

Admissibility of Cut in Coalgebraic Logics

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Abstract

We study sequent calculi for propositional modal logics, interpreted over coalgebras, with admissibility of cut being the main result. As applications we present a new proof of the (already known) interpolation property for coalition logic and establish the interpolation property for the conditional logics CK and CK_{Id} .

Keywords: Modal Logic, Coalgebraic Semantics, Proof Theory, Cut Elimination

1 Introduction

Establishing the admissibility of the cut rule in a modal sequent calculus often allows to establish many other properties of the particular logic under scrutiny. Given that the sequent calculus enjoys the subformula property, the conservativity property is immediate: each formula is provable using only those deductive rules that mention exclusively operators that occur in the formula. As a consequence, completeness of the calculus at large immediately entails completeness of every subsystem that is obtained by removing a set of modal operators and the deduction rules in which they occur. Moreover, cut-free sequent systems admit backward proof search, as the logical complexity of a formula usually decreases when passing from the conclusion to the premise of a deductive rule to the premise. Given that contraction is admissible in the proof calculus, this yields – in presence of completeness – decidability and complexity bounds for the satisfiability problem associated with the logic under consideration [9,2]. Finally, a cut-free system provides the necessary scaffolding to prove interpolation theorems by induction on cut-free proofs.

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For normal modal logics, sequent calculi, often in the guise of tableau systems, have therefore – not surprisingly – received much attention in the literature [1,5,16]. In the context of non-normal logics, sequent calculi have been explored for regular and monotonic modal logics [6], for Pauly’s coalition logic [7] and for a family of conditional logics [13]. All these logics are coalgebraic in nature: their standard semantics can be captured by interpreting them over coalgebras for an endofunctor on sets. This is the starting point of our investigation and we set out to derive sequent systems for logics with coalgebraic semantics and study their properties. Given a (complete) axiomatisation of a logic w.r.t. its coalgebraic semantics, we systematically derive a (complete) sequent calculus. In general, this calculus will only be complete if we include the cut rule. We show that cut free completeness, and therefore eliminability of cut, follows if the axiomatisation is *strictly one-step complete*: every valid clause containing operators applied to propositional variables can be derived using a *single* modal deduction rule. The existence of a cut-free sequent calculus for coalgebraic logics is then exploited to establish conservativity, complexity and interpolation for modal logics in a coalgebraic framework. While conservativity and complexity of coalgebraic logics have already been established in [20] we believe that the results here offer additional conceptual insight. Regarding interpolation, we obtain a new proof of the interpolation property for Coalition Logic [7] while interpolation for the conditional logics CK and CK_{Id} [4] were left as future work in [13] and appear to be new.

On a technical level, we consider modal logics that are built from atomic propositions, propositional connectives and modal operators, that is, in contrast to earlier e.g. [10,14,21,20] we treat propositional variables as first-class citizens. This does not only provide a better alignment with standard texts in modal logic [4,3] but is moreover a prerequisite for formulating the interpolation property.

As a consequence, we are lead to work with coalgebraic models, that is, coalgebras together with a valuation of the propositional variables, right from the start. Completeness and cut-free completeness is then proved via a terminal sequence argument, but over the extension of the signature functor to the slice category $\text{Set}/\mathcal{P}(V)$ where V is the set of propositional variables. This provides an alternative route to the shallow proof property of [20]. In this setting, we observe that strict completeness corresponds to eliminability of cut. As strict completeness can always be achieved by closure of the rule set under rule resolution, this essentially amounts to complementing the rule set so that cuts involving rule conclusions are in fact absorbed in the rule set in strong analogy with Mints’ comparison [12] between resolution and sequent proofs.

We summarise the coalgebraic semantics of modal logics in Section 2 and introduce modal sequent calculi in 3. Section 4 then establishes cut-free completeness and we discuss applications, in particular the interpolation property, in Section 5 before concluding with two open problems.

