Coalgebraic Logics via Categorical Duality

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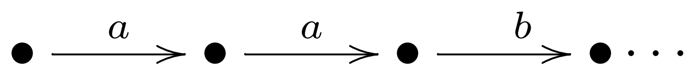
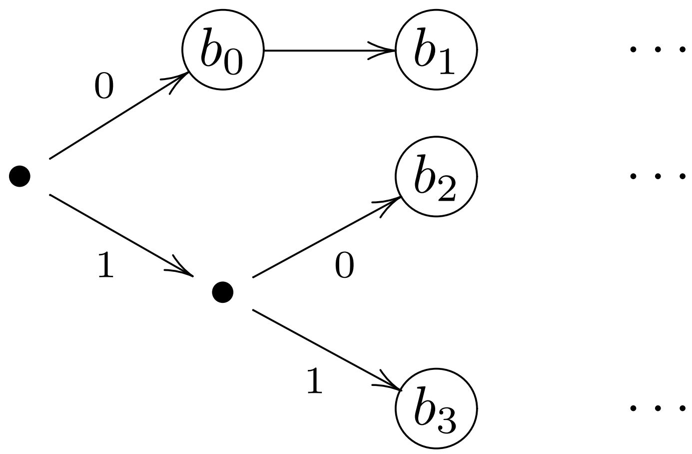
CT 2009, Cape Town

Infinite Streams (over a set A of labels)



- state set C
- dynamics $\gamma : C \to A \times C$

Stream Functions $A^\omega \to B^\omega$ (resp. their representations)

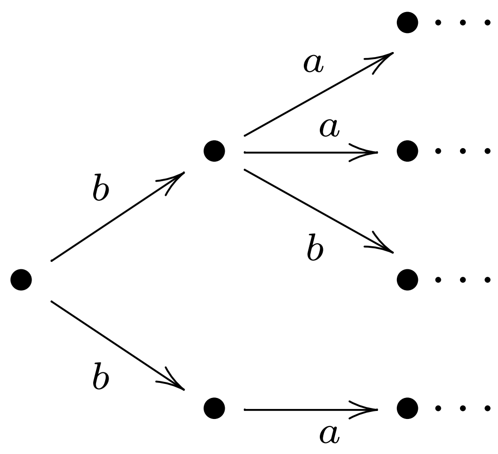


- state set C
- dynamics $\gamma : C \to FC$

$$FC = \mu Y.B \times C + (A \to Y)$$

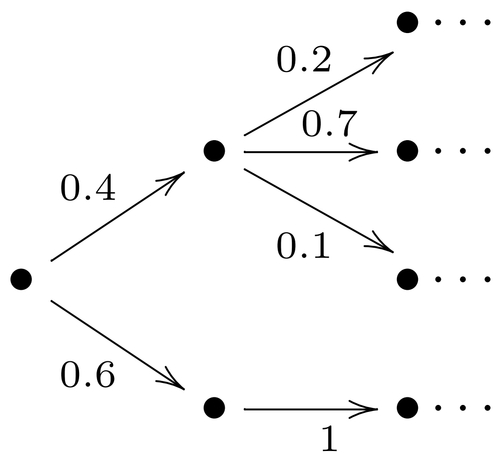
Coalgebras in Computer Science

Nondeterministic Transition Systems (over a set A of labels)



- state set C
- dynamics $\gamma : C \to \mathcal{P}(A \times C)$

Probabilistic Transition Systems



- state set C
- dynamics $\gamma : C \to \mathcal{D}(C)$

$\mathcal{D}(C)$: probability distributions over C

Interlude: the role of Set

Disclaimer. Why restrict the discourse to coalgebras on Set?

- universally understood in Computing
- "standard" systems usually formulated set-theoretically (eg transition systems)
- in applications: systems often have *finite* carrier

Claimer(?)

- but even this leads to (moderately) intersting categorical questions . . .

