Appendix B: Real Options Utility Based Computations for Investment Project Decision making with Risk

1 Introduction

When analysing investment project decision making under risk it is usual to take maximisation of expected profit as the management's objective. Such an approach is standard in microeconomic theory and can be defended by the CAPM (capital asset pricing model), in which unsystematic (i.e. diversifiable) risk should offer no return premium above the risk free rate of return. When the market is complete, i.e. when the cash flows of the project can be replicated by trading in market instruments, such as oil futures etc, even recourse to CAPM is not required as arbitrage argument's, such as Black-Scholes analysis of financial options, demonstrate that the value of the project's cash flows must be their expected present value (under the unique risk-free measure).

While financial markets may be approximately complete, this is not the case for real options: in general there will be no effective method of replicating the cash flows from the project. By CAPM this suggests that the project risks must be unsystematic (or else they would be correlated with the market portfolio and hence hedgable to some degree) which does lend weight to the expected net present value objective.

However it would be simplistic to assume that project managers always follow such a mandate and that they do not take the dispersion of possible project returns into consideration when deciding whether to go ahead with a project and how to allocate resources to a multiple projects. Therefore it may be helpful to take utility maximisation as an alternative management objective. Although this brings about further problems, such as how to select an appropriate utility function, it does generalise the problem and it introduces the idea of risk which we feel is pertinent. Such an approach also highlights that multiple inter-dependent projects can only be managed and valued in combination, and not in isolation (this is true even when risk-neutrality is assumed due to the non-linearity of the maximisation operator).

Thus when maximisation of expected utility is taken as the objective the problem can be formulated in discrete time as:

$$\max_{d \in D} E[U(\sum_{i=1}^{N} \frac{\sum_{j=1}^{n} \Pi_j(t_i, X(t_i), d(t_i))}{(1+r)^{t_i}})] \tag{1}$$

where d is a decision policy in a decision space D, $[t_1, .., t_N]$ are times with t_N being the terminal time, r is a constant continuously compounded discount rate, U is a utility function. There are n projects, indexed by j, and for each project there is a per-period profit function Π_j which is a function of time, of X a vector of underlying stochastic factors (such as output prices, input costs etc) and of the decision policy d. To solve this we need to assume a model for the dynamics of the stochastic factors. A general Markov discrete time stochastic model has the form:

$$X(t_{i+1}) = f(X(t_i), \epsilon_{i+1}) \tag{2}$$

where f can be a linear or non-linear function and $(\epsilon)_i$ is a sequence of random variables with a specified distribution.

The utility function approach is quite general in that it includes other approaches as special cases: Taking a linear (risk-neutral) utility function recovers the original expected present value optimisation, whilst taking a quadratic utility function implies the management are mean-variance optimisers. Higher order polynomial utility functions can be constructed to reflect attitides to skewness and kurtosis. Other utility functions may be chosen based on their risk aversion characteristics. For example the exponential utility function $U(x) = -\exp(-\alpha x)$ with positive α has constant absolute risk aversion, i.e. $U''(x)/U'(x) = constant$ and the power law utility function $U(x) = x^\alpha$, $x \geq 0$ with $\alpha \in (0,1)$ has constant relative risk aversion, i.e. $xU''(x)/U'(x) = constant$.

Rather than explicitly stating a utility function we can also apply worst case analysis. With this approach the management seeks to maximise the objective assuming the stochastic factors take the worst possible values. When the objective is a monotone function of unbounded stochastic factors the "worst case" can be taken to be a realistic number of standard deviations from the mean case.

We can value the projects using the certainty equivalent approach. The value V which solves:

$$U(V) = \max_{d \in D} E[U(\sum_{i=1}^{N} \frac{\sum_{j=1}^{n} \Pi_j(t_i, X(t_i), d(t_i))}{(1+r)^{t_i}})] \tag{3}$$

is the certain value which makes the management indifferent between implementing the projects and receiving V. Denoting the optimal decisions as d^* and the inverse of the utility function as U^{-1} (which we assume exists), this can be written as:

$$V = U^{-1}(E[U(\sum_{i=1}^{N} \frac{\sum_{j=1}^{n} \Pi_j(t_i, X(t_i), d^*(t_i))}{(1+r)^{t_i}})]). \tag{4}$$