2 Coalgebraic and Logical Preliminaries

Given a category \mathbb{C} and an endofunctor $F : \mathbb{C} \rightarrow \mathbb{C}$, an F -coalgebra is a pair (C, γ) where $C \in \mathbb{C}$ is an object of \mathbb{C} and $\gamma : C \rightarrow FC$ is a morphism of \mathbb{C} . A *morphism* between F -coalgebras (C, γ) and (D, δ) is a morphism $m : C \rightarrow D \in \mathbb{C}$ such that $\delta \circ m = Fm \circ \gamma$. The category of F -coalgebras will be denoted by $\text{Coalg}(F)$.

In the sequel, we will be concerned with F -coalgebras both on the category Set of

sets and (total) functions and on the slice category $\text{Set}/\mathcal{P}(V)$, for V a denumerable set of propositional variables that we keep fixed throughout the paper. Working with the slice category $\text{Set}/\mathcal{P}(V)$ allows a convenient treatment of propositional variables. In particular, coalgebras on $\text{Set}/\mathcal{P}(V)$ play the role of Kripke models, i.e. they come equipped with a valuation of propositional variables. Recall that an object of $\text{Set}/\mathcal{P}(V)$ is a function $f : X \rightarrow \mathcal{P}(V)$ and a morphism $m : (X \xrightarrow{f} \mathcal{P}(V)) \rightarrow (Y \xrightarrow{g} \mathcal{P}(V))$ is a commuting triangle, viz a function $m : X \rightarrow Y$ such that $g \circ m = f$. The projection functor mapping $(X \rightarrow \mathcal{P}(V)) \mapsto X$ is denoted by $U : \text{Set}/\mathcal{P}(V) \rightarrow \text{Set}$. For the remainder of the paper, we fix an endofunctor $T : \text{Set} \rightarrow \text{Set}$ and denote its extension to $\text{Set}/\mathcal{P}(V)$ by $T/\mathcal{P}(V) : (\text{Set}/\mathcal{P}(V)) \rightarrow (\text{Set}/\mathcal{P}(V))$; the functor $T/\mathcal{P}(V)$ maps objects $f : X \rightarrow \mathcal{P}(V)$ to the second projection mapping $TX \times \mathcal{P}(V) \rightarrow \mathcal{P}(V)$. Note that an object $M \in \text{Coalg}(T/\mathcal{P}(V))$ is a commuting triangle necessarily of the form

$$\begin{array}{ccc} C & \xrightarrow{\langle \gamma, \vartheta \rangle} & TC \times \mathcal{P}(V) \\ & \searrow \vartheta & \swarrow \pi_2 \\ & & \mathcal{P}(V) \end{array}$$

or equivalently a triple (C, γ, ϑ) where $(C, \gamma) \in \text{Coalg}(T)$ and $\vartheta : C \rightarrow \mathcal{P}(V)$ is a co-valuation of the propositional variables. Passing from the co-valuation $\vartheta : C \rightarrow \mathcal{P}(V)$ to the valuation $\vartheta^\# : V \rightarrow \mathcal{P}(C)$ induced by the self-adjointness of the powerset functor, we can view $T/\mathcal{P}(V)$ -coalgebras as T -coalgebras (C, γ) together with a valuation of propositional variables. The category of $T/\mathcal{P}(V)$ -coalgebras therefore plays the role of T -models (T -coalgebras which we see as frames, together with a valuation of propositional variables). In what follows, we will denote $T/\mathcal{P}(V)$ -coalgebras as triples (C, γ, ϑ) as above and use $\text{Mod}(T)$ to refer to the category $\text{Coalg}(T/\mathcal{P}(V))$ of T -models. If $M = (C, \gamma, \vartheta)$ is a T -model, then we refer to $(C, \gamma) \in \text{Coalg}(T)$ as the *underlying frame* of M .

