy $x_i \leq 0$

(2) If the i^{th} primal c. is an $=$ constraint, the dual variable x_i will be unrs.

(3) If the i^{th} primal variable y_i is unrestricted. the i^{th} dual constraint is an equality constraint.

$$\begin{aligned} \max x_0 &= 2x_1 + x_2 + x_3 + 3x_4 \\ \text{ST} \quad & x_1 + x_2 + 2x_4 = 2 \\ & 2x_1 + x_3 + x_4 \leq 4 \\ & x_1 - x_2 + x_3 \leq 6 \\ & x_1, x_2 \geq 0, x_3 \text{ unrs}, x_4 \leq 0 \end{aligned}$$

4. LINEAR PROGRAMMING & ZERO SUM GAMES

LP can be used to find the value and optimal strategies (for the row (R) and column (C) players) for any 2p0sg.

Example 14: STONE (S), PAPER (P) AND SCISSORS (SR)

Each of the two players simultaneously utters one of the three words S, P or SC. If both players utter the same word, the game is a draw. Otherwise one player wins £1 from the other according to the rules: SC defeats (cuts) P, P defeats (covers) S, S defeats (breaks) SC. Find the value and optimal strategies for this 2p0sg.

R Player	C Player S	P	Sc	Row min
S	0	-1	$+1$	-1
P	$+1$	0	-1	-1
SC	-1	$+1$	0	-1
Col. max	$+1$	$+1$	$+1$	

The game does **not have a saddle** point. Let

x_1 = probability that row player chooses S

x_2 = probability that row player chooses P

x_3 = probability that row player chooses Sc

y_1 = probability that column player chooses S

y_2 = probability that column player chooses P

y_3 = probability that column player chooses Sc

The Row Player's LP

If R chooses the mixed strategy (x_1, x_2, x_3) then R's expected reward against each of the C's strategies are:

C Chooses	R's expected reward if R chooses (x_1, x_2, x_3)
S	$x_2 - x_3$
P	$-x_1 + x_3$
SC	$x_1 - x_2$

By the basic assumption, C will choose a strategy that makes R's expected reward equal to

$$\min(x_2 - x_3, -x_1 + x_3, x_1 - x_2)$$

Then R should choose (x_1, x_2, x_3) to make $\min(x_2 - x_3, -x_1 + x_3, x_1 - x_2)$ as large as possible. To obtain an LP formulation (R's LP) that will yield R's optimal strategy, observe that for any values of x_1, x_2, x_3 the largest value of $\min(x_2 - x_3, -x_1 + x_3, x_1 - x_2)$ is just that *largest* number (say v) that is simultaneously less than or equal to $x_2 - x_3, -x_1 + x_3, x_1 - x_2$. Also, probabilities $x_1, x_2, x_3 \geq 0$ and $x_1 + x_2 + x_3 = 1$

Max $x_0 = v$

S.T. $v \leq x_2 - x_3$ (Stone Constraint)

$v \leq -x_1 + x_3$ (Paper Constraint)

$v \leq x_1 - x_2$ (Scissors Constraint)

$x_1 + x_2 + x_3 = 1$ $x_1, x_2, x_3 \geq 0$, v unrestricted

v in optimal solution is R's “floor". No matter what strategy played by C, R's expected reward is at least v.

The Column Player's LP

Suppose that C has chosen the mixed strategy (y_1, y_2, y_3). For each of R's strategies we may compute R's expected reward if C has chosen (y_1, y_2, y_3).

R. Chooses	Row Player's expected reward if C chooses (y_1, y_2, y_3)
S	$-y_2 + y_3$

P	$y_1 - y_3$
SC	$-y_1 + y_2$

Since R is assumed to know (y_1, y_2, y_3), R will choose a strategy to ensure that R obtains an expected reward of

$$\max(-y_2 + y_3, y_1 - y_3, -y_1 + y_2)$$

Thus, C should choose (y_1, y_2, y_3) to make

$$\max(-y_2 + y_3, y_1 - y_3, -y_1 + y_2)$$

as ***small** as possible*. To obtain an LP formulation, observe that for any choice of (y_1, y_2, y_3)

$$\max(-y_2 + y_3, y_1 - y_3, -y_1 + y_2)$$

will equal to the smallest number that is simultaneously greater than or equal to $-y_2 + y_3$, $y_1 - y_3$, $-y_1 + y_2$ (call this number w). Also $y_1, y_2, y_3 \geq 0, \sum_{i=1}^{3} y_i = 1$ For a mixed strategy:

$$\min \quad y_0 = w$$

$$\text{S.T.} \quad w \geq -y_2 + y_3 \quad ; \quad w \geq y_1 - y_3 \quad , w \geq -y_1 + y_2$$

$$y_1 + y_2 + y_3 = 1 \qquad y_1, y_2, y_3 \geq 0 \qquad \text{w unrestricted.}$$

observe that w is a "ceiling" on C's expected losses (or on R's expected reward) because by choosing a mixed strategy (y_1, y_2, y_3) that solves the LP, C can ensure that C's expected losses will be at most w (whatever R does).

