Generic Cut Elimination Applied to Conditional Logics

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Cut Elimination in Modal Logics

Modal Logics. Classical Propositional Logic + *polyadic* modal operators $\heartsuit \in \Lambda$

$$\mathcal{L}(\Lambda) \ni A, B ::= p \mid \neg A \mid A \wedge B \mid \heartsuit(A_1, \dots, A_n) \qquad (\heartsuit \in \Lambda \; n\text{-ary})$$

Interpretation. For example,

- (standard) modal logic: $\Box A \rightsquigarrow$ necessarily A
- graded modal logic: $(\exists \geq k)A \rightsquigarrow \geq k$ relational successors validate A
- probabilistic modal logic: $A \geq B \rightsquigarrow \mathbb{P}([\![A]\!]) \geq \mathbb{P}([\![B]\!])$
- later: *conditional logic* $A \Rightarrow B \rightsquigarrow A$ under condition B

Crucial.

- no *fixed* set of modal operators (yet)
- *separation* between modalities and propositional connectives

Sequent Calculi for Modal Logics

Sequents. Multisets of Λ-formulas, read disjunctively

$$A \equiv \{A\} \qquad \Gamma, \Delta \equiv \Gamma \cup \Delta$$

Sequent Calculi. Right-Handed Gentzen-Schütte Systems.

$$\frac{}{\Gamma, A, \neg A} \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \qquad \frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)} \qquad \frac{\Gamma, A}{\Gamma, \neg\neg A}$$

plus **Modal Rules** of the form

$$\frac{\Gamma_1 \quad \dots \quad \Gamma_n}{\Gamma_0}$$

where $\Gamma_0, \dots, \Gamma_n$ are Λ-sequents (in examples: generated by schemas).

Idea.

- Propositional Part has cut-elimination $\leadsto$ scrutinize modal rules!
- Similarly for structural rules (weakening, contraction, inversion)

Cut Elimination by (trivial) Example.

Modal Logic K**.** $\Lambda = \{\Box\}$ with $\Box$ unary

Rules. Hilbert-Axiomatisation taken from any textbook

$$(N)\frac{p}{\Box p} \qquad (D)\Box(p \to q) \to \Box p \to \Box q$$

As Sequent Rules, i.e. applying inversion

$$\frac{A}{\Box A} \qquad \frac{}{\neg\Box(A \to B), \neg\Box A, \Box B}$$

Find Occurrences of Cut

$$\frac{\dfrac{A \to B}{\Box(A \to B)} \qquad \dfrac{}{\neg\Box(A \to B), \neg\Box A, \Box B}}{\neg\Box A, \Box B}$$

Idea. Add this as a new rule

$$\frac{\neg A, B}{\neg\Box A, \Box B}$$

More Cuts

New Rule Set.

$$\frac{A}{\Box A} \qquad \frac{\neg A, B}{\neg\Box A, \Box B} \qquad \frac{}{\neg\Box(A \to B), \neg\Box A, \Box B}$$

Find more cuts.

$$\frac{\dfrac{\neg A, B \to C}{\neg\Box A, \Box(B \to C)} \qquad \dfrac{}{\neg\Box(B \to C), \neg\Box B, \Box C}}{\neg\Box A, \neg\Box B, \Box C}$$

Add new Rule.

$$\frac{\neg A, \neg B, \neg C}{\neg\Box A, \neg\Box B, \Box C}$$

After finitely many steps . . .

$$(K_n)\,\frac{\neg A_1, \ldots, \neg A_n, A_0}{\neg\Box A_1, \ldots, \neg\Box A_n, \Box A_0}$$

General Idea. Add cuts between modal rules until this process terminates.