Duality I

Now let $T : \mathsf{Set} \to \mathsf{Set}$ and CABA = completely atomic boolean algebras

First Idea. Extend the duality

$$\mathsf{Set} \underset{\mathrm{Sp}}{\overset{\mathsf{Clp}}{\rightleftarrows}} \mathsf{CABA}$$

between sets and completely atomic boolean algebras to algebras / coalgebras

$$\begin{array}{ccc}
\mathsf{Coalg}(T) & \rightleftarrows & \mathsf{Alg}(T^{\mathrm{op}}) \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\
\mathsf{Set} & \underset{\mathrm{Sp}}{\overset{\mathcal{P}}{\rightleftarrows}} & \mathsf{CABA}
\end{array}$$

Conceptually.

- use "logic of CABA" (infinitary propositional logic) to describe underlying space
- find some extension to describe $\mathsf{Alg}(T^{\mathrm{op}})$ logically

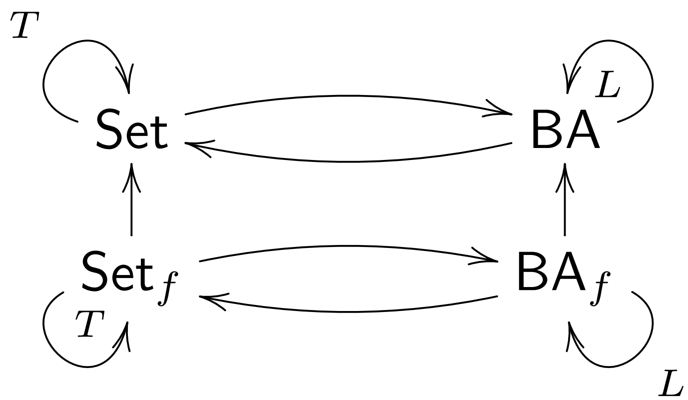
Duality II

Problem. The "logic of CABA" is infinitary

- hard to implement, eg in a theorem prover / model checker
- typical specifications are finitary, anyway

Second Idea. Trash the CA and use BA's instead

- faithful descriptions of *finite* systems
- finite model property / finite algebra property interesting in its own right



But. Functors would have to preserve finite sets! (violated eg for probabilities)

Duality III

Finite Models via Finite Algebras: Restricting Stone-duality

$$\mathsf{BA} \underset{\mathrm{Clp}}{\overset{\mathrm{Sp}}{\rightleftarrows}} \mathsf{Stone}$$

to a duality between *finite* modal algebras and *finite* coalgebras (over Set):

$$\begin{array}{ccc} \mathsf{Alg}_f(L) & \rightleftarrows & \mathsf{Coalg}_f(T) \\ {\scriptstyle \mathrm{forget}}\downarrow & & \downarrow{\scriptstyle \mathrm{forget}} \\ \mathsf{BA}_f & \underset{\mathrm{Clp}}{\overset{\mathrm{Sp}}{\rightleftarrows}} & \mathsf{Set}_f \end{array}$$

The finite model property for coalgebraic logics

- transport a finite *algebraic* model along duality to a (finite) *coalgebraic* one
- investigate the finite algebra property for logics of interest

Coalgebraic Logics, Algebraically

Extensions of propositional logics:

- new (polyadic) operators (modalities) $\heartsuit \in \Lambda$

Algebraic Semantics. boolean algebras with Λ-operators (Λ-BAOs)

- algebras for boolean signature plus operations $\heartsuit \in \Lambda$
- equations: laws of boolean algebras (for the moment)

Observation. $\Lambda - \mathsf{BAO} \cong \mathsf{Alg}(L)$ where

$$L : \mathsf{BA} \to \mathsf{BA}, A \mapsto F\{\heartsuit(a_1, \dots, a_n) \mid \heartsuit \in \Lambda \; n\text{-ary}, a_1, \dots, a_n \in A\}$$

and $F : \mathsf{Set} \to \mathsf{BA}$ is the free construction

Given $\alpha : LA \to A$, have $[\![\heartsuit]\!] : A^n \to A, (a_1, \dots, a_n) \mapsto \alpha(\heartsuit(a_1, \dots, a_n))$