<u>THE RELATION BETWEEN THE LP'S OF R AND C</u>

It is easy to show that C's LP is the DUAL of the R's LP.

R's LP: $\max \quad x_0 = v$

$$\begin{aligned} \text{ST} \quad & -x_2 + x_3 + v \leq 0 \\ & x_1 - x_3 + v \leq 0 \\ & -x_1 + x_2 + v \leq 0 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \quad \text{v unrestricted.} \end{aligned}$$

Let the duals be y_1, y_2, y_3 and w respectively

min \ max	x_1	x_2	x_3	v	
$y_1(\geq 0)$	0	-1	1	1	≤ 0
$y_2(\geq 0)$	1	0	-1	1	≤ 0
$y_3(\geq 0)$	-1	1	0	1	≤ 0
w (unrs)	1	1	1	0	$= 1$
	≥ 0	≥ 0	≥ 0		

We read R's LP across the above table and the dual of R's LP is obtained by reading down each column. Recall that the dual constraint corresponding to v will be an $=$ constraint (as v is unrs) and the dual variable corresponding to the primal $x_1 + x_2 + x_3 = 1$ will be unrs. Thus, the dual can be read down as

$$\min y_0 = w$$

$$\begin{array}{lrrrrl} \text{ST} & & y_2 & -y_3 & +w & \geq 0 \\ & -y_1 & & +y_3 & +w & \geq 0 \\ & y_1 & -y_2 & & +w & \geq 0 \\ & y_1 & +y_2 & +y_3 & & = 1 \\ & \multicolumn{5}{l}{y_1, y_2, y_3 \geq 0 \quad w \text{ unrestricted}} \end{array}$$

which is C's LP.

The DUAL THEOREM $\Rightarrow$ $v = w$ (Both LP's feasible and bounded)
R's "floor" $=$ C's "ceiling"

This is known as the <u>minmax</u> <u>theorem</u>. The common value v and w is the VALUE of the game to R. It can be shown that the optimal strategies obtained via LP represent a stable equilibrium: neither player can improve by a unilateral change in strategy.

For the Stone Paper Scissors game:

$$\text{R's LP} \Rightarrow \ w = 0 \ \ x_1 = \tfrac{1}{3}, x_2 = \tfrac{1}{3}, x_3 = \tfrac{1}{3}; \quad \text{C's LP} \Rightarrow \ v = 0 \ \ y_1 = \tfrac{1}{3}, y_2 = \tfrac{1}{3}, y_3 = \tfrac{1}{3}$$

The fact that v and w are unrestricted can be overcome. Suppose that you add a constant c to every element of A so that all coefficients become nonnegative:

$$\max \quad x_0 = v'$$

$$\begin{aligned} \text{ST} \quad & v' \leq (a_{11} + c)x_1 + (a_{21} + c)x_2 + \ldots + (a_{m1} + c)x_m \\ & v' \leq (a_{12} + c)x_1 + (a_{22} + c)x_2 + \ldots + (a_{m2} + c)x_m \\ & \vdots \\ & v' \leq (a_{1n} + c)x_1 + (a_{2n} + c)x_2 + \ldots + (a_{mn} + c)x_m \\ & x_1 + x_2 + \ldots + x_m = 1 \\ & x_1, x_2, \ldots, x_m \geq 0 \quad v' \geq \, ? \end{aligned}$$

<u>Note:</u>

$$(x_1 + x_2 + \ldots + x_m) = 1$$

$$v' \leq a_{11} x_1 + a_{21} x_2 + \ldots + a_{m1} x_m + c\,(x_1 + x_2 + \ldots + x_m)$$

