

Program Logic and Equivalence in the Presence of Garbage Collection [★]

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Abstract

It is generally thought that reasoning about programs in memory safe, garbage collected languages is much easier than in languages where the programmer has more explicit control over memory. Paradoxically, existing program logics are based on a low level view of storage that is sensitive to the presence or absence of unreachable cells, and Reynolds has pointed out that the Hoare triples derivable in these logics are even incompatible with garbage collection. We present a study of a small language whose operational semantics includes a rule for reclaiming garbage. Our main results include an analysis of propositions that are garbage insensitive, and full abstraction results connecting partial and total correctness to two natural notions of observational equivalence between programs.

1 Introduction

Garbage collection is an essential method used to reclaim heap-allocated objects whose lifetime cannot be easily predicted at compile time. It is most strongly associated with high-level languages such as Lisp, ML and Java, where heap allocation is the norm. It can also be used in a lower level language like C, coexisting with explicit deallocation primitives [10]. In any case, garbage collection relieves the programmer of the burden of explicitly managing dynamically allocated memory. This generally leads to simpler programs, and removes or lessens errors that result from incorrect attempts to access disposed memory, errors that are often difficult to diagnose or even reproduce.

[★] A preliminary version of this paper appeared as “On Garbage and Program Logic” in the proceedings of FOSSACS’01, published as Springer LNCS 2030. This research was supported by the EPSRC under the “Verified Bytecode” and “Local Reasoning about State” projects.

It is commonly thought that reasoning about garbage collected languages (especially memory safe languages) is much easier than reasoning about lower level languages. Paradoxically, however, existing program logics take a view of storage that is sensitive to the presence of unreachable cells, a view that is not invariant under garbage collection.

1.1 A Logical Conundrum

The following problem was raised by Reynolds in [14].

Consider a statement form $x := \text{cons}(E, E')$ that allocates and initializes a new cons cell and places a pointer to that cell in storage variable x . Then the following sequence of instructions creates a new cell, and then makes it garbage by destroying the sole existing pointer to the cell, thus making it unreachable.

$$x := \text{cons}(3, 4); x := z.$$

Now, ask the question: is there a pointer y to a cell in the heap that has 3 in its car and 4 in its cdr, after these statements have been executed? From the point of view of execution the answer is that it depends, on whether a garbage collector has taken control or not.

The conundrum is that program logic seems to take a particular stance, one that is incompatible with garbage collection. That is, previous logics for pointer programs would allow us to derive a Hoare triple

$$\{true\} x := \text{cons}(3, 4); x := z \{\exists y. y \mapsto 3, 4\}.$$

The problem is that on termination there might not actually be a cell whose car is 3 and cdr is 4, if a garbage collector reclaims the detached cell. (Here, $y \mapsto 3, 4$ is a predicate that says that y points to a binary cons cell holding 3 and 4.)

It is worth looking at this example in more detail. After the statement $x := \text{cons}(3, 4)$ has been executed, it has to be admitted that there is such a cell, because it cannot be garbage collected. So we would expect to have a true Hoare triple $\{true\} x := \text{cons}(3, 4) \{\exists y. y \mapsto 3, 4\}$. But then either Hoare's or Floyd's assignment axiom gives us $\{\exists y. y \mapsto 3, 4\} x := z \{\exists y. y \mapsto 3, 4\}$ because x is free in neither the precondition nor the postcondition. And now the die is cast: we obtain the triple above by sequencing.

The conundrum is related to the problem of *full abstraction*. That is, there are Hoare triple contexts that can distinguish programs that are observationally

equivalent, if we take a standard notion of observation (such as termination). For example, one would expect the program fragment given above to be observationally equivalent to $x := z$ on its own, but if the standard assignment axiom is to be believed then the weakest precondition of $\exists y. y \mapsto 3, 4$ is just itself, so we only get $\{\exists y. y \mapsto 3, 4\} x := z \{\exists y. y \mapsto 3, 4\}$. Thus, if there are states where this precondition is false, then the logical context $\{true\} - \{\exists y. y \mapsto 3, 4\}$ will distinguish two “equivalent” commands, since it will not be satisfied by $x := z$. (In this discussion we have ignored the possibility of running out of memory. But a similar point can be made if we bound the number of possible pointers by strengthening the precondition to say that there is at least one cell available, and by using $x := \text{cons}(7, 8); x := z$ in place of the single statement $x := z$.)

The upshot is that previous program logics for pointer programs are unsound in the presence of garbage collection, including all of the pointer logic references in [5,3,9] (with the exception of [7]; see below). We take a few representative examples. In an early work Oppen and Cook described a complete proof system for deriving true Hoare triples about pointer programs [13], but their interpretation of quantifiers is the usual first-order interpretation; as a result their propositions are not garbage insensitive, and the Hoare triple context $\{true\} - \{\exists y. y \mapsto 3, 4\}$ behaves exactly as described above. In a series of papers, the most recent of which is [6], de Boer has advocated an approach where the quantifiers are restricted to range over currently active cells only. However, the currently active cells can contain garbage, and because of this de Boer’s approach falls foul of the conundrum, and fails to characterize observational equivalence, using essentially the same examples we used in this section. Finally, Honsell, Smith, Mason and Talcott [8] have given a characterization of equivalence (the “ciu theorem”) in an expressive language with higher-order store. However, after giving this characterization, a notion of “contextual assertion” is introduced, and an example is given showing a logical context that breaks observational equivalence. The sticking point in all of these approaches is the interpretation of quantifiers. If one considers a quantifier free language then garbage insensitivity is easy to achieve, at the cost of some expressivity. The decidable logic of [1] very nearly qualifies, except for the use of a garbage sensitive atomic predicate hs for describing sharing constraints.

1.2 Reactions

There are several ways to react to the conundrum. One is to lay the blame on the Floyd-Hoare approach to assignment, and its emphasis on substitution. This reaction is difficult to uphold. For, although the soundness of the assignment axiom requires certain assumptions (such as no clashes between named variables), none of these are the essential problem here.

Another reaction is to regard the conundrum as inconsequential. For, the unsoundness phenomenon mentioned above is not disastrous. The various logics are consistent, as they are sound in non-garbage-collecting models, and the unsoundness only seems to show up in “useless” formulae such as $\exists y. y \mapsto 3, 4$. The logic will not lead us wrong on formulae that depend only on reachable elements, such as $x \mapsto 3, 4$.

This reaction has some merit, but is probably too glib. Some analysis should be provided of the propositions which do not lead you wrong.

A third reaction is to maintain the Floyd-Hoare approach, but to enrich the syntax of pre and postconditions with an extra parameter to keep track of the heap and its relation to program variables. This is the approach taken by Hoare and Jifeng [7], where there is an algebraic operation \widehat{state} that removes garbage from the state. Then, the assignment $x := z$ can be modelled with an axiom that keeps track of alterations to heap cells, resulting from garbage collection, as well as stack variables.

$$\{P[\widehat{state}'/state]\} x := z \{P\} \quad state' = (state \mid x \mapsto state(z))$$

By using $\widehat{(\cdot)}$, Hoare and Jifeng can keep the heap garbage collected at all times.

This third approach is certainly theoretically adequate. Its main limitation is the need to carry around an explicit state parameter. However, starting from such an adequate position one might attempt to find situations when one can soundly hide the state parameter, to treat it implicitly in pre and postconditions as is usually done. We emphasize that this step from explicit to implicit state is hazardous, a potential source of unsoundness (as the conundrum shows). The problem is that in the process of hiding the state we might lose substitution-relevant syntactic information, which plays a crucial role in program logic. Indeed, the results of this paper might be viewed as an analysis of situations where implicit state is sound, a hidden-state cousin of the solution of Hoare and Jifeng.

1.3 Overview

The approach we take in this paper concentrates the impact of garbage collection in one place: the properties expressed by pre and postconditions. In proof rules for a program logic, the impact is limited to the interpretation of implications used to strengthen preconditions or weaken postconditions. The idea is that if formulae are restricted, or their semantics altered, to ensure that propositions are insensitive to garbage, then there will be no need to doctor the other rules, for the various statement forms, to take account of the possibility of the collector taking control.

We consider two forms of semantics, one based on total states (which have no dangling pointers) and the other based on partial states (which allow dangling). The partial semantics is the more theoretically cohesive, but is perhaps less approachable initially. After all, in a memory safe language like the one we will consider dangling pointers never arise during program execution. So we begin with the total semantics.

