OPERATIONS RESEARCH: 343

1. LINEAR PROGRAMMING
2. INTEGER PROGRAMMING
3. GAMES

Books:

(i) Intro. to OR $($F.Hillier & J. Lieberman$)$;
(ii) OR $($H. Taha$)$;
(iii) Intro. to Mathematical Prog $($F.Hillier & J. Lieberman$)$;
(iv) Intro. to OR $($J.Eckert & M. Kupferschmid$)$.

LINEAR PROGRAMMING (LP)

LP is an **optimal decision making tool** in which the objective is a linear function and the constraints on the decision problem are **linear equalities and inequalities**. It is a very popular decision support tool: in a survey of Fortune 500 firms, 85% of the responding firms said that they had used LP.

Example 1:

Manufacturer Produces:	A (acid) and C (caustic)
Ingredients used in the production of A & C:	X and Y
Each ton of A requires:	2lb of X; 1lb of Y
Each ton of C requires:	1lb of X ; 3lb of Y
Supply of X is limited to:	11lb/week
Supply of Y is limited to:	18lb/week
1 ton of A sells for:	£1000
1 ton of C sells for:	£1000

Manufacturer wishes to <u>**maximize**</u> weekly value of sales of A & C. Market research indicates no more than 4 tons of acid can be sold each week. How much A & C to produce to solve this problem.
The answer is a pair of numbers:

$$\boxed{\mathbf{x_1 \text{ (weekly production of A)}, x_2 \text{ (weekly p.of C)}}}$$

There are many pairs of numbers (x_1, x_2): (0,0), (1,1), (3,5).... Not all pairs (x_1, x_2) are possible weekly productions (ex. $x_1 = 27, x_2 = 2$ are not possible) ((27, 2) is not a <u>feasible</u> set of production figures). The constraints on x_1, x_2 are such that (x_1, x_2) represent a possible set of production figures:

The amount each product
is produced is non-negative: $\boxed{x_1 \geq 0 \ x_2 \geq 0}$

The amount of ingredient X
required to produce x_1 tons of A
& x_2 tons of C is $2x_1 + x_2$.
As X is limited to 11lb/week: $\boxed{2x_1 + x_2 \leq 11}$

The amount of ingredient Y required
combined with the supply restriction: $\boxed{x_1 + 3x_2 \leq 18}$

We cannot sell more than 4 tons of A/week: $\boxed{x_1 \leq 4}$

A possible set of production figures satisfies these constraints. Conversely any (x_1, x_2) satisfying these constraints are a possible set of production figures: $\boxed{\text{see FIGURE 1}}$

<u>**THE**</u> <u>**FEASIBLE**</u> <u>**REGION**</u> is the intersection of the shaded regions & is given by $\boxed{\text{see FIGURE 2}}$. The feasible region (OPQRS) represents all pairs (x_1, x_2) that satisfy the constraints. The **corners** (vertices) O,P,Q,R,S have a special significance [O=(0,0), P=(0,6), Q=(3,5), R=(4,3), S=(4,0)].

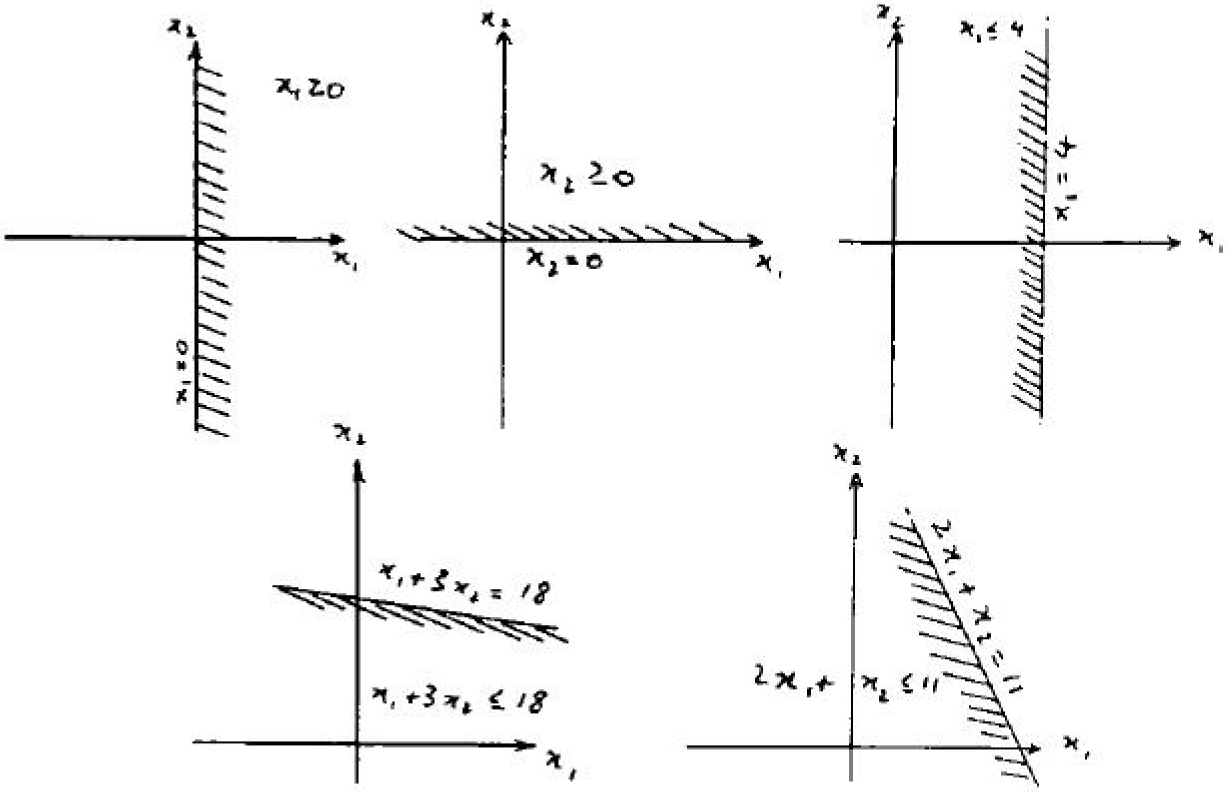

Figure 1

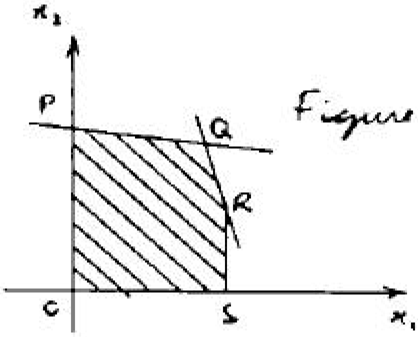

Figure 2.

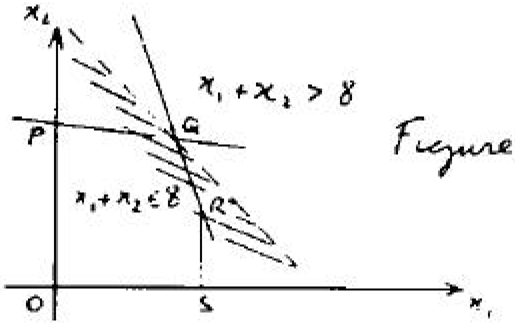

Figure 3

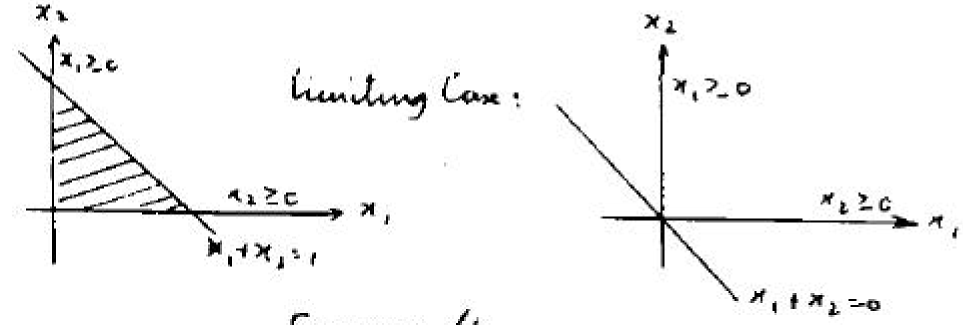

Figure 4

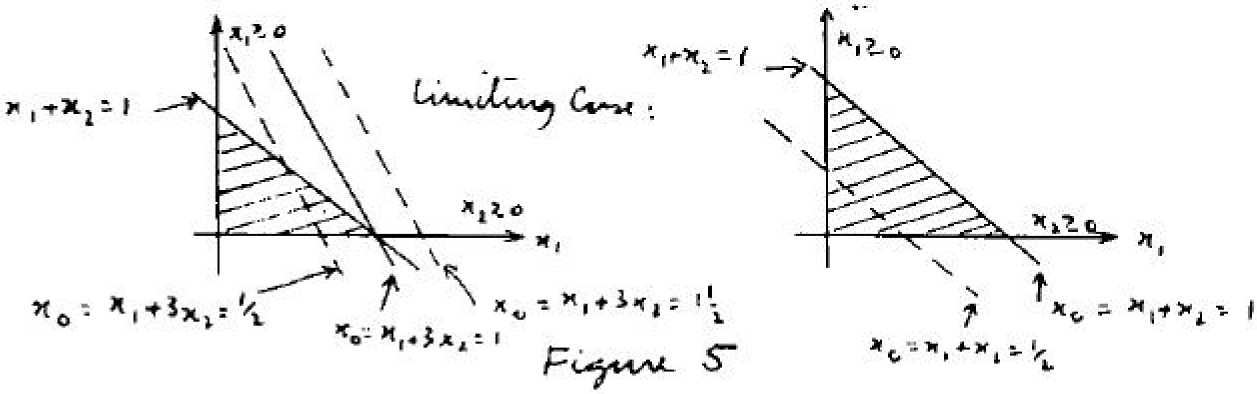

Figure 5

Associated with each feasible (x_1, x_2) is a sales value of £1000 × $(x_1 + x_2)$. Since we wish to maximize this amount, our problem is:

$$\begin{array}{lll} \text{Maximize :} & x_1 + x_2 & \longleftarrow \text{objective function} \\ \text{Subject to} & 2x_1 + x_2 \leq 11 & \longleftarrow \text{constraint} \\ & x_1 + 3x_2 \leq 18 & \longleftarrow \text{constraint} \\ & x_1 \leq 4 & \longleftarrow \text{constraint} \\ & x_1, x_2 \geq 0 & \longleftarrow \text{constraint} \end{array}$$

This is called a **LINEAR PROGRAM (LP)**: A problem of optimizing (maximizing or minimizing) a linear function subject to linear constraints. (Linear: no powers, exponentials or product terms).

PROPERTY (*): Observe that the set {O, P, Q, R, S} contains an optimal solution to our L.P. evaluate the objective function $x_1 + x_2$ at these points: 0,6,8,7,4 $\Rightarrow$ $Q = (3,5)$, $x_1 = 3$, $x_2 = 5$ is the optimal solution. see FIGURE 3

Note: The feasible region (i.e. area described by the polygon OPQRS) lies entirely within that half of the plane for which $x_1 + x_2 \leq 8$. Since $5 + 3 = 8$ no feasible point has a higher objective value than that of Q.

Property (*) holds if we replace $x_1 + x_2$ by any linear function $c_1x_1 + c_2x_2$. e.g. to minimize $3x_1 - x_2$ over points in the polyhedron OPQRS we take the smallest point of 0, $-$ 6, 4, 9, 12 and find that P : $x_1 = 0$, $x_2 = 6$ is an optimal solution.

The **SIMPLEX ALGORITHM**, to be described later, is an efficient method for finding an optimal vertex without necessarily examining all of them.

Property (*) does not imply that points other than vertices cannot be optimal. e.g. if we want to maximize $2x_1 + x_2$ then any point on the segment QR is optimal.

2. STANDARD LP FORM

Any LP can be transformed into **STANDARD FORM**

$$\begin{array}{llll} \text{minimise} & x_0 = & c_1 x_1 + c_2 x_2 + \ldots + c_n x_n & \\ \text{subject to} & & a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n & = b_1 \\ & & a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n & = b_2 \\ & & \vdots & \vdots \\ & & a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n & = b_m \\ \text{and} & & x_1 \geq 0 \quad x_2 \geq 0 \quad \ldots \quad x_n & \geq 0 \end{array}$$

b_i, c_j, a_{ij} : fixed real constants; x_i; i=0, ..., n: real numbers, to be determined.