Thus,

$$v' - c \leq a_{11} x_1 + a_{21} x_2 + \ldots + a_{m1} x_m$$

The same holds for all the constraints. Define v′ such that

$$\boxed{v' = v + c \quad (\text{or } v = v' - c)}$$

Then the above max problem can be written as

$$\max \ x_0 = v'$$

$$\begin{aligned} & v' \leq (a_{11} + c)\, x_1 + \ldots + (a_{m1} + c)\, x_m \\ & \vdots \\ & v' \leq (a_{1n} + c)\, x_1 + \ldots + (a_{mn} + c)\, x_m \end{aligned}$$

$$\sum_{i}^{m} x_i = 1 \quad x_i \geq 0 \quad i = 1, \ldots, m, v' \geq 0.$$

The same can be shown for the dual

$$\min y_o = w'$$

$$w' \geq a_{11} y_1 + \ldots + a_{1n} y_n + c\,(\sum_{i=1}^{n} y_i)$$

$$\vdots$$

$$w' \geq a_{m1} y_1 + \ldots + a_{mn} y_n + c\,(\sum_{i=1}^{n} y_i)$$

$$\sum_{i}^{n} y_i = 1 \quad y_i \geq 0 \quad w' \geq 0.$$

Thus, the solution of the primal LP does not change by adding c to every element of A and $v' = v + c$. The dual solution also does not change and $w' = w + c$.

To introduce ≥ 0 constraints for w, v we therefore have to do the following:

add c = | most negative element of matrix A | to all elements of A.

Let v^* and w^* be the optimal values of the original game ($c = 0$). Let $v^{*'}$, $w^{*'}$ be the optimal values when c is given as above. Since after adding c the matrix will not have any negative elements (rewards) $v^{*'}, w^{*'} \geq 0$ hold. Thus after adding c we may assume that v' and $w' \geq 0$ and ignore v and w unrs. The value v^* of the original game: $v^* = v^{*'} - c$ and $w^* = w^{*'} - c$.

5. TWO-PERSON NON CONSTANT-SUM GAMES

Example 15: Prisoner's Dilemma

Two prisoners who escaped and participated in a robbery have been recaptured and are awaiting trial for their new crime. Although both are guilty, the police chief is not sure whether he has enough evidence to convict them. In order to entice them to testify against each other, the police chief tells each prisoner: “If only one of you confesses and testifies against the other (partner), the person who confesses will go free while the person who does not confess will surely be convicted to a 20 year jail sentence. If both of you confess you will both be convicted and sent to prison for 5 years (because of lack of evidence). If neither confess, I shall convict you of a misdemeanour and you each will get 1 year in prison". What should the prisoners do?

If the prisoners cannot communicate with each other, the strategies and rewards are:

Prisoner 2		
Prisoner 1	Confess	Don't Confess
Confess	$(-5, -5)$	$(0, -20)$
Don't Confess	$(-20, 0)$	$(-1, -1)$
	↑ P 1's reward	↑ P2's reward

Note that sum of rewards in each cell varies from $-2\ (= -1 - 1)$ to $-20(= - 20 + 0) \Rightarrow$ This is not a constant sum game.

Does any strategy "dominate" the other? For each prisoner, "confess" dominates "don't confess" strategy. If each prisoner follows his undominated "confess" strategy, however, each prisoner will get 5 years. On the other hand, if each prisoner chooses the dominated "don't confess" strategy, each prisoner will get 1 year. Thus, if each prisoner chooses his dominated strategy they are better off than if each chooses his undominted strategy.

Definition: As in a 2p0sg a choice of strategy by each player (prisoner) is an EQUILIBRIUM POINT if neither player can benefit by a unilateral change in strategy.

$(-5, -5)$ is an equilibrium point: either prisoner deviating from this decreases his reward to -20 (from -5). However each is better off at $(-1, -1)$. However this is not an equilibrium point because if we are at $(-1, -1)$ each prisoner can increase his reward from -1 to 0 by changing from Don't Confess to Confess (each can benefit from double-crossing his opponent). If the players are cooperating (both don't confess) each player can gain by double-crossing. If both double cross, they will be worse off than the cooperative strategy (don't confess). This cannot occur in a constant sum game.

NC = non-cooperative action

C = cooperative action

P = punishment for non cooperation

S = payoff to person who is double-crossed

R = reward for cooperating if both players cooperate

T = temptation to double-cross the opponent

P2		
P1	NC	C
NC	(P, P)	(T, S)
C	(S, T)	(R, R)

In this game (P, P) is an equilibrium point. This requires $P > S$. For (R, R) not to be an equilibrium point requires $T > R$ (each player has temptation to double-cross). A game is reasonable if $R > P$. Thus, a prisoner's dilemma requires:

$$T > R > P > S$$

This example is of interest because it explains why two adversaries often fail to cooperate with each other.

