

Multi-Stage Imperative Languages: A Conservative Extension Result

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Abstract. This paper extends the recent work [CMT00] on the operational semantics and type system for a core language, called $\text{MiniML}_{\text{ref}}^{\text{BN}}$, which exploits the notion of *closed type* (see also [MTBS99]) to safely combine imperative and multi-stage programming. The main novelties are the identification of a larger set of closed types and the addition of a binder for useless variables. The resulting language is a conservative extension of $\text{MiniML}_{\text{ref}}$, a simple imperative subset of SML.

1 Introduction

This paper extends recent work [CMT00] on the operational semantics and type system for a core language, called $\text{MiniML}_{\text{ref}}^{\text{BN}}$, which exploits the notion of *closed type* (see also [MTBS99]) to safely combining imperative and multi-stage programming. One would expect that the addition of staging constructs to an imperative language should not prevent writing programs like those in normal imperative languages. In fact, a practical multi-stage programming language like MetaML [Met00] is designed to be a *conservative extension* of a standard programming language, like SML, for good pragmatic reasons: to gain acceptance from an existing user community, and to confine the challenges for new users to the staging constructs only.

Unfortunately, $\text{MiniML}_{\text{ref}}^{\text{BN}}$ fails to be a conservative extension of a simple imperative language like $\text{MiniML}_{\text{ref}}$ (i.e. MiniML with ML-style references), because certain well-typed programs in $\text{MiniML}_{\text{ref}}$ fail to be well-typed in $\text{MiniML}_{\text{ref}}^{\text{BN}}$. Technically, the problem is that the *closed types* of $\text{MiniML}_{\text{ref}}^{\text{BN}}$ are not closed under function types (and that locations may store only values of closed types). The best one can do is to define a translation $_*$ from $\text{MiniML}_{\text{ref}}$ to $\text{MiniML}_{\text{ref}}^{\text{BN}}$ respecting typing and operational semantics. The translation uses the closed type constructor $[_]$ and closedness annotations, in particular the translation of a functional type is $(t_1 \rightarrow t_2)* \triangleq [t_1* \rightarrow t_2*]$, which records that a functional type in the source language is a *closed functional type* in the target language.

From a language design perspective the main contribution of this paper is a core language, called $\text{MiniML}_{\text{ref}}^{\text{meta}}$, which extends *conservatively* the simple imperative language $\text{MiniML}_{\text{ref}}$ with the staging constructs of MetaML (and a few

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other features related to closed types). A safe combination of imperative and multi-stage programming in $\text{MiniML}_{\text{ref}}^{\text{meta}}$ is enforced through the use of closed types, as done in [CMT00] for $\text{MiniML}_{\text{ref}}^{\text{BN}}$.

Technically, the main novelty over [CMT00] is the identification of a larger set of closed types, which includes functional types of the form $t \rightarrow c$ where c is a closed type. The closed types of $\text{MiniML}_{\text{ref}}^{\text{BN}}$ enjoy the following property: values of closed types are closed, i.e. have no free variables. The closed types of $\text{MiniML}_{\text{ref}}^{\text{meta}}$ enjoy a weaker property (which is the best one can hope for functional types): the free variables in values of closed types are *useless*, i.e. during evaluation they will never be evaluated (at level 0).

Examples. The restriction of storable values to closed types is motivated by the following MetaML session:

```
-| val a = ref <1>;
val a = ... : ref <int>
-| val b = <fn x => ~(a:=<x>; <2>>);
val b = <fn x => 2> : <int -> int>
-| val c = !a;
val c = <x> : <int>
```

In evaluating the second declaration, the variable x goes outside the scope of the binding lambda, and the result of the third line is wrong, since x is not bound in the environment, even though the session is well-typed according to naive extensions of previously proposed type systems for MetaML. This form of **scope extrusion** is specific to multi-level and multi-stage languages, and it does *not* arise in traditional programming languages, where evaluation is generally restricted to closed terms. The problem lies in the *run-time interaction between free variables and references*. In the type system we propose the above session is not well-typed: $a:=<x>$ cannot be typed, because $<x>$ is not of a closed type.

$\text{MiniML}_{\text{ref}}^{\text{meta}}$ allows among the closed types some functional types, while in $\text{MiniML}_{\text{ref}}^{\text{BN}}$ functional types are never closed. The following interactive session, is typable in $\text{MiniML}_{\text{ref}}^{\text{meta}}$ but not in $\text{MiniML}_{\text{ref}}^{\text{BN}}$.

```
-| val l = ref(fn x => x+1);
val l = (ref fn ... ) : (int -> int) ref
-| <fn x => ~(l := (fn y => ((fn z => y+1) <x>))); <x+1>>;
val it = <(fn x => x+1)> : <int -> int>
```

The first line creates a reference to functions from integers to integers; and the second assigns the function $\text{fn } y \Rightarrow ((\text{fn } z \Rightarrow y+1) \text{ } <x>)$ to it. As a result, the variable x escapes from its binder and leaks into the store. However, this cannot be observed because the variable is “useless”: if we supply an argument to the stored function, the inner application will be evaluated, discarding the term $<x>$. The operational semantics presented here solves the problem with a binder for useless variables, introduced before storing a term.

Relation to MiniML_{ref}^{BN}. There is a significant overlap between MiniML_{ref}^{meta} and MiniML_{ref}^{BN}. We refer to [CMT00] for a broader discussion of related work [DP96, Dav96, TS97, TBS98, MTBS99, BMTS99, Tah99, TS00]. For those familiar with MiniML_{ref}^{BN} (recalled in Appendix A) we summarize the differences:

- MiniML_{ref}^{meta} has no closedness annotation $[c]$, and the closed type constructor $[_]$ cannot be applied to a closed type c . These are *cosmetic* changes, motivated by the following remarks in [CMT00]: closedness annotations play no role in the operational semantics, and a closed type c is *semantically* isomorphic to $[c]$ via the mapping $x \mapsto [x]$. When closedness annotations are removed, the isomorphism becomes an identity, thus the syntax for MiniML_{ref}^{meta} types *forbids* $[c]$, since it is *equal* to c .
- MiniML_{ref}^{meta} has a let-binder $(\text{let}_c x = e_1 \text{ in } e_2)$ corresponding to $(\text{let } [x]: c = [e_1] \text{ in } e_2)$ of MiniML_{ref}^{BN}, for variables of closed type.
- MiniML_{ref}^{meta} has a larger set of closed types, in particular a functional type $t \rightarrow c$ is closed whenever c is closed. This property is essential to prove that every well-formed MiniML_{ref} program is also well-formed in MiniML_{ref}^{meta}.
- MiniML_{ref}^{meta} has a new binder $\bullet e$, called Bullet, which binds all the free variables in e . When all the free variables in e are *useless*, $\bullet e$ and e are *semantically* equivalent. Bullet is used in the operational semantics to prevent scope extrusion (for this purpose it replaces the constant *fault* of [CMT00]), and to annotate terms whose free variables are useless.

In an implementation, Bullet should help improve efficiency, since one knows that $\text{FV}(\bullet e) = \emptyset$ without examining the whole of e . For instance, the function $\bullet \lambda x. e$ does not depend on the environment, only on the argument. Our operational semantics is too abstract to support claims about efficiency, but we expect that a reformulation in terms of weak explicit substitution ([LM99, B97]) could make such claims precise.

