
The Coalgebraic μ -Calculus

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Many Faces of Modal Logic

Modal Logic. Classical Propositional Logic + *Modalities*, e.g.:

Coalition Logic to reason about *Multi-agent systems*.

$\langle a, b \rangle \phi$ \rightsquigarrow Agents a, b can force ϕ

Probabilistic Modal Logic for *Reactive Systems*

$\langle p \rangle \phi$ \rightsquigarrow ϕ holds with probability ≤ 0.2

Graded Modal Logic in *Knowledge Representation*

$\exists \leq 5. \phi$ \rightsquigarrow at most 5 components satisfy ϕ

Historically: **Relational Modal Logic** in *Philosophy*

$\Box \phi$ \rightsquigarrow necessarily ϕ

State of the Art: Different Logics – Different Semantics

Possible World Semantics of standard modal logic

$W \xrightarrow{\gamma} \mathcal{P}(W)$ to interpret $\Box\phi$ as “necessarily ϕ ”

$$w \models \Box\phi \iff \boxed{\forall w' \in \gamma(w) : w' \in \llbracket \phi \rrbracket} \rightsquigarrow \gamma(w) \in \mathbb{P}$$

Distribution Semantics of Probabilistic Logics

$W \xrightarrow{\gamma} \underbrace{D(W)}_{\text{prob. dist.}}$ to interpret $L_p\phi$ as “ ϕ with probability $\geq p$ ”

$$w \models L_p\phi \iff \boxed{\gamma(w)(\llbracket \phi \rrbracket) \geq p} \rightsquigarrow \gamma(w) \in \mathbb{P}$$

Game Frame Semantics of Coalition Logic

$W \xrightarrow{\gamma} ((S_a)_{a \in A}, f : \prod_{a \in A} S_a \rightarrow W)$ to interpret $\langle a, b \rangle\phi$ as “a, b can force ϕ ”

$$w \models \langle a, b \rangle\phi \iff \boxed{\gamma(w) = ((S_a), f) : f(\dots, s_a, s_b, \dots) \in \phi} \rightsquigarrow \gamma(w) \in \mathbb{P}$$

Coalgebras Provide a Semantic Umbrella

Semantic Structures map **States** to **Successors**

$$\text{Coalgebras: } \boxed{W \xrightarrow{\gamma} TW}$$

where $T : \text{Set} \rightarrow \text{Set}$ is a “construction” (technically: a functor) on sets

Modalities express properties of **Successors** in terms of **States**

$$w \models \heartsuit\phi \iff \gamma(w) \in \llbracket \heartsuit \rrbracket(\llbracket \phi \rrbracket)$$