Example 16: **Arms Race**

US and SU are engaged in an arms race. Each have two strategies: develop a new missile (D) or maintain the status-quo. The reward matrix is based on the assumption that if only one nation develops a new missile, the nation with the new missile will conquer the other nations. In this case, the conquering nation earns a reward of 20 and the conquered looses 100 units. Assume cost of developing a missile 10 units. Find the equilibrium point(s).

	SU	
US	D	M
D	$(-10, -10)$	$(10, -100)$
M	$(-100, 10)$	$(0, 0)$

D : Non-cooperative; M : Cooperative. $(-10, -10)$ (both nations non-cooperative) is an equilibrium point. Although $(0, 0)$ leaves both nations better off than $(-10, -10)$, we see that in this situation each nation will gain from a double cross. Thus, $(0, 0)$ is not stable. This example shows how maintaining the balance of power may lead to an arms race.

6. N-PERSON GAME THEORY

In many situations, there are more than two competitors. We therefore consider games with three or more players. Let N = {1, ..., n} be the **set of players** in an **n-person game** which is specified by the game's **characteristic function**.

Definition: For each subset S of N, the **characteristic** function v of a game gives the amount v(S) that the members of S can be sure of receiving if they act together and form a coalition.

Thus, v(S) can be determined by calculating the amount that members of S can get without help from players who are not in S.

Example 17: The drug game

Dr Medicine (player 1) has invented a new drug. Dr Medicine cannot manufacture this drug on his/her own but can sell the formula to company (player) 2 or company (player) 3. The chosen company will split a £1 million with Dr Medicine. The characteristic function of this game is given by

$$v(\{\ \}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2, 3\}) = 0$$

$$v(\{1, 2\}) = v(\{1, 3\}) = v(\{1, 2, 3\}) = £1,000,000.$$

Example 18: The rubbish dumping game

Each property owner has one bag of rubbish and has to dump his/her bag on somebody's property. If b bags of rubbish are dumped on a coalition of property owners, the coalition receives a reward of -b.

The best that the members of any coalition can do is to dump all the rubbish on the property of owners not in S. Thus, the characteristic function of the garbage game ($|S|$ is the number of players in S) is given by

$$v(\{S\}) = -(4 - |S|) \quad (\text{if } |S| < 4)$$

$$v(\{1, 2, 3, 4\}) = -4 \ (\text{if } |S| = 4).$$

The latter follows because if all players are in S, they must dump their garbage on members of S.

Example 19: The land development game

Player 1 owns a piece of land and values it at £10000. Players 2 and 3 are developers who can develop the land and increase its worth to £20000 and £30000, respectively. There are no other prospective buyers.

(any coalition that does not contain 1): $v(\{\ \}) = v(\{2\}) = v(\{3\}) = v(\{2, 3\}) = 0$

(any other coalition value is the maximum value a member of the coalition places on the land

$v(\{1, 2\}) = £20000 \quad v(\{1, 3\}) = £30000 \quad v(\{1, 2, 3\}) = £30,000.$

Consider any subset of player sets A & B: $A \cap B = \emptyset$. For any n-person game the characteristic function satisfies **superadditivity**

$$v(A \cup B) \geq v(A) + v(B).$$

If the players $A \cup B$ band together, one (but not the only) option is to let the players in A fend for themselves and players in B fend for themselves. This would result in the coalition receiving an amount v(A) + v(B). Thus, $v(A \cup B)$ must be at least as large as v(A) + v(B).

Solution concepts for n-person games are related to the reward that each player will receive.

Definition: Reward Vector

Let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

be a vector such that player i receives reward x_i. This is the **reward** vector. A reward vector x (x_i is the i^{th} element of x) is not a reasonable candidate for a solution unless it satisfies

$$v(N) = \sum_{i=1}^{n} x_i \quad \text{(Group Rationality: GR)}$$

i.e. any reasonable reward vector must give all the players an amount that equals the amount that can be attained by the supercoalition consisting of all players, and

$$x_i \geq v(\{i\}) \quad \text{(for each } i \in N) \quad \text{(Individual Rationality: IR)}$$

i.e. player i must receive a reward at least as large as what he can get for himself ($v(\{i\})$).

Definition: If x satisfies both GR and IR, we call that x an **imputation**.