We assume that $b_i > 0$ (each equation may be multiplied by -1 to achieve this).

Compact Notation

$$\begin{array}{ll} \text{minimise} & x_0 = c^T x \\ \text{subject to} & A x = b \\ \text{and} & x \geq 0. \end{array}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$c^T = [c_1, ..., c_n]$$

Example 2: **Slack variables**

$$\begin{array}{lll}
\min & x_0 = & c_1 x_1 + c_2 x_2 + ... + c_n x_n \\
\text{subject to} & & a_{11} x_1 + a_{12} x_2 + ... + a_{1n} x_n \le b_1 \\
 & & a_{21} x_1 + a_{22} x_2 + ... + a_{2n} x_n \le b_2 \\
 & & \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\
 & & a_{m1} x_1 + a_{m2} x_2 + ... + a_{mn} x_n \le b_m \\
\text{and} & & x_1 \ge 0 \quad x_2 \ge 0 \quad ... \quad x_n \ge 0
\end{array}$$

$$\begin{array}{lllllll}
\min & x_0 = & c_1 x_1 + c_2 x_2 + ... + c_n x_n & & & & \\
\text{subject to} & & a_{11} x_1 + a_{12} x_2 + ... + a_{1n} x_n & + \mathbf{y_1} & & & = b_1 \\
 & & a_{21} x_1 + a_{22} x_2 + ... + a_{2n} x_n & & + \mathbf{y_2} & & = b_2 \\
 & & \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots & & & \ddots & \quad \vdots \\
 & & a_{m1} x_1 + a_{m2} x_2 + ... + a_{mn} x_n & & & + \mathbf{y_m} & = b_m \\
\text{and} & & x_1 \ge 0 \quad x_2 \ge 0 \quad ... \quad x_n \ge 0, & \multicolumn{4}{l}{\mathbf{y_1 \ge 0, y_2 \ge 0, ..., y_m \ge 0.}}
\end{array}$$

Total variables: n + m. **Slack variables: $\mathbf{y_1, y_2, ..., y_m}$**

$$m \times (m + n) \text{ matrix:} \quad [\, A \vdots I \,]$$

$$[\, A \vdots I \,] \begin{bmatrix} x \\ ... \\ y \end{bmatrix} = b$$

Example 3: **Surplus variables**

If the inequalities of Example 2 were reversed so that the typical inequality becomes

$$a_{i1} x_1 + a_{i2} x_2 + ... + a_{in} x_n \ge b_i$$

⇒ $$a_{i1} x_1 + a_{i2} x_2 + ... + a_{in} x_n - \underset{\substack{\uparrow \\ \textbf{surplus variable}}}{\mathbf{y_i}} = b_i \, ; \quad y_i \ge 0.$$

By suitably multiplying by (-1) and adjoining slack and surplus variables, any set of linear inequalities can be converted to standard form if the unknown variables are restricted to be nonnegative.

$$\begin{array}{llclll}
\max & 4 x_1 - 3 x_2 + 2 x_3 & \rightarrow & \min & -4 x_1 + 3 x_2 - 2 x_3 & \\
\text{subject to} & 3 x_1 + 2 x_2 - x_3 = 4 & \rightarrow & \text{subject to} & 3 x_1 + 2 x_2 - x_3 & = 4 \\
 & x_1 + x_2 + 2 x_3 \le 5 & \rightarrow & & x_1 + x_2 + 2 x_3 + x_4 & = 5 \\
 & -x_1 + 2 x_2 - x_3 \ge 2 & \rightarrow & & -x_1 + 2 x_2 - x_3 \qquad - x_5 & = 2 \\
 & x_1, x_2, x_3 \ge 0 & \rightarrow & & x_1, x_2, x_3, x_4, x_5 \ge 0 &
\end{array}$$

Example 4: Free variables (I)

Suppose $x_1 \ge 0$ is not present: x_1 may take (+) or (-) values. Substitute x_1 with $x_1 = v_1 - u_1$; $v_1, u_1 \ge 0$. The problem has now (n+1) variables: $v_1, u_1, x_2, ..., x_n$.

Example 5: Free variables (II)

Eliminate x_1 using one of the constraint equations.

$$\begin{array}{ll} \text{min} & x_1 + 3x_2 + 4x_3 \\ \text{subject to} & x_1 + 2x_2 + x_3 = 5 \\ & 2x_1 + 3x_2 + x_3 = 6 \\ & x_2, x_3 \geq 0 \end{array}$$

As x_1 is free, solve for it using the first constraint: $x_1 = 5 - 2x_2 - x_3$. Substitute this in the objective function and the constraint,

$$\min \left\{ x_2 + 3x_3 \;\middle|\; x_2 + x_3 = 4,\ x_2, x_3 \geq 0 \right\}$$

3. EXAMPLES OF LP PROBLEMS

Example 6: The diet problem

To determine the most economical diet that satisfies the basic nutritional requirements for good health.

n different foods: i th sells at price c_i/unit

m basic nutritional ingredients:healthy diet ⇒ daily intake for individual at least b_j units of jth ingredient

each unit of food i contains a_{ji} units of j th ingredient

x_i: number of units of food i in the diet.

$$\begin{array}{llcl} \text{minimise total cost} & x_0 & = & c_1 x_1 + c_2 x_2 + \dots + c_n x_n \\ \text{subject to} & & & a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \geq b_1 \\ & & & \vdots \\ & & & a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \geq b_m \\ \text{and nonnegativity of food quantities} & & & x_1 \geq 0 \quad x_2 \geq 0 \quad \dots \quad x_n \geq 0 \end{array}$$

Example 7: The transportation problem

Quantities $a_1, a_2, \dots, a_m$ of a product are to be shipped from each of m locations and are demanded in amounts $b_1, b_2, \dots, b_n$ at each of n destinations.

c_{ij}: unit cost of transporting product from origin i to destination j

x_{ij}: the amounts to be shipped from i to j (i=1, ..., m; j=1, ..., n)

Determine x_{ij} to satisfy shipping requirements and minimise total cost of transportation.

$$\text{minimise} \quad \sum_{i,j} c_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} = a_i \quad (\text{total shipped from } i^{th} \text{ origin; } i = 1, \dots, m)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad (\text{total required by } j^{th} \text{ destination; } j = 1, \dots, n)$$

$$x_{ij} \geq 0\,;\ i = 1, \dots, m\,;\ j = 1, \dots, n$$

(For consistency we must also have $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$).

4. BASIC SOLUTIONS

To **compute a basic solution**, consider **the system of equalities**

$$\boxed{A\,x = b; \quad x \in \mathbb{R}^n; \quad b \in \mathbb{R}^m; \quad A \in \mathbb{R}^{m\times n}}$$

Select from the n columns of A a set of m **linearly independent columns (exists if rank (A) = m)**. For simplicity, assume that we select the first m columns of A and denote the m × m matrix determined by these columns by $B \in \mathbb{R}^{m \times m}$. B is nonsingular and we may uniquely solve

$$B\, x_B = b; \quad x_B \in \mathbb{R}^m$$

set

$$x = \begin{bmatrix} x_B \\ 0 \end{bmatrix} \text{ i.e. } \begin{bmatrix} \text{the first m components of x are equal to those of } x_B \\ \text{the rest are equal to zero} \end{bmatrix}$$

We thus obtain a solution to A x = b.

Definition: Given A x = b, let B be any nonsingular m × m matrix made up of the columns of A. If all n-m components of x, not associated with the columns of B, are set to zero, the solution to the resulting set of equations is said to be a basic solution (BS) to A x = b, w.r.t. the basis B. The components of x associated with columns of B are basic variables (BV).

B is a basis since it consists of m l.i. columns that can be regarded as a basis for $\mathbb{R}^m$. There may not exist a basic solution to A x = b. To ensure existence we have to assume:

Full rank assumption: The m × n matrix A has m < n and the m rows of A are linearly independent.

Linear dependency among the rows of A ⇒ either contradictory constraints (there is no solution to A x = b : e.g. $x_1 + x_2 = 1$, $x_1 + x_2 = 2$) or to a redundancy that can be eliminated (e.g. $x_1 + x_2 = 1$, $2x_1 + 2x_2 = 2$).

Under the full rank assumption, A x = b will always have at least one basic solution.

Basic variables in a basic solution are not necessarily nonzero:

Definition: If one or more BV in a BS has zero value, then the solution is a **degenerate BS**.

There is an ambiguity in degenerate BS since the zero-valued basic and nonbasic variables can be interchanged.

Definition: x satisfying A x = b and x ≥ 0 is said to be feasible. A feasible solution that is also basic is a **basic feasible solution (BFS)**. If this solution is degenerate, it is called a **degenerate BFS**.

Example 8:

After adding slack variables to the problem of Example 1, we obtain the following equations which `happen' to form an initial basic representation:

$$\begin{array}{rrrrrrl} x_0 & - \; x_1 & - \; x_2 & & & & = 0 \qquad (x_0 = c^T x) \\ & 2x_1 & + \; x_2 & + x_3 & & & = 11 \\ & x_1 & + \; 3x_2 & & + x_4 & & = 18 \\ & x_1 & & & & + x_5 & = 4 \end{array} \tag{1}$$

in basic representation the variables (i.e. the elements of vector x) are divided into basic variables and non-basic variables. In the system of equations given by (1) above

$$\boxed{\text{the basic variables are } \{x_0, x_3, x_4, x_5\}} \text{ and } \boxed{\text{the non basic var's are } \{x_1, x_2\}}.$$

$$\boxed{\begin{array}{c}\text{Each equation in (1) expresses a particular basic variable} \\ \text{as a linear expression in the non-basic variables}\end{array}}.$$

The basic solution of this representation is obtained by setting $x_j = 0$ for each non-basic variable and then solving the equations for the remaining BV's:
Set $x_1 = x_2 = 0 \Rightarrow x_0 = 0, x_3 = 11, x_4 = 18, x_5 = 4$

$$\Rightarrow \text{ BS: } \boxed{(x_0, x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 11, 18, 4)} = \text{BFS}$$

Looking for a better solution than this, we search for a non-basic variable x_j such that increasing x_j (from 0) improves x_0.

$$x_0 = x_1 + x_2$$

$\Rightarrow$ can increase either x_1 or x_2 (increasing both is too complicated). Consider the solutions obtained by increasing x_1 to λ and leaving $x_2 = 0$. In order to satisfy (1) and stay feasible we must ensure that

$$\begin{array}{lllllllll} x_0 & = \lambda & & & & & & & \\ x_3 & = 11 & - & 2\lambda & \geq 0 & \Rightarrow & \lambda & \leq & 11/2 \\ x_4 & = 18 & - & \lambda & \geq 0 & \Rightarrow & \lambda & \leq & 18 \\ x_5 & = 4 & - & \lambda & \geq 0 & \Rightarrow & \lambda & \leq & 4 \end{array} \tag{2}$$

We want the best (largest) λ satisfying (2). As λ takes values between 0-4, the solution defined by (2) has $x_1 = \lambda$ $x_2 = 0$ which corresponds to a point 0S in Figure 2. The solution given by $\lambda = 4$

$$\boxed{\begin{array}{c} (x_0, x_1, x_2, x_3, x_4, x_5) = (4, 4, 0, 3, 14, 0) \\ \text{(point S in fig. 2)} \end{array}}$$

This is also a BFS to (1). BV: $\{x_0, x_1, x_3, x_4\}$ and NBV: $\{x_2, x_5\}$. Note: in a basic solution, the non-basic variables are zero. We need the basic representation, i.e. need to transform (1) so that x_0, x_1, x_3, x_4 are expressed in terms of x_2, x_5. We do this by pivoting (to be discussed later) to get

$$\begin{array}{rrrrrrl} x_0 & & -x_2 & & & +x_5 & = 4 \\ & & x_2 & +x_3 & & -2x_5 & = 3 \\ & & 3x_2 & & +x_4 & -x_5 & = 14 \\ & x_1 & & & & +x_5 & = 4 \end{array} \tag{3}$$

A solution to (3) is a solution to (1) and conversely.