Technically. Modal Operators are *natural transformations*

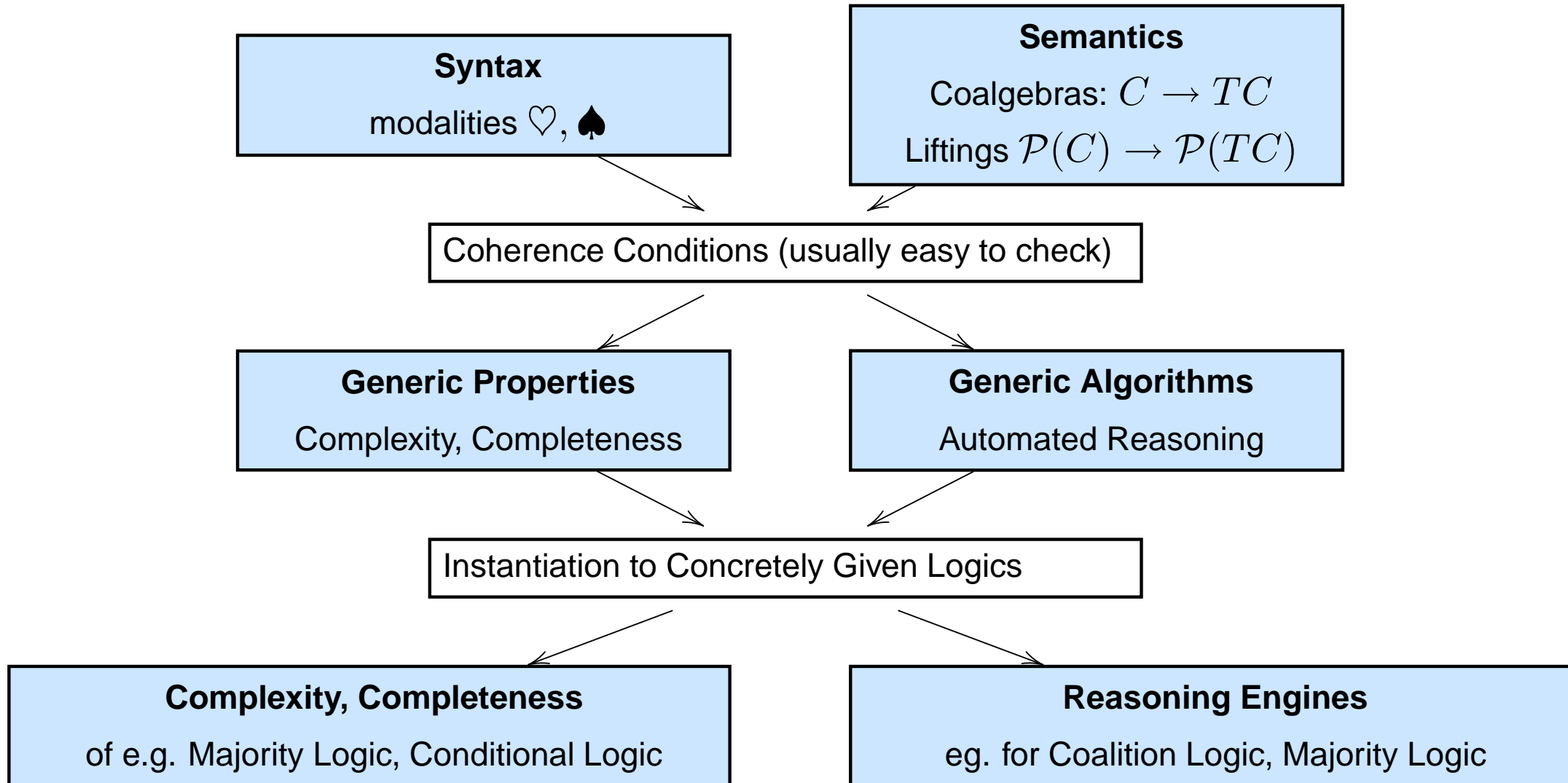
$$\llbracket \heartsuit \rrbracket_X : \mathcal{P}(X) \rightarrow \mathcal{P}(TX)$$

that we add to (classical) propositional logic.

Examples.

- (Standard) modal logic, classical and monotone modal logic
- graded modal logic, probabilistic modal logic, conditional logic, coalition logic

Coalgebraic Semantics is UNIFORM



Generic Completeness

Approach. Find *coherence conditions* between syntax and semantics

Deduction for Coalgebraic Logics: propositional logic plus a set R of

one-step rules ϕ/ψ : ϕ propositional, ψ clause over $\heartsuit a, a \in V$

Intuition. Rules mimic one-step behaviour

One Step Derivability of χ (propositional over $\{\heartsuit x : x \subseteq X\}$) over a set X

- $TX \models \chi$ defined inductively by $\llbracket \heartsuit x \rrbracket = \llbracket M \rrbracket(x)$
- $\mathcal{R}X \vdash \chi$ iff $\{\psi\sigma : X \models \phi\sigma, \phi/\psi \in \mathcal{R}\} \vdash_{\text{PL}} \chi$

R is one-step sound (complete) if $TX \models \chi$ whenever (only if) $\mathcal{R}X \vdash \chi$

Theorem (P, 2003, Schroeder 2006)

Soundness and weak completeness are implied by their one-step counterparts.

Examples of (Cut-Free) Complete Rule Sets

Modal Logic E .

$$\frac{p \leftrightarrow q}{\Box p \rightarrow \Box q}$$

Modal Logic K .

$$\frac{\bigwedge_{i=1,\dots,n} p_i \rightarrow q}{\bigwedge_{i=1,\dots,n} \Box p_i \rightarrow \Box q}$$

Graded Modal Logic.

$$\frac{\sum_{i=1}^n p_i \leq \sum_{j=1}^m q_j}{\bigwedge_{i=1}^n \Diamond_{k_i} p_i \rightarrow \bigvee_{j=1}^m \Diamond_{l_j} q_j}$$

Probabilistic Modal Logic.

