Tutorial 4: Solution

1. (a) The following happens during initialisation:
   - A sends a \( \pi \) message to T, S sends a \( \pi \) message to L and B
   - T,L and B recalculate their posterior probability
   - T sends a \( \pi \) message to X and D, L sends a \( \pi \) message to X and D, B sends a \( \pi \) message to D.
   - X and D recalculate their posterior probabilities (with \( \lambda \) evidence 1 for each state)
   - Propagation finishes
   - (b) S sends a \( \pi \) message to L and B over-riding previous message based on its prior probability.
   - L and B re-calculate their posterior probability (with \( \lambda \) evidence 1 for each state)
   - L sends a \( \pi \) message to X and D, B sends a \( \pi \) message to D
   - X and D recalculate their posterior probabilities (with \( \lambda \) evidence 1 for each state)
   - Propagation finishes
   - (c) X sends a \( \lambda \) message to T and L
   - T and L calculate their posterior probability from both \( \lambda \) and \( \pi \) evidence.
   - T sends a \( \lambda \) message to A and a \( \pi \) message to D
   - A and D recalculate their posterior probabilities. (D still has no \( \lambda \) evidence)
   - L sends a \( \lambda \) message to S (which is ignored) and a \( \pi \) message to D.
   - D recalculates its posterior probability distribution (D still has no \( \lambda \) evidence)
   - Propagation finishes
   - (d) The problem now is that a loop is established. After instantiation D (among other things) sends a \( \lambda \) message to T which sends a \( \pi \) message to X which sends a \( \lambda \) message to L which sends a \( \pi \) message to D, and probability propagation never terminates.

2. T has two states \( t_1 \) means tuberculosis present and \( t_2 \) tuberculosis not present. Hence:
   \[
P(t_1 | t_1 \& l_1) = P(t_1 | t_1 \& l_2) = P(t_1 | t_2 \& l_1) = 1
   \]
   \[
P(t_1 | t_2 \& l_2) = 0
   \]
   The columns of the link matrix sum to 1, hence we can write it as:
   \[
P(LT | T \& L) = \begin{bmatrix}
P(t_1 | t_1 \& l_1) & P(t_1 | t_1 \& l_2) & P(t_1 | t_2 \& l_1) & P(t_1 | t_2 \& l_2) \\
P(t_2 | t_1 \& l_1) & P(t_2 | t_1 \& l_2) & P(t_2 | t_2 \& l_1) & P(t_2 | t_2 \& l_2)
\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1\end{bmatrix}
\]

3. The first step is to calculate the \( \pi \) evidence for \( LT \) (operating equation 4). We can write \( \pi_{LT}(t_1 \& l_1) = \pi_{LT}(t_1) \times \pi_{LT}(l_1) = 0.2 \). Similarly \( \pi_{LT}(t_2 \& l_1) = 0.3, \pi_{LT}(t_2 \& l_2) = 0.3 \). Multiplying this by the link matrix we get \( \pi(l_{t_1}) = 0.7 \) and \( \pi(l_{t_2}) = 0.3 \). Amassing all the evidence at \( LT \) we get \( \epsilon(LT) = (0.21, 0.21) \) and the probability distribution is therefore \((0.5, 0.5)\).

Now tuberculosis is instantiated to state \( t_2 \) so \( \pi_{LT}(T) \) becomes \((0, 1)\). So, \( \pi_{LT}(t_1 \& l_1) = 0, \pi_{LT}(t_1 \& l_2) = 0, \pi_{LT}(t_2 \& l_1) = 0.5, \pi_{LT}(t_2 \& l_2) = 0.5 \) and hence \( \pi(l_{t_1}) = \pi(l_{t_2}) = 0.5 \). The total evidence for \( LT \) is now \((0.15, 0.35)\) and \( P'(LT) = (0.3, 0.7) \).

We can now calculate the \( \pi \) messages to the children. \( \pi_X(LT) = P'(LT) / \lambda_X(LT) = (1, 1) \) (operating equation 2) (notice that because the \( \pi \) evidence for \( LT \) happens to be \((0.5, 0.5)\) the only evidence for one state or the other of \( LT \) is the \( \lambda \) evidence from \( X \), hence the \( \pi \) message to \( X \) correctly conveys no evidence.)

\[
\pi_D(LT) = P'(LT) / \lambda_D(LT) = (0.3, 0.7).
\]

The \( \lambda \) message sent to \( L \) is calculated using operating equation 1.

\[
\lambda_{LT}(l_1) = 0 \times (1 \times 0.3 + 0 \times 0.7) + 1 \times (1 \times 0.3 + 0 \times 0.7) = 0.3 \\
\lambda_{LT}(l_2) = 0 \times (1 \times 0.3 + 0 \times 0.7) + 1 \times (0 \times 0.3 + 1 \times 0.7) = 0.7
\]

4. If \( D \) is not instantiated then propagation will terminate for any other instantiation. If \( D \) is instantiated, then one of \( LT, L, S \) or \( B \) must also be instantiated for the propagation to terminate.