We use a possible-world semantics, where the current heap (a finite collection of cons cells) is the world. The most important case is the interpretation of $\exists x. P$. As usual, x here ranges over values, including pointers and even pointers not in the current heap. If such a new pointer is chosen, then the world is extended as well, to provide a binding for (at least) the new pointer. In this way the total states model adopts a view of the state as finite but arbitrarily extensible. The other connectives are interpreted pointwise, without changing worlds.

This semantics provides the basis for a resolution of the conundrum, but at the price of an unusual semantics of \exists . We also observe that the normal semantics of \exists can be retained, as long as we rule out problematic formulae like $\exists y. y \mapsto 3, 4$. This is done using a form of guarded quantifier, expressed via a syntactic restriction on propositions.

After presenting the model we prove three main results. The first is garbage insensitivity: the semantics of assertions is invariant under the operations of removing or adding garbage cells. The second and third are both full abstraction results, which establish that two programs satisfy the same Hoare triples just when they are observationally equivalent. One of these results connects total correctness specifications with an equivalence obtained from observing termination, and the other connects partial correctness specifications with an equivalence obtained from observing the presence of runtime errors (such as dereferencing `nil`) as well as termination. The first equivalence, for total correctness, conflates divergence and runtime error. Both equivalences equate the program fragments discussed in the conundrum above.

After showing these results we move on to consider partial states. We establish a connection between the total states model and a partial states model based on the celebrated “dense topology” semantics of classical logic [4,11].

We work with a pared down storage model and programming language for this study, where the heap consists of a collection of binary cons cells, and where there are data pointers but no code pointers or closures. Garbage insensitivity appears to extend to higher-order store, but the extension of full abstraction is not obvious.

2 The Storage Model

The storage model we use supports pointer allocation, dereferencing and assignment, and divides the state into *stack* and *heap* components. The stack consists of associations of values to variables, and is altered by the standard assignment statement $x := E$. As is common in Hoare logic, we do not distinguish between a variable name and the l-value it denotes; this is justified because we are not considering aliasing between stack variables. The heap consists of a collection of binary cons cells, which can only be accessed via pointers; the extension to records of other sizes is straightforward.

The basic domains of the model are as follows.

$$\begin{aligned}
 \mathbf{Pointers} &\triangleq \{p, q, \dots\} & \mathbf{Bool} &\triangleq \{false, true\} \\
 \mathbf{Variables} &\triangleq \{x, y, \dots\} & \mathbf{Nat} &\triangleq \{0, 1, \dots\} \\
 \mathbf{Values} &\triangleq \mathbf{Nat} + \mathbf{Bool} + \mathbf{Pointers} + \{nil\} \\
 \mathbf{Stacks} &\triangleq \mathbf{Variables} \rightarrow_{fin} \mathbf{Values} \\
 \mathbf{Heaps} &\triangleq \mathbf{Pointers} \rightarrow_{fin} \mathbf{Values} \times \mathbf{Values} \\
 \mathbf{States} &\triangleq \mathbf{Stacks} \times \mathbf{Heaps}
 \end{aligned}$$

Although there is no ordering on variables, in the programming language to be presented later allocation and deallocation of variables will obey a stack discipline. Stacks and heaps are represented by finite partial functions: we write $dom(s)$ or $dom(h)$ for the domain of definition of a stack or heap.

The collection of **Pointers** can be taken either to be an infinite set, or a finite set if we are concerned with bounded memory. None of the results in this paper depend on the size of **Pointers**.

Notice that a state $s, h \in \mathbf{Stacks} \times \mathbf{Heaps}$ might have dangling pointers, pointers that are reachable but not defined in h . Since we will consider a programming language without memory disposal, total states play a central role. We use the following definitions.

- Pointer q is reachable from p in h if $q = p$ or $h(p) = \langle v_1, v_2 \rangle$ and q is reachable from v_1 or v_2 in h ;
- p is reachable in s, h if there is $x \in dom(s)$ such that p is reachable from $s(x)$ in h ;
- (s, h) is total if every reachable pointer is in the domain of h .

We will also be concerned with the presence or absence of garbage.

- (s, h) is garbage free if every $p \in \text{dom}(h)$ is reachable in s, h .

Garbage collection is intimately connected to the relation of heap extension, which gives us a partial order on **Heaps**.

- $g \sqsubseteq h$ indicates that the graph of heap g is a subset of the graph of heap h .

Note that if $h \sqsubseteq h'$ then (s, h) total implies (s, h') total. From the point of view of this model, given a state s, h the effect of a garbage collector is to shrink h by selecting $g \sqsubseteq h$ in a way that doesn't produce any (new) dangling pointers. In the best case, the collector will remove all garbage.

- $\text{prune}(s, h) = (s, g)$, where $g \sqsubseteq h$ is the subheap of h restricted to those pointers reachable in s, h .

Actually, a relocating collector can move heap cells around. For this the notion of isomorphism is important.

- $=_\alpha$ is equality of states modulo renaming of pointers.
 (To be explicit, if $f : \text{Pointers} \rightarrow \text{Pointers}$ is a permutation, let f^* be the induced function from values to values which is the identity on non-pointer values. Then $s, h =_\alpha s', h'$, where $s' = s; f^*$ and $h' = f^{-1}; h; (f^* \times f^*)$.)

Lemma 1 *The following hold:*

- (1) $\text{prune}(s, h)$ is garbage free.
- (2) If (s, h) is total then so is $\text{prune}(s, h)$.
- (3) The relation $=_\alpha$ preserves totality and garbage freedom: If $(s, h) =_\alpha (s', h')$ and s, h is total (garbage free) then (s', h') is total (garbage free).

3 Expressions

We restrict our attention to expressions that are free from side effects. A programming language for transforming states will be given later in section 7, after we have presented the assertion language.

We include standard operations of arithmetic, along with generic equality testing and pointer dereference. The syntax of expressions is given by the

following grammar:

$$\begin{aligned}
E ::= & x \\
& | 0 | 1 | E + E | E \times E | E - E \\
& | \text{true} | \text{not } E | E \text{ and } E | E == E \\
& | \text{nil}
\end{aligned}$$

We have not included expressions $E.1$ or $E.2$ for dereferencing a pointer. It is theoretically simpler to exclude such operations from expressions, and instead have dereferencing as part of the command forms. In the assertion language dereferencing will be accomplished using an atomic predicate, the points-to relation.

We have avoided all issues of typing. A type system could eliminate type errors in many cases but not, without great complication, for pointer dereferencing, where `nil` or some other value is typically used as a special “pointer” that should not be dereferenced.

So, the semantics of an expression E will determine an element

$$\llbracket E \rrbracket s \in \text{Values} + \{wrong\}$$

where the domain of s includes the free variables of E . A *wrong* result indicates a type error; this is treated essentially as in a partial function semantics, where *wrong* stands for undefined.

Selected semantics definitions are as follows.

$$\begin{aligned}
\llbracket x \rrbracket s &\triangleq s(x) \\
\llbracket E_1 + E_2 \rrbracket s &\triangleq \llbracket E_1 \rrbracket s + \llbracket E_2 \rrbracket s && \text{if } \llbracket E_1 \rrbracket s \in \text{Nat} \text{ and } \llbracket E_2 \rrbracket s \in \text{Nat} \\
& \text{wrong} && \text{otherwise} \\
\llbracket E_1 == E_2 \rrbracket s &\triangleq \text{wrong} && \text{if } \llbracket E_1 \rrbracket s = \text{wrong} \text{ or } \llbracket E_2 \rrbracket s = \text{wrong} \\
& \text{true} && \text{if } \text{wrong} \neq \llbracket E_1 \rrbracket s = \llbracket E_2 \rrbracket s \\
& \text{false} && \text{if } \text{wrong} \neq \llbracket E_1 \rrbracket s \neq \llbracket E_2 \rrbracket s \neq \text{wrong} \\
\llbracket \text{not } E \rrbracket s &\triangleq \text{false} && \text{if } \llbracket E \rrbracket s = \text{true} \\
& \text{true} && \text{if } \llbracket E \rrbracket s = \text{false} \\
& \text{wrong} && \text{otherwise}
\end{aligned}$$

4 Propositions

The grammar for propositions P is as follows.

$$P ::= E = E \mid E \mapsto E, E \mid P \rightarrow P \mid \mathbf{false} \mid \exists x. P$$

We can define various other connectives as usual: $\neg P \triangleq P \rightarrow \mathbf{false}$; $\mathbf{true} \triangleq \neg(\mathbf{false})$; $P \vee Q \triangleq (\neg P) \rightarrow Q$; $P \wedge Q \triangleq \neg(\neg P \vee \neg Q)$; $\forall x. P \triangleq \neg \exists x. \neg P$.