$$\frac{\sum_{i=1}^n p_i + u \leq \sum_{j=1}^m q_j}{\bigwedge_{i=1}^n L_{u_i} p_i \rightarrow \bigvee_{j=1}^m L_{v_j} q_j}$$

Conditional Logic.

$$\frac{\bigwedge_{i=1,\dots,n} q_i \rightarrow q_0 \wedge \bigwedge_{i=1,\dots,n} p_i \leftrightarrow p_0}{\bigwedge_{i=1,\dots,n} (p_i \Rightarrow q_i) \rightarrow (p_0 \Rightarrow q_0)}$$

Coalition Logic.

$$\frac{\bigvee_{i=1}^n \neg p_i}{\bigvee_{i=1}^n \neg [C_i] p_i}$$

$$\frac{\bigwedge_{i=1}^n p_i \rightarrow q \vee \bigvee_{j=1}^m r_j}{\bigwedge_{i=1}^n [C_i] p_i \rightarrow [D] q \vee \bigvee_{j=1}^m [N] r_j}$$

Sequent Calculi for Coalgebraic Logics

Sequents are multisets of formulas. Write Γ, Δ for $\Gamma \cup \Delta$ and Γ, A for $\Gamma, \{A\}$

Propositional Rules (for a right-handed Gentzen-Schuetze System)

$$\frac{}{\Gamma, A, \neg A} \quad \frac{\Gamma, A}{\Gamma, \neg\neg A} \quad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \quad \frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)}$$

Modal Rules from a one-step rule ϕ/ψ where σ ranges over substitutions

$$\frac{\text{Lit}(\phi_1)\sigma \quad \dots \quad \text{Lit}(\phi_n)\sigma}{\text{Lit}(\psi)\sigma, \Delta}$$

and $\text{cnf}(\phi) = \phi_1 \wedge \dots \wedge \phi_n$ and $\text{Lit}(\cdot)$ is the set of literals occurring in a clause.

Notation. $\text{GenR} \vdash \Gamma$ if Γ can be derived using the propositional rules and the “imported” modal rules.

Sequent Proofs vs Hilbert Proofs

Easy Lemma. $R \vdash \bigvee \Gamma$ whenever $\text{Gen}R \vdash \Gamma$.

Cut-Free Complete Rule Sets: Two equivalent definitions

Semantically.

Clauses over successors are derivable using a *single* rule

Syntactically.

Closure under cuts between conclusions of modal rules

Lemma. Suppose R is cut-free complete and contraction closed.

- contraction, cut and weakening are admissible
- the inversion lemma holds for propositional connectives

Thm. Suppose R is strictly complete and contraction closed. Then

$$\text{Coalg}(T) \models \bigvee \Gamma \text{ iff } \text{Gen}R \vdash \Gamma.$$

Complexity

PSPACE Bounds via proof search:

- polynomial bound on the height of the proof tree
- for every sequent Γ , the (codes of) rules that entails Γ can be found in polytime
- for every (code of a) rule, its premises can be found in polytime.

Formally. R is *NPMV* if there exists a finite alphabet Σ such that all sequents can be represented in Σ and a pair

$$f : \Sigma \rightarrow \mathcal{P}(\Sigma) \quad g : \Sigma \rightarrow \mathcal{P}(\Sigma)$$

of nondeterministic polytime functions such that

$$\{\{\Gamma_1, \dots, \Gamma_n\} \mid \frac{\Gamma_1, \dots, \Gamma_n}{\Gamma} \in \text{GenR}\} = \{g(x) \mid x \in f(\Gamma)\}$$

for all sequents Γ .

Thm. If R is NPMV, sound and strictly complete, then R -satisfiability is in PSPACE.

Compositionality By Example: Games and Probabilities

Strategic Games

- *Semantics*: $W \rightarrow GW$
(outcomes of strategic games)
- *Syntax*: $[C]\phi$
(coalition C can force ϕ)

Quantitative Uncertainty

- *Semantics*: $W \rightarrow DW$
(probability distributions)
- *Syntax*: $L_p\phi$
(ϕ with probability $\geq p$)

Taken Together: Games with Uncertain Outcomes

- *Semantics*. $W \longrightarrow D(G(W))$
probability distributions over strategic games
- *Syntax*: $L_p[C]\phi$ (and combinations)
Coalition C can bring about ϕ with probability $\geq p$.