Formal Setup

Given modal similarity type Λ and rule set R

Hilbert-Provability. The predicate $\mathsf{HR} \vdash$ is the least set of formulas that

- contains all propositional tautologies
- is closed under modus ponens and uniform substitution
- contains $\bigvee \Gamma_0$ whenever it contains $\bigvee \Gamma_1, \dots, \bigvee \Gamma_n$ and $\Gamma_1 \dots \Gamma_n / \Gamma_0 \in \mathsf{R}$

Gentzen-Provability. The predicate $\mathsf{GR} \vdash$ is the least set of sequents that

- contains Γ_0 whenever it contains $\Gamma_1, \dots, \Gamma_n$
- is closed under application of the rules

$$\frac{}{\Gamma, A, \neg A} \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \qquad \frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)} \qquad \frac{\Gamma, A}{\Gamma, \neg\neg A}$$

Easy Theorem. $\mathsf{HR} \vdash \bigvee \Gamma$ whenever $\mathsf{GR} \vdash \Gamma$.

Absorption of Structural Rules

Goal. $\mathsf{HR} \vdash \bigvee \Gamma \iff \mathsf{GR} \vdash \Gamma$

Observation. The following rules of weakening, contraction and inversion

$$\frac{\Gamma}{\Gamma, A} \qquad \frac{\Gamma, A, A}{\Gamma, A} \qquad \frac{\Gamma, \neg\neg A}{\Gamma, A} \qquad \frac{\Gamma, \neg(A_1 \wedge A_2)}{\Gamma, \neg A_1, \neg A_2} \qquad \frac{\Gamma, A_1 \wedge A_2}{\Gamma, A_i}(i = 1, 2)$$

should be admissible, as they are in HR.

Idea. Isolate the "essence" of e.g. inversion lemma as a property of rule sets

Defn. For a Λ-sequent Δ, let $\mathsf{A}(\Delta)$ denote the closure of Δ under the above rules.

The rule set R *absorbs structural rules* if

$$\mathsf{GR} + \mathsf{A}(\Gamma_1) \cup \cdots \cup \mathsf{A}(\Gamma_n) \vdash \Gamma$$

for all $\frac{\Gamma_1 \ldots \Gamma_n}{\Gamma_0} \in \mathsf{R}$ and all $\Gamma \in \mathsf{A}(\Gamma_0)$.

Propn. Admissibility of the structural rules follows from their absorption.

(Counter)Examples

For the modal logic K:

Negative. Absorption of weakening fails for

$$\frac{\neg A_1, \ldots, \neg A_n, A_0}{\neg \Box A_1, \ldots, \neg \Box A_n, \Box A_0}$$

For $n = 0$, have $\Box A_0, p \in \mathsf{A}(\Box A)$ but $\mathsf{GR} + \mathsf{A}(A) \not\vdash \Box A_0, p$.

Positive. Absorption holds if weakening is built in:

$$\frac{\neg A_1, \ldots, \neg A_n, A_0}{\neg \Box A_1, \ldots, \neg \Box A_n, \Box A_0, \Delta}$$

(Note that inversion is trivial)

More (Counter)Examples

For the modal logic T = K + $\Box A \to A = K + \neg\Box A, A$

Find Cuts between K and T

$$\dfrac{\dfrac{\neg A, B}{\neg\Box A, \Box B} \qquad \dfrac{}{\neg\Box B, B}}{\neg\Box A, B}$$

Negative. Absorption of inversion fails if we adopt this as a new rule

$$\frac{\neg A, B}{\neg\Box A, B} \qquad (\text{take } B = \neg(B_0 \wedge B_1))$$

Negative. Inversion works, but contraction fails for

$$\frac{\neg A, \Gamma}{\neg\Box A, \Gamma} \qquad (\text{take } \Gamma = \neg\Box A)$$

Positive. Absorption of structural rules holds for

$$\frac{\neg A, \neg\Box A, \Gamma}{\neg\Box A, \Gamma}$$

(which is the version of the T-rule that we know and like)

Absorption of the Cut-Rule

(Same) Idea. Distill the "essence" of cut-elimination proofs into rule properties