In general, checking whether a variable is useless requires a static analysis (preferably of the whole program, see [WS99]). The MiniML_{ref}^{meta} type system has a simple rule to infer $\bullet e: c^n$, namely $e: c^n$ when all the free variables in e have level $> n$. This rule makes sense only in the context of multi-level languages, but it infers $\bullet(!l \langle x \rangle): c^n$, where l is a location of closed type $\langle t \rangle \rightarrow c$, which is beyond conventional analyses.

Structure of the paper. Section 2 introduces MiniML_{ref}, which is MiniML of [CDDK86] with ML-style references. Section 3 introduces MiniML_{ref}^{meta}, which extends MiniML_{ref} with

- The three staging constructs of MetaML [TS97, TBS98, Met00]: Brackets $\langle e \rangle$, Escape $\sim e$ and Run $\text{run } e$.
- A let-binder $(\text{let}_c x = e_1 \text{ in } e_2)$ for variables of closed type.
- A binder $\bullet e$, called Bullet, of all the free variables in a term e of closed type.

We also prove type safety along the lines of [CMT00]. Section 4 shows that MiniML_{ref}^{meta} is a conservative extension of MiniML_{ref}. Section 5 discusses improvements to the type system through the addition of sub-typing, alternatives to Bullet, and variation to the syntax and operational semantics of MiniML_{ref}^{meta}.

2 MiniML_{ref}

This section describes the syntax, type system and operational semantics of MiniML_{ref}, an extension of MiniML ([CDDK86]) with ML-style references. Types t are defined as

$$t \in \mathbb{T} ::= \text{nat} \mid \text{ref } t \mid t_1 \rightarrow t_2$$

The sets of MiniML_{ref} terms and values are parametric in an infinite set of variables $x \in \mathbb{X}$ and an infinite set of locations $l \in \mathbb{L}$

$$\begin{aligned} e \in \mathbb{E} ::= & x \mid \lambda x.e \mid e_1 e_2 \mid \text{fix } x.e \mid \mathbf{z} \mid \mathbf{s} e \mid (\text{case } e \text{ of } \mathbf{z} \rightarrow e_1 \mid \mathbf{s} x \rightarrow e_2) \mid \\ & \text{ref } e \mid !e \mid e_1 := e_2 \mid l \\ v \in \mathbb{V} ::= & \lambda x.e \mid \mathbf{z} \mid \mathbf{s} v \mid l \end{aligned}$$

The first line lists the MiniML terms: variables, abstraction, application, fix-point for recursive definitions, zero, successor, and case-analysis on natural numbers. The second line lists the three SML operations on references, and constants l for locations. These constants are not allowed in user-defined programs, but they are instrumental to the operational semantics of MiniML_{ref}.

Note 1. We will use the following notation and terminology

- Term equivalence, written \equiv , is α -conversion. $\text{FV}(e)$ is the set of variables free in e . \mathbb{E}_0 indicates the set of terms without free variables. Substitution of e_1 for x in e_2 (modulo \equiv) is written $e_2[x := e_1]$.
- m, n range over the set \mathbb{N} of natural numbers. Furthermore, $m \in \mathbb{N}$ is identified with the set $\{i \in \mathbb{N} \mid i < m\}$ of its predecessors.
- $f: A \xrightarrow{\text{fin}} B$ means that f is a partial function from A to B with a finite domain, written $\text{dom}(f)$.
- $\Sigma: \mathbb{L} \xrightarrow{\text{fin}} \mathbb{T}$ is a *signature* (for locations only), written $\{l_i: \text{ref } t_i \mid i \in m\}$.
- $\Gamma: \mathbb{X} \xrightarrow{\text{fin}} \mathbb{T}$ is a type assignment, written $\{x_i: t_i \mid i \in m\}$.
- $\mu \in \mathbb{S} \triangleq \mathbb{L} \xrightarrow{\text{fin}} \mathbb{V}_0$ is a store, where \mathbb{V}_0 is the set of closed values.
- $\Sigma, l: \text{ref } t$, $\Gamma, x: t$ and $\mu\{l = v\}$ denote extension/update of a signature, assignment and store respectively.

Type System. The type system of MiniML_{ref} is given in Figure 1, and it enjoys the following basic properties:

Lemma 1 (Weakening).

1. $\Sigma; \Gamma \vdash e: t_2$ and x fresh imply $\Sigma; \Gamma, x: t_1 \vdash e: t_2$
2. $\Sigma; \Gamma \vdash e: t_2$ and l fresh imply $\Sigma, l: \text{ref } t_1; \Gamma \vdash e: t_2$

Lemma 2 (Substitution).

$\Sigma; \Gamma \vdash e_1: t_1$ and $\Sigma; \Gamma, x: t_1 \vdash e_2: t_2$ imply $\Sigma; \Gamma \vdash e_2[x := e_1]: t_2$

We say that a store μ is well-formed for Σ (and write $\Sigma \models \mu \iff$)

$$\text{dom}(\Sigma) = \text{dom}(\mu) \text{ and } \Sigma \vdash v: t \text{ whenever } \mu(l) = v \text{ and } \Sigma(l) = \text{ref } t.$$

$$\begin{array}{c}
\frac{}{\Sigma; \Gamma \vdash x: t} \quad \frac{\Sigma; \Gamma, x: t_1 \vdash e: t_2}{\Sigma; \Gamma \vdash \lambda x. e: t_1 \rightarrow t_2} \quad \frac{\Sigma; \Gamma \vdash e_1: t_1 \rightarrow t_2 \quad \Sigma; \Gamma \vdash e_2: t_1}{\Sigma; \Gamma \vdash e_1 e_2: t_2} \\
\frac{\Sigma; \Gamma, x: t \vdash e: t}{\Sigma; \Gamma \vdash \text{fix } x. e: t} \quad \frac{}{\Sigma; \Gamma \vdash z: \text{nat}} \quad \frac{\Sigma; \Gamma \vdash e: \text{nat}}{\Sigma; \Gamma \vdash \text{s } e: \text{nat}} \\
\frac{\Sigma; \Gamma \vdash e: \text{nat} \quad \Sigma; \Gamma \vdash e_1: t \quad \Sigma; \Gamma, x: \text{nat} \vdash e_2: t}{\Sigma; \Gamma \vdash (\text{case } e \text{ of } z \rightarrow e_1 \mid \text{s } x \rightarrow e_2): t} \quad \frac{\Sigma; \Gamma \vdash e: t}{\Sigma; \Gamma \vdash \text{ref } e: \text{ref } t} \\
\frac{\Sigma; \Gamma \vdash e: \text{ref } t}{\Sigma; \Gamma \vdash !e: t} \quad \frac{\Sigma; \Gamma \vdash e_1: \text{ref } t \quad \Sigma; \Gamma \vdash e_2: t}{\Sigma; \Gamma \vdash e_1 := e_2: \text{ref } t} \quad \frac{}{\Sigma; \Gamma \vdash l: \text{ref } t} \quad \Sigma(l) = \text{ref } t
\end{array}$$

Fig. 1. Type System for MiniML_{ref}

Operational Semantics. The operational semantics of MiniML_{ref} is given in Figure 2. The semantics is non-deterministic because of the rule for evaluating $\text{ref } e$. Evaluation of a term $e \in \mathbf{E}_0$ with an initial store μ_0 can lead to

- a *result* v and a new store μ_1 , when we can derive $\mu_0, e \hookrightarrow \mu_1, v$, or
- a *run-time error*, when we can derive $\mu_0, e \hookrightarrow \text{err}$.