From $x_0 = 4 + x_2 - x_5$ we see that to increase x_0, we should increase x_2 (& keep $x_5 = 0$).

Set $x_2 = \lambda$, $x_5 = 0$. To satisfy (3) and stay feasible the other variables must satisfy

$$\begin{array}{lllllllll} x_0 & = 4 & + & \lambda & & & & & \\ x_3 & = 3 & - & \lambda & \geq 0 & \Rightarrow & \lambda & \leq & 3 \\ x_4 & = 14 & - & 3\lambda & \geq 0 & \Rightarrow & \lambda & \leq & 14/3 \\ x_1 & = 4 & & & \geq 0 & & & & \end{array}$$

The best value for $\lambda = 3$. As $\lambda \in [0, 3]$, the solution has $x_1 = 4$, $x_2 = \lambda$ which corresponds to a point on SR in Figure 2. The solution given by $\lambda = 3$ is:

$$\boxed{\begin{array}{c} (x_0, x_1, x_2, x_3, x_4, x_5) = (7, 4, 3, 0, 5, 0) \\ \text{(point R in fig. 2)} \end{array}}$$

This is also a BFS to (1). the BV: $\{x_0, x_1, x_2, x_4\}$; NBV: $\{x_3, x_5\}$. The basic representation:

$$\begin{array}{rrrrrrl} x_0 & & & +x_3 & & -x_5 & = 7 \\ & & x_2 & +x_3 & & -2x_5 & = 3 \\ & & & -3x_3 & +x_4 & +5x_5 & = 5 \\ & x_1 & & & & +x_5 & = 4 \end{array} \tag{4}$$

A solution to (4) is a solution to (1) and conversely. From $x_0 = 7 - x_3 + x_5$, to increase x_0 we should increase x_5. Set $x_5 = \lambda$ $x_3 = 0$ to satisfy (4) and stay feasible, the other variables must satisfy

$$\begin{array}{lllllllll} x_0 & = 7 & + & \lambda & & & & & \\ x_2 & = 3 & + & 2\lambda & \geq 0 & \Rightarrow & \lambda & \leq & \infty \\ x_4 & = 5 & - & 5\lambda & \geq 0 & \Rightarrow & \lambda & \leq & 1 \\ x_1 & = 4 & - & \lambda & \geq 0 & \Rightarrow & \lambda & \leq & 4 \end{array}$$

$$\boxed{\Rightarrow \lambda = 1}$$

As λ takes values from 0 to 1, the solution defined by (12) has $x_1 = 4 - \lambda$, $x_2 = 3 + 2\lambda$ corresponding to a point on RQ in fig. 2. The solution given by $\lambda = 1$:

$$\boxed{\begin{array}{c}(x_0, x_1, x_2, x_3, x_4, x_5) = (8, 3, 5, 0, 0, 1)\\ \text{(point Q in fig. 2)}\end{array}}$$

This is also a BFS to (1). BV : $\{x_0, x_1, x_2, x_5\}$; NBV: $\{x_3, x_4\}$. Basic representation:

$$\begin{array}{llllllll}
x_0 & & + \frac{2}{5} x_3 & + \frac{1}{5} x_4 & & = 8 \\
 & x_2 & - \frac{1}{5} x_3 & + \frac{2}{5} x_4 & & = 5 \\
 & & - \frac{3}{5} x_3 & + \frac{1}{5} x_4 & + x_5 & = 1 \\
x_1 & & + \frac{3}{5} x_3 & - \frac{1}{5} x_4 & & = 3
\end{array} \tag{5}$$

A solution of (5) is a solution of (1), and conversely. Thus, any solution to (1) satisfies

$$x_0 = 8 - \tfrac{2}{5} x_3 - \tfrac{1}{5} x_4$$

Any feasible solution has $x_3, x_4 \geq 0$ and hence by (5) $x_0 \leq 8$. $(8, 3, 5, 0, 0, 1)$ has $x_0 = 8$ and so this solution is maximal.

SUMMARY

1. Among the FS to min $\{ x_0 = c^T x \mid A x = b, x \geq 0 \}$ there is an important <u>finite</u> subset: **BFS**.

2. Each BFS is associated with a **basic representation**: A set of equations <u>equivalent</u> to
$$\min \{ x_0 = c^T x \mid A x = b, x \geq 0 \}$$
that expresses each BV in terms of the NBV's.

3. By looking at a basic representation **we can see if increasing any NBV will improve the objective**. If there is one, we can increase it until a new, better, BFS is reached (usual case). If there does not exist such a NBV, we have the optimal solution.

BASIC FEASIBLE SOLUTIONS

Let $S_n = \{ 1 ,...., n \}$, $I \subseteq S_n$ have m elements. Set of basic variables: $\{x_i \mid i \in I\} \cup \{x_0\}$. Let a_j denote the column of A corresponding to x_j, $j \in S_n$. Associated with I is an $m \times m$ matrix $B = B(I)$ where the columns of B are made up from $\{a_i \mid i \in I\}$.

Example 9:

If $$A = \begin{bmatrix} 2 & 4 & 3 & 3 & 1 & 0 \\ 3 & -3 & 4 & 2 & 0 & 1 \\ -1 & 2 & 1 & 2 & 0 & 0 \end{bmatrix}$$

Then $$I = \{1, 5, 2\} \Rightarrow B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & -3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$I = \{6, 3, 4\} \Rightarrow B = \begin{bmatrix} 0 & 3 & 3 \\ 1 & 4 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

The remaining columns a_j for $j \notin I$ form matrix N and so (after shuffling the columns of A) we may assume

$$A = [\, B \,\vdots\, N \,]$$

we then conformably partition c, x into (c_B, c_N) and (x_B, x_N) respectively. i.e.

$$c_B = (c_i \mid i \in I); \quad c_N = (c_j \mid j \notin I); \quad x_B = (x_i \mid i \in I); \quad x_N = (x_j \mid j \notin I)$$

Example 10: $n = 6; \; I = \{5, 3, 2\}$

$$c_B = [c_5, c_3, c_2]; \quad c_N = [c_1, c_4, c_6]$$

$$x_B = [x_5, x_3, x_2]; \quad x_N = [x_1, x_4, x_6]$$

Given this partition,

$$\min \left\{ x_0 = c_B^T x_B + c_N^T x_N \;\middle|\; B x_B + N x_N = b; x_B, \; x_N \geq 0 \right\} \tag{6,a}$$

$$x_0 - c_B^T x_B - c_N^T x_N = 0 \tag{6,b}$$

$$B x_B + N x_N = b \tag{6,c}$$

As B is assumed to be nonsingular (i.e B^{-1} exists) then a solution to (6,c) satisfies

$$x_B + B^{-1} N x_N = B^{-1} b \tag{7,a}$$

and conversely. Using (7,a) to eliminate x_B from (6,a) yields

$$x_0 = c_B^T B^{-1} b + \left(c_N^T - c_B^T B^{-1} N \right) x_N \tag{7,b}$$

Note that (7) expresses the BV's (x_0, x_B) in terms of the NBV's x_N. The vector

$$r^T = \left(c_N^T - c_B^T B^{-1} N \right) \tag{8}$$

is the relative (or **reduced**) cost vector (for NBV's). It is the components of r that determine which vector can be brought into the basis.

Example 11:

$$c = \begin{bmatrix} 6 \\ 3 \\ 4 \\ 2 \\ -3 \\ 4 \\ 0 \\ 0 \end{bmatrix}; \quad A = \left[\begin{matrix} 2 & -1 & 3 & 2 & 3 & 2 & 1 & 0 \\ 3 & 4 & 2 & 2 & 3 & 0 & 0 & 1 \end{matrix} \, 0 \right]; \quad b = \begin{bmatrix} 4 \\ 2 \end{bmatrix};$$

$$I = \{4, 3\} \Rightarrow B = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}; \text{det } B = -2 \neq 0 \quad B^{-1} = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}$$

$$\text{(7,a)} \Rightarrow \quad \begin{array}{l} \frac{5}{2} x_1 + 7x_2 + x_4 + \frac{3}{2} x_5 - 2x_6 - x_7 + \frac{3}{2} x_8 = -1 \\ - x_1 - 5x_2 + x_3 + 2x_6 + x_7 - x_8 = 2 \end{array}$$

$$\text{(7,b)} \Rightarrow \quad x_0 - 5x_1 + 9x_2 - 6x_5 - 2x_7 + x_8 = 6$$

The Importance of BFS

It is necessary only to consider BFS's when seeking an optimal solution to an LP because the optimal value is always achieved at such a solution.

Definition: Given an LP in standard form, a feasible solution to the constraints $\{A\,x = b;\ x \geq 0\}$ that achieves the minimum value of the objective function subject to those constraints is said to be an optimal feasible solution. If this solution is basic then it is an optimal BFS.

Theorem 1: Fundamental theorem of LP

Given an LP in standard form where A is an $m \times n$ matrix of rank m:

(i) if there is a feasible solution, then there is a BFS (see Figure 4).

(ii) if there is an optimal solution, then there is an optimal BFS (see Figure 5).

Theorem 1 reduces the task of solving an LP to that of searching over BFS's. Since for a problem having n variables and m constraints, there are at most

$$\boxed{\binom{n}{m} = \frac{n!}{m!\,(n-m)!}}$$

basic solutions (corresponding to the number of ways of selecting m on n columns), there are only a finite number of possibilities. Thus Theorem 1 yields an obvious but terribly inefficient way of computing the optimum through a finite search technique.

Example 12: $\binom{n}{m}$ **for small problem**

$$m = 30\,,\, n = 100. \text{ as } \binom{100}{30} = \frac{100!}{30!\,70!} \approx 2.9 \text{ x } 10^{25}.$$

This would take approximately two years assuming we could check 10^6 sets of I / second.

The set of basic variables: $\{x_i \mid i \in I\} \cup \{x_o\}$ where $I \subset S_n$ and has m elements.

The set of non-basic variables: $x_j \notin I$ and $j \neq 0$.

The BS corresponding to I is given by:

(i) $x_j = 0$ for $j \notin I$ and $j \neq 0 \Rightarrow x_N = 0$ (in (7,a))

(ii) $x_B = B^{-1}\,b - B^{-1}\,N\,x_N = B^{-1}\,b$

This is a feasible solution iff $B^{-1}\,b \geq 0$ in which case it is a BFS.

Example 13: A, b, c given as in Example 11, $I = \{4, 3\}$. BS: $(x_0, x_1, \ldots, x_8) = (6,0,0,2,-1,0,0,0,0)$ is not feasible.

Exercise: Find some BFS for this example.

Note The number of distinct BFS's to (1) is $\leq \binom{n}{m} =$ the number of sets $I \subseteq S_n$ with $|I| = m$. $\Rightarrow$ This number is finite. This number is usually $< \binom{n}{m}$ because for a given I, (i) B(I) may be singular, (ii) the basic solution may not be nonnegative. Also it is possible that two (or more) distinct I_1, I_2 can lead to the same BFS:

Example 14:

$$\left.\begin{array}{l} x_1 - x_2 + 2x_3 = 1 \\ 2x_1 + x_2 - x_3 = 2 \end{array}\right\} I_1 = \{1, 2\}\,, I_2 = \{1, 3\}$$

Both I_1 and I_2 lead to the same BFS: $(1, 0, 0)$.

Example 15: To demonstrate that we cannot simply state "an LP has an optimal BFS"

Infeasible $\{FR\} = \emptyset$: $\min \left\{ x_0 = 2x_1 + x_2 \;\middle|\; x_1 + x_2 \leq 1\,;\ \ x_1 + x_2 \geq 2;\ x_1\,,\, x_2 \geq 0 \right\}$

Unbounded: $\max \left\{ x_0 = x_1 \;\middle|\; x_1 + x_2 \geq 1;\ x_1\,,\, x_2 \geq 0 \right\}$

We can make x_1 arbitrarily large. i.e. there is no maximum value to x_0 and so no optimal solution.

Note: The problem min x_0 above has a solution $x_1 = 0\,, x_2 = 1$ and so unbounded refers to the objective value and not to the `size' of $\{FR\}$ (which is unbounded)

5.THE SIMPLEX ALGORITHM

Convention: Indexing the rows of a BV: **A row is indexed by the basic variable in that row**.