The assertion language looks rather sparse, but using the expressions a number of properties can be defined. For example, $E = E + 0$ says that E is a number, and $\exists yz. x \mapsto y, z$ says that x is a pointer. In practice, one would also want atomic predicates for describing reachability properties (e.g. [1]), or a definitional mechanism for defining them. Any such predicates could be included, as long as they satisfy the properties Growth, Shrinkage and Renaming described in the next section.

The semantics is described in terms of a judgement of the form

$$s, h \models P$$

which asserts that P holds of total state s, h . We require $\text{Free}(P) \subseteq \text{dom}(s)$, where $\text{Free}(P)$ indicates the set of variables occurring free in P .

The quantifier-free fragment is straightforward.

$$\begin{aligned} s, h \models E = E' & \iff \llbracket E \rrbracket s = \llbracket E' \rrbracket s = v \in \mathbf{Values} \\ s, h \models E \mapsto E_1, E_2 & \iff h(\llbracket E \rrbracket s) = \langle \llbracket E_1 \rrbracket s, \llbracket E_2 \rrbracket s \rangle \\ s, h \models \mathbf{false} & \text{ never} \\ s, h \models P \rightarrow Q & \iff \text{if } s, h \models P \text{ then } s, h \models Q \end{aligned}$$

The only point to note is the interpretation of equality¹, which requires that neither side is wrong.

Normally, one would expect the following interpretation of \exists .

EXISTENTIAL: FIRST TRY

$$s, h \models \exists x. P \iff \exists v \in \mathbf{Values}. (s \mid x \mapsto v), h \models P$$

¹ The symbol for equality is $==$ for expressions and $=$ for propositions; the former is wrong when either side is wrong, and the second asserts that both sides are defined and equal.

(($s \mid x \mapsto v$) is the stack like s except that x maps to v .) The immediate problem is that this is not even well defined: there is no guarantee that ($s \mid x \mapsto v$), h be a total state, so the right-hand side is not well formed. We might attempt to allow partial states and maintain this interpretation, but then we run into the conundrum mentioned in the Introduction.

Another approach that is often suggested is to restrict quantifiers so that they range over reachable elements only.

EXISTENTIAL: SECOND TRY

$$s, h \models \exists x.P \stackrel{\Delta}{\iff} \exists v \in \mathbf{Values}. (s \mid x \mapsto v), h \models P \text{ and} \\ v \in \mathbf{Pointers} \text{ implies } v \text{ is reachable in } s, h$$

The right-hand side is now well-defined, as the reachability requirement ensures that ($s \mid x \mapsto v$) is total. Although tantalizing, this notion has several obstacles to overcome. It invalidates Hoare's assignment axiom, since $\{\exists y. y \mapsto 3, 4\} x := z \{\exists y. y \mapsto 3, 4\}$ fails for it. This happens when $x := z$ detaches a cell with car 3 and cdr 4. Just as significantly, it invalidates the rule of Weakening of contexts in first-order logic. This rule says that if P holds in a context with a collection X of variables, then we know that P holds for all bigger collections. These points do not erect a comprehensive roadblock to the "reachable elements" approach, but they do indicate that such a suggestion requires careful analysis and development.

The solution we adopt in this paper is to accompany the selection of a value with a change of world. That is, if v is a pointer whose value is not determined in the current heap, then we look for a bigger heap where the value is determined. And in doing so, we must also ensure that no dangling pointers are introduced by following v further into the heap. Thus, we look for an extended heap, which maintains total state status.

EXISTENTIAL: OFFICIAL VERSION

$$s, h \models \exists x.P \stackrel{\Delta}{\iff} \exists v \in \mathbf{Values}. \exists g \sqsupseteq h. \\ (s \mid x \mapsto v), g \text{ is total and } (s \mid x \mapsto v), g \models P$$

It is worth spelling out the induced semantics of \forall , that obtains from the $\neg\exists\neg$ encoding:

$$s, h \models \forall x.P \stackrel{\Delta}{\iff} \forall v \in \mathbf{Values}. \forall g \sqsupseteq h. \text{ if } (s \mid x \mapsto v), g \text{ is total} \\ \text{then } (s \mid x \mapsto v), g \models P$$

The semantics handles the example from section 1.1 as follows. Given any

heap, the semantics of \exists allows a new pointer to be selected, along with a heap extension (change of possible world) mapping the pointer into 3, 4; in particular $\exists y. y \mapsto 3, 4$ is always true. So, both of the Hoare triple contexts in section 1.1 boil down to $\{true\}-\{true\}$. And, because of the soundness results in sections 8 and 9, we know that the semantics is not *too* abstract: it makes *enough* distinctions to distinguish genuinely inequivalent commands.

One property of the model is worth noting, the truth of

$$\forall xy. \exists z. z \mapsto x, y$$

This says that for any pair of values we consider, there is some cell we can find which has the pair as its contents. This illustrates the extensible heap view taken by the semantics, where there is an inexhaustible stock of pointers, or locations, with arbitrary possible contents, to choose from. (This formula is true in all states only if `Pointers` is infinite. With a finite collection of pointers, $\forall xy$ could saturate the set of available pointers leaving none for $\exists z. z \mapsto x, y$.)

Many specifications one uses when reasoning about pointer programs are garbage insensitive. For example: x points to a linked list; or x points to a linked list representing the sequence L . With the aid of recursive definitions, these properties can be expressed in our language. For example,

$$list(x) \Leftrightarrow x = \text{nil} \vee (\exists z. x \mapsto -, z \wedge list(z))$$

can be regarded as a definition of “ x points to a linked list”, where $x \mapsto -, z$ means that x points to a cell where the second component contains z , but the first component could contain anything. (We will not give a formal treatment of recursive definitions in this paper; the example is used just to illustrate a garbage insensitive predicate.)

Examples of properties that are not garbage insensitive include some that are invariant under pointer renaming (relocation) and some that are not. An example of the former is “the heap has exactly n cells”, while an example of the latter is “if we add 2 to pointer p , we get pointer q ”.

The semantics of \exists might seem forbidding. When working with a memory safe language, and a total states model, it is hard to imagine an intuitive reason to quantify over non-reachable elements in program specifications that typically arise. Furthermore, the “problem” with the first and second attempts at a semantic definition arise from a seemingly silly example: why would one ever want to write an assertion of the form $\exists y. y \mapsto 3, 4$?

We can partially relieve this bother by observing that, in a variety of naturally occurring cases, the official definition dovetails with either of the first or the second. Put briefly, if we consider a guarded form of quantifier, where we know

not just that the quantified variable is reachable but also how to reach it, then there is no need to select a larger heap.

We formulate a notion of guarding which involves looking only one-deep into the heap; deeper looking could also be considered.

Lemma 2 (Guarding Lemma) *Suppose y is a variable different from x and G is of the form $y \mapsto x, E$ or $y \mapsto E, x$. Then*

$$s, h \models \exists x. G \wedge P \iff \exists v \in \text{Values}. (s \mid x \mapsto v), h \models G \wedge P$$

Note that $(s \mid x \mapsto v), h$ is a total state and the value v of x is uniquely determined by $y \mapsto x, E$ or $y \mapsto E, x$. The \Leftarrow direction of the Lemma is immediate, and the \Rightarrow direction follows from the Garbage Insensitivity Theorem introduced in the next section.

This lemma can be used to justify a simple semantics in many examples. But we will work with the more general semantics, the “official version”, when proving theoretical properties. Also, the official semantics will prove useful later when we show a link to a semantics based on partial states, where quantifying over unreachable locations is more natural.

5 Garbage Insensitivity

There are two aspects to garbage insensitivity, which we label growth and shrinkage.

Growth. If $s, g \models P$, $g \sqsubseteq h$, and s, h is total then $s, h \models P$.

Shrinkage. If $s, h \models P$, $g \sqsubseteq h$, and s, g is total then $s, g \models P$.