Evaluation of a term may also lead to divergence, although a big-step operational semantics can express this third possibility only indirectly. One would have to adopt a reduction semantics (as advocated by [WF94]) to achieve a more accurate classification of the possible computations. In our setting, Type Safety means that evaluation of a well-typed program cannot lead to a run-time error, namely

Theorem 1 (Safety). $\mu_0, e \hookrightarrow d$ and $\Sigma_0 \models \mu_0$ and $\Sigma_0 \vdash e: t$ imply that there exist μ_1 and v and Σ_1 such that $d \equiv (\mu_1, v)$ and $\Sigma_0, \Sigma_1 \models \mu_1$ and $\Sigma_0, \Sigma_1 \vdash v: t$.

3 MiniML_{ref}^{meta}

This section describes the syntax, type system and operational semantics of MiniML_{ref}^{meta}, and establishes Type Safety. Types t , closed types c and open types o are defined as

$$\begin{aligned}
t \in \mathbf{T} &::= c \mid o \\
c \in \mathbf{C} &::= \text{nat} \mid t_1 \rightarrow c_2 \mid [o] \mid \text{ref } c \\
o \in \mathbf{O} &::= t_1 \rightarrow o_2 \mid \langle t \rangle
\end{aligned}$$

Intuitively, a term can be assigned a closed type c only when its free variables are *useless*. The set of MiniML_{ref}^{meta} terms is parametric in an infinite set of variables $x \in \mathbf{X}$ and an infinite set of locations $l \in \mathbf{L}$

$$\begin{aligned}
e \in \mathbf{E} &::= x \mid \lambda x. e \mid e_1 e_2 \mid \text{fix } x. e \mid z \mid \text{s } e \mid (\text{case } e \text{ of } z \rightarrow e_1 \mid \text{s } x \rightarrow e_2) \mid \\
&\langle e \rangle \mid \tilde{e} \mid \text{run } e \mid (\text{let}_{c,x} e = e_1 \text{ in } e_2) \mid \bullet e \mid \\
&\text{ref } e \mid !e \mid e_1 := e_2 \mid l
\end{aligned}$$

$$\begin{array}{c}
\mu_0, \lambda x.e \hookrightarrow \mu_0, \lambda x.e \quad \frac{\mu_0, e_1 \hookrightarrow \mu_1, \lambda x.e \quad \mu_1, e_2 \hookrightarrow \mu_2, v_2 \quad \mu_2, e[x:=v_2] \hookrightarrow \mu_3, v}{\mu_0, e_1 e_2 \hookrightarrow \mu_3, v} \\
\frac{\mu_0, e_1 \hookrightarrow \mu_1, v \not\equiv \lambda x.e \quad \mu_0, e[x:=\text{fix } x.e] \hookrightarrow \mu_1, v}{\mu_0, e_1 e_2 \hookrightarrow \text{err}} \quad \frac{\mu_0, \text{fix } x.e \hookrightarrow \mu_1, v \quad \mu_0, z \hookrightarrow \mu_0, z}{\mu_0, e \hookrightarrow \mu_1, v} \\
\frac{\mu_0, e \hookrightarrow \mu_1, v}{\mu_0, s e \hookrightarrow \mu_1, s v} \quad \frac{\mu_0, e \hookrightarrow \mu_1, z \quad \mu_1, e_1 \hookrightarrow \mu_2, v}{\mu_0, (\text{case } e \text{ of } z \rightarrow e_1 \mid s x \rightarrow e_2) \hookrightarrow \mu_2, v} \\
\frac{\mu_0, e \hookrightarrow \mu_1, v \not\equiv z \mid s e'}{\mu_0, (\text{case } e \text{ of } z \rightarrow e_1 \mid s x \rightarrow e_2) \hookrightarrow \text{err}} \quad \frac{\mu_0, e \hookrightarrow \mu_1, s v \quad \mu_1, e_2[x:=v] \hookrightarrow \mu_2, v_2}{\mu_0, (\text{case } e \text{ of } z \rightarrow e_1 \mid s x \rightarrow e_2) \hookrightarrow \mu_2, v_2} \\
\frac{\mu_0, e \hookrightarrow \mu_1, v}{\mu_0, \text{ref } e \hookrightarrow \mu_1\{l=v\}, l} \quad \frac{l \notin \text{dom}(\mu_1) \quad \mu_0, e \hookrightarrow \mu_1, l}{\mu_0, !e \hookrightarrow \mu_1, v} \quad \mu_1(l) \equiv v \\
\frac{\mu_0, e \hookrightarrow \mu_1, v \not\equiv l \in \text{dom}(\mu_1)}{\mu_0, !e \hookrightarrow \text{err}} \quad \frac{\mu_0, e_1 \hookrightarrow \mu_1, l \quad \mu_1, e_2 \hookrightarrow \mu_2, v}{\mu_0, e_1 := e_2 \hookrightarrow \mu_2\{l=v\}, l} \\
\frac{\mu_0, e_1 \hookrightarrow \mu_1, v \not\equiv l \in \text{dom}(\mu_1)}{\mu_0, e_1 := e_2 \hookrightarrow \text{err}} \quad \mu_0, l \hookrightarrow \mu_0, l
\end{array}$$

The rules for error propagation follow the ML-convention, i.e. for every normal evaluation rule $\frac{\{\mu_i, e_i \hookrightarrow \mu_{i+1}, v_i \mid i \in n\}}{\mu_0, e \hookrightarrow \mu_n, v}$ and every $m \in n$ one should add an error propagation rule $\frac{\{\mu_i, e_i \hookrightarrow \mu_{i+1}, v_i \mid i \in m\} \quad \mu_m, e_m \hookrightarrow \text{err}}{\mu_0, e \hookrightarrow \text{err}}$.

Fig. 2. Operational Semantics for MiniML_{ref}

The second line lists the three multi-stage constructs of MetaML [TS97]: *Brackets* $\langle e \rangle$ and *Escape* $\sim e$ are for building and splicing code, and *Run* is for executing code. The second line lists also a let-binder ($\text{let}_c x = e_1 \text{ in } e_2$) for variables of closed type, and a binder *Bullet* $\bullet e$, which binds all the free variables of e , hence $FV(\bullet e) = \emptyset$, and $(\bullet e)[x:=e_1] \equiv \bullet e$.

Note 2. We will use the following notation and terminology (see also Note 1)

- w ranges over terms not of the form $\bullet e$, while $\circ w$ can be either w or $\bullet w$.
 - $\Sigma: \mathbb{L} \xrightarrow{\text{fin}} \mathbb{T}$ is a *signature* (for locations only), written $\{l_i: \text{ref } c_i \mid i \in m\}$.
 - $\Delta: \mathbb{X} \xrightarrow{\text{fin}} (\mathbb{C} \times \mathbb{N})$ and $\Gamma: \mathbb{X} \xrightarrow{\text{fin}} (\mathbb{T} \times \mathbb{N})$ are type-and-level assignments, written $\{x_i: c_i^{n_i} \mid i \in m\}$ and $\{x_i: t_i^{n_i} \mid i \in m\}$ respectively.
- We use the following operations on type-and-level assignments:
- $\{x_i: t_i^{n_i} \mid i \in m\}^{+n} \triangleq \{x_i: t_i^{n_i+n} \mid i \in m\}$ adds n to the level of the x_i ;
 - $\{x_i: t_i^{n_i} \mid i \in m\}^{\leq n} \triangleq \{x_i: t_i^{n_i} \mid n_i \leq n \wedge i \in m\}$ removes the x_i with level $> n$.
 - $\Gamma, x: t^n$ and $\Delta, x: c^n$ denote the extension of type-and-level assignments.