The simplex algorithm is based on the fact that if a BFS x is not optimal, then there is some neighbouring basic solution y with a better objective value. If we examine the sequence of BFS's found when solving Example 8 we see that the sequence of index sets I (and x_0) of basic variables was

$$\begin{array}{ccccccc} \{0, 3, 4, 5\} & \to & \{0, 3, 4, 1\} & \to & \{0, 2, 4, 1\} & \to & \{0, 2, 5, 1\} \\ \\ I_1 & & I_2 & & I_3 & & I_4 \end{array}$$

Notice that for t = 1, 2, 3 ..., I_{t+1} is obtained from I_t by removing one element and replacing it by a new element i.e. $| I_{t+1} \setminus I_t | = | I_t \setminus I_{t+1} | = 1$.

If I, I′ are such that B(I), B(I′) are non-singular, then I, I′ are said to be neighbours if $| I \setminus I' | = | I' \setminus I | = 1$.

Pivots

At each stage of the simplex algorithm we have a basic representation and its BFS. We then use the reduced costs to see if there is some neighbouring representation that has a better solution. Should such a neighbour exist, we construct this representation by pivoting.

Consider the system A x = b ($A \in \mathbb{R}^{m\times n}$, $m \leq n$)

$$\begin{array}{lllll} a_{11} & x_1 + a_{12} & x_2 + \ldots + a_{1n} & x_n & = b_1 \\ & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & x_1 + a_{m2} & x_2 + \ldots + a_{mn} & x_n & = b_m \end{array} \tag{9}$$

If the equations (9) are linearly independent, we may replace a given equation by a nonzero multiple of itself plus any linear combination of the other equations. This leads to the well known **Gaussian reduction** schemes, whereby multiples of equations are systematically subtracted from one another to yield a canonical form. If the first m columns of A are linearly independent, the system (9) can, by a sequence of such multiplications and subtractions, be converted to the following **canonical form:**

$$\begin{array}{llllll} x_1 & & & + \; y_{1,m+1}\, x_{m+1} + y_{1,m+2}\, x_{m+2} + \ldots + y_{1,n} x_n & = & y_{10} \\ & x_2 & & + \; y_{2,m+1}\, x_{m+1} + y_{2,m+2}\, x_{m+2} + \ldots + y_{2,n} x_n & = & y_{20} \\ & & \ddots & \quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots & & \vdots \\ & & x_m & + \; y_{m,m+1}\, x_{m+1} + y_{m,m+2}\, x_{m+2} + \ldots + y_{m,n} x_n & = & y_{m0} \end{array} \tag{10}$$

Example 16:

$$\left.\begin{array}{lllll} x_1 & + \; 3 \; x_2 & + & x_3 = & 1 \\ \\ & \quad\;\; x_2 & + \; 3 & x_3 = & 3 \end{array}\right\} \text{eq. 1 - 3} \times \text{eq. 2}$$

$$\to \begin{array}{lllll} x_1 & & - \; 8 & x_3 = & - 8 \\ \\ & x_2 & + \; 3 & x_3 = & 3 \end{array}$$

According to this canonical representation:

$$\boxed{\begin{array}{c} \text{Basic variables: } x_1 = y_{10},\, x_2 = y_{20},\, \ldots,\, x_m = y_{m0} \\ \text{Non-Basic variables: } x_{m+1} = 0,\, x_{m+2} = 0,\, \ldots,\, x_n = 0. \end{array}}$$

We relax our definition and consider a system to be in canonical form if, among n variables, there are m basic ones with the property that each appears in only one equation, its coefficient in that equation is unity, and no two of these m variables occur in any one equation. This is equivalent to saying that a system is in canonical form if by some reordering of the equations and variables, it takes the form (10). (10) is also represented by its corresponding coefficients or tableau:

$$\begin{array}{cccccccc}
1 & 0 & \dots & 0 & y_{1,m+1} & y_{1,m+2} & \cdots & y_{1,n} & y_{10} \\
0 & 1 & \dots & 0 & y_{2,m+1} & y_{2,m+2} & \cdots & y_{2,n} & y_{20} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & 1 & y_{m,m+1} & y_{m,m+2} & \cdots & y_{m,n} & y_{m0}
\end{array}$$

The question solved by pivoting is this: given a system in canonical form, suppose a non-basic variable is to be made basic and a basic variable is to be made nonbasic. What is the new canonical form corresponding to the new set of basic variables? The procedure is quite simple. Suppose in (10) we wish to replace the basic variable x_p, $1 \le p \le m$, by the nonbasic variable x_q. This can be done iff $y_{pq} \neq 0$ in (10). It is accomplished by <u>dividing the row p by y_{pq} to get unit coefficient for x_q in the p th equation, then subtracting suitable multiples of row p from each of the other rows in order to get zero coefficient for x_q in all other equations</u>. This transforms the q th column of the tableau so that it is zero except its p th entry, which is 1 and does not affect the columns of the other basic variables. Denoting the coefficients of the new canonical form by y'_{ij}:

$$\boxed{y'_{ij} = y_{ij} - \frac{y_{pj}}{y_{pq}} y_{iq}, \quad i \neq p \quad \text{and} \quad y'_{pj} = \frac{y_{pj}}{y_{pq}}, \quad j = 0, ..., n} \tag{11}$$

(11) are the pivot equations in LP. y_{pq} is the pivot element.

Example 17:

$$\begin{array}{lllllll}
x_1 & & & + \ x_4 & + \ x_5 & - x_6 & = 5 \\
& x_2 & & + 2\ x_4 & -3\ x_5 & + \ x_6 & = 3 \\
& & x_3 & - \ x_4 & + 2\ x_5 & - x_6 & = -1
\end{array}$$

Find the basic solution with basic variables x_4, x_5, x_6.

x_1	x_2	x_3	x_4	x_5	x_6	
1	0	0	$\boxed{1}$	1	-1	5
0	1	0	2	-3	1	3
0	0	1	-1	2	-1	-1

1	0	0	1	1	-1	5
-2	1	0	0	$\boxed{-5}$	3	-7(replacing x_2 by x_5 as BV)
1	0	1	0	3	-2	4

3/5	1/5	0	1	0	-2/5	18/5
2/5	-1/5	0	0	1	-3/5	7/5
-1/5	3/5	1	0	0	$\boxed{-1/5}$	-1/5

1	-1	-2	1	0	0	4 (New basic solution:
1	-2	-3	0	1	0	2 x_4=4, x_5=2, x_6=1)
1	-3	-5	0	0	1	1

Example 18:

Using the Example 11, $I = \{4,3\}$

$$\begin{array}{l} x_0 - 5x_1 + 9x_2 - 6x_5 - 2x_7 + x_8 = 0 \\ \tfrac{5}{2}x_1 + 7x_2 + x_4 + \tfrac{3}{2}x_5 - \boxed{2x_6} - x_7 + \tfrac{3}{2}x_8 = -1 \\ -x_1 - 5x_2 + x_3 + 2x_6 + x_7 - x_8 = 2 \end{array}$$

and pivoting on (4,6) yields

$$\begin{array}{l} x_0 - 5x_1 + 9x_2 - 6x_5 - 2x_7 + x_8 = 0 \\ -\tfrac{5}{4}x_1 - \tfrac{7}{2}x_2 - \tfrac{1}{2}x_4 - \tfrac{3}{4}x_5 + x_6 + \tfrac{1}{2}x_7 - \tfrac{3}{4}x_8 = \tfrac{1}{2} \\ \tfrac{3}{2}x_1 + 2x_2 + x_3 + x_4 + \tfrac{3}{2}x_5 + \tfrac{1}{2}x_8 = 1 \end{array}$$

which is the basic representation for $I + 6 - 4 = \{6, 3\}$.

The **simplex algorithm** starts with a **BFS** and a **basic representation** and proceeds by a **sequence of pivots** to find a **BFS which is also optimal**. For most problems, finding an initial BFS is not easy and this is discussed later. However, for those problems in which the constraints are

$$\boxed{\sum_{j=1}^{n} a_{ij}\, x_j \le b_i \;; \quad i = 1,\dots,m \;; \quad x_j \ge 0 \;; \quad j = 1,\dots,n}$$

and where $b_i \ge 0$, $i = 1, \dots, m$ it is straightforward. On adding **slack variables, $x_{n+1}, \dots, x_{n+m}$**, we find that

$$\boxed{x_0 - \sum_{j=1}^{n} c_j\, x_j = 0 \;; \quad x_{n+i} + \sum_{j=1}^{n} a_{ij}\, x_j = b_i \;; \quad i = 1,\dots,m}$$

is itself a basic representation with $I = \{n+1, \dots, n+m\}$ and $x_j = 0$ $j=1, \dots, n$ (non basic), $x_{n+i} = b_i$, $i=1, \dots, m$ (Basic) is feasible *as long as* $b_i \ge 0$.

We can now develop the simplex algorithm. Assume that we have some basic representation (7, a) - (7, b). Notice that, because of equivalence the original problem (i.e. minimise $x_0 = c^T x$, subject to $A x = b$; $x \ge 0$) and (7, a) - (7, b) have the same set of solutions.

The goal of the simplex algorithm is to produce a basic representation whose basic solution is optimal. This is done by satisfying the conditions *of* **Theorem 2**

Theorem 2 (*optimalty*)

If $r = c_N - N^T B^{-T} c_B \ge 0$ (see (7, b)), then the associated basic feasible solution minimizes x_0.

Proof

For the given basic solution (assumed feasible)

$$x_0 = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N = c_B^T B^{-1} b$$

For any other solution to (7, a)-(7, b) we have

$$x_0 = c_B^T B^{-1} b + r^T x_N$$

$$r^T x_N = r_{m+1} x_{m+1} + r_{m+2} x_{m+2} + \dots + r_n x_n \ge 0$$

since $r \ge 0$ and $x_N \ge 0$. It then follows that

$$x'_0 = c_B^T B^{-1} b + r^T x_N \ge c_B^T B^{-1} b = x_0. \qquad \square$$

Suppose now that our basic representation does not satisfy the conditions of Theorem 2. In the simplex algorithm we try to choose a pivot so that new basic representation is (a) feasible and (b) $x'_0 < x_0$ (unfortunately, because of degeneracy, we can only guarantee $x'_0 \leq x_0$ - more on this later).

The motivation for the pivot choice is given in two ways. We assume that one of the elements of r is negative. At this stage we need to introduce an alternative representation using

$$x_0 - (c_N^T - c_B^T B^{-1}N)x_N = c_B^T B^{-1}b$$

$$x_0 + \beta^T x = \beta_0 \ (= c_B^T B^{-1}b)$$

or

$$x_0 + \sum_{i=1}^{n} \beta_i x_i = \beta_0$$

where $\beta_i = 0, \forall i \in I$, and β_i = corresponding element of $(-r)$ $\forall i \in S_n - I$ (non-basic variables). If the basic variables are $x_1, x_2, ..., x_m$, then $\beta_1 = \beta_2 = ... = \beta_m = 0$ and corresponding to the non-basic variables $x_{m+1}, ..., x_n$ we have $\beta_{m+1} = -r_{m+1}$, $\beta_{m+2} = -r_{m+2}$, ..., $\beta_n = -r_n$.

Thus, assume that for some $k \in S_n - I$ we have $\beta_k > 0$. We can first look at the current BFS and examine how increasing x_k will lead to a better solution.

Example 19:

$$\begin{array}{llcl}
x_0 & + 6x_4 - 5x_5 + x_6 & = & 26 \\
\quad x_1 & + 2x_4 + 2x_5 - x_6 & = & 7 \\
\qquad x_2 & - 3x_4 - 3x_5 - 3x_6 & = & 5 \\
\qquad\quad x_3 & \boxed{+3}x_4 - x_5 + x_6 & = & 6
\end{array}$$

now $\beta_4 > 0$. We consider increasing x_4 while keeping $x_5 = x_6 = 0$. The values of the basic variables will become

$$\begin{aligned}
x_0 &= 26 - 6x_4 \\
x_1 &= 7 - 2x_4 \\
x_2 &= 5 + 3x_4 \\
x_3 &= 6 - 3x_4
\end{aligned}$$

The larger x_4, the smaller x_0

Must ensure $x_1, x_2, x_3 \geq 0$

The larger x_4, the smaller x_0 will become. We must, however, ensure that x_1, x_2, x_3 remain non-negative. We can see that x_2 actually increases and remains non-negative. However, if $x_4 > 7/2$, x_1 will become negative. Thus, the best (feasible) solution is

$$x_4 = \min\left\{ \tfrac{7}{2}, \tfrac{6}{3} \right\} = 2 \quad \text{or}$$

$$\boxed{(x_0, x_1, x_2, x_3, x_4, x_5, x_6) = (14, 3, 11, 0, 2, 0, 0)}$$

Of key importance is the fact that this latter solution is also a basic solution. It is that associated with the new - basic representation obtained by a pivot on the circled $+3$.