- think double induction on cut rank and proof size
- allow cuts on smaller proofs and smaller cut formulas

Defn. A ruleset R *absorbs cut*, if for all rules $(r_1)\frac{\Gamma_1,\dots,\Gamma_n}{A,\Gamma_0}$, $(r_2)\frac{\Delta_1,\dots,\Delta_k}{\neg A,\Delta_0} \in$ R

$$\mathsf{GR} + \mathsf{Cut}(A, r_1, r_2) \vdash \Gamma_0, \Delta_0$$

where $\mathsf{Cut}(A, r_1, r_2)$ contains structural rules, cut on formulas $< A$, the premises $\Gamma_1, \dots, \Gamma_n, \Delta_1, \dots, \Delta_k$ and Γ, Δ where, for some formula B,

- Γ, B and $\Delta, \neg B \in \{\Gamma_1, \dots, \Gamma_n, \Delta_1, \dots, \Delta_k\}$ (cut premise/premise), or
- $\Gamma, B = \Gamma_0, A$ and $\Delta, \neg B \in \{\Delta_1, \dots, \Delta_k\}$ (cut conclusion/premise), or
- $\Gamma, B = \Delta_0, \neg A$ and $\Delta, \neg B \in \{\Gamma_1, \dots, \Gamma_n\}$ (cut premise/conclusion).

A rule set that absorbs structural rules and the cut rule is called *absorbing*.

Cut Absorption implies Cut Elimination

Thm. If R absorbs structural rules and cut, then cut is admissible.

Proof. (Sketch) Double induction on cut rank and proof size, case modal / prop. rule:

$$\dfrac{\dfrac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \qquad \dfrac{\Delta_1 \qquad \dots \qquad \Delta_n}{\neg(A \wedge B), \Delta}}{\Gamma, \Delta}$$

- closure under inversion gives proof of $\neg A, \neg B, \Delta$
- cut on (smaller) formulas A, B gives

$$\dfrac{\dfrac{\Gamma, A \qquad \Delta, \neg A, \neg B}{\Gamma, \Delta, \neg B} \qquad \Gamma, B}{\Gamma, \Gamma, \Delta}$$

- closure under contraction gives proof of Γ, Δ

Cor. If R is closed under uniform substitution and absobrs cut and structural rules, then GR $\vdash \Gamma \iff$ HR $\vdash \bigvee \Gamma$.

Construction of Cut-Free Rule Sets

Conceptually. Absorption by addition of cuts + equivalence transformations

Admissibility. A rule $(r)\Gamma_1 \dots \Gamma_n/\Gamma_0$ is *admissible* in HR if, for all formulas A,

$$\mathsf{HR} \vdash A \iff \mathsf{H}(\mathsf{R} \cup \{r\}) \vdash A.$$

Simple Lemma 1. (Propositional weakening / strengthening is sound in HR)

The rule $\Delta_1 \dots \Delta_k/\Delta_0$ is admissible, if there is $\Gamma_1, \dots, \Gamma_n/\Gamma_0 \in \mathsf{R}$ s.t.

$$\{\bigvee \Delta_1, \dots, \bigvee \Delta_k\} \vdash_{\mathsf{PL}} \bigvee \Gamma_i (1 \leq i \leq n) \text{ and } \bigvee \Gamma_0 \vdash_{\mathsf{PL}} \bigvee \Delta_0.$$

Simple Lemma 2. (Cut is sound in HR)

If $\Gamma_1 \dots \Gamma_n/A, \Gamma_0$ and $\Delta_1 \dots \Delta_k \neg A, \Delta_0 \in \mathsf{R}$ then

$$\frac{\Gamma_1 \dots \Gamma_n \quad \Delta_1 \dots \Delta_k}{\Gamma_0, \Delta_0}$$

is admissible in HR.