Remark 1. The new binder *Bullet* $\bullet e$ serves many purposes, which the constant *fault* of [CMT00] can fulfill only in part (e.g. *fault* is not typable). Intuitively, $\bullet e$ is like a closure (e, ρ) , where ρ is the environment (explicit substitution) mapping all variables to *fault*, and in addition it records that e should have a closed type.

$$\begin{array}{c}
\frac{}{\Sigma; \Delta; \Gamma \vdash x: t^n} \quad (\Delta, \Gamma)(x) = t^n \quad \frac{\Sigma; \Delta; \Gamma, x: t_1^n \vdash e: t_2^n}{\Sigma; \Delta; \Gamma \vdash \lambda x. e: t_1 \rightarrow t_2^n} \\
\frac{\Sigma; \Delta; \Gamma \vdash e_1: t_1 \rightarrow t_2^n \quad \Sigma; \Delta; \Gamma \vdash e_2: t_1^n}{\Sigma; \Delta; \Gamma \vdash e_1 e_2: t_2^n} \quad \frac{\Sigma; \Delta; \Gamma, x: t^n \vdash e: t^n}{\Sigma; \Delta; \Gamma \vdash \text{fix } x. e: t^n} \quad \frac{}{\Sigma; \Delta; \Gamma \vdash z: \text{nat}^n} \\
\frac{\Sigma; \Delta; \Gamma \vdash e: \text{nat}^n}{\Sigma; \Delta; \Gamma \vdash s e: \text{nat}^n} \quad \frac{\Sigma; \Delta; \Gamma \vdash e: \text{nat}^n \quad \Sigma; \Delta; \Gamma \vdash e_1: t^n \quad \Sigma; \Delta; \Gamma, x: \text{nat}^n \vdash e_2: t^n}{\Sigma; \Delta; \Gamma \vdash (\text{case } e \text{ of } z \rightarrow e_1 \mid s x \rightarrow e_2): t^n} \\
\frac{\Sigma; \Delta; \Gamma \vdash e: c^n}{\Sigma; \Delta; \Gamma \vdash \text{ref } e: \text{ref } c^n} \quad \frac{\Sigma; \Delta; \Gamma \vdash e: \text{ref } c^n}{\Sigma; \Delta; \Gamma \vdash ! e: c^n} \quad \frac{\Sigma; \Delta; \Gamma \vdash e_1: \text{ref } c^n \quad \Sigma; \Delta; \Gamma \vdash e_2: c^n}{\Sigma; \Delta; \Gamma \vdash e_1 = e_2: \text{ref } c^n} \\
\frac{}{\Sigma; \Delta; \Gamma \vdash l: \text{ref } c^n} \quad \Sigma(l) = \text{ref } c \quad \frac{\Sigma; \Delta; \Gamma \vdash e: t^{n+1} \quad \Sigma; \Delta; \Gamma \vdash e: \langle t \rangle^n}{\Sigma; \Delta; \Gamma \vdash \langle e \rangle: \langle t \rangle^n} \quad \frac{}{\Sigma; \Delta; \Gamma \vdash \sim e: t^{n+1}} \\
\frac{\Sigma; \Delta; \Gamma \vdash e: \langle t \rangle^n}{\Sigma; \Delta; \Gamma \vdash \text{run } e: t^n} \quad \frac{\Sigma; \Delta^{\leq n}; \emptyset \vdash e: o^n}{\Sigma; \Delta; \Gamma \vdash e: [o]^n} \quad \frac{\Sigma; \Delta; \Gamma \vdash e: [o]^n}{\Sigma; \Delta; \Gamma \vdash e: o^n} \\
\frac{\Sigma; \Delta; \Gamma \vdash e_1: c^n \quad \Sigma; \Delta, x: c^n; \Gamma \vdash e_2: t^n}{\Sigma; \Delta; \Gamma \vdash (\text{let}_c x = e_1 \text{ in } e_2): t^n} \quad \frac{\Sigma; \Delta_1^{+(n+1)}; \Gamma_1^{+(n+1)} \vdash e: c^n}{\Sigma; \Delta; \Gamma \vdash \bullet e: c^n}
\end{array}$$

Fig. 3. Type System for MiniML_{ref}^{meta}

The typing rule for Bullet, in combination with Type Safety (Theorem 2), formalizes the property that in a term of closed type (at level n) all the free variables (at level $> n$) are *useless*. In fact, during evaluation a variable bound by Bullet (unlike variables captured by other binders) cannot get instantiated, thus its occurrences must disappear before reaching level 0 (otherwise they will cause a run-time error).

The operational semantics of Figure 2 uses Bullet to prevent scope extrusion when a location l is initialized or assigned. In fact, what gets stored in l is the closed value $\bullet w$, instead of the value w . Therefore, if a free variable in w was in the scope of an enclosing binder, e.g. x in $\langle \lambda x. \sim (l := w; \langle x \rangle) \rangle$, it is caught by Bullet, instead of becoming free.

Unlike locations (which exist only at execution time) and *fault* (which is not typable), Bullet could be used in user-defined programs to record that a term has a closed type. The operational semantics uses such information when evaluating an application (if $\lambda x. e$ has a closed type, then e must have a closed type) and a let-binder (the let must bind x to a term of closed type) for capturing free variables. For instance, during evaluation of $\bullet(\lambda x. e) v$ the free variables of v get captured in $\bullet(e[x := v])$.

3.1 Type System

Figure 3 gives the type system of MiniML_{ref}^{meta}. A typing judgement has the form $\Sigma; \Delta; \Gamma \vdash e: t^n$, read “ e has type t and level n under the assignment $\Sigma; \Delta; \Gamma$ ”. Σ gives the type of locations which can be used in e , Δ and Γ (must have disjoint domains and) give the type and level of variables which may occur free in e .

Remark 2. All typing rules, except the last four, are borrowed from [CMT00]. The introduction and elimination rules for $[o]$ say that $[o]$ is a sub-type of o . The rule for $(\text{let}_c x = e_1 \text{ in } e_2)$ incorporates the typing rule (close*) of [CMT00]. The rule for $\bullet e$ says that Bullet binds all the free variables in e . One can think of $\bullet e$ as the closure (e, ρ) , where ρ is the environment (explicit substitution) mapping all variables to **fault**.

The type system enjoys the following basic properties (see also [CMT00]):

Lemma 3 (Weakening).

1. $\Sigma; \Delta; \Gamma \vdash e: t_2^n$ and x fresh imply $\Sigma; \Delta; \Gamma, x: t_1^m \vdash e: t_2^n$
2. $\Sigma; \Delta; \Gamma \vdash e: t_2^n$ and x fresh imply $\Sigma; \Delta, x: c^m; \Gamma \vdash e: t_2^n$
3. $\Sigma; \Delta; \Gamma \vdash e: t_2^n$ and l fresh imply $\Sigma, l: \text{ref } c_1; \Delta; \Gamma \vdash e: t_2^n$

Lemma 4 (Substitution).