Compositionality by Uniformity

Semantics: Defined by Operations (Functors) $T, S : \text{Set} \rightarrow \text{Set}$

- **Combination** of Semantical Structures: **Functor Composition**

$$T \circ S, \quad T + S, \quad T \times S : \quad \text{Set} \rightarrow \text{Set}$$

- **Synthesis** of Logics, Proof Calculi and Algorithms

$$\heartsuit \rightsquigarrow T\text{-successors} \quad \spadesuit \rightsquigarrow S\text{-successors}$$

- **Induced Combinations**

- $(\heartsuit \spadesuit) \rightsquigarrow T \circ S$ -successors describe *sequencing*
- $(\heartsuit \times \spadesuit) \rightsquigarrow T \times S$ -successors describe *fusion*
- $(\heartsuit + \spadesuit) \rightsquigarrow T + S$ -successors describe *choice*

Main Results. Combinations preserve *completeness* and *PSPACE-decidability*

Extensions: Hybrid Coalgebraic Logic

Extend modal logics with *nominals* $i \in N$ and *satisfaction operators* $@_i$

$$\mathcal{L} \ni \phi, \psi ::= a \mid \perp \mid \phi \rightarrow \psi \mid \heartsuit(\phi_1, \dots, \phi_n) \mid @_i \phi$$

for $a \in N \cup V$, \heartsuit n -ary and V a set of propositional variables.

Hybrid Valuations $\pi : N \cup V \rightarrow \mathcal{P}(C)$ assign singleton sets to nominals

Intuition. Nominals are *names* of individual entities in models (like Henry VIII)

Coalgebraic Semantics $\llbracket \phi \rrbracket_{(C, \gamma)}^\pi$ of $\phi \in \mathcal{L}$ over (C, γ) wrt hybrid valuation π :

- as before for modal operators and nominals (given structure for the modalities)
- and $\llbracket @_i \phi \rrbracket_{(C, \gamma)}^\pi = \{c \in C \mid \pi(i) \models \phi\}$

NB. The semantics of $@$ -formulas is either empty or the carrier of the model.

Hybrid Completeness

Axioms for Nominals

$$\begin{array}{ll} (K@) & @_i(\phi \rightarrow \psi) \rightarrow (@_i\phi \rightarrow @_i\psi) \\ (sd) & @_i\phi \leftrightarrow \neg @_i\neg\phi \\ (in) & i \wedge \phi \rightarrow @_i\phi \\ (@\perp) & @_i\perp \rightarrow \perp \\ (ref) & @_ii \\ (sym) & @_ij \leftrightarrow @_ji \\ (nom) & @_ij \wedge @_jp \rightarrow @_ip \\ (ag) & @_j@_ip \rightarrow @_ip \end{array}$$

Interaction Axiom

$$(mob) \quad @_ia \rightarrow (\heartsuit b \leftrightarrow \heartsuit(b \wedge @_ia)) \quad (\heartsuit \in \Lambda)$$

Thm. (Myers/P/Schröder 2008) Hybrid coalgebraic modal logic, i.e. (mob) + nominal axioms + a set of complete one-step rules is complete up to the finite model property.

Sequent Calculi for Hybrid Logics

Sequents. Multisets of formulas of the form $@_t A$ for formulas A

Static Rules. @-prefixed versions of propositional and nominal rules

$$\begin{array}{ll}
 (\text{Ax}) \quad @_t \neg A, @_t A, \Gamma & (\text{Ref}) \quad @_t t, \Gamma \quad (\text{@}\top) \quad @_t \top, \Gamma \\
 (\neg\neg) \quad \frac{@_t A, \Gamma}{@_t \neg\neg A, \Gamma} & (\wedge) \quad \frac{@_t A, \Gamma \quad @_t B, \Gamma}{@_t (A \wedge B), \Gamma} \\
 (\neg\wedge) \quad \frac{@_t \neg A, @_t \neg B, \Gamma}{@_t \neg (A \wedge B), \Gamma} & (\text{At}) \quad \frac{@_t A, \Gamma}{@_s @_t A, \Gamma} \\
 (\text{Sd}) \quad \frac{@_s \neg A, \Gamma}{@_t \neg @_s A, \Gamma} & (\text{Eq}) \quad \frac{\Gamma[t := i]}{@_t \neg i, \Gamma}
 \end{array}$$

Modal Rules.

$$(R) \quad \frac{@_n \Gamma_1 \sigma, @_t \Gamma_0 \sigma, \Delta \quad \dots \quad @_n \Gamma_k \sigma, @_t \Gamma_0 \sigma, \Delta}{@_t \Gamma_0 \sigma, \Delta} (n \text{ fresh})$$

for all sequent rules(!) $\Gamma_1 \dots \Gamma_n / \Gamma_0$

Cut-Elimination in Coalgebraic Hybrid Logic

Thm. The (@-prefixed version of the) cut rule

$$\frac{@_t A, \Gamma \quad @_t \neg A, \Delta}{\Gamma, \Delta}$$

is admissible.