Example 20: Consider the payoff vectors of Example 19. Any solution concept for n-person games chooses some subset of the set of imputations (possibly empty) as the solution to the n-person game.

x	is x an imputation?
(£10000, £10000, £10000)	Yes
(£5000, £2000, £5000)	No: $x_1 \leq v(\{1\})$, so IR violated
(£12000, £19000, -£1000)	No: IR violated
(£11000, £11000, £11000)	No: GR violated

THE CORE OF AN n-PERSON GAME

Definition: Given an imputation $x = [x_1, x_2, ..., x_n]^T$, we say that the imputation $y = [y_1, y_2, ..., y_n]^T$ **dominates** x through a coalition S, i.e.

$$\boxed{y >^{S} x}$$

if

$$\sum_{i \in S} y_i \leq v(S) \quad \text{and for all } i \in S,\ y_i > x_i$$

If $y >^{S} x$, then both of the following must be true:

→ Each member of S prefers y to x

→ Since $\sum_{i \in S} y_i \leq v(S)$, the members of S can attain the rewards given by y

Thus, $y >^{S} x$, then x should not be considered as a possible solution to the game, because the players in S can object to the rewards given by x and enforce their objection by banding together and thereby receiving the rewards given by y (since the members of S can surely receive an amount equal to v(s)).

John von Neumann and Oskar Morgenstern argued that a reasonable solution concept for an n-person game was the set of all undominated imputations.

Definition: The **core** of an n-person game is the set of all undominated imputations. The following two examples illustrate domination.

Example 21: Consider a three person game with the following characteristic function:

$$v(\{\ \}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$
$$v(\{1,2\}) = 0.1,\ v(\{1, 3\}) = 0.2,\ v(\{2, 3\}) = 0.2,\ v(\{1, 2, 3\}) = 1$$

Let $x = [.05, .90, .05]^T$, $y = [.10, .80, .10]^T$.

To show $y >^{\{1,3\}} x$, note that both x and y are imputations. Next, observe that with y, players 1 and 3 receive more than they receive with x. Also, y gives players in {1, 3} a total of .1 + .1 = .2. Since .2 does not exceed v({1, 3}) = .2, it is reasonable to assume that players 1 and 3 can band together and receive a total reward of .2. Thus, players 1 and 3 will never allow the rewards given by x to occur.

Example 22: For the land development game in Example 19, let $x = [£19000, £100, £10100]^T$, $y = [£19800, £100, £10100]^T$. To show $y >^{\{1,3\}} x$, we need only observe that players 1 and 3 from y (£29900) does not exceed v({1, 3}). If x were proposed as a solution, player 1 would sell the land to player 3 and y (or some other imputation that dominates x) would result. The important point is that x cannot occur because players 1 and 3 would not allow it.

THEOREM: DETERMINATION OF THE CORE

An imputation $x = [x_1, x_2, ..., x_n]^T$ is in the core of an n-person game if and only if for each subset (coalition) S of N we have

$$\sum_{i \in S} x_i \geq v(S).$$

(for the proof see, for example, P. Morris, *Introduction to Game Theory*, Springer, 1994)

This theorem states that an imputation x is in the core (ie x is undominated) iff for every coalition S the total of the rewards for each player in S (according to x) is at least as large as v(S).

We consider the core of the three games discussed so far.

Example 23: The drug game (continued). $x = [x_1, x_2, x_3]^T$ will be an imputation iff

$$x_1 \geq 0 \; ; \; x_2 \geq 0 \; ; \; x_3 \geq 0 \; ; \; x_1 + x_2 + x_3 = £1000000$$

The theorem shows that $x = [x_1, x_2, x_3]^T$ will be in the core iff x_1, x_2, x_3 satisfy the above conditions and

$$x_1 + x_2 \geq £1000000 \; ; \; x_1 + x_3 \geq £1000000 \; ; \; x_2 + x_3 \geq £0; \; x_1 + x_2 + x_3 \geq £1000000$$

If $x = [x_1, x_2, x_3]^T$ is in the core, then all the above inequalities must hold simultaneously. As $x_1 + x_2 + x_3 =$ £1000000, we must also have $x_1 + x_2 = £1000000$; $x_1 + x_3 = £1000000$. Hence, $x_1 = £1000000$ and $x_2 = x_3 =$ £0. We see that this satisfies all the inequalities and thus the core of the game is the imputation x = (£1000000, £0, £0). The core emphasises the importance of player 1.

An alternative solution concept is the Shapley value. This would give player 1 less than £1000000 and players 2 and 3 some money.