1. $\Sigma; \Delta; \Gamma \vdash e_1: t_1^m$ and $\Sigma; \Delta; \Gamma, x: t_1^m \vdash e_2: t_2^n$ imply $\Sigma; \Delta; \Gamma \vdash e_2[x := e_1]: t_2^n$
2. $\Sigma; \Delta^{\leq m}; \emptyset \vdash e_1: c^m$ and $\Sigma; \Delta, x: c^m; \Gamma \vdash e_2: t_2^n$ imply $\Sigma; \Delta; \Gamma \vdash e_2[x := e_1]: t_2^n$

3.2 Operational Semantics

The operational semantics of $\text{MiniML}_{\text{ref}}^{\text{meta}}$ is given in Figure 4. The rules derive evaluation judgements of the form $\mu, e \xrightarrow{n} d$, where $\mu \in \mathcal{S}$ is a *value store* (see below). In the rules v ranges over terms, but *a posteriori* one can show that v ranges over *values at level n* (see below). We will show that evaluation of a well-typed program cannot lead to a run-time error (Theorem 2).

Definition 1. *The set $\mathcal{V}^n \subset \mathcal{E}$ of values at level n is defined by the BNF*

$$\begin{aligned}
v^n \in \mathcal{V}^n &::= w^n \mid \bullet w^n \\
w^0 \in \mathcal{W}^0 &::= \lambda x. e \mid \mathbf{z} \mid \mathbf{s} v^0 \mid \langle v^1 \rangle \mid l \\
w^{n+1} \in \mathcal{W}^{n+1} &::= x \mid \lambda x. v^{n+1} \mid v_1^{n+1} v_2^{n+1} \mid \text{fix } x. v^{n+1} \mid \mathbf{z} \mid \mathbf{s} v^{n+1} \mid \\
&\quad (\text{case } v^{n+1} \text{ of } \mathbf{z} \rightarrow v_1^{n+1} \mid \mathbf{s} x \rightarrow v_2^{n+1}) \mid \langle v^{n+2} \rangle \mid \text{run } v^{n+1} \mid \\
&\quad (\text{let}_c x = v_1^{n+1} \text{ in } v_2^{n+1}) \mid \\
&\quad \text{ref } v^{n+1} \mid !v^{n+1} \mid v_1^{n+1} := v_2^{n+1} \mid l \\
w^{n+2} \in \mathcal{W}^{n+2}_+ &= \sim v^{n+1}
\end{aligned}$$

$\mu \in \mathcal{S} \stackrel{\Delta}{=} \mathcal{L} \xrightarrow{\text{fin}} \mathcal{V}_0^0$ is a **value store**, where \mathcal{V}_0^0 is the set of closed values at level 0. We write $\Sigma \models \mu \stackrel{\Delta}{\iff} \text{dom}(\Sigma) = \text{dom}(\mu)$ and $\Sigma; \emptyset \vdash v: c^0$ whenever $\mu(l) = v$ and $\Sigma(l) = \text{ref } c$.

The following result establishes basic facts about the operational semantics, similar to those established for $\text{MiniML}_{\text{ref}}^{\text{BN}}$ (see [CMT00]).

Lemma 5 (Values). $\mu_0, e \xrightarrow{n} \mu_1, v$ and μ_0 is value store imply μ_1 is a value store, $\text{dom}(\mu_0) \subseteq \text{dom}(\mu_1)$, $v \in \mathcal{V}^n$ and $\text{FV}(v) \subseteq \text{FV}(e)$.

In the rules below $\circ w$ is a meta-expression ranging over terms of the form w and $\bullet w$.

$$\begin{array}{c}
\text{Normal Evaluation} \\
\frac{\mu_0, \lambda x.e \xrightarrow{0} \mu_0, \lambda x.e \quad \frac{\mu_0, e_1 \xrightarrow{0} \mu_1, \lambda x.e \quad \mu_1, e_2 \xrightarrow{0} \mu_2, v_2 \quad \mu_2, e[x:=v_2] \xrightarrow{0} \mu_3, v}{\mu_0, e_1 e_2 \xrightarrow{0} \mu_3, v}}{\mu_0, x \xrightarrow{0} \text{err}} \quad \frac{\mu_0, e_1 \xrightarrow{0} \mu_1, \bullet \lambda x.e \quad \mu_1, e_2 \xrightarrow{0} \mu_2, v_2 \quad \mu_2, \bullet(e[x:=v_2]) \xrightarrow{0} \mu_3, v}{\mu_0, e_1 e_2 \xrightarrow{0} \mu_3, v}}{\mu_0, x \xrightarrow{0} \text{err}} \\
\frac{\mu_0, e_1 \xrightarrow{0} \mu_1, v \not\equiv \circ \lambda x.e \quad \mu_0, e[x:=\text{fix } x.e] \xrightarrow{0} \mu_1, v \quad \mu_0, z \xrightarrow{0} \mu_0, z}{\mu_0, e_1 e_2 \xrightarrow{0} \text{err}} \quad \frac{\mu_0, \text{fix } x.e \xrightarrow{0} \mu_1, v}{\mu_0, e \xrightarrow{0} \mu_1, v} \quad \frac{\mu_0, e \xrightarrow{0} \mu_1, \circ z \quad \mu_1, e_1 \xrightarrow{0} \mu_2, v_1}{\mu_0, s e \xrightarrow{0} \mu_1, s v} \quad \frac{\mu_0, (\text{case } e \text{ of } z \rightarrow e_1 \mid s x \rightarrow e_2) \xrightarrow{0} \mu_2, v_1}{\mu_0, e \xrightarrow{0} \mu_1, v \not\equiv \circ z \mid \circ s v} \quad \frac{\mu_0, e \xrightarrow{0} \mu_1, \circ s v \quad \mu_1, e_2[x:=\circ v] \xrightarrow{0} \mu_2, v_2}{\mu_0, (\text{case } e \text{ of } z \rightarrow e_1 \mid s x \rightarrow e_2) \xrightarrow{0} \text{err}} \quad \frac{\mu_0, (\text{case } e \text{ of } z \rightarrow e_1 \mid s x \rightarrow e_2) \xrightarrow{0} \mu_2, v_2}{\mu_0, e \xrightarrow{0} \mu_1, \circ w} \quad \frac{l \notin \text{dom}(\mu_1) \quad \frac{\mu_0, e \xrightarrow{0} \mu_1, \circ l \quad \mu_1(l) \equiv v}{\mu_0, !e \xrightarrow{0} \mu_1, v}}{\mu_0, \text{ref } e \xrightarrow{0} \mu_1\{l = \bullet w\}, l} \\
\frac{\mu_0, e \xrightarrow{0} \mu_1, \circ w \quad w \notin \text{dom}(\mu_1)}{\mu_0, !e \xrightarrow{0} \text{err}} \quad \frac{\mu_0, e_1 \xrightarrow{0} \mu_1, \circ l \quad \mu_1, e_2 \xrightarrow{0} \mu_2, \circ w \quad l \in \text{dom}(\mu_1)}{\mu_0, e_1 := e_2 \xrightarrow{0} \mu_2\{l = \bullet w\}, l} \\
\frac{\mu_0, e_1 \xrightarrow{0} \mu_1, \circ w \quad w \notin \text{dom}(\mu_1)}{\mu_0, e_1 := e_2 \xrightarrow{0} \text{err}} \quad \frac{\mu_0, l \xrightarrow{0} \mu_0, l}{\mu_0, \langle e \rangle \xrightarrow{0} \mu_1, \langle v \rangle} \quad \frac{\mu_0, e \xrightarrow{1} \mu_1, v}{\mu_0, \tilde{e} \xrightarrow{0} \text{err}} \quad \frac{\mu_0, e \xrightarrow{0} \mu_1, \circ \langle v \rangle \quad \mu_1, \bullet v \xrightarrow{0} \mu_2, v_0}{\mu_0, \text{run } e \xrightarrow{0} \mu_2, v_0} \quad \frac{\mu_0, e \xrightarrow{0} \mu_1, v \not\equiv \circ \langle e' \rangle}{\mu_0, \text{run } e \xrightarrow{0} \text{err}} \\
\frac{\mu_0, e_1 \xrightarrow{0} \mu_1, \circ w \quad \mu_1, e_2[x:=\bullet w] \xrightarrow{0} \mu_2, v_2}{\mu_0, (\text{let}_c x = e_1 \text{ in } e_2) \xrightarrow{0} \mu_2, v_2} \quad \frac{\mu_0, e \xrightarrow{n} \mu_1, \circ w}{\mu_0, \bullet e \xrightarrow{n} \mu_1, \bullet w}
\end{array}$$