2 Pharmaceutical Company Research Spending

Consider a pharmaceutical company that has developed a patented, working drug. As this company has a finite time $[0,T] = [t_0, t_N]$ within which it has an effective monopoly on this drug it can set a fixed price P for the drug. However it can continue to research methods of producing the drug more cheaply. The cost of such research for the ith time period is denoted by d_i. Thus the objective is to dynamically set the level of R and D spending so as to maximise the expected profits over this finite time from sales of the drug. Although the price level is set by the company, and we take it to be constant for simplicity, the level of demand for the drug, Q, is stochastic. We choose a simple stochastic model:

$$\begin{aligned} Q_i &= Q_{i-1}(1+\epsilon_i) \\ \epsilon_i &\sim NID(0, \Delta t) \end{aligned} \tag{5}$$

The effectiveness of the research spending is also stochastic. We model the effect of research spending on costs by the following dynamics:

$$\begin{aligned} C_i &= C_{i-1}(1-a\eta_i) \\ \eta_i &\sim Poisson(d_i) \end{aligned} \tag{6}$$

The current period cost per unit of drug produced, C_i, is equal to last period's cost, unless the random variable η_i is different from zero. η_i is a "breakthrough" variable modelling successes in research that effect reductions in production costs. The η_i are modelled as independent Poisson random variables with intensity parameter d_i, that is the rate of research breakthroughs is positively dependent on the level of research spending. We assume the sequences ϵ and η are independent. An appropriate choice of the constant a, and the time step Δt can be made according to past experience of such research spending.

Using the discrete time formulation of the general problem we have:

$$\max_d E[U(\sum_{i=1}^{i=N} \frac{Q_i(P - C_i) - d_i}{(1+r)^i})] \tag{7}$$

where r is the constant Δt compounded discount rate.

2.1 Mean-Variance Analysis

We now analyse the above problem assuming a quadratic utility function. For this we require the first two moments of the expected profit function.

The models for the quantity demanded, equation (5), and the production cost, equation (6), have the solutions:

$$\begin{aligned} Q_i &= Q_1 \Pi_{j=2}^{i}(1+\epsilon_j) \\ C_i &= C_1 \Pi_{j=2}^{i}(1 - a\eta_j) \end{aligned}$$

Moreover since $E[\epsilon_i] = 0$ and $E[\eta_i] = d_i$ we have:

$$\begin{aligned} E[Q_i] &= Q_1 \\ E[C_i] &= C_1 \Pi_{j=2}^{i}(1 - ad_j) \end{aligned}$$

This enables us to write the first two moments of the profit function more explicitly: The expected profit is:

$$\begin{aligned} M_1(d) := E[\sum_{i=1}^{i=N} \frac{Q_i(P-C_i) - d_i}{(1+r)^i}] &= E[\sum_{i=1}^{i=N} \frac{Q_1 \Pi_{j=2}^{i}(1+\epsilon_j)(P - C_1 \Pi_{j=2}^{i}(1-a\eta_j)) - d_i}{(1+r)^i}] \\ &= \sum_{i=1}^{i=N} \frac{Q_1(P - C_1 \Pi_{j=2}^{i}(1 - ad_j)) - d_i}{(1+r)^i}]. \end{aligned}$$

And the second moment of the profit is:

$$\begin{aligned} M_2(d) &:= E[(\sum_{i=1}^{i=N} \frac{Q_i(P-C_i) - d_i}{(1+r)^i})^2] \\ &= E[(\sum_{i=1}^{i=N} \frac{Q_1 \Pi_{j=2}^{i}(1+\epsilon_j)(P - C_1 \Pi_{j=2}^{i}(1-a\eta_j)) - d_i}{(1+r)^i})^2] \\ &= E[\sum_{k=1}^{k=N} (\sum_{i=1}^{i=N} \frac{Q_1 \Pi_{j=2}^{i}(1+\epsilon_j)(P - C_1 \Pi_{j=2}^{i}(1-a\eta_j)) - d_i}{(1+r)^i} \\ &\quad \frac{Q_1 \Pi_{j=2}^{k}(1+\epsilon_j)(P - C_1 \Pi_{j=2}^{k}(1-a\eta_j)) - d_i}{(1+r)^i})] \\ &= \sum_{k=1}^{k=N} \sum_{i=1}^{k=N} E[Q_1^2 \Pi_{j=1}^{i}(1+\epsilon_j) \Pi_{j=1}^{k}(1+\epsilon_j) \\ &\quad \times (P - C_1 \Pi_{j=1}^{i}(1-\eta_j))((P - C_1 \Pi_{j=1}^{k}(1-\eta_j)) \\ &\quad - Q_1 \Pi_{j=1}^{i}(1+\epsilon_j)(P - C_1 \Pi_{j=1}^{i}(1-\eta_j)) d_k \\ &\quad - Q_1 \Pi_{j=1}^{k}(1+\epsilon_j)(P - C_1 \Pi_{j=1}^{k}(1-\eta_j)) d_i \\ &\quad + d_i d_k)/(1+r)^{2i}] \end{aligned}$$