Applications: Conditional Logics

Similarity Type. $\Lambda = \{\Rightarrow\}$ with $\Rightarrow$ binary (nonmonotonic conditional)

Basic Rules.

$$(\mathsf{RCEA})\frac{A \leftrightarrow A'}{(A \Rightarrow B) \leftrightarrow (A' \Rightarrow B)} \qquad (\mathsf{RCK})\frac{B_1 \wedge \cdots \wedge B_n \rightarrow B}{(A \Rightarrow B_1) \wedge \cdots \wedge (A \Rightarrow B_n) \rightarrow (A \Rightarrow B)}$$

Additional Axioms.

$$(\mathsf{ID}) A \Rightarrow A \qquad (\mathsf{MP})(A \Rightarrow B) \rightarrow (A \rightarrow B) \qquad (\mathsf{CEM})(A \Rightarrow B) \vee (A \Rightarrow \neg B)$$

Terminology. $\mathsf{CK} = (\mathsf{RCEA}) + (\mathsf{RCK})$, additional axioms juxtaposed, e.g. $\mathsf{CKCEMMP} = \mathsf{CK} + (\mathsf{CEM}) + (\mathsf{MP})$.

Logics without (CEM)

Notation. $A_0 = \cdots = A_n$ contains $\neg A_i, A_0$ and $A_i, \neg A_0$ for $i = 1, \ldots, n$.

Basic Conditional Logic CK.

$$(\mathsf{CK}_g)\frac{A_0 = \cdots = A_n \quad \neg B_1, \ldots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma}$$

CK + (ID). Axiom schema (CK_g) plus

$$(\mathsf{ID}_g)\frac{A = B}{A \Rightarrow B, \Gamma}$$

CK + (MP). Axiom schema (CK_g) plus

$$(\mathsf{MP}_g)\frac{A, \neg(A \Rightarrow B), \Gamma \quad \neg B, \neg(A \Rightarrow B), \Gamma}{\neg(A \Rightarrow B), \Gamma}$$

In all cases. Cut elimination holds, and provability is unchanged.

Logics with (CEM)

CK + (CEM). New rule schema

$$(\mathsf{CKCEM}_g)\,\frac{A_0 = \cdots = A_n \quad B_0, \ldots, B_j, \neg B_{j+1}, \neg B_n}{(A_0 \Rightarrow B_0), \ldots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \ldots, \neg(A_n \Rightarrow B_n), \Gamma}$$

for $1 \leq j \leq n$.

CK + CEM + MP. Rule schemas $(\mathsf{CKCEM}_g) + (\mathsf{MP}_g)$ and new schema

$$(\mathsf{MPEM}_g)\,\frac{A, (A \Rightarrow B), \Gamma \qquad B, (A \Rightarrow B), \Gamma}{(A \Rightarrow B), \Gamma}.$$

In all cases. Cut elimination holds, and provability is unchanged.

Complexity.

Conditional Logics without (CEM).

- Polynomial bound on height and branching of proof tree

Thm. Provability for CK, CKID, CKMP is in $PSPACE$.

Conditional Logics with (CEM).

- not (significantly) harder than propositional logic
- the rule

$$(\mathsf{CKCEM}_g)\frac{A_0 = \cdots = A_n \quad B_0, \ldots, B_j, \neg B_{j+1}, \neg B_n}{(A_0 \Rightarrow B_0), \ldots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \ldots, \neg(A_n \Rightarrow B_n), \Gamma}$$

doesn't create branching in the second argument

Thm. Provability for CKCEM and CKCEMMP is in $coNP$.

Summary and Conclusions

Generic Cut Elimination.

- absorbing rule sets: inductive steps for cut elimination
- construction of absorbing sets: add cuts until saturated

Conditional Logics.

- New (internalised, cut-free) sequent calcul for extensions of CK
- Cut-elimination for $\mathsf{CKCEMMP}$.

Questions and Further Work.

- Fine tuning: preservation of proof height for structural rules
- Other base logics (e.g. FOL)
- Other flavours (e.g. intuitionistic)