Algebraic vs Coalgebraic Semantics

Interlude. Suppose there are finitely many(!) new modalities. Then

$$L : \mathsf{BA} \to \mathsf{BA}, A \mapsto F\{\heartsuit(a_1, \dots, a_n) \mid \heartsuit \in \Lambda \; n\text{-ary}, a_1, \dots, a_n \in A\}$$

preserves finite boolean algebras, and induces

$$L^{\mathrm{op}} : \mathsf{Set}_f \to \mathsf{Set}_f, X \mapsto \mathrm{Sp} \circ L \circ \mathcal{P}(X)$$

Side by side. Suppose $(A, \alpha) = (\mathcal{P}(C), \gamma^{-1})$ comes from (C, γ) under duality

Algebraic Semantics	**Coalgebraic Semantics**
• Structures: $(A, \alpha) \in \mathsf{Alg}(L)$.	• Structures: $(C, \gamma) \in \mathsf{Coalg}(L^{\mathrm{op}})$
• Operators: $[\![\heartsuit]\!] : A^n \to LA$	• Operators: $[\![\heartsuit]\!] : \mathcal{P}(C)^n \to \mathcal{P}L^{\mathrm{op}}(C)$
• Semantics: $[\![\heartsuit(\vec{a})]\!] = \alpha \circ [\![\heartsuit]\!](\vec{a})$	• Semantics: $[\![\heartsuit(\vec{p})]\!] = \gamma^{-1} \circ [\![\heartsuit]\!](\vec{p})$

Conclusion. Modalities for L^{op}-coalgebras are natural maps $\mathcal{P}^n \to \mathcal{P} \circ L^{\mathrm{op}}$.

Coalgebraic Semantics of Modal Logics

Structures for T coalgebras determine the semantics of modal operators: they

assign a *nbhd frame translation*

$$[\![M]\!] : TC \to \mathcal{P}\mathcal{P}(C)$$

or, equivalently, a *predicate lifting*

$$[\![M]\!] : \mathcal{P}(C) \to \mathcal{P}(TC)$$

to every modal operator M of the language, parametric in C.

Together with a T-coalgebra (C, γ) this gives a

neighbourhood frame

$$C \xrightarrow{\gamma} TC \xrightarrow{[\![M]\!]} \mathcal{P}\mathcal{P}(C)$$

boolean algebra with operator

$$\mathcal{P}(C) \xrightarrow{[\![M]\!]} \mathcal{P}(TC) \xrightarrow{\gamma^{-1}} \mathcal{P}(C)$$

Induced **Coalgebraic Semantics** $[\![\phi]\!] \subseteq C$ of a modal formula

from a *nbhd perspective*

$$c \in [\![M\phi]\!] \text{ iff } [\![\phi]\!] \in [\![M]\!] \circ \gamma([\![\phi]\!])$$

equivalent *predicate lifting view*

$$c \in [\![M\phi]\!] \iff \gamma(c) \in [\![M]\!]([\![\phi]\!])$$

Examples

Neighbourhood Frames, i.e. coalgebras $C \to \mathcal{P}\mathcal{P}(C)$

$$[\![\Box]\!] = \mathrm{id} : \underbrace{\mathcal{P}\mathcal{P}(C)}_{TC} \to \mathcal{P}\mathcal{P}(C)$$

(identical nbhd frame translation)

Kripke Frames, ie. coalgebras $C \to \mathcal{P}(C)$

viewed as neighbourhood frames

$$\begin{aligned} [\![\Box]\!] : \overbrace{\mathcal{P}(C)}^{TC} &\to \mathcal{P}\mathcal{P}(C) \\ c &\mapsto \{c' : c' \supseteq c\} \end{aligned}$$

via boolean algebras with operators

$$\begin{aligned} [\![\Box]\!] : \mathcal{P}(C) &\to \mathcal{P}\overbrace{\mathcal{P}(C)}^{TC} \\ c &\mapsto \{c' : c' \subseteq c\} \end{aligned}$$