Growth says just that if a proposition is true then it remains true if we add extra cells to the heap. *Shrinkage* says the opposite: truth is preserved by removing cells from the heap. In both cases there is the proviso that the cells added or removed do not result in dangling pointers, as our semantics is defined only for total states.

Growth is essentially Kripke’s monotonicity property for intuitionistic logic, but for one difference: the side condition on totality is a property that refers to both the stack and heap components. Conventional Kripke monotonicity would consider all larger heaps (possible worlds), without any side conditions that refer to stacks/environments.

Shrinkage looks like a backwards form of Kripke monotonicity, but is not. Because of the restriction to total states, we will only be able to go back as far as the heap obtained by pruning, which is dependent on s . (In the dense semantics presented later, which allows partial states, forwards Kripke monotonicity does hold, while backwards does not.)

Theorem 3 (Garbage Insensitivity Theorem) *All the propositions satisfy Growth and Shrinkage. As a consequence, each proposition is completely determined by its meaning at garbage-free states:*

$$s, h \models P \text{ iff } \text{prune}(s, h) \models P.$$

The proof of the theorem relies on the Renaming Lemma below. The need arises when proving the Growth case of existentials, because the heap extension in the definition of Growth can overlap with the heap extension in the semantics of existentials, and a renaming is needed to shift one of the heap extension so that they do not overlap. (As a consequence, monotone renamings would be sufficient for the proof.)

Lemma 4 (Renaming Lemma) *All propositions are invariant under pointer renaming. If $s, h \models P$ and $s, h =_{\alpha} s', h'$ then $s', h' \models P$.*

Proof: By induction on the structure of P . ■

If we combine this with Garbage Insensitivity then we obtain a high level correspondent to the idea that propositions are invariant under the operation of a relocating garbage collector. The Renaming Lemma is true because we have provided no facilities for pointer arithmetic among the expressions we consider. This fact is not represented explicitly in the semantic domains (it could be by incorporating naturality conditions), but holds for all definable propositions. If we were to allow unrestricted pointer arithmetic, then propositions (or programs) *would* be sensitive to the effects of a garbage collector; renaming is thus a property that is closely linked (though not in a logically necessary way) with memory safety.

The theorem holds for any collection of atomic predicates that are closed under Growth, Shrinkage and Renaming.

Proof: [of Garbage Insensitivity] By structural induction on P .

The case of **false** is trivial. For $P \rightarrow Q$ we consider *Shrinkage*. Suppose $s, h \models P \rightarrow Q$ and consider $g \sqsubseteq h$ where s, g is total. Either $s, h \not\models P$ or $s, h \models Q$ must hold. If the former, we use the *Growth* part of the induction hypothesis for P to conclude $s, g \not\models P$, and thus $s, g \models P \rightarrow Q$. If the latter we use the *Shrinkage* induction hypothesis to conclude $s, g \models Q$, and thus $s, g \models P \rightarrow Q$. The proof of *Growth* for $P \rightarrow Q$ is symmetric.

For the *Growth* case of $\exists x. P$, if $s, g \models \exists x. P$ then there is v and $k \sqsupseteq g$ with $(s \mid x \mapsto v), k \models P$. Consider a permutation that renames the pointers in the domain of k but not g to be different from all pointers in h , leaving pointers in g fixed. Let k' be the heap obtained from k via this permutation. Then $(s \mid x \mapsto v'), k' \models P$ by the Renaming Lemma, where v has been perhaps renamed to v' (if v is in the domain of k but not g). Let j denote the lub of h and k' . j exists: it is just the union of h and k' , which have g in common but which are otherwise defined on distinct pointers. It is not difficult to see that $(s \mid x \mapsto v'), k'$ is total. (Since the permutation fixes g , no renaming is required in s .) Since $j \sqsupseteq k'$, we can apply the *Growth* induction hypothesis, to obtain $(s \mid x \mapsto v'), j \models P$. Since $j \sqsupseteq h$ we satisfied the right-hand-side of the definition of $s, h \models \exists x. P$.

The *Shrinkage* case of $\exists x. P$ follows immediately from the clause for \exists and the transitivity of \sqsubseteq .

The cases of the equality predicates and points-to relation are immediate from the definition. ■

An amusing point is that the proof of *Shrinkage*, the most unusual property of the semantics, is straightforward; the real work was in the *Growth* case of \exists . However, if we were to take \forall rather than \exists as primitive, then *Growth* would have been trivial, and the work would have had to be put in for *Shrinkage*.

6 Quantifier Laws

The semantics of section 4 clearly satisfies all of the laws of classical propositional logic. Close consideration of quantifier laws is essential, however, to justify the semantic clause for \exists .

We did not consider the quantifier laws before now because the validity of one of them, the elimination rule, is by no means immediate: it depends on Garbage Insensitivity. (To be precise, the proof of the elimination rule will refer only to Growth, but Growth itself depends on Shrinkage in the case of \rightarrow .)

To state the quantifier laws we set up a notion $P \models Q$ of logical consequence, where P and Q are propositions.

Semantic Consequence.

$$P \models Q \stackrel{\Delta}{\iff} \text{for all total } s, h \text{ if } s, h \models P \text{ then } s, h \models Q.$$

(If we were being pedantic, we would parameterize the judgement form with a context X of variables containing those free in P and Q , and require that

all stacks in the definition have domain X . To avoid notational clutter, we simply say that all semantic judgements are assumed to be well formed in our definitions; moreover the premises and conclusions of inference rules are implicitly quantified over their free variables. Similar remarks apply to the definitions of Hoare triples and observational equivalence given later.)

Theorem 5 (Quantifier Laws Theorem) *The semantics validates the usual rule of \exists elimination*

$$\frac{Q \wedge P \models R \quad Q \models \exists x.P}{Q \models R} \quad (x \text{ not free in } Q \text{ or } R)$$

and the following version of \exists introduction:

$$\frac{Q \models P[E/x] \quad Q \models E = E}{Q \models \exists x.P}$$

where $P[E/x]$ is the usual capture-free substitution.

Recall that our interpretation of equality is like in a partial function logic, where $E = F$ can hold only when neither side is *wrong*. Thus, the premise $E = E$ is not trivially true; it ensures that E denotes a genuine value.

In the standard semantics of classical logic, the validity of \exists elimination rests on a weakening lemma:

$$\text{If } s \models P \text{ then } (s \mid x \mapsto v) \models P$$

where v is a value and x is not free in P . Our semantics has a world component as well, so we might attempt:

$$\text{If } s, h \models P \text{ then } (s \mid x \mapsto v), h \models P.$$

But now we run into the problem of total states again, since if v is not defined in h , then $(s \mid x \mapsto v), h$ will not be total. The correct property is as follows.

Lemma 6 (Variable Weakening Lemma) *If $s, h \models P$, $g \sqsupseteq h$, x is not free in P , and $(s \mid x \mapsto v), g$ is total, then $(s \mid x \mapsto v), g \models P$.*

Proof: First, we consider the following

Claim. Suppose x is not free in P and $v \in \mathbf{Values}$. If $v \in \text{dom}(h)$ or $v \notin \mathbf{Pointers}$ then

$$s, h \models P \iff (s \mid x \mapsto v), h \models P$$

The proof of the claim is by induction on P . The cases of equality and *false*

are straightforward, and the two subcases of \rightarrow each exercise both directions of the \iff . The case \mapsto is immediate.

We consider \exists in detail, where we α -rename the bound variable to $\exists y.P$, where y is different from x . For the \Rightarrow direction, if $s, h \models \exists y.P$ then $(s \mid y \mapsto v_y), g \models P$ for some v_y and $g \sqsupseteq h$. Now, since $v \in \text{dom}(h)$ or $v \notin \text{Pointers}$, we have that $((s \mid y \mapsto v_y) \mid x \mapsto v), g$ is total. Also, $((s \mid y \mapsto v_y) \mid x \mapsto v) = ((s \mid x \mapsto v) \mid y \mapsto v_y)$. So, by the induction hypothesis, we get $((s \mid x \mapsto v) \mid y \mapsto v_y), g \models P$, and by the definition of \exists this means $(s \mid x \mapsto v), h \models \exists y.P$

For the \Leftarrow direction, assume $(s \mid x \mapsto v), h \models \exists y.P$. Then $((s \mid x \mapsto v) \mid y \mapsto v_y), g \models P$ for some v_y and $g \sqsupseteq h$. By induction we obtain $(s \mid y \mapsto v_y), g \models P$ and by the definition of \exists this implies $s, h \models \exists y.P$. This completes the proof of the Claim.