Returning to the general case with $\beta_k > 0$, if we put $x_k = \lambda > 0$ and $x_j = 0, j \notin I \cup \{k\}$, then the value of basic variable x_i must satisfy

$$x_i = y_{i0} - y_{ik}\lambda \ \text{ for } i \in I \cup \{0\}$$

in order that (7, a, b) still hold.

Now we have assumed that $\beta_k > 0$ and so x_0 decreases monotonically as λ increases. We thus increase x_k as much as possible while ensuring that all variables except x_0 <u>remain</u> <u>nonnegative</u> in value. The variables which are non-basic (currently) other than x_k remain at zero. So we have only to consider variables which are currently basic.

Rules (12)

If $\mathbf{y_{ik} \leq 0}$ **then $\boldsymbol{\lambda}$ is unrestricted** for that equation

$$y_{i0} - y_{ik}\lambda \geq y_{i0} \geq 0, \quad \forall \lambda \geq 0 \text{ (here, } i \neq 0)$$

$$\Rightarrow x_i \geq 0 \text{ no matter how large } \lambda \text{ becomes.}$$

$$\text{(Example: } x_2 = 5 + 3x_4 \geq 0, \forall x_4 = \lambda \geq 0.)$$

If $\mathbf{y_{ik} > 0}$ **then** $\mathbf{y_{i0} - y_{ik}\boldsymbol{\lambda} \geq 0 \Leftrightarrow \boldsymbol{\lambda} \leq y_{i0}/y_{ik}}$

$\Rightarrow$ to ensure that all variables remain non-negative we need only to ensure

$$\lambda \leq y_{i0}/y_{ik} \ \forall i \in I \text{ such that } y_{ik} > 0.$$ ☐

As a consequence, we can show:

<u>Theorem</u> <u>3</u>

If for some basic feasible representation $\beta_k > 0$ and $y_{ik} \leq 0$ for $i \in I$ then the problem is <u>unbounded</u> (below) i.e. there is no minimum value for x_0.

Proof

From (12) we see that we can make λ above arbitrarily large and still have a feasible solution. The objective value for this solution is $x_0 = \beta_0 - \beta_k\lambda \to -\infty$. ☐

If $\exists\, i \in I$ such that $y_{ik} > 0$ then the best solution is obtained by making λ as large as possible i.e.

$$\lambda = \min\left\{ y_{i0}/y_{ik} \mid y_{ik} > 0, i \in I \right\} \text{ (the ratio test).}$$

If θ denotes this value of λ, then the solution obtained is

$$\boxed{\text{(i) } x_k = \theta;\ \text{(ii) } x_i = y_{i0} - y_{ik}\theta,\ \forall i \in I \cup \{0\};\ \text{(iii) } x_j = 0, \forall j \notin I \cup \{k\}} \qquad (13)$$

Suppose that $\theta = y_{\ell 0}/y_{\ell k}$. Then we see from the pivot formulae (11) that the solution given in (13) is the basic solution obtained after pivoting on (ℓ,k).

The pivot choice (ℓ,k) above has the following characteristics:

$$\beta_k > 0 \qquad (14, a)$$

$$y_{\ell 0}/y_{\ell k} = \min\left\{ y_{i0}/y_{ik} \mid y_{ik} > 0 \right\} \qquad (14, b)$$

This choice of pivot can also be justified from the pivot formulae. Assume now that our current BFS is non-degenerate (i.e. $y_{i0} > 0,\ i \in I$). We seek a pivot that produces a new BFS which is

$$\boxed{\text{FEASIBLE and } \beta'_0 < \beta_0} \qquad (15)$$

<u>Theorem</u> <u>4</u>

Assuming non-degeneracy, (15) holds iff (14) hold.

Proof

For feasibility we must have

$$\text{(i) } y'_{\ell 0} = y_{\ell 0}/y_{\ell k} \geq 0$$

$y_{\ell 0} \geq 0$ from BFS; thus (i) is true iff $y_{\ell k} > 0$

$$\text{(ii) } y'_{i0} = y_{i0} - y_{ik}\left(y_{\ell 0}/y_{\ell k}\right) \geq 0$$

and (ii) holds trivially if $y_{ik} \leq 0$ as then

$$y'_{i0} \geq y_{i0} > 0.$$

For $y_{ik} > 0$, (ii) holds iff

$$y_{i0} - y_{ik}\left(y_{\ell 0}/y_{\ell k}\right) \geq 0 \quad \Rightarrow \quad y_{\ell 0}/y_{\ell k} \leq y_{i0}/y_{ik}$$

which justifies (14, b).

To obtain

$$\beta'_0 = \beta_0 - \beta_k\left(y_{\ell 0}/y_{\ell k}\right) < \beta_0,$$

(this is an application of the pivot equation to the equation $x_0 + \beta^T x = \beta_0$; specifically, we are evaluating the new value of β_0) we must have

$$\beta_k\left(y_{\ell 0}/y_{\ell k}\right) > 0.$$

Since $y_{\ell 0}, y_{\ell k} > 0$, this is possible if and only if (14, a) holds. ☐

We have so far only considered minimization problems. *A **maximization problem can be dealt with** by noting that*

$$\mathbf{max}\ \mathbf{x_0} = -\ \mathbf{min}\ (-\ \mathbf{x_0})$$

or by looking for positive reduced costs rather than negative reduced costs in the simplex method.

In general there will be more than one non-basic variable with $\beta_k > 0$. One reasonable policy used to choose k is

$$\beta_k = \max_j \left\{ \beta_j \right\}.$$

i.e. choose the variable which produces the greatest decrease in x_0 per unit increase in the variable.

The Simplex Algorithm (for minimization problems)

Step 0: Find an initial basic feasible solution and construct its basic representation.

Step 1: If $\beta_k \leq 0$ for $k \notin I$ stop, the current basis is optimal. Else

Step 2: If ∃ k such that $\beta_k > 0$ and $y_{ik} \leq 0$ for $i \in I$, stop. There is no finite minimum. Else

Step3: Choose x_k such that $\beta_k > 0$ (entry criterion) − x_k enters basis

Step 4: Let $y_{\ell 0}/y_{\ell k} = \min_{i \in I} \left\{ y_{i0}/y_{ik} \mid y_{ik} > 0 \right\}$ (Exit criterion) − x_ℓ leaves basis.

Step 5: Pivot on $y_{\ell k}$ and go to step 1.

The procedures Step 1 - 5 define what is called a **Simplex iteration.**

Iterations can be effectively carried out using a Simplex tableau (Extended)

			Basic		Non-Basic			
Basic Variables	x_1	...	x_i	...	x_j	...	x_n	R.H.S.
x_0	0		0		β_j		β_n	β_0
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_i	0		1		y_{ij}		y_{in}	y_{i0}
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$

Example 20:

Minimise $x_0 = -4x_1 - 2x_2 + x_3$

Subject to
$$x_1 + x_2 + x_3 \leq 4$$
$$x_1 - x_2 - 2x_2 \leq 3$$
$$3x_1 + 2x_2 + x_3 \leq 12$$
$$x_1, x_2, x_3 \geq 0$$

Adding slack variables $\mathbf{x_4, x_5, x_6}$, the constraints become

$$\begin{aligned} x_1 + x_2 + x_3 + \mathbf{x_4} \qquad\qquad &= 4 \\ x_1 - x_2 - 2x_3 \qquad + \mathbf{x_5} \qquad &= 3 \\ 3x_1 + 2x_2 + x_3 \qquad\qquad + \mathbf{x_6} &= 12 \\ x_1, \ldots, x_6 \geq 0 & \end{aligned}$$

We thus have an initial (all slack) basic feasible solution with basic variables x_4, x_5, x_6.

Basic Variables	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0	4	2	-1				0
x_4	1	1	1	1			4
x_5	$1*$	-1	-2		1		3
x_6	3	2	1			1	12
x_0		6	7		-4		-12
x_4		2	$3*$	1	-1		1
x_1	1	-1	-2		1		3
x_6		5	7		-3	1	3
x_0		$\frac{4}{3}$		$-\frac{7}{3}$	$-\frac{5}{3}$		$-\frac{43}{3}$
x_3		$\frac{2}{3}*$	1	$\frac{1}{3}$	$-\frac{1}{3}$		$\frac{1}{3}$
x_1	1	$\frac{1}{3}$		$\frac{2}{3}$	$\frac{1}{3}$		$\frac{11}{3}$
x_6		$\frac{1}{3}$		$-\frac{7}{3}$	$-\frac{2}{3}$	1	$\frac{2}{3}$
x_0			-2	-3	-1		-15
x_2		1	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$		$\frac{1}{2}$
x_1	1		$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{7}{2}$
x_6			$-\frac{1}{2}$	$-\frac{5}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$

Finiteness of the Algorithm:

Theorem 5

If all basic solutions are non-degenerate then the simplex algorithm described above **must terminate after a finite number of steps** with an optimal solution or with proof that no finite optimum exists.

Proof

Since no basis is degenerate, $y_{\ell 0} > 0$ at each step and hence $\beta'_0 < \beta_0$ at each step (remember Theorem 4) i.e. the sequence of objective values obtained by the algorithm is a strictly monotonically decreasing ($\beta''_0 < \beta'_0 < \beta_0$). Therefore, no basic solution can be repeated.

Since there are a finite number of basic solutions the process cannot continue indefinitely and so must terminate at step 1 or step 2 after a finite number of iterations. ☐

Theorem 6

In the absence of degeneracy a necessary condition for a basis to be minimal is that $\beta_j \leq 0$.

Proof

(Same as Theorem 2). If $\beta_k > 0$ for some k, then either there is no finite minimum or by pivoting on $y_{\ell k}$ defined in step 4, we can strictly reduce the value of the objective function. ☐

6. DEGENERACY

We have discussed the simplex algorithm under assumptions of non-degeneracy. We can say that a basic solution is degenerate if it has more than n-m zero valued components.

Lemma

A basic solution x to the LP problem is associated with more than one index set iff it is degenerate (under a 'mild' assumption).

Proof

Suppose first that x can be obtained from I_1 and I_2 where $I_1 \neq I_2$. Then $x_j = 0$ for $j \in (S_n - I_1) \cup (S_n - I_2) = S_n - I_1 \cap I_2$. As $I_1 \neq I_2$, $| I_1 \cap I_2 | < m$ and so x is degenerate. (Note, for example, this means $y_{i0} = 0$ for $i \in I_1 - I_2$ using the representation defined by I_1.)

Suppose now that x is a degenerate basic solution. Let I be an index set which produces x and suppose for some $\ell \in I$, $y_{\ell 0} = 0$.

`Mild' assumption: for each $j \in S_n$ there is a solution to LP with $x_j \neq 0$.

Under this assumption, $\exists\, k \notin I$ such that $y_{\ell k} \neq 0$ otherwise the equation $x_\ell = 0$ is part of the representation. If we pivot on (ℓ,k) the new basic solution is identical to the current one - substitute $y_{\ell 0} = 0$ into (11) with $j = 0$. Thus x can be produced from index set I and $I + k - \ell$. ☐

How does degeneracy affect the simplex algorithm?

We have seen that if pivot (ℓ,k) satisfies $y_{\ell 0} = 0$ then the new basic solution obtained is identical to the old one. In particular, $\beta'_0 = \beta_0$ and the proof of the Theorem (finite termination) breaks down.