Probabilistic Transition Systems, i.e. coalgebras $C \to \mathcal{D}C$

$$\begin{aligned} [\![L_p]\!] : \mathcal{P}(C) &\to \mathcal{P}\overbrace{\mathcal{D}(C)}^{TC} \\ c &\mapsto \{\mu : C \to [0,1] : \mu(c) \geq p\} \end{aligned}$$

(algebraic perspective)

Axiomatisation of Coalgebraic Logics

What Axioms? are needed to have $\phi = \top \iff [\![\phi]\!]_{(C,\gamma)} = 1$ for all (C,γ)?

Crucial. model class that furnishes the structures, here *all* of $\mathsf{Coalg}(T)$

- *one-step axioms* completely describe *one-step behaviour* $C \to TC$.
- in analogy to $\mathsf{Alg}(F) \cong \mathsf{Alg}(\Sigma, E)$ for E "very simple" and $F : \mathsf{Set} \to \mathsf{Set}$

One-step axioms. Propositional formulas over $\heartsuit(\vec{\phi})$ where $\vec{\phi}$ are propostional formulas over V.

Example. Kripke Frames $C \to \mathcal{P}(C)$ with $[\![\Box]\!]_X(A) = \{B \in \mathcal{P}(X) \mid B \subseteq A\}$

$$\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q \qquad\qquad \Box\top \leftrightarrow \top$$

Frame Conditions encode models with specific properties:

Coalgebraic Logics	=	One-Step Axioms	+	Frame Conditions

Algebraic Semantics of Modal Logics

Starting Point. Complete Axiomatisation $\mathcal{L} = (\Lambda, \mathcal{A})$ of $\mathsf{Coalg}(T)$

- $T : \mathsf{Set} \to \mathsf{Set}$ and interpretations $[\![\heartsuit]\!] : \mathcal{P}^n \to \mathcal{P} \circ T$ for $\heartsuit \in \Lambda$ n-ary
- one-step axioms $\mathcal{A}$ so that $\phi = \top \iff [\![\phi]\!]_{\mathbb{C}} = 1$ for all $\mathbb{C} \in \mathsf{Coalg}(T)$

Goal. Extend $\mathcal{A}$ with frame conditions

Defn. The *functorial presentation* of $\mathcal{L}$ is the endofunctor

$$L : \mathsf{BA} \to \mathsf{BA}, \quad A \mapsto F(\{\heartsuit a \mid a \in A, \heartsuit \in \Lambda\})/\sim$$

where $\sim$ is the equivalence induced by the axioms $\mathcal{A}$ and F is free construction.

Observation. $\mathsf{Alg}(L) \cong \Lambda$-BAO$(\mathcal{A})$ are the Λ-BAOs satisfying all axioms in $\mathcal{A}$.

3 Different Types of Semantics

Given modalities Λ, T : $\mathsf{Set} \to \mathsf{Set}$, interpretation $[\![\heartsuit]\!] : \mathcal{P}^n \to \mathcal{P} \circ T$, axioms $\mathcal{A}$

Algebraic Semantics. in the category $\mathsf{Alg}(L) \simeq \mathsf{BAO}(\Lambda, \mathcal{A})$

$$[\![\heartsuit]\!]_{(A,\alpha)}(a_1, \ldots, a_n) = \alpha(\underbrace{\heartsuit(a_1, \ldots, a_n)}_{\in LA})$$