Note that from the independence assumptions and the properties of the Normal and Poisson distributions we have

$$\begin{aligned} E[\Pi_{j=1}^{i}(1+\epsilon_j)\Pi_{j=1}^{k}(1+\epsilon_j)] &= E[\Pi_{j=1}^{i\wedge k}(1+\epsilon_j)^2\Pi_{j=i\wedge k+1}^{i\vee k}(1+\epsilon_j)] \\ &= E[\Pi_{j=1}^{i\wedge k}(1+2\epsilon_j+\epsilon_j^2)\Pi_{j=i\wedge k+1}^{i\vee k}(1+\epsilon_j)] \\ &= \Pi_{j=1}^{i\wedge k}(1+\Delta t) \end{aligned}$$

and that

$$\begin{aligned} E[\Pi_{j=1}^{i}(1-\eta_j)\Pi_{j=1}^{k}(1-\eta_j)] &= E[\Pi_{j=1}^{i\wedge k}(1-\eta_j)^2\Pi_{j=i\wedge k}^{i\vee k}(1-\eta_j)] \\ &= \Pi_{j=1}^{i\wedge k}(1+3d_j+d_j^2)\Pi_{j=i\wedge k+1}^{i\vee k}(1+d_j). \end{aligned}$$

Hence we obtain that the second moment of the profit is:

$$\begin{aligned} M_2(d) = \sum_{k=1}^{k=N}\sum_{i=1}^{k=N}(Q_1^2\Pi_{j=1}^{i\wedge k}(1+\Delta t) \\ (P^2 - PC_1(\Pi_{j=1}^{i}(1+d_j)+\Pi_{j=1}^{k}(1+d_j)) + C_1^2\Pi_{j=1}^{i\wedge k}(1+3d_j+d_j^2)\Pi_{j=i\wedge k+1}^{i\vee k}(1+d_j)) \\ -Q_1(P - C_1\Pi_{j=1}^{i}(1+d_j))d_k \\ -Q_1(P - C_1\Pi_{j=1}^{k}(1+d_j))d_i \\ +d_id_k)/(1+r)^{2i}) \end{aligned}$$

The mean of the objective function is simply $M_1(d)$ while the variance is $Var(d) = M_2(d) - M_1(d)^2$. Thus using the constant b as a measure of the management's aversion to risk, in the form of variance, the problem can be stated as the deterministic non-linear programming problem:

$$\max_d M_1(d) - bVar(d) \tag{8}$$

with the constraint that the vector of research spending is positive:

$$d \geq 0. \tag{9}$$

Let the solution to this problem be denoted by d^*. Then as shown above in section 1 the value of this project can be computed using the certainty equivalent approach:

$$\begin{aligned} V &= U^{-1}(M_1(d^*) - bVar(d^*)) \\ &= \frac{1-\sqrt{1-4b(M_1(d^*)-bVar(d^*))}}{2b} \end{aligned} \tag{10}$$

2.2 Worst Case Analysis

Using the worst case approach the problem is formulated as:

$$\max_d \min_{(\epsilon,\eta)} \sum_{i=1}^{N} \frac{Q_i(P-C_i)-d_i}{(1+r)^i} \tag{11}$$

Since whatever the value of d_i the worst value of $\eta_i = 0$ is still possible, i.e. meaning there are no research breakthroughs, and since d_i does not influence the value of ϵ_i, it is optimal to minimise d_i and hence not to perform any research. Then the realised profit will be:

$$\sum_{i=1}^{N} \frac{Q_i(P-C_i)}{(1+r)^i} \tag{12}$$

3 Compound IT Investment Real Option

This example illustrates the way many investment projects can be modelled by compound options, that is they offer the management options to invest in projects that themselves can be valued using the real option method. This example is partly based on Benaroch and Kauffman's [1999] real option analysis of Yankee 24's deferral option on providing POS debit electronic banking services to retail institutions.

Consider an IT company that has developed proprietary technology to provide network financial services. Due to the proprietary nature of the technology it is confident that it will have no competitors in this market for two years. However the implementation cost of providing the technology is high and current potential customers are reticent about subscribing to the service. It is clear that the IT company has a two year option to defer investment and wait for more information concerning the potential uptake of its technology.