$$\begin{array}{c}
\text{Symbolic Evaluation} \\
\frac{\mu_0, x \xrightarrow{n+1} \mu_0, x \quad \frac{\mu_0, e \xrightarrow{n+2} \mu_1, v}{\mu_0, \langle e \rangle \xrightarrow{n+1} \mu_1, \langle v \rangle} \quad \frac{\mu_0, e \xrightarrow{0} \mu_1, \langle v \rangle}{\mu_0, \tilde{e} \xrightarrow{1} \mu_1, v} \quad \frac{\mu_0, e \xrightarrow{0} \mu_1, \bullet \langle \circ w \rangle}{\mu_0, \tilde{e} \xrightarrow{1} \mu_1, \bullet w}}{\mu_0, e \xrightarrow{0} \mu_1, v \not\equiv \circ \langle e' \rangle} \quad \frac{\mu_0, e \xrightarrow{n+1} \mu_1, v}{\mu_0, \tilde{e} \xrightarrow{n+2} \mu_1, \tilde{v}}
\end{array}$$

In all other cases symbolic evaluation is applied to the immediate sub-terms from left to right without changing level.

Error Propagation

The rules for error propagation follow the ML-convention (see Figure 2).

Fig. 4. Operational Semantics for MiniML_{ref}^{meta}

Proof. By induction on the derivation of the evaluation judgement $\mu_0, e \xrightarrow{n} \mu_1, v$. Notice that in the rules evaluating $\text{ref } e$ and $e_1 := e_2$ it is important that we store $\bullet w$, since w may have free variables.

The following lemma is used to prove type safety in the case for evaluating $\text{run } e$ at level 0. The result holds also for closed types of the form nat and $\text{ref } c$.

Lemma 6 (Closedness). *If $\Sigma; \Delta^{+1}; \Gamma^{+1} \vdash \circ w^0: [o]^0$, then $\text{FV}(w^0) = \emptyset$.*

Proof. By induction on the derivation of $\Sigma; \Delta^{+1}; \Gamma^{+1} \vdash \circ w^0: [o]^0$.

Evaluation of $\text{run } e$ at level 0 requires to view a value v at level 1 as a term to be evaluated at level 0. The following lemma says that this confusion in the levels is compatible with the type system.

Lemma 7 (Demotion). *$\Sigma; \Delta^{+1}; \Gamma^{+1} \vdash v^{n+1}: t^{n+1}$ implies $\Sigma; \Delta; \Gamma \vdash v^{n+1}: t^n$.*

Proof. By induction on the derivation of $\Sigma; \Delta^{+1}; \Gamma^{+1} \vdash v^{n+1}: t^{n+1}$.

The *reflective* nature of $\text{MiniML}_{\text{ref}}^{\text{meta}}$ is fully captured by the Demotion Lemma and the following Promotion Lemma (which is not relevant to the proof of Type Safety).

Lemma 8. *$\Sigma; \Delta; \Gamma \vdash e: t^n$ implies $e \in V^{n+1}$ and $\Sigma; \Delta^{+1}; \Gamma^{+1} \vdash e: t^{n+1}$.*

Finally, we establish the key result relating the type system to the operational semantics. This result entails that evaluation of a well-typed program $\emptyset; \emptyset \vdash e: t^0$ cannot raise an error, i.e. $\emptyset, e \xrightarrow{0} \text{err}$ is not derivable.

Theorem 2 (Safety). *$\mu_0, e \xrightarrow{n} d$ and $\Sigma_0 \models \mu_0$ and $\Sigma_0; \Delta^{+1}; \Gamma^{+1} \vdash e: t^n$ imply that there exist μ_1 and v^n and Σ_1 such that $d \equiv (\mu_1, v^n)$ and $\Sigma_0, \Sigma_1 \models \mu_1$ and $\Sigma_0, \Sigma_1; \Delta^{+1}; \Gamma^{+1} \vdash v^n: t^n$.*

Proof. By induction on the derivation of the evaluation judgement $\mu_0, e \xrightarrow{n} d$.

4 Conservative Extension Result

This section shows that $\text{MiniML}_{\text{ref}}^{\text{meta}}$ is a *conservative extension* of $\text{MiniML}_{\text{ref}}$ w.r.t. typing and operational semantics. When we need to distinguish the syntactic categories of $\text{MiniML}_{\text{ref}}^{\text{meta}}$ from those of $\text{MiniML}_{\text{ref}}$ we use a superscript $^{\text{meta}}$ for the formers, e.g. \mathbf{E}^{meta} denotes the set of $\text{MiniML}_{\text{ref}}^{\text{meta}}$ terms, while \mathbf{E} denotes the set of $\text{MiniML}_{\text{ref}}$ terms. We have the following inclusions between the syntactic categories of the two languages:

Lemma 9. *$\mathbf{T} \subseteq \mathbf{C}^{\text{meta}}$ and $\mathbf{E} \subseteq \mathbf{E}^{\text{meta}}$ and $\mathbf{V} \subseteq \mathbf{V}^{0\text{meta}}$.*

Proof. Easy induction on the structure of $t \in \mathbf{T}$, $e \in \mathbf{E}$ and $v \in \mathbf{V}$.

There are minor mismatches between the typing and evaluation judgements of the two languages, thus we introduce three derived predicates, which simplify the formulation of the conservative extension result:

- $e: t$, i.e. e is a program of type t ;
- $e \Downarrow$, i.e. evaluation of e may lead to a value;
- $e \Downarrow \text{err}$, i.e. evaluation of e may lead to a run-time error.

The following table defines the three predicates in $\text{MiniML}_{\text{ref}}$ and $\text{MiniML}_{\text{ref}}^{\text{meta}}$:

predicate	meaning in $\text{MiniML}_{\text{ref}}$	meaning in $\text{MiniML}_{\text{ref}}^{\text{meta}}$
$e: t$	$\emptyset; \emptyset \vdash e: t$	$\emptyset; \emptyset; \emptyset \vdash e: t^0$
$e \Downarrow$	$\exists \mu, v. \emptyset, e \hookrightarrow \mu, v$	$\exists \mu, v. \emptyset, e \xrightarrow{0} \mu, v$
$e \Downarrow \text{err}$	$\emptyset, e \hookrightarrow \text{err}$	$\emptyset, e \xrightarrow{0} \text{err}$

The conservative extension result can be stated as follows (the rest of the section establishes several facts, which combined together imply the desired result)

Theorem 3 (Conservative Extension). *MiniML_{ref} and MiniML_{ref}^{meta} agree on the validity of the assertions $e: t$, $e \Downarrow$ and $e \Downarrow \text{err}$, whenever $e \in \mathbf{E}$ and $t \in \mathbf{T}$.*

A typing judgement $\Sigma; \Gamma \vdash e: t$ for $\text{MiniML}_{\text{ref}}$ it is not appropriate for $\text{MiniML}_{\text{ref}}^{\text{meta}}$, because $\Gamma: \mathbf{X} \xrightarrow{\text{fin}} \mathbf{T}$ and e lack the level information. Therefore, we introduce the following operation to turn a type assignment into a type-and-level assignment

$$\{x_i: t_i | i \in m\}^n \triangleq \{x_i: t_i^n | i \in m\}$$

i.e. Γ^n assigns level n to all variables declared in Γ .