On the syntactic side, we work with modal logics over an arbitrary modal similarity type (set of modal operators with associated arities) Λ . The set of Λ -formulas given by the grammar

$$\mathcal{F}(\Lambda) \ni \phi, \psi ::= p \mid \perp \mid \phi \rightarrow \psi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \heartsuit(\phi_1, \dots, \phi_n)$$

where $A \in V$ and $\heartsuit \in \Lambda$ is n -ary. We abbreviate $\phi \rightarrow \perp$ by $\neg\phi$ and $\perp \rightarrow \perp$ by \top . Note that, in contrast to the earlier treatments of coalgebraic modal logic [10,14,21,20], the definition above includes propositional variables as first-class citizens. If S is a set (of formulas, or variables) then $\Lambda(S)$ denotes the set $\{\heartsuit(s_1, \dots, s_n) \mid \heartsuit \in \Lambda \text{ is } n\text{-ary}, s_1, \dots, s_n \in S\}$ of formulas comprising exactly one application of a modality to elements of S . We denote the set of propositional formulas over a set S by $\text{Prop}(S)$ and write $\text{Cl}(S)$ for the set of clauses over elements of S , i.e. formulas of the form $s_1 \wedge \dots \wedge s_n \rightarrow s'_1 \vee \dots \vee s'_k$ for $n, k \in \omega$ and $s_1, \dots, s_n, s'_1, \dots, s'_k \in S$. If $\phi \in \text{Prop}(S)$, then every valuation $\tau : S \rightarrow \mathcal{P}(X)$ inductively defines a subset $\llbracket \phi \rrbracket_X^\tau \subseteq X$ by evaluation in the boolean algebra $\mathcal{P}(X)$ and we write $X, \tau \models \phi$ if $\llbracket \phi \rrbracket_X^\tau = X$.

An S -substitution is a partial mapping $\sigma : V \rightarrow S$. We denote the result of simultaneously substituting $\sigma(x)$ for every element $x \in \text{dom}(\sigma)$ in a formula $\phi \in \mathcal{F}(\Lambda)$ by $\phi\sigma$.

Formulas of $\mathcal{F}(\Lambda)$ are interpreted over T -coalgebras provided that T extends to a Λ -structure, i.e. comes equipped with an assignment of predicate liftings (natural transformations)

$$\llbracket \heartsuit \rrbracket : 2^n \rightarrow 2 \circ T$$

to every n -ary modal operator $\heartsuit \in \Lambda$. Here $2 : \text{Set} \rightarrow \text{Set}$ is the contravariant powerset functor, and for any functor F , F^n denotes the n -fold product of F with itself, i.e. $F^n(X) = FX \times \cdots \times FX$. We usually leave the assignment of predicate liftings to modal operators implicit and simply use T to refer to the entire Λ -structure.

Given a Λ -structure T and $M = (C, \gamma, \vartheta) \in \text{Mod}(T)$, the semantics of $\phi \in \mathcal{F}(\Lambda)$ is inductively given by

$$\llbracket \heartsuit(\phi_1, \dots, \phi_n) \rrbracket_M = \gamma^{-1} \circ \llbracket \heartsuit \rrbracket_C(\llbracket \phi_1 \rrbracket_M, \dots, \llbracket \phi_n \rrbracket_M)$$

and

$$\llbracket p \rrbracket_M = \{c \in C \mid p \in \vartheta(c)\}$$

for $p \in V$ together with the standard clauses for the propositional connectives.

If $M = (C, \gamma, \vartheta)$ is a T -model, semantic validity $\llbracket \phi \rrbracket_M = C$ is denoted by $M \models \phi$. We write $\text{Mod}(T) \models \phi$ if $M \models \phi$ for all $M \in \text{Mod}(T)$.

