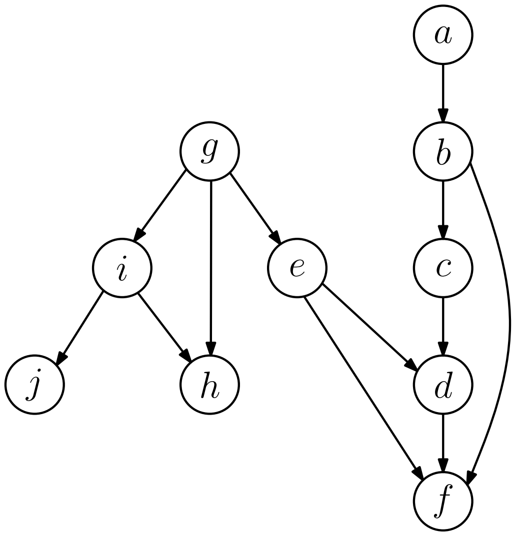

Figure 1: Directed graphical model.

1. Given the graphical model in Figure 1, which of the following conditional independence statements are correct?

   (a) $a \perp\!\!\!\perp f$

   (b) $a \perp\!\!\!\perp g$

   (c) $b \perp\!\!\!\perp i|f$

   (d) $d \perp\!\!\!\perp j|g, h$

   (e) $i \perp\!\!\!\perp b|h$

   (f) $j \perp\!\!\!\perp d$

   (g) $i \perp\!\!\!\perp c|h, f$

2. Consider two random variables x, y with joint distribution $p(x, y)$. Show that:

$$\mathbb{E}_x[x] = \mathbb{E}_y\Big[\mathbb{E}_x[x|y]\Big]$$

   Here, $\mathbb{E}_x[x|y]$ denotes the expected value of x under the conditional distribution $p(x|y)$, with a similar notation for the conditional variance.

3. Consider a Gaussian random variable $\boldsymbol{x} \sim \mathcal{N}\big(\boldsymbol{x}\,|\,\boldsymbol{\mu}_x, \Sigma_x\big)$, where $\boldsymbol{x} \in \mathbb{R}^D$.

   The random variable $\boldsymbol{x}$ is transformed according to

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} + \boldsymbol{w}\,, \tag{1}$$

   where $\boldsymbol{y} \in \mathbb{R}^E$, $\boldsymbol{A} \in \mathbb{R}^{E\times D}$, $\boldsymbol{b} \in \mathbb{R}^E$, and $\boldsymbol{w} \sim \mathcal{N}\big(\boldsymbol{w}\,|\,\boldsymbol{0}, \boldsymbol{Q}\big)$ is independent Gaussian noise. "Independent" means that $\boldsymbol{x}$ and $\boldsymbol{w}$ are independent random variables.

   (a) Write down $p(\boldsymbol{y}|\boldsymbol{x})$.

(b) $p(\boldsymbol{y}) = \int p(\boldsymbol{y}|\boldsymbol{x})p(\boldsymbol{x})d\boldsymbol{x}$ is Gaussian.[1] Compute the mean $\boldsymbol{\mu}_y$ and the covariance Σ_y. Derive your result in detail.

(c) Now, a value $\hat{\boldsymbol{y}}$ is measured. Compute the posterior distribution $p(\boldsymbol{x}|\hat{\boldsymbol{y}})$.[2]
Hint for solution: Start by explicitly computing the joint Gaussian $p(\boldsymbol{x}, \boldsymbol{y})$. This also requires to compute the cross-covariances $\mathrm{Cov}_{x,y}[\boldsymbol{x}, \boldsymbol{y}]$ and $\mathrm{Cov}_{y,x}[\boldsymbol{y}, \boldsymbol{x}]$. Then, apply the rules for Gaussian conditioning.

[1] affine transformation of the Gaussian random variable $\boldsymbol{x}$

[2] This posterior is also Gaussian, i.e., we need to determine only its mean and covariance matrix.