Let $t_0, t_1, .., t_N = T$ be a set of decision times for the management: i.e. at each t_i the management chooses the value of the decision variable $C(t_i) \in \{0, K_1, K_2\}$ where $0 < K_1 << K_2$, K_1 is the cost of keeping the option to implement the project at a later date alive (e.g. costs of keeping staff on the potetntial project etc.) and K_2 is the cost of actually implementing the project. Thus at each time t_i the management decides whether to:

1. give up the opportunity of implementing the project ($C(t_i) = 0$), or
2. defer the decision until the next decision time (buy a compound option on the option to implement the project at a later date) ($C(t_i) = K_1$), or
3. invest in the project (exercise the underlying option) ($C(t_i) = K_2$).

Therefore if $C(t_i) = 0$ or $C(t_i) = K_2$ then $C(t_j) = 0$ for all $j > i$. To formulate these constraints on $C(t)$ explicitly let

$$C(t) = \alpha_0(t)0 + \alpha_1(t)K_1 + \alpha_2(t)K_2 \tag{13}$$

with $\alpha_j(t) \in \{0, 1\}$ for $j = 0, 1, 2$. Then $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ is constrained by:

$$\begin{aligned} \forall i &\in \{1, \ldots, N\} : \\ \alpha_0(t_i) + \alpha_1(t_i) + \alpha_2(t_i) &= 1 \\ \alpha_0(t_{i-1}) = 1 \vee \alpha_2(t_{i-1}) = 1 &\Rightarrow \alpha_0(t_i) = 1 \end{aligned} \tag{14}$$

Given the above constraints on α the problem is formulated as:

$$\max_{\alpha} E[U(\sum_{t=t_0}^{T}((V(t) - K_2)\alpha_2(t) - K_1\alpha_1(t))e^{-rt})] \tag{15}$$

For concreteness we specify a simple model for the value of the project if implemented as a function of customer demand. Let $S(t_i)$ be the level of demand during the period $(t_{i-1}, t_i]$ and p be the fixed price the firm will charge to customers using the technology. Assume a fixed horizon $T^* > T$ and for simplicity assume maintence costs are minimal and can be ignored. Then the value of the implemented project $V(t)$ is the present value of expected future revenue:

$$V(t) = E[\sum_{t_i=t}^{T^*} p.S(t_i)e^{-r(t_i-t)}] \tag{16}$$

To complete the model we define the dynamics of S as

$$S(t_i) = S(t_{i-1})(1+\epsilon_i) \tag{17}$$
$$\epsilon_i \sim N(0,c) \tag{18}$$

where c is constant.

3.1 Worst case Analysis

The worst case approach is formulated as:

$$\max_{\alpha} \min_{\epsilon} \sum_{t=t_0}^{T} ((V(t)-K_2)\alpha_2(t) - K_1\alpha_1(t))e^{-rt} \tag{19}$$

with the constraints (14). However since ϵ_i is unbounded below we take a two standard deviation move in each ϵ_i as a realistic lower bound. Since the variance of ϵ_i is c, once the project has been implemented the worst case is $\epsilon_i = -2\sqrt{c}$. Assuming each $\epsilon_i = -2\sqrt{c}$ the problem would be:

$$S(t_i) = S(t_0)(1-2\sqrt{c})^i$$
$$V(t) = \sum_{t_i=t}^{T^*} p.S(t_i)e^{-r(t_i-t)}$$

$$\max_{\alpha} \sum_{t=t_0}^{T} ((V(t)-K_2)\alpha_2(t) - K_1\alpha_1(t))e^{-rt}.$$

This may not, however, be the overall worst case, as in this problem it is likely that the project would not be implemented. In fact the worst case may be if ϵ and thus S achieve high values early on, so as to tempt the management into implementing the project, but after implementation the values of ϵ and thus S fall quickly making the implementation an expensive mistake with hindsight. Therefore further analysis of the actual worst case is required to develop the worst case approach for this problem.