Synthetic Coalgebraic Semantics in the category $\mathsf{Coalg}(L^\star)$ for

$$L^\star : \mathsf{Set}_f \to \mathsf{Set}_f, \quad X \mapsto \mathrm{Sp} \circ L \circ \mathcal{P}(X)$$

$$[\![\heartsuit]\!]_X(A_1, \ldots, A_n) = \underbrace{\heartsuit(A_1, \ldots, A_n)}_{\in L\mathcal{P}(X) \cong \mathcal{P}L^\star X}$$

Intended Coalgebraic Semantics in the category $\mathsf{Coalg}(T)$ via given $[\![\heartsuit]\!]$

Relating the different semantics

Algebraic vs Synthetic. $\mathsf{Alg}(L)_f \simeq \mathsf{Coalg}(L^\star)_f$ preserving semantics.

Synthetic vs Coalgebraic. Semantically equivalent, i.e.

For every $L^\star$-coalgebra $\mathbf{C}$ there exists a T-coalgebra $\mathbf{D}$ with the same carrier s.t.

$$c \models_{\mathbf{C}} \phi \iff c \models_{\mathbf{D}} \phi$$

for all modal formulas ϕ, and vice versa (assuming completeness for T).

Corollary. For frame conditions *over finitely many modalities* in addition to rank-1 axioms:

Finite L-algebras	$\iff$	Finite $L^\star$-Coalgebras	$\iff$	Finite T-Coalgebras

(use proof-theoretic wizardry, aka cut elimination, for infinitely many modalities)

The Finite Algebra Property

Given. Coalgebraic Logic $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ where $\mathcal{A}$ are rank-1 and Θ are arbitrary.

Question. Does $\mathcal{L}$ have the finite algebra property?

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Take 1. Let's ask the algebraists – but very little seems to be known.

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Take 2. Roll your own.

Thm. If the modal rank of all $\phi \in \Theta$ is ≤ 1 then $\mathcal{L}$ has the finite algebra property.

General Axioms. On a case-by-case basis.

Applications: Logics of Knowledge and Uncertainty

Semantics. Coalgebras for the functor

$$TX = \prod_{i \in N} \mathcal{D}(X) \times \mathcal{P}(X)$$

Interpretation.

$$[\![\mathsf{K}_i]\!]_X(A) = \{(\mu_i, S_i)_{i \in N} \mid S_i \subseteq A\}$$

$$[\![L^i_p]\!]_X(A) = \{(\mu_i, S_i)_{i \in N} \mid \textstyle\sum_{x \in A} \mu_i(x) \geq p\}.$$

Axiomatisation. Axioms for K and L_p plus a subset of

(4)	$\mathsf{K}_i\mathsf{K}_i p \to \mathsf{K}_i p$	(C)	$\mathsf{K}_i p \to L^i_1 p$
(5)	$\mathsf{K}^\star_i p \to \mathsf{K}_i\mathsf{K}^\star_i p$	(U)	$p \to L^i_1 p$
(B)	$p \to \mathsf{K}_i\mathsf{K}^\star_i p$	(S)	$p \to \mathsf{K}_i p$

Thm. The logic of uncertainty and knowledge, augmented with any subset of the above frame conditions, is complete.

Applications: Extensions of Coalition Logic

Semantics. Coalgebras for the game-frame functor

$$TX = \{(S_1, \dots, S_n, f) \mid \emptyset \neq S_i \in \mathsf{Set}, f : \prod_{i \in N} S_i \to X\}.$$

Interpretation.

$$[\![[C]]\!]_X(A) = \{(S_1, \dots, S_n, f) \in TX \mid \exists \sigma_C \in S_C. \forall \sigma_{\bar{C}} \in S_{\bar{C}}. f(\sigma_C, \sigma_{\bar{C}}) \in A\}.$$

Axiomatisation. Axioms of Coalition Logic plus possibly

$$(C) \quad [C]\phi \to \phi \qquad\qquad (F) \quad [C]\phi \to [C][C]\phi$$

$$(P) \quad \phi \to [C]\phi$$

Thm. Coalition Logic, extended with any subset of the frame conditions above, is complete up to the fmp.