This definition of coalgebraic semantics relativises to one-step formulas, i.e. clauses $\chi \in \text{Cl}(\Lambda(S))$: every valuation $\tau : S \rightarrow \mathcal{P}(X)$ induces a subset $\llbracket \chi \rrbracket_{TX}^\tau \subseteq TX$ defined inductively by $\llbracket \heartsuit(s_1, \dots, s_n) \rrbracket_{TX}^\tau = \llbracket \heartsuit \rrbracket_S(\tau(s_1), \dots, \tau(s_n))$ and we write $TX, \tau \models \chi$ if $\llbracket \chi \rrbracket_{TX}^\tau = TX$. Our techniques will be illustrated by the following two running examples:

Example 2.1 (Coalition Logic and Conditional Logic) (i) *Coalition logic [15] allows to reason about the coalitional power in games. We take $N = \{1, \dots, n\}$ to be a fixed set of agents, subsets of which are called coalitions. The similarity type Λ of coalition logic contains a unary modal operators $[C]$ for every coalition $C \subseteq N$. Informally, $[C]\phi$ expresses that coalition C has a collaborative strategy to force ϕ . The coalgebraic semantics for coalition logic is based on the (class-valued) signature functor \mathbf{C} defined by*

$$\mathbf{C}X = \{(S_1, \dots, S_n, f) \mid \emptyset \neq S_i \in \text{Set}, f : \prod_{i \in N} S_i \rightarrow X\}.$$

The elements of $\mathbf{C}X$ are understood as strategic games with set X of states, i.e. tuples consisting of nonempty sets S_i of strategies for all agents i , and an outcome function $(\prod S_i) \rightarrow X$. A \mathbf{C} -coalgebra is a game frame [15]. We denote the set $\prod_{i \in C} S_i$ by S_C , and for $\sigma_C \in S_C, \sigma_{\bar{C}} \in S_{\bar{C}}$, where $\bar{C} = N - C$, $(\sigma_C, \sigma_{\bar{C}})$ denotes the obvious element of $\prod_{i \in N} S_i$. A Λ -structure over \mathbf{C} is defined by the predicate liftings

$$\llbracket [C] \rrbracket_X(A) = \{(S_1, \dots, S_n, f) \in \mathbf{C}X \mid \exists \sigma_C \in S_C. \forall \sigma_{\bar{C}} \in S_{\bar{C}}. f(\sigma_C, \sigma_{\bar{C}}) \in A\}.$$

(ii) *The similarity type of the conditional logics CK and CK_{Id} contains the single binary modal operator \multimap (usually written as \Rightarrow which would conflict with the sequent notation introduced later) that represents a non-monotonic conditional. The selection function semantics of CK is captured coalgebraically via the functor $\mathbf{CK}X = (2(X) \rightarrow \mathcal{P}(X))$ with \multimap representing function space, and \mathbf{CK} -coalgebras are standard conditional models [4]. We extend \mathbf{CK} to a Λ -structure by virtue of the predicate lifting*

$$\llbracket \multimap \rrbracket_X(A, B) = \{f : 2X \rightarrow \mathcal{P}X \mid f(A) \subseteq B\}$$

which induces the standard semantics of CK. The conditional logic CK_{Id} additionally obeys the (rank-1) axiom $A \multimap A$ and is interpreted over the functor $CK_{Id}X = \{f : 2(X) \rightarrow \mathcal{P}(X) \mid \forall A \subseteq X. f(A) \subseteq A\}$; note that CK_{Id} is a sub-functor of CK. The functor CK_{Id} extends to a Λ -structure by relativizing the interpretation of \multimap given above, i.e.

$$\llbracket \multimap \rrbracket_X(A, B) = \{f \in CK_{Id}X \mid f(A) \subseteq B\}$$

for subsets $A, B \subseteq X$.

