Tutorial 5: Solution

1. $$P(B|A) \quad \begin{array}{c|cc} & a_1 & a_2 \\ \hline b_1 & 0.5 & 0.5 \\ b_2 & 0.5 & 0.5 \end{array} \qquad P(C|A) \quad \begin{array}{c|cc} & a_1 & a_2 \\ \hline c_1 & 0.5 & 0.5 \\ c_2 & 0.5 & 0.5 \end{array} \qquad P(B\&C|A) \quad \begin{array}{c|cc} & a_1 & a_2 \\ \hline b_1c_1 & 0.25 & 0 \\ b_1c_2 & 0.25 & 0.5 \\ b_2c_1 & 0.25 & 0.5 \\ b_2c_2 & 0.25 & 0 \end{array}$$

   The joined link matrix describes a relationship not seen in the independent matrices.

2. (a) Joined

   $$\begin{bmatrix} 0.25 & 0 \\ 0.25 & 0.5 \\ 0.25 & 0.5 \\ 0.25 & 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.025 \\ 0.125 \\ 0.125 \\ 0.025 \end{bmatrix}$$

   (b) Independent (both B and C)

   $$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.15 \\ 0.15 \end{bmatrix}$$

   The joined network propagates information, but the independent does not.

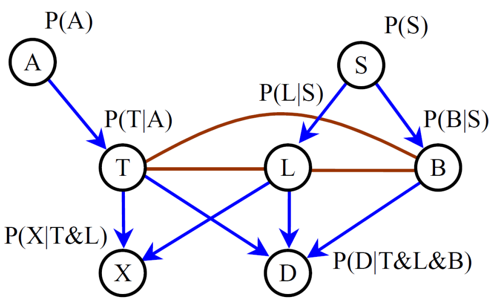
3. (a) Joined

   $$[1\ 1\ 0\ 0] \begin{bmatrix} 0.25 & 0 \\ 0.25 & 0.5 \\ 0.25 & 0.5 \\ 0.25 & 0 \end{bmatrix} = [0.5\ 0.5]$$

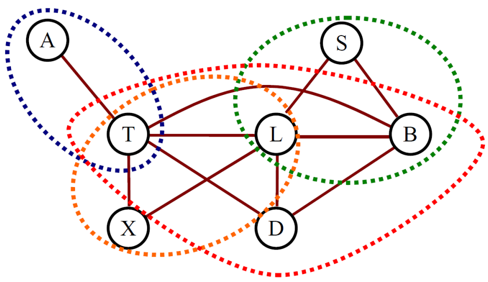
   (b) Independent

   $$[1\ 0] \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = [0.5\ 0.5]$$

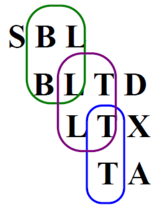
4. The moral graph has three new arcs. There is no need for further triangulation.



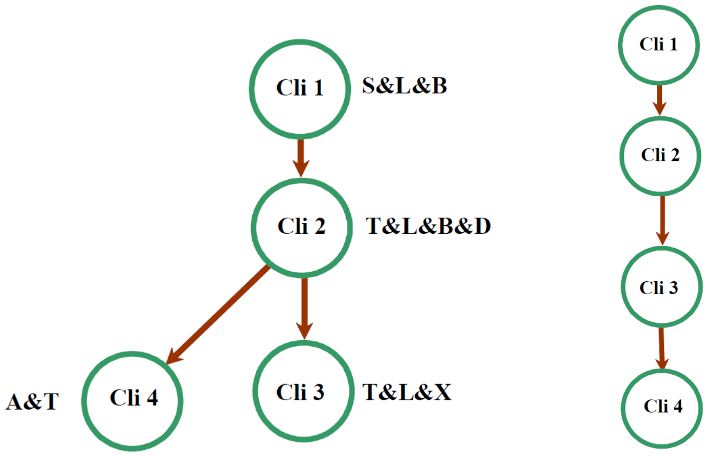
   The cliques are as follows:



Grouping the nodes in cliques gives us four sets as follows: [AT], [SLB], [TLBD], [TLX]. We now need to define the running intersection property. This means joining the cliques such that for any clique those variables which appear higher up in the tree appear in its immediate parent node. We can do this by breadth first search starting from a clique containing a root of the original network. There is no unique solution, here is one ordering:



There are several possible join trees that obey the running intersection property with this ordering



From now on we use the first tree.
We now need to determine the X set variables. We do this by finding the variables in each clique for which the parents are in the same clique. The potential functions are allocated on the basis of the X variables in each clique. Prior probabilities go to the clique where their variable appears. As these are roots of the original tree they will belong to exactly one clique. Conditional probabilities go to the clique containing both the parents and the children.