Proof. Triple(!) induction over modal rank, size of cut formula and proof size.

Complexity. Proof search is not necessarily terminating:

$$(R) \quad \frac{@_n \Gamma_1 \sigma, @_t \Gamma_0 \sigma, \Delta \quad \dots \quad @_n \Gamma_k \sigma, @_t \Gamma_0 \sigma, \Delta}{@_t \Gamma_0 \sigma, \Delta} (n \text{ fresh})$$

but polynomially many applications of (R) suffice on every branch.

Corollary. Given that the original rule set is NPMV, cut-free complete and contraction closed, satisfiability in Coalgebraic Hybrid Logic is PSPACE-decidable.

Extensions II: Fixpoints

Extend basic logics with least and greatest *fixpoints*: use *negation normal form*

$A, B ::= p \mid \bar{p} \mid A \vee B \mid A \wedge B \mid \heartsuit(A_1, \dots, A_n) \mid \bar{\heartsuit}(A_1, \dots, A_n) \mid \mu p.F \mid \nu p.F$

where $p \in V$, \heartsuit is n -ary and \bar{p} does not occur in F .

Dual Operators. For \heartsuit n -ary, $\bar{\heartsuit} = \neg \heartsuit \neg$ (semantically)

Intuition. μ is *finite unfolding* (safety) and ν is *infinite recurrence* (liveness)

Coalgebraic Semantics $\llbracket A \rrbracket_M^\pi$ where $M = (C, \gamma)$ and π a valuation

- as before for propositional connectives and modalities
- $\llbracket \mu p.F \rrbracket_M = \text{LFP}(F_p^M)$ and $\llbracket \nu p.F \rrbracket_M = \text{GFP}(F_p^M)$

where $F_p^M(X) = \llbracket F \rrbracket_{M, \sigma'}$ with $\sigma'(q) = \sigma(q)$ for $q \neq p$ and $\sigma'(p) = X$.

Semantically, this is very easy indeed ...

(Dual) Axiomatisation

Here. Easier to use *Tableaux* than *Sequent Calculi*

Tableau Sequents. Finite sets of formulas $\Gamma = \{A_1, \dots, A_n\}$ read *conjunctively*

Tableau Rules. As before, with modal rules dualised

$$\begin{array}{c} (\wedge) \frac{\Gamma; A \wedge B}{\Gamma; A; B} \quad (\vee) \frac{\Gamma; A \vee B}{\Gamma; A \quad \Gamma; B} \quad (\text{f}) \frac{\Gamma; \eta p.A}{\Gamma; A[p := \eta p.A]} \quad (\text{m}) \frac{\Gamma_0 \sigma, \Delta}{\Gamma_1 \sigma \dots \Gamma_n \sigma} \quad (\text{Ax}) \frac{\Gamma, A, \bar{A}}{} \end{array}$$

Note. Applying rules starting from Γ only creates a *finite* set of sequents, $\text{Cl}(\Gamma)$

Remarks.

- *rule application* preserves and reflects *satisfiability*
- *No distinction* between *least* and *greatest* fixpoints
- naive *unfolding* leads to infinite *loops*

Construction of Tableaux

Idea. Tableaux as *finite graphs* that may not unfold *least* fixpoints *infinitely* often

Definition. A *Tableaux* for a sequent Γ is a *finite*, *rooted* and *labelled* graph (N, K, R, ℓ) where

- N is the set of nodes and $K \subseteq N \times N$ is the set of edges,
- R is the root node and $\ell : N \rightarrow S(\Gamma)$ is a labelling with $\ell(R) = \Gamma$
- if a rule can be applied to $\ell(n)$ then n has appropriately labelled successors
- otherwise, n may not have any successors

Closed Tableaux. axioms on all end nodes, infinite unfolding of lfp's on infinite paths