3 Sequent Systems for Coalgebraic Logics

Previous work on deduction in coalgebraic logics has focused on languages without propositional variables and deduction was formalised using Hilbert-style proof systems where propositional variables were simulated using nullary modalities. This contrasts with our treatment here where we treat propositional variables as first-class citizens in a Gentzen-style sequent calculus. By a $\mathcal{F}(\Lambda)$ -sequent we mean a pair (Γ, Δ) (written $\Gamma \Rightarrow \Delta$) where $\Gamma, \Delta \subseteq \mathcal{F}(\Lambda)$ are multisets of formulas. The sequent $\Gamma_0 \Rightarrow \Delta_0$ is a *sub-sequent* of $\Gamma \Rightarrow \Delta$ if $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$ as multisets; this is denoted by $\Gamma_0 \Rightarrow \Delta_0 \subseteq \Gamma \Rightarrow \Delta$. If $\Gamma \subseteq \mathcal{F}(\Lambda)$ is a multiset, we write $\text{supp}(\Gamma)$ for its set of elements, disregarding multiplicities. This is extended to sequents $\Gamma \Rightarrow \Delta$ by stipulating that $\text{supp}(\Gamma \Rightarrow \Delta) = \text{supp}(\Gamma) \cup \text{supp}(\Delta)$. We identify a formula ϕ with the singleton multiset $\{\phi\}$ whenever convenient and denote the multiset union of $\Gamma, \Delta \subseteq \mathcal{F}(\Lambda)$ by Γ, Δ .

Substitutions are applied pointwise to sequents: if σ is a substitution and $\Gamma \Rightarrow \Delta$ is a sequent, $(\Gamma \Rightarrow \Delta)\sigma = \{\phi\sigma \mid \phi \in \Gamma\} \Rightarrow \{\phi\sigma \mid \phi \in \Delta\}$. In our terminology, a *sequent rule* is a tuple (E_1, \dots, E_n, E_0) of sequents, usually written in the form

$$\frac{E_0 \quad \dots \quad E_n}{E_0} \quad \text{or} \quad E_1, \dots, E_n / E_0$$

where we use captial letters A, B, C, \dots from the beginning of the alphabet to indicate propositional variables. We silently identify sequent rules modulo reordering of the sequents in the premise. To make the notation bearable, we allow ourselves to use additional meta-variables Γ, Δ, \dots to represent multisets of elements of V so that for example the introduction of conjunction on the right

$$R \wedge \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

is in fact a rule schema, each instance of which is given by a choice of multisets $\Gamma, \Delta \subseteq V$.

Given a set \mathbf{R} of sequent rules, the notion of deduction is standard: the set D of \mathbf{R} -*derivable sequents* is the least set that is closed under the rules in \mathbf{R} , i.e. the least set D of sequents that satisfies $E_0\sigma \in D$ whenever $E_1, \dots, E_n/E_0 \in \mathbf{R}$, $\sigma : V \rightarrow \mathcal{F}(\Lambda)$ is a substitution and $E_1\sigma, \dots, E_n\sigma \in D$. We write $\mathbf{R} \vdash \Gamma \Rightarrow \Delta$ if $\Gamma \Rightarrow \Delta$ is an \mathbf{R} -derivable sequent. A sequent rule $E_1, \dots, E_n/E_0$ is \mathbf{R} -*admissible* if $\mathbf{R} \vdash E_0\sigma$ whenever $\mathbf{R} \vdash E_i\sigma$ for all $i = 1, \dots, n$.

We use the following system \mathbf{G} to account for the propositional part of our calculus:

$$\begin{array}{ll}
 (Ax) & \Gamma, A \Rightarrow \Delta, A \\
 (L\perp) & \perp, \Gamma \Rightarrow \Delta \\
 (L\wedge) & \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \\
 (R\wedge) & \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \\
 (L\vee) & \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \\
 (R\vee) & \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \\
 (L\rightarrow) & \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \\
 (R\rightarrow) & \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow A \rightarrow B, \Delta}
 \end{array}$$

where Ax and $L\perp$ are assumed to have empty premise. This is a slight modification of the system $\mathbf{G3c}$ of [22] where only $A \in V$ is permitted in (Ax) and (Ax) as formulated here is admissible. Note that exchange rules are not needed as \mathbf{G} is formulated in terms of multisets.