Proposition 1. *$\Sigma; \Gamma \vdash e: t$ in $\text{MiniML}_{\text{ref}}$ implies $\Sigma; \emptyset; \Gamma^0 \vdash e: t^0$ in $\text{MiniML}_{\text{ref}}^{\text{meta}}$.*

Proof. Easy induction on the derivation of $\Sigma; \Gamma \vdash e: t$.

An immediate consequence of Proposition 1 is that $e: t$ in $\text{MiniML}_{\text{ref}}$ implies $e: t$ in $\text{MiniML}_{\text{ref}}^{\text{meta}}$. For the converse, we need to define a translation from \mathbf{T}^{meta} to \mathbf{T} .

Definition 2. *The function $\|\cdot\|$ from \mathbf{T}^{meta} to \mathbf{T} is defined as*

$$\begin{aligned} \|[o]\| &\triangleq \|o\| \\ \|[t]\| &\triangleq \|t\| \end{aligned}$$

and it commutes with all other type-constructs of $\text{MiniML}_{\text{ref}}^{\text{meta}}$. The extension to signatures Σ is point-wise; $\|\Gamma\|(x) = \|t\|$ when $\Gamma(x) = t^n$ and similarly for Δ .

Proposition 2. *$\Sigma; \Delta; \Gamma \vdash e: t^n$ implies $\|\Sigma\|; \|\Delta\|; \|\Gamma\| \vdash e: \|t\|$, provided $e \in \mathbf{E}$.*

Proof. By induction on the derivation of $\Sigma; \Delta; \Gamma \vdash e: t^n$.

The operational semantics of $\text{MiniML}_{\text{ref}}^{\text{meta}}$ may introduce Bullet (e.g. when manipulating the store), even when the evaluation starts in a configuration (μ, e) without occurrences of \bullet . Therefore, to relate the operational semantics of $\text{MiniML}_{\text{ref}}^{\text{meta}}$ and $\text{MiniML}_{\text{ref}}$, we introduce a partial function on \mathbf{E}^{meta} which erases Bullet from $\bullet e$ when $\text{FV}(e) = \emptyset$.

Definition 3 (Erasure). The partial function $|\cdot|$ on E^{meta} is defined as

$$|\bullet e| \triangleq \begin{cases} |e| & \text{if } \text{FV}(e) = \emptyset \\ \text{undefined} & \text{otherwise} \end{cases}$$

and it commutes with all other term-constructs of $\text{MiniML}_{\text{ref}}^{\text{meta}}$.

Lemma 10. The erasure enjoys the following properties:

- If $\Sigma; \Delta; \Gamma \vdash e: t^n$ and $|e|$ is defined, then $\Sigma; \Delta; \Gamma \vdash |e|: t^n$;
- if $|e_2| \equiv e'_2$ and $|e_1| \equiv e'_1$ then $|e_2[x: e_1]| \equiv e'_2[x: e'_1]$.

Proof. The first part is by induction on the derivation of $\Sigma; \Delta; \Gamma \vdash e: t^n$; the second is by induction on the structure of e_2 .

Definition 4 (Bisimulation). The relation $R \subseteq E^{\text{meta}} \times E_0$ is given by

$$e R e' \iff \text{FV}(e) = \emptyset \text{ and } |e| \equiv e'.$$

The relation is extended to stores μ and configurations d as follows:

$$\mu R \mu' \iff \text{dom}(\mu) = \text{dom}(\mu') \text{ and } \mu(l) R \mu'(l) \text{ when } l \in \text{dom}(\mu);$$

$$d R d' \iff d = \text{err} = d' \text{ or } (d = (\mu, e) \text{ and } d' = (\mu', e') \text{ where } \mu R \mu' \text{ and } e R e').$$

The following proposition says that R is a bisimulation between the operational semantics of $\text{MiniML}_{\text{ref}}^{\text{meta}}$ and $\text{MiniML}_{\text{ref}}$.

Proposition 3. If $\mu R \mu'$ and $e R e'$, then

1. $\mu, e \xrightarrow{0} d$ implies there exist (unique) d' such that $d R d'$ and $\mu', e' \hookrightarrow d'$;
2. $\mu', e' \hookrightarrow d'$ implies there exist d such that $d R d'$ and $\mu, e \xrightarrow{0} d$.

Proof. The first part is by induction on the derivation of $\mu, e \xrightarrow{0} d$. The second part is by lexicographic induction on the derivation of $\mu', e' \hookrightarrow d'$ and the number of top-level Bullets in e (i.e. n such that $e \equiv \bullet^n w$).

This implies the conservative extension result for the predicates $e \Downarrow$ and $e \Downarrow \text{err}$.

5 Conclusions and Further Research

In this section we discuss possible improvements to the type system and variations to the syntax and operational semantics of $\text{MiniML}_{\text{ref}}^{\text{meta}}$.

Sub-typing. In $\text{MiniML}_{\text{ref}}^{\text{meta}}$ sub-typing arises naturally, e.g. one expects $[o] \leq o$ for any open type $o \in \mathcal{O}$.

Before adding a sub-sumption rule $\frac{\Sigma; \Delta; \Gamma \vdash e: t_1^n}{\Sigma; \Delta; \Gamma \vdash e: t_2^n} t_1 \leq t_2$ to $\text{MiniML}_{\text{ref}}^{\text{meta}}$,

it is better to adopt a more general syntax for types t and closed types c

$$\begin{aligned} t \in \mathcal{T} &::= \text{nat} \mid t_1 \rightarrow t_2 \mid \text{ref } c \mid \langle t \rangle \mid [t] \\ c \in \mathcal{C} &::= \text{nat} \mid t_1 \rightarrow c_2 \mid \text{ref } c \mid [t] \end{aligned}$$

and let the sub-typing rule derive $[c] = c$. One expects the usual sub-typing rules for functional and references types, and it seems natural to require the Code and Closed type constructors to be covariant, i.e.

$$\frac{t'_1 \leq t_1 \quad t_2 \leq t'_2}{t_1 \rightarrow t_2 \leq t'_1 \rightarrow t'_2} \quad \frac{c' \leq c \quad c \leq c'}{\text{ref } c \leq \text{ref } c'} \quad \frac{t \leq t'}{\langle t \rangle \leq \langle t' \rangle} \quad \frac{t \leq t'}{[t] \leq [t']}$$

while sub-typing axioms, which generate non trivial relations, are

$$[t] \leq t \quad c \leq [c] \quad [t_1 \rightarrow t_2] \leq [t_1] \rightarrow [t_2] \quad [\langle t \rangle] \leq \langle [t] \rangle$$

From the sub-typing axioms and rules above one can derive the following facts:

- $t \leq c$ implies $t \in \mathbf{C}$, by induction on the derivation of $t \leq c$;
- $[c] = c$ and $c \rightarrow [t] = [c \rightarrow t]$, while the following sub-typing are strict $[\langle t \rangle] < \langle [t] \rangle$ and $[t_1] \rightarrow [t_2] < [t_1 \rightarrow t_2]$.