To prove the lemma, suppose $s, h \models P$, $g \sqsupseteq h$ and $(s \mid x \mapsto v), g$ is total. Then s, g is also a total state, and by the *Growth* part of Garbage Insensitivity we obtain $s, g \models P$. Since $(s \mid x \mapsto v), g$ is total, we have that either $v \in \text{dom}(g)$ or $v \notin \text{Pointers}$. By the Claim for s, g we obtain $(s \mid x \mapsto v), g \models P$. ■

We can now prove the validity of \exists elimination as follows. If $s, h \models Q$ then $(s \mid x \mapsto v), g \models P$ for some $g \sqsupseteq h$ by the top right premise and the definition of \exists . By the Weakening Lemma, $(s \mid x \mapsto v), g \models Q$. By the top left premise, $(s \mid x \mapsto v), g \models R$, and by Weakening and Garbage Insensitivity $s, h \models R$.

The validity of the \exists introduction rule is comparatively straightforward. If $s, h \models Q$ then $\llbracket E \rrbracket s = v \in \text{Values}$ is not wrong, by the assumption $E = E$, and $s, h \models P[E/x]$ by the top left premise, and $(s \mid x \mapsto v), h \models P$ by the Substitution Lemma to follow. (Note that no change of world is needed here.)

Lemma 7 (Substitution Lemma) *If $\llbracket E \rrbracket s = v \in \text{Values}$ then the following hold*

- (1) $\llbracket E'[E/x] \rrbracket s = \llbracket E' \rrbracket (s \mid x \mapsto v)$
- (2) $s, h \models P[E/x]$ iff $(s \mid x \mapsto v), h \models P$.

The proof of the Substitution Lemma is not difficult, and does not much exercise the change of world used to interpret \exists because the value v is reachable in s, h (since it is defined by a term, E).

7 A Programming Language and Garbage Collecting Semantics

We define a small programming language for altering stacks and heaps.

$$C ::= x := E \mid x.i := E \mid x := E.i \mid x := \text{cons}(E_1, E_2)$$

$$\mid \text{skip} \mid \text{if } E \text{ then } C \mid C; C \mid \text{while } E \text{ do } C \text{ od} \mid \text{newvar } x. C \text{ end}$$

The binding form `newvar` $x. C$ `end` declares a new stack variable, which is de-allocated on block exit. This is why we refer to the s component of the semantics as the stack.

The commands are interpreted using a relation \rightsquigarrow on configurations. Configurations include terminal configurations s, h as well as triples C, s, h . Also, in order to treat `newvar` we introduce a new command `dealloc`(x); this command form is used only in the operational semantics, and is not part of the language. In other sections metavariable C will refer exclusively to the unextended language, while here it includes `dealloc`.

The \rightsquigarrow relation is specified by the following rules.

$$\frac{[[E]]s = v \in \text{Values}}{x := E, s, h \rightsquigarrow (s \mid x \mapsto v), h} \quad \frac{[[E]]s = v \in \text{Values} \quad s(x) = p \in \text{dom}(h)}{x.i := E, s, h \rightsquigarrow s, (h \mid p.i \mapsto v)}$$

$$\frac{[[E]]s = p \in \text{dom}(h) \quad h(p) = \langle v_1, v_2 \rangle}{x := E.i, s, h \rightsquigarrow (s \mid x \mapsto v_i), h}$$

$$\frac{p \notin \text{dom}(h) \quad p \in \text{Pointers} \quad [[E_1]]s = v_1 \in \text{Values} \quad [[E_2]]s = v_2 \in \text{Values}}{x := \text{cons}(E_1, E_2), s, h \rightsquigarrow (s \mid x \mapsto p), (h \mid p \mapsto \langle v_1, v_2 \rangle)}$$

$$\frac{}{\text{skip}, s, h \rightsquigarrow s, h} \quad \frac{[[E]]s = \text{true}}{\text{if } E \text{ then } C, s, h \rightsquigarrow C, s, h} \quad \frac{[[E]]s = \text{false}}{\text{if } E \text{ then } C, s, h \rightsquigarrow s, h}$$

$$\frac{C_1, s, h \rightsquigarrow C'_1, s', h'}{(C_1; C_2), s, h \rightsquigarrow (C'_1; C_2), s', h'} \quad \frac{C_1, s, h \rightsquigarrow s', h'}{(C_1; C_2), s, h \rightsquigarrow C_2, s', h'}$$

$$\frac{[[E]]s = \text{false}}{\text{while } E \text{ do } C \text{ od}, s, h \rightsquigarrow s, h}$$

$$\frac{[[E]]s = \text{true} \quad (C; \text{while } E \text{ do } C \text{ od}), s, h \rightsquigarrow K}{\text{while } E \text{ do } C \text{ od}, s, h \rightsquigarrow K}$$

$$\frac{}{\text{newvar } x. C \text{ end}, s, h \rightsquigarrow C; \text{dealloc}(x), (s \mid x \mapsto \text{nil}), h} \quad x \notin \text{dom}(s)$$

$$\frac{}{\text{dealloc}(x), s, h \rightsquigarrow s - x, h}$$

$$\frac{g \sqsubseteq h, \quad s, g \text{ total}}{s, h \rightsquigarrow s, g} \quad \frac{s, h =_\alpha s, g}{s, h \rightsquigarrow s, g}$$

In the third-last rule $s - x$ is the stack like s but undefined on x . $(h \mid p.i \mapsto v)$ is the heap like h except that the i 'th component of the $h(p)$ cell is v .

From a logical point of view, the final two rules are essentially forcing the Shrinkage and Renaming properties of assertions. If we did not have these

properties, then the interpretations of specifications in the following two sections would not fit well with the operational semantics. From an implementation point of view, these two rules are abstractions of parts of a relocating garbage collector, the first being a rule for collection and the second for relocation.

We say that

- A configuration K is either an intermediate configuration C, s, h or a final configuration s, h ;
- “ C, s, h is stuck” in case there is no configuration K such that $C, s, h \rightsquigarrow K$;
- “ C, s, h goes wrong” when there exists a configuration K such that $C, s, h \rightsquigarrow^* K$ and K is stuck;
- “ C, s, h terminates normally” just if there is s', h' such that $C, s, h \rightsquigarrow^* s', h'$ where \rightsquigarrow^* is the reflexive and transitive closure of \rightsquigarrow .

It is possible to prove a number of properties about this operational semantics, such as that normal termination, going wrong and divergence are mutually exclusive, invariance under pointer renaming, and a kind of determinacy, where the prunes of any two output states gotten from the same initial state must be isomorphic.

Although we will not attempt a precise formulation, it should be clear that the language is memory safe: no execution sequence ever goes wrong from an attempt to dereference a pointer not in the domain of the current heap, as long as we start from a total state whose stack component binds all variables free in the command.

8 Partial Correctness

If P and Q are propositions and C is a command then we have the specification form $\{P\}C\{Q\}$.

PARTIAL CORRECTNESS

$\{P\}C\{Q\}$ is true just if for all total states s, h with $\text{dom}(s) \supseteq \text{Free}(P, C, Q)$, if $s, h \models P$ then

- C, s, h doesn't go wrong, and
- if $C, s, h \rightsquigarrow^* s', h'$ then $s', h' \models Q$.

We have arranged the definition of partial correctness so that well specified programs don't go wrong. That is, if $\{P\}C\{Q\}$ holds then executing C in a state satisfying P never leads to a runtime error.

Using partial correctness assertions it is possible to distinguish between programs that differ in when they go wrong, but that agree on all outputs states when they are reached. As an extreme example, consider a command *diverge* that always diverges without getting stuck (e.g. `while true do $x := x$ od`) and another command `$x := \text{nil}; x.1 := 17$` that always gets stuck. The former satisfies the triple $\{true\} - \{true\}$ while the latter does not. This indicates that the notion of observational equivalence appropriate to partial correctness must distinguish divergence and stuckness.

These considerations lead to a notion \cong_{pc} of observational equivalence based on observing both normal termination and stuckness. In formulating observational equivalence, we will only consider the execution of closed commands, ones where all variables have been bound by `newvar`.

