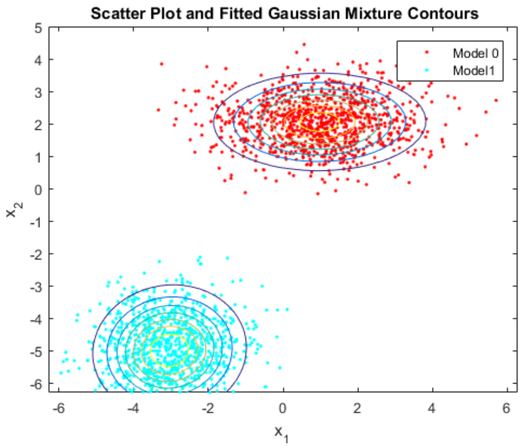

Figure 1: Joint distribution $p(x_1, x_2)$ over two parameters

1. Rejection sampling in high dimensions: Consider a true distribution $p(\boldsymbol{x}) = \mathcal{N}\left(\mathbf{0}, \sigma_p^2 \boldsymbol{I}\right)$ and a proposal distribution $q(\boldsymbol{x}) = \mathcal{N}\left(\mathbf{0}, \sigma_q^2 \boldsymbol{I}\right)$ where $\sigma_q = 1.01\sigma_p$.
    - What is the value of k such that $kq \geq p$, if $\boldsymbol{x} \in \mathbb{R}^{1000}$?
    - What is the acceptance rate of rejection sampling with this proposal distribution?
2. Why may Gibbs sampling have a hard time exploring the parameter distribution in Fig. 1?
3. In Metropolis-Hasting, we accept samples with probability α, where

$$\alpha(x', x^{(t)}) = \min\left(1, \frac{\tilde{p}(x')q(x^{(t)}|x')}{\tilde{p}(x^{(t)})q(x'|x^{(t)})}\right).$$

   Show that $p(x)$ is an invariant distribution of the Markov chain by showing detailed balance.

4. Consider the following model:

$$p(y) = \mathcal{N}\left(\mu, \tau^{-1}\right)$$
$$p(\mu) = \mathcal{N}\left(\mu_0, s_0\right)$$
$$p(\tau) = \mathrm{Gamma}(a, b)$$

   Here, τ is the precision (inverse variance) and y is an observed data point.
    - Draw the corresponding directed graphical model, including all deterministic parameters
    - Write down the expressions for the joint distribution $p(y, \mu, \tau)$
    - Write down expressions for the conditional distributions $p(\mu|y, \tau)$ and $p(\tau|\mu, y)$, which are required to apply Gibbs sampling to the posterior distribution $p(\mu, \tau|y)$