We plan to investigate the addition of sub-typing and its effects on type safety.

Useless-variable annotation. The binder $\bullet e$ of $\text{MiniML}_{\text{ref}}^{\text{meta}}$ takes an all or nothing approach. One could provide a more fine-grained annotation $(x)e$, which allows to name a useless variable. The typing rules for $(x)e$ are the obvious one:

$$\frac{\Sigma; \Delta; \Gamma, x: t^m \vdash e: c^n}{\Sigma; \Delta; \Gamma \vdash (x)e: c^n} \quad m > n \quad \frac{\Sigma; \Delta, x: t^m; \Gamma \vdash e: c^n}{\Sigma; \Delta; \Gamma \vdash (x)e: c^n} \quad m > n$$

One can define the derived notation $(X)e$, where X is a finite set/sequence of variables, by induction on the cardinality of X : $(\emptyset)e \triangleq e$, $(x, X)e \triangleq (x)(X)e$. One might identify $\bullet e$ with $\tilde{e} \triangleq (X)e$, where $X = \text{FV}(e)$. However, at the operational level such identification is not right. In fact, the rule

$$\frac{\mu_0, e_1 \xrightarrow{0} \mu_1, \bullet \lambda x. e \quad \mu_1, e_2 \xrightarrow{0} \mu_2, v_2 \quad \mu_2, \bullet(e[x := v_2]) \xrightarrow{0} \mu_3, v}{\mu_0, e_1 \quad e_2 \xrightarrow{0} \mu_3, v}}$$

is not an instance of

$$\frac{\mu_0, e_1 \xrightarrow{0} \mu_1, (X)\lambda x. e \quad \mu_1, e_2 \xrightarrow{0} \mu_2, v_2 \quad \mu_2, ((X)e)[x := v_2] \xrightarrow{0} \mu_3, v}{\mu_0, e_1 \quad e_2 \xrightarrow{0} \mu_3, v}}$$

since the free variables in v_2 are bound by \bullet , but not by (X) . This seems to suggest that one might want to maintain $\bullet e$ even in the presence of $(x)e$. On the other hand, the conservative extension of $\text{MiniML}_{\text{ref}}$ into $\text{MiniML}_{\text{ref}}^{\text{meta}}$ seems to become simpler if we use $(x)e$ (and a suitable adaptation of the operational semantics) instead of $\bullet e$.

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$$\begin{array}{c}
\text{(case}^*\text{)} \frac{\Sigma; \Delta; \Gamma \vdash e: \text{nat}^n \quad \Sigma; \Delta; \Gamma \vdash e_1: t^n \quad \Sigma; \Delta, x: \text{nat}^n; \Gamma \vdash e_2: t^n}{\Sigma; \Delta; \Gamma \vdash (\text{case } e \text{ of } z \rightarrow e_1 \mid s x \rightarrow e_2): t^n} \\
\frac{\Sigma; \Delta^{\leq n}; \emptyset \vdash e: t^n \quad \Sigma; \Delta; \Gamma \vdash e_1: [t_1]^n \quad \Sigma; \Delta, x: t_1^n; \Gamma \vdash e_2: t_2^n}{\Sigma; \Delta; \Gamma \vdash [e]: [t]^n} \\
\text{(fix}^*\text{)} \frac{\Sigma; \Delta^{\leq n}, x: t^n; \emptyset \vdash e: t^n}{\Sigma; \Delta; \Gamma \vdash \text{fix } x.e: t^n} \quad \text{(close}^*\text{)} \frac{\Sigma; \Delta; \Gamma \vdash e: c^n}{\Sigma; \Delta; \Gamma \vdash [e]: [c]^n}
\end{array}$$

Fig. 5. Type System for $\text{MiniML}_{\text{ref}}^{\text{BN}}$

A $\text{MiniML}_{\text{ref}}^{\text{BN}}$

This section recalls the syntax and type system of $\text{MiniML}_{\text{ref}}^{\text{BN}}$, to help in a comparison with $\text{MiniML}_{\text{ref}}^{\text{meta}}$. The types t and closed types c of $\text{MiniML}_{\text{ref}}^{\text{BN}}$ are defined as

$$t \in \mathbb{T} ::= c \mid t_1 \rightarrow t_2 \mid \langle t \rangle \quad c \in \mathbb{C} ::= \text{nat} \mid [t] \mid \text{ref } c$$

Remark 3. Function types are never closed, the types $[c]$ and c are not identified.

The set of $\text{MiniML}_{\text{ref}}^{\text{BN}}$ terms is defined as

$$\begin{aligned}
e \in \mathbb{E} ::= & x \mid \lambda x.e \mid e_1 e_2 \mid \text{fix } x.e \mid z \mid s e \mid (\text{case } e \text{ of } z \rightarrow e_1 \mid s x \rightarrow e_2) \mid \\
& \langle e \rangle \mid \sim e \mid \text{run } e \mid [e] \mid (\text{let } [x] = e_1 \text{ in } e_2) \mid \\
& \text{ref } e \mid !e \mid e_1 := e_2 \mid l \mid \text{fault}
\end{aligned}$$

Remark 4. The constant `fault` leads to a run-time error when evaluated at level 0, and evaluates to itself at higher levels. Operationally, `fault` is equivalent to the $\text{MiniML}_{\text{ref}}^{\text{meta}}$ term $\bullet x$. There is an explicit closed construct $[e]$, and one let-binder ($\text{let } [x] = e_1 \text{ in } e_2$).

Figure 5 summarizes the typing rules of $\text{MiniML}_{\text{ref}}^{\text{BN}}$ which differ from those of $\text{MiniML}_{\text{ref}}^{\text{meta}}$. The main differences are:

- (case^{*}) corresponds to declare the bound variable in Δ , instead of Γ , and is only used to simplify the translation of $\text{MiniML}_{\text{ref}}^{\text{BN}}$ in $\text{MiniML}_{\text{ref}}^{\text{BN}}$.
- (close^{*}) is necessary because there is no identification of $[c]$ with c .
- (fix^{*}) can type recursive definitions (e.g. of closed functions) that are not typable with (fix). For instance, from $\emptyset; f': [t_1 \rightarrow t_2]^n, x: t_1^n \vdash e: t_2^n$ one cannot derive $\text{fix } f'. [\lambda x.e]: [t_1 \rightarrow t_2]^n$, while the following modified term $\text{fix } f'. (\text{let } [f] = f' \text{ in } [\lambda x.e[f' := [f]]])$ has the right type, but the wrong behavior (it diverges!). The rule (fix^{*}) allows to type $[\text{fix } f. \lambda x.e[f' := [f]]]$, which has the desired operational behavior.

In $\text{MiniML}_{\text{ref}}^{\text{meta}}$ the (fix^{*}) rule is not necessary: assuming a unit type $()$, one could write the term $\text{fix } f'. (\text{let}_c f = \lambda(). f' \text{ in } \lambda x.e[f' := f()])$ which has the desired type and does not diverge; this term is not typable in $\text{MiniML}_{\text{ref}}^{\text{BN}}$ because $f: () \rightarrow (t_1 \rightarrow t_2)$ would not have a closed type.