Summary

Coalgebraic Logics

- specialise to the logic of many types of systems studied in CS
- nearly always presented in set theoretic context

Algebraic Semantics

- helps answer questions about coalgebraic logics (completeness)
- BUT: had to do most algebra ourselves!

Current Work

- Bye bye, categories: decidability and complexity bounds
- Hello, categories: logics for bialgebras

Deleted Slides

Deleted slides to follow

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Given. Coalgebraic Logic $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ where $\mathcal{A}$ are rank-1 and Θ are arbitrary.

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Answer. It depends . . . but we can try a filtration-style argument.

The Finite Algebra Property

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Question. Does $\mathcal{L}$ have the finite algebra property?

Answer. It depends . . . but we can try a filtration-style argument.

Defn. Suppose Δ is closed under negation and $\mathcal{M} = \{\Phi \subseteq \Delta \mid \Phi \text{ max. cons.}\}$.

A Δ-*filtration* is an algebra structure $\mathbb{F} = (\mathcal{P}(\mathcal{M}), (f_M)_{M \in \Lambda})$ on $\mathcal{P}(M)$ with

$$f_M(\{\Phi \in \mathcal{M} \mid \phi \in \Phi\}) = \{\Phi \in \mathcal{M} \mid M\phi \in \Phi\}$$

whenever $M\phi \in \Delta$. The filtration $\mathbb{F}$ is *safe* for ϕ if $\mathbb{F} \models \phi = \top$.

Note. Existence of safe $\mathcal{A} \cup \Theta$-filtrations imply the finite algebra property.

Generic Results: Standard Filtrations

Defn. For Δ closed, finite and $\mathcal{M} = \{\Phi \subseteq \Delta \mid \Phi \text{ max. cons.}\}$, let

$$f_M(A) = \{\Phi \in \mathcal{M} \mid \nu(\Phi) \vdash M(\bigvee_{\Psi \in A} \bigwedge \Psi)\}$$

where $\nu(\Phi)$ is a fixed maximally consistent extension of Φ.

Filtration Lemma. The standard filtration (above) is safe for $\mathcal{A} \cup \Theta$ provided the modal rank of all $\phi \in \Theta$ is ≤ 1.

Corollary. All logics with frame conditions of rank ≤ 1 have the finite modal property.

Examples.

- Conservative Coalitions: coalition logic plus $[C]\phi \to \phi$
- Known facts are extremely likely: proabilistic K plus $\Box\phi \to L_1\phi$

Variations: Frame Conditions of rank ≥ 2

Symmetry. $\phi \to MM^\star\phi \; (M^\star = \neg M \neg)$

$$f_M^B(A) = \{\Psi \in f_M(A) \mid \forall \Phi \in \mathcal{M}(\Phi \rightsquigarrow_M \Psi => \Psi \in A)\}$$

where $\Phi \rightsquigarrow \Psi$ iff $\nu(\Phi) \vdash M^\star \bigwedge \Psi$.

Euclidean-ness. $M^\star\phi \to MM^\star\phi$

$$f_M^5(A) = \{\Psi \in f_M(A) \mid \forall \Phi \in \mathcal{M}(\Phi \leftrightsquigarrow_M \Psi => \Phi \in A)\}$$

where $\Phi \leftrightsquigarrow_M \Psi$ iff there are $\Omega_0, \ldots, \Omega_n$ such that

- $\Omega_0 \rightsquigarrow_M \Phi$ and $\Omega_n \rightsquigarrow_M \Psi$
- $\Omega_i \rightsquigarrow_M \Omega_{i-1}$ or $\Omega_{i-1} \rightsquigarrow_M \Omega_i$ for all $i = 1, \ldots, n$.

Generalised Transitivity. $M\phi \to MM\phi$

$$f_M^{\bullet 4}(A) = \mathrm{gfp}(X \mapsto f_M^\bullet(A) \cap f_M^\bullet(X))$$