A program context $G[\cdot]$ is a command with a hole \cdot , and $G[C]$ is obtained by replacing \cdot with C , possibly capturing some free variables of C ; a closing context is one which captures all the free variables.

We define $C \cong_{pc} C'$ to hold just if

for all closing contexts $G[\cdot]$,

- $G[C], (), ()$ goes wrong just if $G[C'], (), ()$ does, and
- $G[C], (), ()$ terminates normally just if $G[C'], (), ()$ does.

Theorem 8 (Full Abstraction Theorem, Partial Correctness) $C \cong_{pc} C'$ iff C and C' satisfy the same partial correctness assertions.

The proof of the theorem occupies the remainder of this section. It goes by first relating \cong_{pc} to a semantics of commands which ignores garbage and renaming of locations, and then relating this semantics to partial correctness.

Consider a semantics $\llbracket - \rrbracket_{pc}$ of commands

$$\llbracket C \rrbracket_{pc} \subseteq \mathbf{TStates} \times \mathbf{TStates}_*$$

where $\mathbf{TStates}$ is the set of total states, and $\mathbf{TStates}_* \triangleq \mathbf{TStates} \cup \{wrong\}$. The definition of $\llbracket C \rrbracket_{pc}$ is as follows

$$\begin{aligned} ((s, h), (s', h')) \in \llbracket C \rrbracket_{pc} &\iff C, s, h \rightsquigarrow^* s', h' \\ ((s, h), wrong) \in \llbracket C \rrbracket_{pc} &\iff C, s, h \text{ goes wrong} \end{aligned}$$

The equivalence relation $\sim \subseteq \mathbf{TStates} \times \mathbf{TStates}$ is defined as

$$(s, h) \sim (s', h') \iff \text{prune}(s, h) =_{\alpha} \text{prune}(s', h')$$

where $=_\alpha$ is equality modulo renaming of locations.

Lemma 9 $\llbracket C \rrbracket_{pc}$ induces a partial function between quotients

$$\llbracket C \rrbracket_{pc}^\sim : (\mathbf{TStates}/\sim) \rightarrow (\mathbf{TStates}_*/\sim_*)$$

where $-/-$ is the quotient operation and \sim_* extends \sim with (wrong, wrong). Moreover, $\llbracket C \rrbracket_{pc}^\sim$ respects the structure of C .

Lemma 10 $C \cong_{pc} C' \Leftrightarrow \llbracket C \rrbracket_{pc}^\sim = \llbracket C' \rrbracket_{pc}^\sim$.

Proof: In the \Leftarrow direction, assume $\llbracket C \rrbracket_{pc}^\sim = \llbracket C' \rrbracket_{pc}^\sim$. Then (lemma 9) $\llbracket G[C] \rrbracket_{pc}^\sim = \llbracket G[C'] \rrbracket_{pc}^\sim$ for all closing contexts $G[-]$. The conclusion $C \cong_{pc} C'$ follows by observing that the properties “goes wrong” and “terminates normally” in the definition of \cong_{pc} are preserved by $\llbracket - \rrbracket_{pc}^\sim$.

For the \Rightarrow direction, assume $\llbracket C \rrbracket_{pc}^\sim \neq \llbracket C' \rrbracket_{pc}^\sim$. By symmetry of C and C' we can assume w.l.o.g. that there exist $c \in \mathbf{TStates}/\sim$ and $d \in \mathbf{TStates}_*/\sim_*$ such that

$$\llbracket C \rrbracket_{pc}^\sim(c) = d \quad \llbracket C' \rrbracket_{pc}^\sim(c) \neq d$$

In general there exists a command $Create(c)$ that creates a store in class c from the empty store. $Create(c)$ uses **cons** to create appropriate new cells, and extends the initial stack and heap. There are two cases:

- (1) Case $d = \text{wrong}$. Consider the following context

$$G[-] \triangleq Create(c); -$$

Then $G[C]$ “goes wrong” and $G[C']$ does not, thus $C \not\cong_{pc} C'$.

- (2) Case $d \neq \text{wrong}$. In general there exists a command ending in a boolean expression $Check(d)$ that evaluates to *true* if the current state is in class d , and to *wrong* otherwise (to define an analogous command returning *false* instead of *wrong* one would need language constructs to test if $x.i$ is defined before de-referencing).

To define $Check(d)$, consider a garbage-free store (s, h) in class d . For each $p \in \text{dom}(h)$, there exists a path in the heap reaching p from a stack variable x , let this path be represented by the sequence $i_1 \cdot \dots \cdot i_n$ where each i_j represents the component followed at step j . Define the command

$$C_p \triangleq x_p := x; x_p := x_p.i_1; \dots; x_p := x_p.i_n; x_p^1 := x_p.1; x_p^2 := x_p.2$$

which introduces new free variables x_p, x_p^1, x_p^2 .

Define the expression $E_p \triangleq x_p$, extended to values with $E_v \triangleq v$ if v is a number, or a boolean, or *nil*. We can now define $Check(d)$, for some (s, h) in class d :

$$\begin{aligned}
CH_1 &\triangleq \bigwedge_{p \neq q \in \text{dom}(h)} E_p \neq E_q \\
CH_2 &\triangleq \bigwedge_{(x,v) \in s} x == E_v \\
CH_3 &\triangleq \bigwedge_{(p, \langle v_1, v_2 \rangle) \in h} (x_p^1 == E_{v_1}) \wedge (x_p^2 == E_{v_2}) \\
\text{Check}(d) &\triangleq \text{newvar } \vec{x}_p, \vec{x}_p^1, \vec{x}_p^2. \left(\text{Seq}_{p \in \text{dom}(h)} C_p \right); \\
&\quad \text{if } (CH_1 \wedge CH_2 \wedge CH_3) \text{ then true else not nil}
\end{aligned}$$

where $\vec{x}_p, \vec{x}_p^1, \vec{x}_p^2$ are sequences of variables for $p \in \text{dom}(h)$ and Seq is sequencing of commands. Note that `not nil` is an expression which always evaluates to *wrong*.

Finally, consider the context

$$G[-] \triangleq \text{Create}(c); -; \text{if } \text{Check}(d) \text{ then skip}$$

Then $G[C]$ terminates normally, and $G[C']$ does not, thus $C \not\approx_{pc} C'$.

■

Lemma 11 $\llbracket C \rrbracket_{pc}^{\sim} = \llbracket C' \rrbracket_{pc}^{\sim} \Leftrightarrow C$ and C' satisfy the same partial correctness assertions.

Proof: In the \Rightarrow direction, assume $\llbracket C \rrbracket_{pc}^{\sim} = \llbracket C' \rrbracket_{pc}^{\sim}$. Since the semantics of propositions respects \sim (immediate consequence of garbage insensitivity and invariance under renaming), and since the definition of partial correctness involves the properties “terminates normally” and “goes wrong”, we can conclude that C and C' satisfy the same partial correctness assertions.

For the \Leftarrow direction, suppose that $\llbracket C \rrbracket_{pc}^{\sim} \neq \llbracket C' \rrbracket_{pc}^{\sim}$. For each $c \in (\text{TStates}/\sim)$ there exists a proposition P_c such that P_c is true if and only if the current state is in class c .

$$P_c \triangleq \exists \vec{x}_p. CH'_1 \wedge CH'_2 \wedge CH'_3$$

where the propositions CH'_i are analogous to the expressions CH_i in the definition of $\text{Check}(c)$ in the proof of lemma 10:

$$\begin{aligned}
CH'_1 &\triangleq \bigwedge_{p \neq q \in \text{dom}(h)} E_p \neq E_q \\
CH'_2 &\triangleq \bigwedge_{(x,v) \in s} x = E_v \\
CH'_3 &\triangleq \bigwedge_{(p, \langle v_1, v_2 \rangle) \in h} E_p \mapsto E_{v_1}, E_{v_2}
\end{aligned}$$

Note that the existential does not exercise the change of world since the value of all the quantified variables is determined by assertions $- \mapsto -, -$. There are three cases modulo symmetry of C and C' :

- (1) Case $\llbracket C \rrbracket_{pc}^{\sim}(c) = \text{wrong}$ and $\llbracket C' \rrbracket_{pc}^{\sim}(c) \neq \text{wrong}$. Then $\{P_c\} C \{true\}$ is false and $\{P_c\} C' \{true\}$ is true.
- (2) Case $\llbracket C \rrbracket_{pc}^{\sim}(c) = d \neq \text{wrong}$ and $\llbracket C' \rrbracket_{pc}^{\sim}(c)$ is undefined. Then $\{P_c\} C \{false\}$ is false and $\{P_c\} C' \{false\}$ is true.
- (3) Case $\llbracket C \rrbracket_{pc}^{\sim}(c) = d$ and $\llbracket C' \rrbracket_{pc}^{\sim}(c) = d'$ with $d, d' \neq \text{wrong}$ and $d \neq d'$. Then $\{P_c\} C \{P_d\}$ is true and $\{P_c\} C' \{P_d\}$ is false.