Note that \mathbf{G} is complete w.r.t. propositional validity, i.e. $\mathbf{G} \vdash \Gamma \Rightarrow \Delta$ iff $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is a propositional tautology. Our next task is to extend \mathbf{G} with additional sequent rules to account for modal deduction. It has been shown in [17] that coalgebraic logics can always be completely axiomatised in *rank 1*, in particular, by a (possibly infinite) number of *one-step rules*:

Definition 3.1 A one-step rule over a modal similarity type Λ is a pair (ϕ, ψ) , usually written ϕ/ψ , where $\phi \in \text{Prop}(V)$ is a propositional formula and $\psi \in \text{Cl}(\Lambda(V))$ is a clause over $\Lambda(V)$ where every propositional variable occurs at most once in ψ .

The restriction on occurrences of propositional variables in the conclusion of one-step rules is unproblematic and can always be satisfied by adding new variables, e.g. by passing from the rule $\phi/\heartsuit a \rightarrow \spadesuit a$ to the rule $\phi \wedge a \leftrightarrow b/\heartsuit a \rightarrow \spadesuit b$ if b does not occur in ϕ .

We are now associating a sequent rule $(E_1, \dots, E_n/E_0)$ to every one-step rule ϕ/ψ . The sequents E_1, \dots, E_n occurring in the premise essentially represent the minimal conjunctive normal form of the premise ϕ of the one-step rule. The following notions are convenient for this purpose.

Definition 3.2 Suppose $\phi \in \text{Prop}(V)$ is a propositional formula. A valuation for ϕ is a partial function $\tau : V \rightarrow \{\perp, \top\}$; valuations are partially ordered by graph inclusion denoted by \subseteq . If $\tau : V \rightarrow \{\perp, \top\}$ is a partial valuation, we define $\phi\tau = \perp$ if $\phi\tau' = \perp$ for all total valuations $\tau' \supseteq \tau$. The set $\min(\phi)$ of minimal falsifications of ϕ is the set of minimal elements of $\{\sigma : V \rightarrow \{\perp, \top\} \mid \phi\sigma = \perp\}$. To every valuation σ with finite domain we associate a clause $\text{cl}(\sigma) = \bigwedge_{\sigma(A)=\top} A \rightarrow \bigvee_{\sigma(A)=\perp} A$.

If for example $\phi = p \wedge (q \rightarrow p \wedge r)$ then $\min(\phi) = \{\sigma_1, \sigma_2\}$ with $\sigma_1(p) = \perp$ and $\sigma_2(q) = \top, \sigma_2(r) = \perp$ and both σ_1, σ_2 undefined, otherwise. We obtain $\text{cl}(\sigma_1) = p$ and $\text{cl}(\sigma_2) = q \rightarrow r$ resulting in the conjunctive normal form $p \wedge (q \rightarrow r)$ of ϕ . We obtain:

Lemma 3.3 If $\phi \in \text{Prop}(V)$, then $\min(\phi)$ is finite and ϕ and $\bigwedge\{\text{cl}(\sigma) \mid \sigma \in \min(\phi)\}$ are propositionally equivalent. Moreover $\phi \rightarrow \psi$ is a propositional tautology iff every $\sigma \in \min(\psi)$ extends some $\sigma' \in \min(\phi)$.

This allows us to associate sequents to formulas, and more generally sequent rules to one-

step rules as follows:

Definition 3.4 (Associated Sequents) (i) If $\chi = \bigwedge_{i=1}^n \Phi_i \rightarrow \bigvee_{j=1}^k \Psi_j \in \text{Cl}(\Lambda(V))$ is a clause, then the associated sequent of χ is the sequent $\mathcal{S}(\chi) = \Phi_1, \dots, \Phi_n \Rightarrow \Psi_1, \dots, \Psi_k$.