Conceptually. Closed tableaux are *finitely represented* witnesses for unsatisfiability

Example: Coalition Logic with Fixpoints

Recall. $[C]\phi \rightsquigarrow$ “Coalition C can force ϕ ”

Closed Tableau for $[C]\nu X.(p \wedge [\overline{N}]X) \wedge [D]\mu Y.(\overline{p} \vee [D]Y)$:

$$\begin{array}{c}
 \frac{[C]B \wedge [D]A}{[C]B; [D]A} \\
 \frac{[C]B; [D]A}{B; A} \leftarrow \\
 \frac{p \wedge [\overline{N}]B; A}{p \wedge [\overline{N}]B; \overline{p} \vee [D]A} \\
 \frac{p \wedge [\overline{N}]B; \overline{p} \vee [D]A}{p; [\overline{N}]B; \overline{p} \vee [D]A} \\
 \hline
 \frac{p; [\overline{N}]B; \overline{p} \quad p; [\overline{N}]B; [D]A}{p; [\overline{N}]B; \overline{p}}
 \end{array}$$

“ C can perpetuate p indefinitely whereas D can achieve \overline{p} in finitely many steps”

Games for the coalgebraic μ -calculus

Main Technical Tool. Two-Player Parity Games

- every board position b has a priority $\Omega(b)$
- \exists wins (and \forall loses) a play if largest infinitely occurring priority is even
- unfolding of *least fixpoints* gives *odd* priorities

Model Checking Game

- modal satisfiability game
- played on state/formula pairs
- unfolding of fixpoints

Tableaux Game

- played on sequents and rules
- \forall chooses rule
- \exists chooses conclusion

Crucially. *formulas* in the model checking game vs *formula sets* in tableaux

Goal. Γ has closed tableau $\iff \forall$ wins tableau game $\iff \Gamma$ unsatisfiable

Traces: The Good, The Bad And The Ugly

Traces, or: how to align the two games

- in the tableau game: plays are sequences Γ_1, Γ_2 of *formula sets*
- in the tableau game: plays are sequences A_1, A_2, \dots of *formulas*

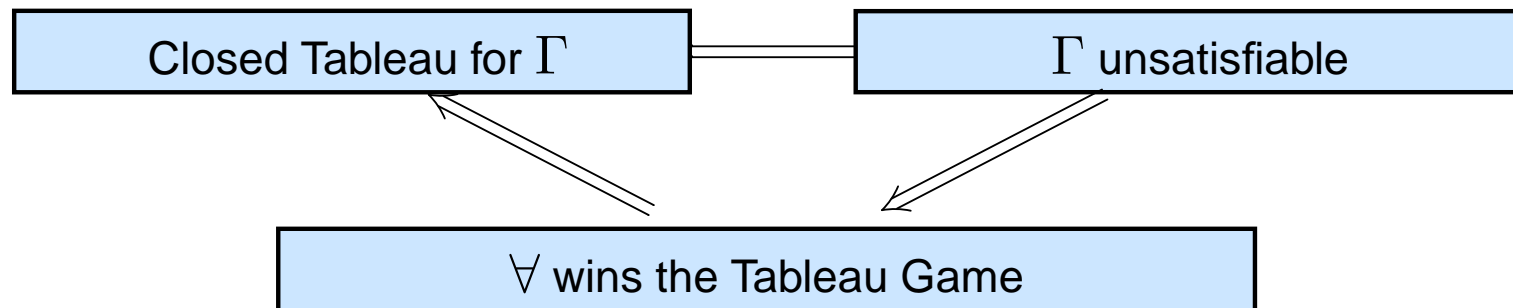
Intuition. \forall loses play (Γ_i) if it carries (A_i) which loses model checking game.

Fortunately. There is a (det.) parity word A automaton that rejects all such (Γ_i) .

The Tableau Game.

- board positions: $(\Gamma, a) \cup (\{\Gamma_1, \dots, \Gamma_n\}, a)$ where $a \in A$
- \forall chooses tableau rule, \exists chooses conclusions
- $(\Gamma, a) \rightarrow_{\forall} (\{\Gamma_1, \dots, \Gamma_n\}, a) \rightarrow_{\exists} (\Delta, a')$ if $a \xrightarrow{\Delta} a'$ in the automaton
- priorities given by automaton

Soundness, Completeness and Decidability



Proof Ideas.