In all the cases C and C' do not satisfy the same partial correctness assertions.

■

So far we have considered a notion of observational equivalence, but a natural question is if the results can be generalized to a notion of observational preorder. From the program logic viewpoint, a natural generalization is to ask whether one command satisfies *at least* the partial correctness assertions that the other does. As for equivalence, we proceed through the semantics $\llbracket - \rrbracket_{pc}$, so we define a preorder on the set of partial functions $(\mathbf{TStates}/\sim) \rightarrow (\mathbf{TStates}_*/\sim_*)$:

$$f \leq f' \iff \text{for each } c \in \text{dom}(f), c \in \text{dom}(f') \text{ and } f(c) \leq f'(c)$$

where $d \leq d' \iff d = d' \in (\mathbf{TStates}/\sim)$ or $d' = \text{wrong}$.

We define $C \sqsubseteq_{pc} C'$ to hold just if

for all closing contexts $G[\cdot]$,

- if $G[C], (), ()$ goes wrong then $G[C'], (), ()$ does
- if $G[C], (), ()$ terminates normally then $G[C'], (), ()$ terminates normally or goes wrong.

Intuitively, the order is as follows:

$$\text{diverge} < \text{terminate normally} < \text{wrong}$$

The following theorem justifies the above orders.

Theorem 12 (Ordered Full Abstraction Theorem, Partial Correctness)

The following are equivalent:

- (1) For each P, Q , if $\{P\} C \{Q\}$ is true then $\{P\} C' \{Q\}$ is true.
- (2) $\llbracket C' \rrbracket_{pc} \leq \llbracket C \rrbracket_{pc}$.
- (3) $C' \sqsubseteq_{pc} C$.

9 Total Correctness

If P and Q are assertions and C is a command then we have the specification form $[P] C [Q]$.

TOTAL CORRECTNESS We say that $[P] C [Q]$ is true just if

- $\{P\} C \{Q\}$ is true, and
- for all s, h , if $s, h \models P$ then C, s, h terminates normally.

Using total correctness assertions we can no longer distinguish between divergence and going wrong in the way that we did with partial correctness. That is, neither divergence nor an always sticking command satisfies $[true] - [true]$, because neither terminates normally. This suggests to define a notion of observational equivalence that conflates divergence and going wrong. Since, in our language, there is no way to recover from, or handle, a runtime error, we can do this by observing termination only (ignoring wrongness).

We define $C \cong_{tc} C'$ to hold just if

for all closing contexts $G[\cdot]$,

- $G[C]()()$ terminates normally just if $G[C']()$ does.

Theorem 13 (Full Abstraction Theorem, Total Correctness) $C \cong_{tc} C'$ iff C and C' satisfy the same total correctness assertions.

Similarly to what we did with partial correctness, we define a total semantics on quotients

$$\llbracket C \rrbracket_{tc}^{\sim} : (\mathbf{TStates} / \sim) \rightarrow (\mathbf{TStates}_{\perp} / \sim_{\perp})$$

where $\mathbf{TStates}_{\perp} \triangleq \mathbf{TStates} \cup \{\perp\}$, \sim_{\perp} extends \sim with (\perp, \perp) , and \perp conflates non-termination and getting stuck. The formal definition of $\llbracket C \rrbracket_{tc}$ is as follows

$$\begin{aligned} \llbracket C \rrbracket_{tc}^{\sim}(c) = d &\iff \llbracket C \rrbracket_{pc}^{\sim}(c) = d \quad \text{when } d \in \mathbf{TStates} / \sim \\ \llbracket C \rrbracket_{tc}^{\sim}(c) = \perp &\iff \llbracket C \rrbracket_{pc}^{\sim}(c) = \{wrong\} \text{ or } \llbracket C \rrbracket_{pc}^{\sim}(c) \text{ undefined} \end{aligned}$$

To justify the definition of $\llbracket C \rrbracket_{tc}^{\sim}$ we observe that the identification of non-termination and stuckness is compatible with the (operational) semantics of commands, hence it is compositional.

Lemma 14 $C \cong_{tc} C' \iff \llbracket C \rrbracket_{tc}^{\sim} = \llbracket C' \rrbracket_{tc}^{\sim}$.

Proof: In the \Leftarrow direction, assume $\llbracket C \rrbracket_{tc}^{\sim} = \llbracket C' \rrbracket_{tc}^{\sim}$. Then $\llbracket G[C] \rrbracket_{tc}^{\sim} = \llbracket G[C'] \rrbracket_{tc}^{\sim}$ for all closing contexts $G[-]$. The conclusion $C \cong_{tc} C'$ follows by observing

that the property “terminates normally” in the definition of \cong_{tc} is preserved by $\llbracket - \rrbracket_{tc}^{\sim}$ (the same would not hold of property “goes wrong”).

For the \Rightarrow direction, assume $\llbracket C \rrbracket_{tc}^{\sim} \neq \llbracket C' \rrbracket_{tc}^{\sim}$. Then there exist c, d, d' such that $\llbracket C \rrbracket_{tc}^{\sim}(c) = d$ and $\llbracket C' \rrbracket_{tc}^{\sim}(c) = d'$ and $d \neq d'$, hence one of d, d' must be different from \perp , say d . Consider the context

$$G[-] \triangleq \text{Create}(c); -; \text{Check}(d)$$

where $\text{Create}(c)$ and $\text{Check}(d)$ are like in the proof of Lemma 10. Then $G[C]$ terminates normally while $G[C']$ does not, hence $C \not\cong_{tc} C'$. ■

Lemma 15 $\llbracket C \rrbracket_{tc}^{\sim} = \llbracket C' \rrbracket_{tc}^{\sim} \Leftrightarrow C$ and C' satisfy the same total correctness assertions.

Proof: In the \Rightarrow direction, assume $\llbracket C \rrbracket_{tc}^{\sim} = \llbracket C' \rrbracket_{tc}^{\sim}$. Since the semantics of propositions respects \sim , and since the definition of total correctness involves the property “terminates normally” (and not “goes wrong”), we can conclude that C and C' satisfy the same total correctness assertions.

For the \Leftarrow direction, assume $\llbracket C \rrbracket_{tc}^{\sim} \neq \llbracket C' \rrbracket_{tc}^{\sim}$. Then there exist c, d, d' such that $\llbracket C \rrbracket_{tc}^{\sim}(c) = d$ and $\llbracket C' \rrbracket_{tc}^{\sim}(c) = d'$ and $d \neq d'$, hence one of d, d' must be different from \perp , say d . Now, $[P_c]C[P_d]$ is true and $[P_c]C'[P_d]$ is false, where P_c and P_d are like in the proof of Lemma 11. This concludes the proof that C and C' do not satisfy the same total correctness assertions. ■

As for partial correctness, the above results generalize to preorders. The partial order on functions $(\text{TStates}/\sim) \rightarrow (\text{TStates}_{\perp}/\sim_{\perp})$ is defined as

$$f \leq f' \triangleLeftrightarrow f(c) \leq f'(c) \text{ for each } c \in (\text{TStates}/\sim)$$

where $d \leq d' \triangleLeftrightarrow d = d' \in (\text{TStates}/\sim)$ or $d = \perp$.

We define $C \sqsubseteq_{tc} C'$ to hold just if

for all closing contexts $G[\cdot]$,

- if $G[C], (), ()$ terminates normally then $G[C'], (), ()$ does.

The following theorem relates the above orders.

Theorem 16 (Ordered Full Abstraction Theorem, Total Correctness)

The following are equivalent:

- (1) For each P, Q , if $[P]C[Q]$ is true then $[P]C'[Q]$ is true.
- (2) $\llbracket C \rrbracket_{tc} \leq \llbracket C' \rrbracket_{tc}$.
- (3) $C \sqsubseteq_{tc} C'$.