Let us call a pivot (ℓ,k) degenerate if $y_{\ell 0} = 0$ and non-degenerate otherwise. An instance of simplex algorithm can now be decomposed into:

$$\begin{bmatrix}\text{sequence of}\\ \text{degenerate}\\ \text{pivots}\end{bmatrix} \begin{matrix}\text{non-degenerate}\\ \text{pivot}\end{matrix} \begin{bmatrix}\text{sequence of}\\ \text{degenerate}\\ \text{pivots}\end{bmatrix} \begin{matrix}\text{non-degenerate}\\ \text{pivot}\end{matrix} \quad .....$$

Note that some or all of these sequences of degenerate pivots may be empty.

Geometrically speaking, the current BFS remains unchanged throughout a sequence of degenerate pivots and then a non-degenerate pivot 'moves us' to a neighbouring BFS.

We know that the number of non-degenerate pivots is $\leq \binom{n}{m}$. However, suppose that $I_1, I_2, \ldots I_k, \ldots$ denotes a sequence of basic index sets produced during some sequence of degenerate pivots. Suppose that $I_k = I_{k+\ell}$ for $\ell \geq 3$ (ℓ cannot be 1 or 2) then, assuming a given index set determines a unique pivot we will have

$$I_k = I_{k+\ell} = I_{k+2\ell} = \ldots.$$

$$I_{k+1} = I_{k+\ell+1} = I_{k+2\ell+1} = \ldots$$

$$I_{k+2} = I_{k+\ell+2} = I_{k+2\ell+2} = \cdots$$

and so the algorithm will cycle and never terminate.

Example 21:

$$\min \quad x_0 = -\tfrac{3}{4}x_4 + 20x_5 - \tfrac{1}{2}x_6 + 6x_7$$

$$\text{subject to} \quad x_1 + \tfrac{1}{4}x_4 - 8x_5 - x_6 + 9x_7 = 0$$

$$x_2 + \tfrac{1}{2}x_4 - 12x_5 - \tfrac{1}{2}x_6 + 3x_7 = 0$$

$$x_3 + x_6 = 1$$

We have the following sequence of tableaus choosing $\beta_k = \max(\beta_j)$ and choose first ℓ that minimizes the ratio y_{i0}/y_{ik} for $y_{ik} > 0$.

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_0	0	0	0	$\frac{3}{4}$	-20	$\frac{1}{2}$	-6	0
x_1	1	0	0	$\frac{1}{4}^*$	-8	-1	9	0
x_2	0	1	0	$\frac{1}{2}$	-12	$-\frac{1}{2}$	3	0
x_3	0	0	1	0	0	1	0	1

x_0	-3	0	0	0	4	$\frac{7}{2}$	-33	0
x_4	4	0	0	1	-32	-4	36	0
x_2	-2	1	0	0	4^*	$\frac{3}{2}$	-15	0
x_3	0	0	1	0	0	1	0	1

x_0	-1	-1	0	0	0	2	-18	0
x_4	-12	8	0	1	0	8^*	-84	0
x_5	$-\frac{1}{2}$	$\frac{1}{4}$	0	0	1	$\frac{3}{8}$	$-\frac{15}{4}$	0
x_3	0	0	1	0	0	1	0	1

x_0	2	-3	0	$-\frac{1}{4}$	0	0	3	0
x_6	$-\frac{3}{2}$	1	0	$\frac{1}{8}$	0	1	$-\frac{21}{2}$	0
x_5	$\frac{1}{16}$	$-\frac{1}{8}$	0	$\frac{3}{64}$	1	0	$\frac{3}{16}^*$	0
x_3	$\frac{3}{2}$	-1	1	$-\frac{1}{8}$	0	0	$\frac{21}{2}$	1

x_0	1	-1	0	$\frac{1}{2}$	-16	0	0	0
x_6	2^*	-6	0	$-\frac{5}{2}$	56	1	0	0
x_7	$\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{4}$	$\frac{16}{3}$	0	1	0
x_3	-2	6	1	$\frac{5}{2}$	-56	0	0	1

x_0	0	2	0	$\frac{7}{4}$	-44	$-\frac{1}{2}$	0	0
x_1	1	-3	0	$-\frac{5}{4}$	20	$\frac{1}{2}$	0	0
x_7	0	$\frac{1}{3}^*$	0	$\frac{1}{6}$	-4	$-\frac{1}{6}$	1	0
x_3	0	0	1	0	0	1	0	1

x_0	0	0	0	$\frac{3}{4}$	-20	$\frac{1}{2}$	-6	0
x_1	1	0	0	$\frac{1}{4}$	-8	-1	9	0
x_2	0	1	0	$\frac{1}{2}^*$	-12	$-\frac{1}{2}$	3	0
x_3	0	0	1	0	0	1	0	1

We can avoid the possibility of cycling by tightening pivot choice rule. There are several possibilities. We give one of the simplest, prove its validity and then discuss whether in practice any such rule is necessary!

Bland's Rule

(i) Pivot column choice: $k = \min \{ j \neq 0 \mid \beta_j > 0 \}$

(ii) Pivot row choice: Let $\rho = \min \{ y_{i0}/y_{ik} \mid y_{ik} > 0 \}$, $\ell = \min \{ i \mid y_{i0}/y_{ik} = \rho$ and $y_{ik} > 0 \}$

Theorem 7 (Degeneracy)

With Bland's rule the simplex algorithm cannot cycle and hence is finite.

Degeneracy in Practice

Until recently, cycling only occurred in contrived examples (as the one given above). It has therefore been the practice to ignore it in commercial codes.

More recent experience with larger and larger problems indicates that cycling is now considered a rare possibility. Rigorous methods such as Bland's rule are not satisfactory in practice as they increase in practice the number of (or work per) iterations in the vast majority of problems which would not cycle anyway.

It has also been suggested that it is perfectly satisfactory to replace $y_{i0} = 0$ by $y_{i0} = \epsilon > 0$ ($\epsilon = 10^{-2}$ or 10^{-3}) and then continue.

7. SHADOW PRICES (SP)

Shadow prices are important accounting prices in decision making and in sensitivity analysis. Suppose that we have solved problem

$$\min \left\{ x_0 = c^T x \,\middle|\, A x = b ,\ x \geq 0 \right\}$$

and found an optimal basis matrix B

$$x_B = B^{-1} b \geq 0 \ \text{(The basis is feasible)} \tag{16, a}$$

$$r = c_N - N^T B^{-T} c_B \geq 0 , \ \text{(All reduced costs are non-negative)} \tag{16, b}$$

The shadow prices Π for this problem are defined by $\Pi = B^{-T} c_B$ (or $\Pi^T = c_B^T B^{-1}$) (If there is more than one optimal basis there may be more than one set of SP.)

These 'prices' give information about the objective value if we alter the RHS of the constraints. Let $p \in \mathbb{R}^m$ denote a general RHS and define the perturbation function $v(p): \mathbb{R}^m \to \mathbb{R}$ by

$$v(p) = \min \left\{ c^T x \,\middle|\, A x = p ; x \geq 0 \right\} \tag{17}$$

Thus, solving $\min \{ x_0 = c^T x \mid A x = b ,\ x \geq 0\}$ computes $v(b)$ (to be rigorous $v: \mathbb{R}^m \to \mathbb{R} \cup \{ -\infty, +\infty\}$ using $-\infty$ for unbounded problems and $+\infty$ for infeasible problems.)

Theorem 8

If $B^{-1} p \geq 0$ then $v(p) = v(b) + \Pi^T (p - b)$

Proof

If $B^{-1} p \geq 0$ then B is an optimal basis for (17) as (16, b) is not affected by changing b to p. Thus,

$$\begin{aligned} v(p) &= c_B^T B^{-1} p \qquad \left(v(p) = x_0(p) = c_B^T B^{-1} p + r^T x_N = c_B^T B^{-1} p\right) \\ &= c_B^T B^{-1} b + c_B^T B^{-1} (p - b) \\ &= v(b) + \Pi^T (p - b) \end{aligned}$$

□

This is a local result i.e. p must not differ *substantially* from b *so* $B^{-1} p \geq 0$ *is maintained*. We also have the following global result.

Theorem 9

$$v(p) \geq v(b) + \Pi^T (p - b); \quad \forall p \in \mathbb{R}^m$$

Proof

$$v(p) = \min_{x \geq 0;\, A x = p} \left\{ c^T x - \Pi^T \left(A x - p \right) \right\}$$

$$\geq \min_{x \geq 0} \left\{ c^T x - \Pi^T \left(A x - p \right) \right\}$$

$$= \min_{x \geq 0} \left\{ \left(c^T - \Pi^T A \right) x + \Pi^T p \right\}$$

$$\geq \Pi^T p$$

As

$$[c^T - \Pi^T A]x = [[c_B^T \vdots c_N^T] - c_B^T B^{-1} [B \vdots N]] \begin{bmatrix} x_B \\ x_N \end{bmatrix}$$

$$= [c_B^T \vdots c_N^T] \begin{bmatrix} x_B \\ x_N \end{bmatrix} - c_B^T [I \vdots B^{-1} N] \begin{bmatrix} x_B \\ x_N \end{bmatrix}$$

$$= c_B^T x_B - c_B^T x_B + [c_N^T - c_B^T B^{-1} N] x_N$$

$$= r^T x_N \quad (r \geq 0, x_N \geq 0)$$

$$\geq 0$$

$$\Pi^T p = \Pi^T b + \Pi^T (p - b) = c_B^T B^{-1} b + \Pi^T (p - b) = v (b) + \Pi^T (p - b). \qquad \square$$

Why Π is called the vector of SP?

Suppose $b_1, \dots , b_m$ represent demands for certain products and $c_1, \dots , c_n$ are the costs of certain activities which produce these products. Suppose there is an increase in demand of ξ for product t i.e. $b_t := b_t + \xi$ and suppose that a small firm offers to produce the extra demand at price μ_t. Should one accept or decide to produce more oneself?

$$\left[\text{Note: } p = b + \xi e_t, \text{ where } e_t = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow t \right]$$

Accept the offer ⇒ total production cost = $v (b) + \mu_t \xi$

$$\text{Produce extra} \Rightarrow \text{total production cost :} \begin{cases} = v (b) + \Pi_t \xi & \text{if } B^{-1} (b + \xi e_t) \geq 0 \\ \geq v (b) + \Pi_t \xi, & \text{in general} \end{cases}$$

Thus, if $\mu_t < \Pi_t$ one should definitely accept the offer. If $\mu_t > \Pi_t$ and if $B^{-1} (b + \xi e_t) \geq 0$ one should definitely reject the offer. In this case Π_t is the maximum price one should pay.

Maximization Problems

For maximization problems, Theorem 8 is unchanged and the inequality is reversed in the statement of Theorem 9.

Evaluation of Shadow Prices

In certain circumstances the shadow prices for a particular row can be read off from the final tableau. Suppose that row t was initially a $\leq$ constraint and a slack variable x_s was added. The objective row coefficient β_s for this variable in the final tableau is given by

$$\beta_s = -c_s + \Pi^T a_s = 0 + \Pi^T e_t = \Pi_t$$

Therefore Π_t can be read off. If x_s is the slack variable for a $\geq$ constraint then we get $\beta_s = -\Pi_t$.

Example 22:

Consider Example 20 reading off the final top row coefficients we get

$$\Pi = \begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix}$$

Note that in this case $\Pi \leq 0$ which makes sense. If the R.H. sides <u>increase</u> to $(4+\xi_1, 3+\xi_2, 12+\xi_3)$ then the minimum obtainable objective function value will <u>decrease</u> to $-15 + \Pi_1 \xi_1 + \Pi_2 \xi_2 + \Pi_3 \xi_3 = -15 - 3 \xi_1 - \xi_2$ (for `small' positive ξ_1, ξ_2, ξ_3).

8. INITIAL BASIC FEASIBLE SOLUTION : The two-phase SIMPLEX

If no BFS is known for the problem one can create one by adding **artificial variables**. (Previously, in Section 4, we constructed a feasible all slack basis. Alternatively, one may know a BFS because a similar problem has been solved previously. We consider below, the situation when neither is possible).