For this example, if we choose an imputation that is not in the core, we can show how it is dominated. Consider imputation x = (£900000, £50000, £50000). If we let y = (£925000, £75000, £0), then $y >^{\{1,2\}} x$.

Example 24: The Garbage Game (Continued). $x = [x_1, x_2, x_3, x_4]^T$ is an imputation iff

$$x_1 \geq -3 \; ; \; x_2 \geq -3 \; ; \; x_3 \geq -3 \; ; \; x_4 \geq -3 \; ; \; x_1 + x_2 + x_3 + x_4 = -4$$

Applying the theorem to all three-player coalitions, for $x = [x_1, x_2, x_3, x_4]^T$ to be in the core, it is necessary to satisfy

$$x_1 + x_2 + x_3 \geq -1 \; ; \; x_1 + x_2 + x_4 \geq -1 \; ; \; x_1 + x_3 + x_4 \geq -1 \; ; \; x_2 + x_3 + x_4 \geq -1$$

Adding these four inequalities, we find that $3(x_1 + x_2 + x_3 + x_4) \geq -4$, we find that this contradicts the equality $x_1 + x_2 + x_3 + x_4 = -4$. Thus, no imputation can satisfy the conditions for the core. Hence, the core of the garbage game is empty.

To understand the reason for the empty core, consider the imputation $x = [-2, -1, -1, 0]^T$, which treats players 1, 2, 3 unfairly. Players 1 and 2 could, for example, join together to ensure the imputation $x = [-1.5, -.5, -1, -1]^T$. Thus, $y >^{\{1,2\}} x$. In a similar fashion any imputation could be dominated by another imputation. The two player version of the game has a core consisting of (-1, -1) and for $n > 2$, the core is empty.

Example 25: The Land Development Game (Continued). $x = [x_1, x_2, x_3]^T$ is an imputation iff

$$x_1 \geq £10000 \; ; \; x_2 \geq £0 \; ; \; x_3 \geq £0 \; ; \; x_1 + x_2 + x_3 = £30000$$

The theorem shows that $x = [x_1, x_2, x_3]^T$ will be in the core iff x_1, x_2, x_3 satisfy the above conditions and

$$x_1 + x_2 \geq £20000 \; ; \; x_1 + x_3 \geq £30000 \; ; \; x_2 + x_3 \geq £0 \; ; \; x_1 + x_2 + x_3 \geq £30000$$

Considering $x_1 + x_3 \geq £30000$ and $x_1 + x_2 + x_3 = £30000$ leads to $x_2 = £0$ and $x_1 + x_3 = £30000$. By $x_1 + x_2 \geq £20000$, we have $x_1 \geq £20000$. Any x in the core must also satisfy $x_3 \geq £0$ and $x_1 \leq £30000$ and any x satisfying $x_2 = £0$ and $x_1 + x_3 = £30000$ in the core. Thus, if $£20000 \leq x_1 \leq £30000$ and any vector of the form $(x_1, £0, £(30000 - x_1))$ will be in the core.

Player 3 outbids player 2 and purchases the land from player 1 for price x_1($£20000 \leq x_1 \leq £30000$). Then player 1 receives x_1, player 3 receives $30000 - x_1$ and player 2 receives nothing. In this example, the core contains an infinite number of points.

THE SHAPLEY VALUE

In the drug game, we found that the core of the drug gave all the benefits or rewards to the game's most important player (the inventor of the drug). An alternative concept for n-person games is the **Shapley value** which in general gives more equitable solutions than the core does.
(see: L. Shapley "Quota Solutions of n-Person Games" in: Contributions to the Theory of Games, eds. H. Kuhn and A. Tucker, Princeton University Press, Princeton, NJ, 1953 and Owen, G. "Game Theory" Academic Press, Florida, 1982).

For any characteristic function, Lloyd Shapley showed that there is a unique reward vector $x = [x_1, x_2, ..., x_n]^T$ satisfying the following axioms:

Axiom 1 Relabelling of players interchanges the players' rewards. Suppose the Shapley value of a three person game is $x = [10, 15, 20]^T$. If we interchange the roles of player 1 and 3 (for example, if originally v({1}) = 10, and v({3}) = 15, we would make v({1}) = 15 and v({3}) = 10) then the Shapley value for the new game would be $x = [20, 15, 10]^T$.

Axiom 2 Group rationality. $\sum_{i=1}^{n} x_i = v(N)$.

Axiom 3 If v(S − {i}) = v(S) holds for all coalitions S, then the Shapley value has $x_i = 0$. If player i adds no value to any coalition, player i receives reward 0 from the Shapley value.