A difference emerges with respect to partial correctness: while more divergent commands prove more partial correctness assertions, they prove fewer total correctness assertions.

10 Partial States and Density

Focussing on partial states opens up possibilities for connecting the semantics to other, more general, forms of interpretation. In this section we establish a connection to the dense semantics of classical logic [4,11]. This connection provides welcome theoretical support for the clauses in the total states model, since utmost care is needed whenever the semantics of quantifiers is altered.

We begin by defining a notion of support, which works for partial states. The idea is that a state s, h supports P if it contains enough information to conclude P . Technically, this means that, for any total state we wish to extend the partial state to, P will hold.

Suppose s, h is a state, perhaps partial. Then

$$s, h \Vdash P \stackrel{\Delta}{\iff} \forall g \sqsupseteq h. \text{ if } s, g \text{ is total then } s, g \models P$$

Although \Vdash is defined in terms of \models , it can be regarded as a semantics in its own right, in that it can be defined by using semantic clauses, without reference to \models . However, a pointwise characterization of the connectives does not work. To see why, consider that $s, [] \Vdash (x \mapsto 3, 4) \rightarrow \text{false}$ does not hold, while the property “ $s, [] \Vdash x \mapsto 3, 4$ implies $s, [] \Vdash \text{false}$ ” does. Thus, for the partial state semantics we need to consider changes of world to interpret implication.

A first point to note is that all assertions satisfy the monotonicity condition from Kripke’s interpretation of intuitionistic logic.

Monotonicity Condition:

$$s, h \Vdash P \implies \forall g \sqsupseteq h. s, g \Vdash P.$$

This follows from the definition of \Vdash in terms of \models .

The Monotonicity Condition is not the whole story, in that it does not account for excluded middle, garbage, or the central role of total states. These are summed up in further properties satisfied by \Vdash .

Totality Condition:

$$s, h \Vdash P \iff \forall g \sqsupseteq h. \text{ if } s, g \text{ is total then } s, g \Vdash P.$$

Partial Shrinkage Condition:

If $s, h \Vdash P$, $g \sqsubseteq h$, and the domain of g contains all those pointers reachable in s, h , then $s, g \Vdash P$.

Density Condition:

$s, h \Vdash P \iff \forall h' \sqsupseteq h. \exists h'' \sqsupseteq h'. s, h'' \Vdash P$.

These conditions are not independent. In particular, Density follows from Monotonicity and Partial Shrinkage. Totality and Shrinkage are of interest from the perspective of pointers, while Density is more of a logical property.

Density is a famous property, which can be understood from several perspectives (see [11]). One is the perspective of sheaves, where propositions satisfying density are those from a $\neg\neg$ topology. Totality and Partial Shrinkage are also reminiscent of topological properties, in what is known as the atomic topology (indeed, the results of this paper were obtained in a topological setting first, and then simplified by restricting to the total states model, as justified by the theorem below.) Another way to view density is from the perspective of Kripke's semantics of intuitionistic logic, applied to the result of a $\neg\neg$ translation from classical into intuitionistic logic. In either case, the result is a possible world semantics satisfying Kripke's monotonicity property, but also all of the laws of classical logic.

Lemma 17 *Suppose $g \sqsubseteq h$, and the domain of g contains all those pointers reachable in s, h . Then for all $g' \sqsupseteq g$ such that s, g' is total there exists $h' \sqsupseteq h$ such that s, h' is total and $\text{prune}(s, g') =_{\alpha} \text{prune}(s, h')$.*

Theorem 18 (Dense Semantics Theorem) *All propositions satisfy Totality, Partial Shrinkage and Density. Furthermore, the standard clauses of dense semantics all hold.*

$s, h \Vdash \text{false} \quad \text{never}$

$s, h \Vdash P \rightarrow Q \iff \forall h' \sqsupseteq h. \text{if } s, h' \Vdash P \text{ then } s, h' \Vdash Q$

$s, h \Vdash \exists x.P \iff \forall h' \sqsupseteq h, \exists h'' \sqsupseteq h'. \exists v \in \text{Values}. (s \mid x \mapsto v), h'' \Vdash P$

As a result, the forcing relation \Vdash satisfies all the laws of classical logic ² (with the appropriate side-condition on \exists -intro to account for definedness).

² The reader may wonder whether such an unusual semantics of \rightarrow provides a good basis for programmers to specify programs. This worry is a good one, and we would say that the dense semantics is mostly of theoretical interest, as an analysis of the fine structure of the total states semantics. But, we also emphasize that the "unusual" semantics of \rightarrow corresponds to the typical, pointwise, semantics when evaluated at total states.

Proof: The \Rightarrow direction of Density follows at once from the Monotonicity Condition. For the \Leftarrow direction, suppose that the right-hand side holds but $s, h \not\Vdash P$. This means that there is a total state s, g extending s, h where $s, g \not\Vdash P$. By the Shrinkage part of the Garbage Insensitivity Theorem, it follows that $s, g' \not\Vdash P$ for all $g' \sqsupseteq g$ where s, g' is total. This contradicts the assumed right-hand side of Density: g is a heap that has no superheap supporting P . This shows Density. Totality is straightforward, and Partial Shrinkage follows immediately from Lemma 17, Garbage Insensitivity and the definition of \Vdash .

To check the semantic clauses we consider implication only. The \Rightarrow direction is immediate. For \Leftarrow , suppose the right-hand side holds and $s, h \not\Vdash P \rightarrow Q$. Then there is a larger total state s, g where $s, g \models P$ and $s, g \not\Vdash Q$. But this contradicts the rhs, because if we pick $g \sqsupseteq h$ then we must have that $s, g \Vdash P$ implies $s, g \Vdash Q$ (note that \Vdash and \models agree on total states). This shows \rightarrow . ■

The conditions in this theorem can be taken as the basis of an alternate definition of \Vdash , one that does not appeal to \models . As we intimated above, it is standard that the semantic clauses above validate all of classical logic, as long as all atomic propositions satisfy density [4,11]. One way to present classical logic is as the $(\rightarrow, false)$ fragment of intuitionistic logic, with the addition of reductio ad absurdum: $((P \rightarrow false) \rightarrow false) \rightarrow P$. The reader may enjoy verifying this law from the forcing clause for \rightarrow in the theorem, assuming that P is dense. From this one can obtain all boolean algebra laws for other classical connectives and universal quantification, using their usual abbreviations in terms of $\rightarrow, false$ and $\exists x.P$.

In summary, because \Vdash and \models agree on total states, the total state semantics can be viewed as a specialization of an established semantics, the dense semantics.

11 Final Remarks

In this paper we have presented a semantic model based on total states, and interpretations of Hoare triples. The main results are garbage insensitivity and full abstraction with respect to natural notions of observational equivalence.

It is possible, though by no means trivial, to use these results to account for proof rules for reasoning about programs in the presence of a garbage collector. Separation logic [14,9,12,15] is a recent development that permits reasoning about imperative programs that use shared mutable data structures. One of the main features of separation logic is the presence of a separating conjunction $P * Q$, holding of a heap when it can be split into two disjoint subheaps

where P , respectively Q , hold; this has been used to give simplified rules for pointer allocation, update, and even disposal. A separating conjunction that is compatible with garbage collection can be formulated using the partial states model of this paper: partial states are important because the operation of partitioning a heap into disjoint parts often results in a partial state, even when the beginning state is total. A detailed account of the interaction between garbage insensitivity and separation logic proof rules may be found in [2].

We began the paper by noting a logical conundrum, and then provided an analysis of assertions that provides a technical resolution to it. The price to pay for this resolution is an unusual and unfamiliar semantics, certainly in the total states form, but even in the partial states form (the dense semantics will be unfamiliar to the average programmer, or even average program prover). It may be that the price is too high to justify a separate assertion language for practical purposes. However, the garbage insensitive assertions can be seen as special cases of the standard (intuitionistic and classical) semantics of separation logic, using $\neg\neg$ and modal embeddings; see [2]. So a less radical reaction to the conundrum would be to work with a standard semantics, and simply keep an eye out for propositions that are garbage insensitive. From this perspective the results here can be viewed as showing some of the structure that these propositions possess. Incidentally, embedding garbage insensitive predicates into a larger language might help in the treatment of a runtime system that includes both a compiled user program and a collector, where the compiled program is expected to work in a way that is garbage insensitive, while the collector itself is lower level.

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