Suppose the constraints are

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\geq 6 \\ 2x_1 - x_2 - x_3 &= 4 \\ x_1 + 2x_2 - x_3 &\leq 3 \\ x_i \geq 0, i = 1, ..., 3. \end{aligned}$$

After adding slack variables x_4, x_5 we add **artificials ξ_1, ξ_2** to give

$$\begin{aligned} x_1 + 2x_2 + 3x_3 - x_4 + \xi_1 &= 6 \\ 2x_1 - x_2 - x_3 + \xi_2 &= 4 \\ x_1 + 2x_2 - x_3 + x_5 &= 3 \\ x_i \geq 0, i = 1, ..., 5 ; \ \xi_1, \xi_2 \geq 0. \end{aligned}$$

In general, we first add slack variables to obtain equations and then ensure that RHS's b_i are non-negative by multiplying through by -1 where necessary.

The aim next is to construct an enlarged system of equalities which is itself a basic representation. Some rows will contain slack variables. Other rows will contain an artificial and some other rows will contain both artificial and slack variables. This produces a basic representation whose BFS consists of artificials and slacks (in rows which do not have an artificial).

In general one needs an artificial variable for each equality constraint and one for each inequality $\geq b_i$ where $b_i > 0$. The basic solution constructed will be feasible as we have ensured non-negative RHS's

After adding slacks and artificials, we have the augmented system:

$$x_0 - c^T x = 0, \quad Ax + I_m \xi = b \tag{18}$$

$$\xi^T = [\ldots..\overset{\substack{\text{feasible}\\ \downarrow}}{0}..\overset{\substack{\text{infeasible}\\ \downarrow}}{\xi_p} \ldots]^T$$

Clearly, a solution (x^*, ξ^*) to (18) gives a solution to min $\left\{ x_0 = c^T x \;\middle|\; A\,x = b,\, x \geq 0 \right\}$ iff $\xi^* = 0$.

We note that by construction a BFS to (18) is known. The problem of finding a BFS to min $\{ x_0 = c^T x \mid A\,x = b,\, x \geq 0 \}$ has now been replaced by that of finding a BFS to (18) with $\xi = 0$. To do this we solve the linear programming problem

$$\min \zeta = \boldsymbol{\xi_1} + \boldsymbol{\xi_2} + \boldsymbol{\ldots} + \boldsymbol{\xi_a} \qquad (19)$$

$$\text{S.T. } x_0 - c^T x = 0, \qquad Ax + \mathbf{I_m}\,\boldsymbol{\xi} = b, \qquad x, \xi \geq 0$$

If a feasible solution to min $\{ x_0 = c^T x \mid A\,x = b,\, x \geq 0 \}$ exists then the minimum value of ζ is zero with $\xi_1 = \ldots = \xi_a = 0$. We can apply the simplex method directly to (19) since by construction we have an initial BFS to the problem. If having solved (19) we find $\xi = 0$ then current values of $x_0, x_1, \ldots x_n$ will constitute a BFS to min $\left\{ x_0 = c^T x \;\middle|\; A\,x = b,\; x \geq 0 \right\}$.

Infeasiblity ζ can be expressed in terms of the initial non-basic variables in the following way. Suppose that the row containing artificial ξ_i is

$$\xi_i + \sum_{j=1}^{n} a_{ij}\, x_j = b_i$$

adding the infeasible rows we obtain

$$\zeta + \sum_{i \in P} (\Sigma\, x_j\, a_{ij}) = \sum_{i \in P} b_i \qquad (20)$$

where P is the set of indices of infeasible rows (i.e. those with an artificial variable). (20) expresses ζ in terms of the non-basic x_j. The coefficients of x_j being given by the sum of the coefficients of x_j in the infeasible rows. (Note that the basic x_i (the slack variables for the equations that do not need an artificial) do not exist in the rows in which an artificial occurs. This is why (20) expresses ζ in terms of the non-basic x_j.)

Example 23:

$$\max\; x_0 = 3x_1 + x_2 - x_3$$

$$\begin{array}{ll} \text{S.T.} & x_1 + x_2 + x_3 = 10 \\ & 2x_1 - x_2 \geq 2 \\ & x_1 - 2x_2 + x_3 \leq 6; \qquad x_i \geq 0, i = 1, \ldots, 3. \end{array}$$

adding slacks and artificials where necessary, the constraints become

$$\begin{array}{rl} x_0 - 3x_1 - x_2 + x_3 & = 0 \\ x_1 + x_2 + x_3 \qquad + \xi_1 & = 10 \\ 2x_1 - x_2 \qquad - x_4 \quad + \xi_2 & = 2 \\ x_1 - 2x_2 + x_3 \qquad\qquad + x_5 & = 6 \end{array}$$

$$x_i \geq 0, i = 1, \ldots, 5\;;\; \xi_1, \xi_2 \geq 0.$$

Note that artificial columns are ignored after the corresponding variables is made non-basic.

Basic Variables	x_1	x_2	x_3	x_4	x_5	ξ_1	ξ_2	RHS

ζ	3		1	-1				12
x_0	-3	-1	1					0
ξ_1	1	1	1			1		10
ξ_2	$\boxed{2}$	-1		-1			1	2
x_5	1	-2	1		1			6

ζ		$1\frac{1}{2}$	1	$\frac{1}{2}$				9
x_0		$-2\frac{1}{2}$	1	$-1\frac{1}{2}$				3
ξ_1		$\boxed{1\frac{1}{2}}$	1	$\frac{1}{2}$		1		9
x_1	1	$-\frac{1}{2}$		$-\frac{1}{2}$				1
x_5		$-1\frac{1}{2}$	1	$\frac{1}{2}$	1			5

ζ								0
x_0			$\frac{8}{3}$	$-\frac{2}{3}$				18
x_2		1	$\frac{2}{3}$	$\frac{1}{3}$				6
x_1	1		$\frac{1}{3}$	$-\frac{1}{3}$				4
x_5			2	$\boxed{1}$	1			14

---

x_0			4		$\frac{2}{3}$			$\frac{82}{3}$
x_2		1			$-\frac{1}{3}$			$\frac{41}{3}$
x_1	1		1		$\frac{1}{3}$			$\frac{26}{3}$
x_4			2	1	1			14

Description of the Two-Phase Method

Phase 1

Step 1: Modify the constraints so that the RHS of each constraint is non-negative. This requires that each constraint with negative RHS be multiplied through by (-1).

Step 1′: Identify each constraint that is now (after Step 1) an equality or $\geq$ constraint. In Step 3 we shall add an artificial variable to such constraints.

Step 2: Convert each inequality constraint to standard form. If i is a $\leq$ constraint, add a slack variable. If constraint i is a $\geq$ constraint, subtract an excess variable.

Step 3: If (after Step 1′) constraint i is a $\geq$ or an equality constraint, add an artificial variable ξ_i to constraint i.

Step 4: Find the minimum value of ζ using the simplex algorithm. Each excess and artificial variable is restricted to be ≥ 0.

Phase 1 ends when ζ has been minimized. This phase will result in the following three cases which are dealt with in Phase 2.

Phase 2

Case 1: $\zeta^* > 0$. The original LP problem has no feasible solution.

Case 2: The optimal value $\zeta^* = 0$ and no artificial variables are in the optimal Phase 1 basis. (The final phase 1 basis contains no basic artificial variables at zero value.) $\Rightarrow$ Drop all columns in the optimal Phase 1 tableau that correspond to the artificial variables. We now combine the original objective function with the constraints from Phase 1 tableau. The final basis of Phase 1 is the initial basis of the Phase 2 LP. The optimal solution to the Phase 2 LP is the optimal solution to the original LP problem.

Case 3 The optimal value $\zeta^* = 0$ and at least one artificial variable (at zero value) is in the optimal Phase 1 basis. (When this occurs, it indicates that the original LP had at least one redundant constraint.) Again, continue by optimizing x_0 but have to ensure that no artificial variables becomes non-zero again. We note first that we will not make an artificial variable non-zero by allowing it to enter the basis once it becomes non-basic. The problem occurs when x_k is to enter the basis and $y_{\ell k} < 0$ where $y_{\ell k}$ is the coefficient in column k for some row ℓ with an artificial ξ in it. But as $y_{\ell 0} = 0$ (the coefficient of the RHS of row ℓ), necessarily we can pivot on $y_{\ell k}$ (an unusual pivot). The basic solution does not change but the artificial is pivoted out of the basis. One thus applies the normal simplex criteria for choice of pivot except that the above case $y_{\ell k} < 0$ causes a "non-standard" pivot selection.

We can see now how we have sidestepped the earlier assumption that the rows the A matrix were linearly independent - we have ensured this by adding artificial variables.

If the rows of the original A matrix are in fact linearly dependent then even when a feasible solution is found there will be artificial basic variables (at zero value, of course).

We briefly discuss why $\zeta^* > 0 \Rightarrow$ the original LP has no feasible solution and $\zeta^* = 0 \Rightarrow$ the original LP has at least one feasible solution.

Suppose the original LP is infeasible. Then, the only way to obtain a feasible solution to the Phase 1 LP is to let at least one artificial variable to be positive $\Rightarrow \zeta^* > 0 \Rightarrow$ Case 1.

If the original LP has a feasible solution, this feasible solution (with all $\xi_i = 0$) is feasible in the Phase 1 LP and leads to $\zeta^* = 0 \Rightarrow$ if the original LP has a feasible solution, the optimal Phase 1 solution will have $\zeta^* =$ 0.

Example 24: (Case 2)

$$
\begin{aligned}
\min\ x_0 = {} & 2x_1 + 3x_2 \\
& \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 \le 4 \\
& x_1 + 3x_2 \ge 20 \\
& x_1 + x_2 = 10 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

Steps 1 − 3 transform the constraints into

$$
\begin{array}{rrrrrrrrrrl}
\tfrac{1}{2}x_1 & + & \tfrac{1}{4}x_2 & + & s_1 & & & & & = & 4 \\
x_1 & + & 3x_2 & & & - & e_2 & +\xi_2 & & = & 20 \\
x_1 & + & x_2 & & & & & & +\ \xi_3 & = & 10
\end{array}
$$

(s = slack, e = excess, ξ = artificial)

Step 4 yields the Phase 1 LP

$$\min \zeta = \xi_2 + \xi_3$$

subject to:

$$
\begin{array}{rrrrrrrrrrl}
\tfrac{1}{2}x_1 & + & \tfrac{1}{4}x_2 & + & s_1 & & & & & = & 4 \\
x_1 & + & 3x_2 & & & - & e_2 & +\xi_2 & & = & 20 \\
x_1 & + & x_2 & & & & & & +\ \xi_3 & = & 10
\end{array}
$$

Initial BFS for Phase 1: $s_1 = 4$, $\xi_2 = 20$, $\xi_3 = 10$. ξ_2 and ξ_3 must be eliminated from the objective function ζ before solving Phase 1:

$$
\begin{array}{llrrrrrrl}
 & \text{Row } 0 & \zeta & & & & -\ \xi_2 & -\ \xi_3 & = 0 \\
+ & \text{Row } 2 & & x_1 & +\ 3x_2 & -\ e_2 & +\ \xi_2 & & = 20 \\
+ & \text{Row } 3 & & x_1 & +\ x_2 & & & +\ \xi_3 & = 10 \\
= & \text{New Row } 0 & \zeta & +2x_1 & +4x_2 & -\ e_2 & & & = 30
\end{array}
$$

Combining new row 0 with the Phase 1 constraints yields the initial Phase 1 tableau. Since Phase 1 is always a minimisation problem (even if the original LP is a maximisation problem) we enter x_2 into the basis. The ratio test indicates that x_2 will enter the basis in row 2. Then ξ_2 will exit from the basis.

In the second tableau, since $2/3 > 1/3$, x_1 enters the basis. The ratio test indicates that ξ_3 should leave the basis. Since ξ_2 and ξ_3 will be nonbasic in the next tableau, we know that the third tableau is optimal Phase 1.