- closed tableaux “guide” \forall in the model checking game
- tableau-winning strategies for \forall yield closed tableaux
- tableau-winning strategies for \exists code satisfying models

Crucial Step. use of coalgebraic *coherence conditions* in model construction

EXPTIME Complexity Bounds

Idea. Decidability via parity games

Parity Word Automaton (via Safra construction)

- exponentially many states, *polynomial* index (in $|\Gamma|$)

Rules and Conclusions. Require *exponentially tractable* rules (cf NPMV earlier):

- polynomial rule codes, premise and conclusions in EXPTIME

Game Board of the parity game

- exponential in $|\Gamma|$ (word automaton \times (rule codes \cup sequents))

Bean Counting Theorem. Assuming exponential tractability, satisfiability is in EXPTIME.

Examples and New Results

Observation. All naturally occurring rule sets are tractable (even in NP)

The modal and graded μ -calculus. $TX = \mathcal{P}(X) / TX = B(X)$

- known EXPTIME bounds by automata-theoretic methods (Emerson/Jutla and Kupferman/Sattler/Vardi)
- alternative proof via tableaux (Niwinski/Walukiewicz)

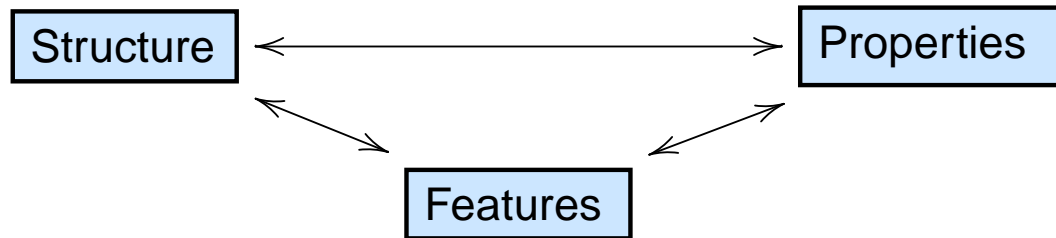
New EXPTIME bounds

- the probabilistic μ -calculus: $TX = DX$
- the coalitional μ -calculus: $TX = GX$
- the monotone μ -calculus: $TX = MX$

Proof. Using (known) exponential tractability of rule sets.

Via Compositionality. E.g. EXPTIME for probabilistic games: $TX = D \circ GX$

Long-Term Goal: Bespoke Logics

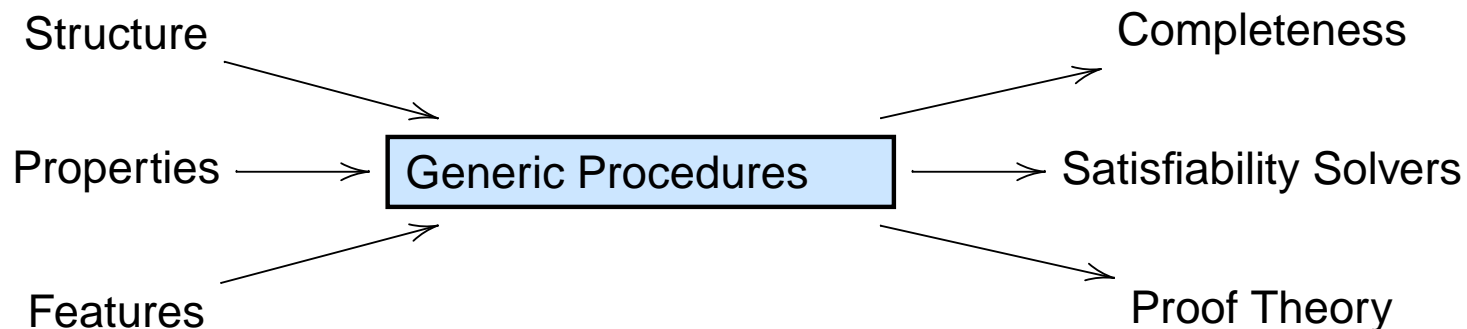


Structures. Basic semantic model – e.g. probabilistic systems or game frames

Properties. Additional frame conditions, e.g. generalised transitivity

Features. Logical means of expressivity, e.g. fixpoints, nominals

Long term goal: Pick and Choose Approach to Modal Logics



Application Pull: Logic-Based Knowledge Representation