4 Option to Abandon Machine Investment

We now consider a fixed investment into a machine producing a commodity. We take the output capacity of the machine to be fixed at level Q, but take the commodity price P and variable costs of production C to be positive discrete time stochastic processes:

$$P_{t_i} = P_{t_{i-1}} \exp(\mu\Delta t + \sigma\sqrt{\Delta t}\epsilon_i) \tag{20}$$
$$C_{t_i} = C_{t_{i-1}} \exp(\xi\Delta t + \gamma\sqrt{\Delta t}\eta_i) \tag{21}$$
$$(\epsilon,\eta) \sim N(0,I_2) \tag{22}$$

Therefore the operating profit for any time period is

$$Q(P-C) \tag{23}$$

The management have the option to abandon the investment incurring a one off fixed cost A for severing contracts, and receiving a one-off payment S from selling the machine. The lifetime of the machine is n time periods of length Δt so we have n equally spaced decision times $\{t_1,\ldots,t_n\}$. Let $\alpha_0(t), \alpha_1(t), \alpha_2(t) \in \{0,1\}$ be binary decision variables where:

- $\alpha_0(t_i) = 1$ indicates that the machine will operate normally for the time period $(t_i, t_{i+1}]$
- $\alpha_1(t_i) = 1$ indicates that the machine is abandoned at time t_i
- $\alpha_2(t_i) = 1$ indicates that the machine was abandoned at a time prior to t_i

Clearly we have constraints on α of:

$$\begin{aligned} \forall i &\in \{1, \dots, n\} : \\ \alpha_0(t_i) + \alpha_1(t_i) + \alpha_2(t_i) &= 1 \\ \alpha_1(t_{i-1}) = 1 \vee \alpha_2(t_{i-1}) = 1 &\Rightarrow \alpha_2(t_i) = 1 \end{aligned} \tag{24}$$

Then the profit realised at each time t_i from sales at t_i and costs incurred over $(t_{i-1}, t_i]$ is:

$$\alpha_0(t_{i-1})Q(P_{t_i} - C_{t_i}) + \alpha_1(t_{i-1})(S - A) + \alpha_2(t_{i-1})0. \tag{25}$$

The optimisation problem is thus:

$$\max_{\alpha} E[U(\sum_{i=1}^{n} \frac{\alpha_0(t_i)Q(P_{t_{i+1}} - C_{t_{i+1}}) + \alpha_1(t_i)(S - A)}{(1+r)^{t_{i+1}}})] \tag{26}$$

4.1 Worst Case Analysis

The worst case approach is formulated as:

$$\max_{\alpha} \min_{(\epsilon,\eta)} \sum_{i=1}^{n} \frac{\alpha_0(t_i)Q(P_{t_{i+1}} - C_{t_{i+1}}) + \alpha_1(t_i)(S - A)}{(1+r)^{t_{i+1}}} \tag{27}$$

with the constraints (24).

5 Example of Inter-dependent Projects: Optimal Resource Split Between two Projects

Suppose a firm has two projects producing outputs Q_1 and Q_2 whose market prices P_1 and P_2 follow the random processes:

$$\begin{aligned} P_1(t_i) &= P_1(t_{i-1}) + \epsilon_i \\ P_2(t_i) &= P_2(t_{i-1}) + \eta_i \\ (\epsilon_i, \eta_i) \sim NID(0, I) & \end{aligned}$$

Assume the firm has a fixed amount of labour L and capital K and that the projects have Cobb-Douglas production functions of the form:

$$\begin{aligned} Q_1 &= K_1^{\theta} L_1^{1-\theta} \\ Q_2 &= K_2^{\alpha} L_2^{1-\alpha} \end{aligned} \tag{28}$$

where θ and α are constants in $(0, 1)$ and the resources are constrained by

$$\begin{aligned} L_1 + L_2 &= L \\ K_1 + K_2 &= K. \end{aligned} \tag{29}$$

In addition assume that there are adjustment costs proportional to the amount of resource switched from one project to another. Thus the cost of switching labour and capital respectively in the jth period is:

$$a|L_1(t_j) - L_1(t_{j-1}|$$
$$b|K_1(t_j) - K_1(t_{j-1}|$$

where a and b are constant.

The value to the firm of each project and the optimal way to split capital and labour between the two projects can be analysed in the framework proposed above in section 1:

$$\begin{aligned} V(t,c) &= \max_{K_j,L_j} E[U(\sum_{t_0}^{t_N}[P_1(t_j)K_1(t_j)^\theta L_1(t_j)^{1-\theta} + P_2(t_j)K_2(t_j)^\alpha L_1(t_j)^{1-\alpha} \\ & -a|L_1(t_j) - L_1(t_{j-1}| - b|K_1(t_j) - K_1(t_{j-1}|] - c_1 - c_2)] \end{aligned}$$

Adjustment costs complicate the problem in that the optimal choice of resource allocation between projects will not only depend on current profitability, but also on the costs of adjusting resources to expand production of the currently more profitable output and on the expectation of future prices.

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