BV	x_1	x_2	s_1	e_2	ξ_2	ξ_3	RHS	ratio
ζ	2	4		-1			30	
s_1	1/2	1/4	1				4	16
ξ_2	1	$\boxed{3}$		-1	1		20	20/3
ξ_3	1	1				1	10	10
ζ	2/3			1/3	-4/3		10/3	
s_1	5/12		1	1/12	-1/12		7/3	28/5
x_2	1/3	1		-1/3	1/3		20/3	20
ξ_3	$\boxed{2/3}$			1/3	-1/3	1	10/3	5
ζ					-1	-1	0	
s_1			1	-1/8	1/8	5/8	1/4	
x_2		1		-1/2	1/2	-1/2	5	
x_1	1			1/2	-1/2	3/2	5	

$\zeta^* = 0 \Rightarrow$ Phase 1 concluded. BFS: $s_1 = 1/4$, $x_2 = 5$, $x_1 = 5$. No artificial variables in the basis: this is an example for case 2. We now drop the columns of the artificial variables ξ_2, ξ_3 (we no longer need them) and reintroduce the original objective function:

$$\min x_0 = 2x_1 + 3x_2 \quad \text{or} \quad x_0 - 2x_1 - 3x_2 = 0.$$

Since x_1 and x_2 are in the optimal Phase 1 basis, they must be eliminated from Phase 2, row zero (i.e. the objective function x_0). This is normally done implicitly, or automatically, as Phase 1 progresses, as in Example 23. The purpose of this explicit illustration is to highlight the underlying mechanics of the process.

$$\begin{array}{llcl}
\text{Phase 2 Row 0:} & x_0 - 2x_1 - 3x_2 & = & 0 \\
+ 3 \times \text{(Row 2):} & 3x_2 - \frac{3}{2} e_2 & = & 15 \\
+ 2 \times \text{(Row 3):} & 2x_1 + e_2 & = & 10 \\
= \text{New Phase 2 Row 0:} & x_0 - \frac{1}{2} e_2 & = & 25
\end{array}$$

We now begin Phase 2 with the following:

$$\begin{array}{rcl}
\min \; x_0 - \frac{1}{2} e_2 & = & 25 \\
s_1 - \frac{1}{8} e_2 & = & \frac{1}{4} \\
x_2 - \frac{1}{2} e_2 & = & 5 \\
x_1 + \frac{1}{2} e_2 & = & 5
\end{array}$$

This is optimal. In this problem, Phase 2 requires no further pivots. If Phase 2 row 0 does not indicate an optimal tableau, simply continue with the simplex algorithm until an optimal row 0 (i.e. objective function) is obtained.

Example 25: (Case 1)

$$\begin{array}{lrl} \min\ x_0 = & 2x_1 + 3x_2 & \\ & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 & \le 4 \\ & x_1 + 3x_2 & \ge 36 \\ & x_1 + x_2 & = 10 \\ & x_1, x_2 & \ge 0 \end{array}$$

After Steps 1 - 4,

$\min \zeta = \xi_2 + \xi_3$

subject to:

$$\begin{array}{rcrcrcrcrcrcl} \tfrac{1}{2}x_1 & + & \tfrac{1}{4}x_2 & + & s_1 & & & & & & & = & 4 \\ x_1 & + & 3x_2 & & & - & e_2 & + & \xi_2 & & & = & 36 \\ x_1 & + & x_2 & & & & & & & + & \xi_3 & = & 10 \end{array}$$

Initial BFS for Phase 1: $s_1 = 4$, $\xi_2 = 36$, $\xi_3 = 10$. Again, ξ_2 and ξ_3 must be eliminated from the objective function ζ before solving Phase 1:

$$\begin{array}{lllcl} & \text{Row 0} & \zeta & - \xi_2 - \xi_3 & = 0 \\ + & \text{Row 2} & x_1 + 3x_2 - e_2 + \xi_2 & & = 36 \\ + & \text{Row 3} & x_1 + x_2 & + \xi_3 & = 10 \\ = & \text{New Row 0} & \zeta + 2x_1 + 4x_2 - e_2 & & = 46 \end{array}$$

Since $4 > 2$, x_2 enters the basis and replaces ξ_3. In the second tableau, no variable in Row 0 has a positive coefficient: optimal Phase 1 tableau with $\zeta^* = 6 > 0 \Rightarrow$ no feasible solution to this problem.

BV	x_1	x_2	s_1	e_2	ξ_2	ξ_3	RHS	ratio
ζ	2	4		-1			46	
s_1	1/2	1/4	1				4	16
ξ_2	1	3		-1	1		36	12
ξ_3	1	$\boxed{1}$				1	10	10
ζ	-2			-1		-4	6	
s_1	1/4		1			-1/4	3/2	
ξ_3	-2			-1	1	-3	6	
x_2	1	1				1	10	

9. EXTENSIONS OF LP

Some optimization problems can be converted to an LP, a sequence of LP's or be solved by modifying the simplex algorithm.

EXTENSION 1: Min-Max with LP

Let $c^{(1)}, \dots, c^{(p)} \in \mathbb{R}^n$ and let $\phi(x) = \max\limits_{t=1,\dots,p} \left\{ (c^{(t)})^T x \right\}$

The min-max problem

$$\min \left\{ \phi(x) \,\middle|\, Ax = b;\ x \ge 0 \right\} \tag{21}$$

can be converted to the LP

$$\min \left\{ x_0 \,\middle|\, x_0 - (c^{(t)})^T x \ge 0,\ t = 1,\dots,p;\ Ax = b;\ x \ge 0 \right\} \tag{22}$$

Theorem 10

If (x_0^*, x^*) solve (22) then x^* solves (21) and $x_0^* = \phi(x^*)$.

Proof

If x is a feasible solution (21) then $(\phi(x), x)$ is a feasible solution to (22) (since x satisfies $Ax = b$, $x \geq 0$ and $\phi(x) - (c^{(t)})^T x \geq 0, t = 1,..., p$.)

Thus, $x_0^* \leq \phi(x)$ which, in particular, implies that $x_0^* \leq \phi(x^*)$. But as $x_0^* \geq (c^{(t)})^T x^*$ for $t = 1, ..., p$ in (22), we have $x_0^* \geq \phi(x^*)$ and hence $x_0^* = \phi(x^*)$.

It then follows that $x_0^* = \phi(x^*)$ and $\phi(x) \geq x_0^* = \phi(x^*)$ for any feasible solution (21). ☐

EXTENSION 2: Min-min problems

Let $c^{(1)}, ... c^{(p)}$ be as in (1) and let

$$\psi(x) = \min_{t = 1, ..., p} \left\{ (c^{(t)})^T x \right\}.$$

We consider the problem

$$\min \left\{ \psi(x) \,\middle|\, A x = b;\ x \geq 0 \right\} \tag{23}$$

This can be tackled by solving the p LP's:

$$\min \left\{ (c^{(t)})^T x \,\middle|\, A x = b ; x \geq 0 \right\} \tag{24,t}$$

t=1, ..., p. Let $x^{(t)}$, $t = 1, ... , p$, denote an optimum solution to (24,t) and let $z^{(t)} = (c^{(t)})^T x^{(t)}$.

<u>Theorem</u> 11

If $z^{(t^*)} = \min_{t = 1,..., p} \left\{ z^{(t)} \right\}$ then $x^{(t^*)}$ is an optimal solution to (23) and $z^{(t^*)} = \psi(x^{(t^*)})$.

Proof

If x is a feasible solution to (23) then for some q

$$\begin{aligned} \psi(x) &= (c^{(q)})^T x \\ &\geq z^{(q)} \\ &\geq z^{(t^*)} \end{aligned}$$

Now for $t \neq t^*$ we have

$$\begin{aligned} (c^{(t)})^T x^{(t^*)} &\geq z^{(t)} \\ &\geq z^{(t^*)} \\ &= (c^{(t^*)})^T x^{(t^*)} \end{aligned}$$

and hence $\psi(x^{(t^*)}) = z^{(t^*)}$. ☐

EXTENSION 3: Goal Programming and Approximation Problems:

$$\text{minimise} \sum_{i=1}^{p} | (c^{(i)})^T x - b |$$

A typical function $(c^{(i)})^T x - b$ may be split into negative and positive parts by writing

$$(c^{(i)})^T x - b = x_i^+ - x_i^- \ ; \qquad x_i^+, x_i^- \geq 0.$$

The relationship

$$| (c^{(i)})^T x - b | \leq x_i^+ + x_i^-$$

leads to the formulation

$$\min_{x,\, x^+,\, x^-} \left\{ \sum_{i-1}^{p} (x_i^+ + x_i^-) \;\middle|\; (c^{(i)})^T x - b = x_i^+ - x_i^- , i = 1, ..., p \,;\, x_i^+, x_i^- \geq 0 \right\}$$

for $x \in \mathbb{R}^n, x^+, x^- \in \mathbb{R}^p$.

Example 26

$$\min \left\{ | x_1 - 2 x_2 + x_3 | \;\middle|\; 2 x_1 + 3 x_2 + 4 x_3 \geq 60; 7 x_1 + 5 x_2 + 3 x_3 \geq 105; x_1, x_2, x_3 \geq 0 \right\}$$

The LP formulation (written as a max problem)

$$\begin{array}{llllllll}
\max & & & & - & (x_1^+ + x_1^-) & & \\
 & x_1 & -2\, x_2 & + \quad x_3 & - & (x_1^+ - x_1^-) & = & 0 \\
 & 2\, x_1 & +3\, x_2 & +4\, x_3 & & & \geq & 6 \\
 & 7\, x_1 & +5\, x_2 & +3\, x_3 & & & \geq & 105 \\
 & & \multicolumn{4}{l}{x_1, x_2, x_3, x_1^+, x_1^- \geq 0} & &
\end{array}$$

EXTENSION 4: Fractional LP

$$\min \left\{ \frac{\alpha_0 + \alpha_1 x_1 + ... + \alpha_n x_n}{\beta_0 + \beta_1 x_1 + ... + \beta_n x_n} \;\middle|\; A x = b;\ x \geq 0 \right\} \tag{25}$$

Where the set $P = \{ x \mid Ax = b ,\ x \geq 0 \}$ is bounded i.e $\exists\, L > 0$ such that

$$P \subseteq \{ x \mid \| x \| \leq L \}.$$

We first make the transformation

$$x_j = \frac{y_j}{y_0} \text{ for } j = 1, ... , n$$

and assume that $y_0 \geq 0$. Problem (25) then becomes

$$\min \left\{ \frac{\alpha_0 y_0 + \alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_n y_n}{\beta_0 y_0 + \beta_1 y_1 + \beta_2 y_2 + ... + \beta_n y_n} \;\middle|\; \begin{array}{c} b_i y_0 - \sum_{j=1}^{n} a_{ij} y_j = 0, \quad i = 1 , ..., m \\ y_0 > 0, y_1 , ... , y_n \geq 0 \end{array} \right\} \tag{26}$$

Note next that if $(y_0, y_1 , ... , y_n)$ is feasible for (26) then $\lambda\, (y_0 , y_1 , ... , y_n)$ is also feasible for any $\lambda > 0$ and further has the same objective value (as $\lambda = 1$). We can thus restrict our attention to $(y_0, y_1 , ... , y_n)$ satisfying

$$\beta_0 y_0 + ... + \beta_n y_n = 1 \text{ or } - 1$$

Because given an optimal solution to (26) which does not satisfy one of these equations we can positively scale it to one that does. Thus, (26) can be solved by solving the <u>two</u> problems.

$$\min \left\{ \alpha_0 y_0 + \alpha_1 y_1 + ... + \alpha_n y_n \;\middle|\; \begin{array}{c} b_i y_0 - \sum_{j=i}^{n} a_{ij} y_j = 0, i = 1, ... , m \\ \beta_0 y_0 + \beta_1 y_1 + ... + \beta_n y_n = \delta \\ y_0 , ... y_n \geq 0 \end{array} \right\} \tag{27}$$

Where in one problem $\delta = 1$ and the other $\delta = -1$. We then choose the better of the two solutions, $(y_0^* , y_1^*, ... , y_n^*)$ and the $(y_1^*/y_0^* , ... , y_n^*/y_0^*)$ is optimal for (25). This will only be valid if we know that $y_0^* > 0$. Suppose that to the contrary $y_0^* = 0$. Then setting

$$\xi = \begin{bmatrix} y_1^* \\ \vdots \\ y_n^* \end{bmatrix},$$

we have

$$A\, \xi = 0, \xi \geq 0 \text{ and } \xi \neq 0,$$

as $\beta_1 \xi_1 + \ldots + \beta_n \xi_n = 1$ or -1. It follows that if $x \in P$ then $x + \lambda \xi \in P$ for <u>any</u> $\lambda > 0$. This contradicts the fact that P is bounded (why?).