Clique	W_i	S_i	R_i	X_i	Ψ
1	SLB		SLB	SLB	$P(S)P(L\|S)P(B\|S)$
2	LBTD	BL	TD	D	$P(D\|T\&L\&B)$
3	LTX	LT	X	X	$P(X\|T\&L)$
4	TA	T	A	AT	$P(A)P(T\|A)$

Propagation with nothing instantiated:

Clique	S	R	Ψ (initially)	$P(R\|S)$	$P(S)$
1		SLB	$P(S)P(L\|S)P(B\|S)$	$P(S\&L\&B)$	
2	BL	TD	$P(D\|B\&L\&T)$	$P(T\&D\|B\&L)$	$P(B\&L)$
3	TL	X	$P(X\|T\&L)$	$P(X\|T\&L)$	$P(T\&L)$
4	T	A	$P(A)P(T\|A)$	$P(A\|T)$	$P(T)$

We note that during initialisation we can calculate $P(R|S)$ using $\Psi/\Sigma_R\Psi$ in every clique.
Clique 1: $P(R|S) = P(SLB) = \Psi$ (since $\Sigma_{SLB}P(SLB)$ evaluates to 1)

Clique 3: $P(R|S) = P(X|TL)$, (this is the same as $\Psi/\Sigma_R\Psi$ since $\Sigma_X P(X|TL)$ evaluates to 1 everywhere)
Clique 4: $P(R|S) = P(A|T) = P(A)P(T|A)/P(T)$ (Bayes Theorem) = $\Psi/\Sigma_A\Psi$
Clique 2: We want to show that the potential function initialises to $P(R|S) = P(TD|BL)$.
We know $P(DBLT) = P(TD|BL)P(BL) = P(D|BLT)P(BLT) = P(D|BLT)P(BL)P(T|BL) = P(D|BLT)P(BL)P(T)$
Remember that we are considering the special case where nothing has been instantiated. Thus T is not dependent on BL. This allows us to substitute $P(T) = P(T|BL)$ in this case.
Thus we have that $P(TD|BL) = P(D|BLT)P(T)$.
Although we set the potential function to $P(D|BLT)$ which is not the same as $P(TD|BL)$, during the initialisation there will be a λ message arriving from clique 4. This λ evidence is $\Sigma_A P(A)P(T|A) = P(T)$. So once the λ evidence has been propagated the potential function of clique 2 will be $P(D|BLT)P(T)$ and it will be equal to $P(TD|BL)$.

To conclude the initialisation there is no λ evidence from clique 3 since $\Sigma_X P(X|T\&L)$ evaluates to all 1s. The λ evidence from 2 to 1 $\Sigma_{TD} P(T\&D|B\&L)$ again evaluates to no evidence.

π evidence: This is the $P(S)$ value and is calculated by marginalising the joint probability of the parent node $P(R\&S) = P(R|S)P(S)$, starting at the top and working down.

Propagation with D instantiated:

Clique	S	R	Ψ (initially)	$P(R\|S)$	$P(S)$
1		SLB	$P(S)P(L\|S)P(B\|S)$	$P(S\&L\&B)$	
2	BL	T	$P(B\&L\&T)$	$P(T\|B\&L)$	$P(B\&L)$
3	TL	X	$P(X\|T\&L)$	$P(X\|T\&L)$	$P(T\&L)$
4	T	A	$P(A)P(T\|A)$	$P(A\|T)$	$P(T)$

The only change is in clique 2. With D instantiated the given conditional probability matrix reduces into a joint evidence $P(B\&L\&T)$ which depends on the state of D. We can now calculate $P(T|B\&L)$ by dividing out the evidence for $B\&L$ thus $P(T|B\&L) = P(B\&L\&T)/P(B\&L) = \Psi/\Sigma_R\Psi$ (as normal)

The λ evidence from below is the same as before. However, this time there will also be λ evidence from clique 2 to clique 1. This is computed as $\Sigma_T\Psi(Cl2)$.

The π evidence is calculated as before.