Axiom 4 Let x be the Shapley value vector of game v and let y be the Shapley value vector for game $\hat{v}$. Then, the Shapley value vector for the game $(v + \hat{v})$ is x + y.

The validity of Axiom 4 is often questioned: adding up rewards from two different games may be like adding up apples and oranges. If Axioms 1-4 are assumed to be valid, however, Shapley proved:

THEOREM (SHAPLEY VALUE)

Given any n-person game with characteristic function v, there is a unique reward vector $x = [x_1, x_2, ..., x_n]^T$ satisfying axioms 1-4. The reward to the i^{th} player (x_i) is given by

$$x_i = \sum_{\substack{\text{all S for which} \\ \text{is not in S}}} p_n(S)\,[v(S \cup \{i\}) - v(s)] \;; \qquad p_n(S) = \frac{|S|!\,(n - |S| - 1)!}{n!}$$

where | S | is the number of players in coalition S and for n ≥ 1, n! = n (n − 1)(n − 2)...(2)(1) (with 0! = 1).

Although the above formulae seem complex, they have a simple interpretation. Suppose players 1, 2, ..., n arrive in a random order. That is, for any of the n! permutations of 1, 2, ..., n has a 1/(n!) chance of being in which the players arrive. For example, if n = 3, then there is a 1/3! = 1/6 probability that the players will arrive in any one of the following sequences

1	2	3		2	3	1
1	3	2		3	1	2
2	1	3		3	2	1

Suppose that when player i arrives, she finds players in the set S have already arrived. If player i forms a coalition with the players who are present when he arrives, player adds v(S ∪ {i}) − v(S) to coalition S. The probability that when player i arrives the players in the coalition S are present is $p_n(S)$. Then the formulae imply that player i's reward should be the expected amount that player i adds to the coalition made up of the players who are present when player i arrives.

To derive the formula for We now show that $p_n(S)$ given above is the probability that when player i arrives, the players in the subset S will be present. Observe that the number of permutations of 1, 2, ..., n that result in player i's arriving when players in the coalition S are present is given by

$$\underbrace{|S|(|S|-1)(|S|-2)\dots(2)(1)}_{\text{S arrives}} \quad \underbrace{(1)}_{\text{i arrives}} \quad \underbrace{(n - |S| - 1)(n - |S| - 2)\dots(2)(1)}_{\text{Players not in } S \cup \{i\} \text{ arrive}}$$

$$= |S|!\,(n - |S| - 1)!$$

Since there are a total of n! permutations of 1, 2, ..., n, the probability that player i will arrive and see the players in S is

$$\frac{|S|!\,(n-|S|-1)!}{n!} = p_n(S)\ .$$

Thus, the definition of x_i is this: calculate [v (S ∪ {i}) − v(s)] for each of the n! possible orderings of the players, weight each one with probability 1/(n!) of that ordering occurring, add the results. Among the n! terms in the sum which defines x_i there are many duplications. Indeed, suppose that we have an ordering in which {i} occurs at position k. With S being the set of players in this ordering, if we permute the part of the ordering coming before {i} and the part coming after it, we obtain a new ordering in which x_i is again in the k^{th} position. Moreover, for both the original and the permitted orderings, the term [v (S ∪ {i}) − v(s)] is the same. There are | S | ! permutations of the players coming before, and (n − | S | − 1)! permutations coming after, {i}. Thus, the term [v (S ∪ {i}) − v(s)] occurs | S | !(n − | S | − 1)! times. This explains the probability $p_n(S)$, given that any ordering has probability 1/(n!) and associated with [v (S ∪ {i}) − v(s)], there are | S | !(n − | S | − 1)! orderings.

Example 26 : The drug game (continued). The Shapley value.

For x_1, the reward player 1 should receive, we list all coalitions S in which player 1 is not a member. For each such coalition, we compute v(S ∪ {i}) − v(S) and $p_3(S)$:

S	$p_3(S)$	v(S ∪ {1}) − v(S)
{ }	2/6	£0
{2}	1/6	£1000000
{2, 3}	2/6	£1000000
{3}	1/6	£1000000

Since player 1 adds on the average

$$(2/6)(0) + (1/6)(1000000) + 2/6(1000000) + (1/6)(1000000) = £\frac{4000000}{6}$$

the Shapley value concept recommends that player 1 receives a reward of $£\frac{4000000}{6}$. To compute the Shapley value for player 2, we require the information:

S	$p_3(S)$	v(S ∪ {2}) − v(S)
{ }	2/6	£0
{1}	1/6	£1000000
{3}	1/6	£0
{1, 3}	2/6	£0

Thus, the Shapley value for player 2 is

$$(1/6)(£1000000) = £\frac{1000000}{6}.$$

Since, the Shapley value must allocate a total of v({1, 2, 3}) = £1000000 to the players, the Shapley value for player 3 is

$$£1000000 - x_1 - x_2 = £\frac{1000000}{6}.$$

Shapley value is essentially computed using the fact that player i should receive the expected amount that she adds to the coalition present when she arrives. In this example, this method yields the computation below

	AMOUNT (£) ADDED BY PLAYER'S ARRIVAL		
ORDER OF ARRIVAL	Player 1	Player 2	Player 3
1, 2, 3	0	1000000	0
1, 3, 2	0	0	1000000
2, 1, 3	1000000	0	0
2, 3, 1	1000000	0	0
3, 1, 2	1000000	0	0
3, 2, 1	1000000	0	0

Since each of the six orderings are equally likely, we find that the Shapley values are given as above.

The Shapley value can be used as a measure of the power of individual members of a political or business organisation. For example, the UN Security Council consists of five permanent members (who have the veto power over any resolution) and ten nonpermanent members. For a resolution to pass the Security Council, it must receive at least nine votes, including the votes of all permanent members. Assigning a value 1 to all coalitions that can pass a resolution, and a value 0 to all those that cannot defines a characteristic function. For this characteristic function, it can be shown that the Shapley value for each permanent member is .1963 and the Shapley value for each nonpermanent member is .001865 and $5 \times (.1963) + 10 \times (.001865) = 1$. Thus, the Shapley value indicates that $5 \times (.1963) = 98\%$ of the power in the Security Council resides with the permanent members.

Example 27: Suppose three types of planes use an airport. A Piper Cub (player 1) requires a 100-yd runway, a DC-10 (player 2) requires a 150-yd runway and a 747 (player 3) requires a 400 yd runway. Suppose the cost in pounds of maintaining a runway for a year is equal to the length of the runway. since 747's land at the airport, the airport will have a 400-yd runway. For simplicity, suppose that each year only one plane lands at the airport. How much of the £400 annual maintenance cost should be charged to each plane?

We define a three-player game in which the value to a coalition is the cost associated with the runway length needed to service the largest plane in the coalition. Thus, the characteristic function for this game (cost $\Rightarrow$ negative revenue) would be

$$v(\{\ \}) = 0,\ v(\{1\}) = -£100,\ v(\{1, 2\}) = v(\{2\}) = -£150$$
$$v(\{3\}) = v(\{2, 3\}) = v(\{1, 3\}) = v(\{1, 2, 3\}) = -£400$$

To find the Shapley value (cost) to each player, we assume that the three planes land in a random order and we determine how much cost (on the average) each plane adds to the cost incurred by the planes that are already present:

		COST ADDED BY PLAYER'S ARRIVAL(£)		
ORDER OF ARRIVAL	PROBABILITY OF ORDER	Player 1	Player 2	Player 3
1, 2, 3	1/6	100	50	250
1, 3, 2	1/6	100	0	300
2, 1, 3	1/6	0	150	250
2, 3, 1	1/6	0	150	250
3, 1, 2	1/6	0	0	400
3, 2, 1	1/6	0	0	400

Player 1 cost = (1/6)(100 + 100) = £200/6
Player 2 cost = (1/6)(50 + 150 + 150) = £350/6
Player 3 cost = (1/6)(250 + 300 + 250 + 250 + 400 + 400) = £1850/6

In general, even if more than one plane of each type lands, it has been shown that the Shapley value allocates runway operating cost as follows: all planes that use a portion of the runway should divide the cost of that portion of the runway (S. Littlechild and G. Owen (1973). "A simple expression for the Shapley value in a special case" Management Science, Vol. 20, 370-372). Thus, all planes should cover the cost of the first 100 yd of runway, the DC10's and 747's should pay the next 150 − 100=50 yd of runway and the 747's should pay the last 400 − 150 = 250 yd of runway. If there were ten Piper Cub, five DC10 and two 747 landings, the Shapley value concept would recommend that each Piper Cub pay £100/(10 + 5 + 2)= £5.88, each DC10 pay £(5.88 + (150 − 100)/(5 + 2)) = £13.03 and each 747 pay £(13.03 + (400 − 150)/2)=£138.03.