(ii) If $\phi \in \text{Prop}(V)$ then associated sequent set of ϕ is the set of sequents $\mathcal{S}(\phi) = \{\mathcal{S}(\text{cl}(\sigma)) \mid \sigma \in \text{min}(\phi)\}$.

(iii) Finally, if ϕ/ψ is a one-step rule, then $\mathcal{S}(\phi/\psi)$ denotes the set of sequent calculus rules consisting of all rules $E_1, \dots, E_n, \Gamma, F \Rightarrow \Delta, G$ where the E_i are pairwise distinct with $\mathcal{S}(\phi) = \{E_1, \dots, E_n\}$ and $\mathcal{S}(\psi) = F \Rightarrow G$.

If \mathbf{R} is a set of one-step rules, then \mathbf{GR} is the system consisting of the rules of \mathbf{G} together with the rules $\mathcal{S}(\phi/\psi)$ for all $\phi/\psi \in \mathbf{R}$.

We note that – by virtue of the convention that Γ, Δ stand for multisets of variables $A, B, C \subseteq V$ – every one-step rule gives rise to a set of sequent calculus rules. This is needed to absorb the structural rules of weakening. It is easy to see that all sequent rules obtained from one-step rules have a very specific form: all premises are sequents consisting only of variables $A, B, C \in V$ whereas conclusions consists of modal operators applied to meta-variables $\heartsuit(A_1, \dots, A_n)$ together with context formulas.

The sole purpose of the above definitions is to define sequent calculi associated with sets of one-step rules. For our two running examples, the situation is as follows.

Example 3.5 (Coalition Logic and Conditional Logic) (i) Consider the set $\mathbf{R}_{\mathbf{C}}$ consisting of the one-step rules

$$\frac{\bigwedge_{i=1}^k A_i \rightarrow \perp}{\bigwedge_{i=1}^k [C_i]A_i \rightarrow \perp} \quad \frac{\bigwedge_{i=1}^k A_i \rightarrow B \vee \bigvee_{j=1}^l A'_j}{\bigwedge_{i=1}^k [C_i]A_i \rightarrow [D]B \vee \bigvee_{j=1}^l [N]A'_j}$$

subject to the side condition that the C_i are pairwise disjoint; the second rule additionally requires that $C_i \subseteq D$ for all $i = 1, \dots, k$. The sequent rules associated to this set are most economically presented if we abbreviate $\mathbf{A} = A_1, \dots, A_k$ for $A_1, \dots, A_k \subseteq V$ and $\mathbf{C} = (C_1, \dots, C_k)$ for $C_1, \dots, C_k \subseteq C$; in this case $[\mathbf{C}]\mathbf{A}$ represents the multiset $[C_1]A_1, \dots, [C_k]A_k$ of formulas. Using this notation, we obtain the following two associated sequent rules

$$(A) \frac{\mathbf{A} \Rightarrow \emptyset}{\Gamma, [\mathbf{C}]\mathbf{A} \Rightarrow \Delta} \quad (B) \frac{\mathbf{A} \Rightarrow B, \mathbf{A}'}{\Gamma, [\mathbf{C}]\mathbf{A} \Rightarrow \Delta, [D]B, [\mathbf{N}]\mathbf{A}'}$$

where $\mathbf{N} = N, \dots, N$; both rules are subject to the side condition that the coalitions appearing in \mathbf{C} are disjoint; rule (B) moreover requires that their union is a subset of D . We refer to the calculus containing \mathbf{G} and the rules (A), (B) above as \mathbf{GC} .

(ii) The set $\mathbf{R}_{\mathbf{CK}}$ contains the one-step rules

$$\frac{\bigwedge_{i=1}^n B_i \rightarrow B_0 \wedge \bigwedge_{i=0}^n (A_i \leftrightarrow A_0)}{\bigwedge_{i=1}^n (A_i \multimap B_i) \rightarrow (A_0 \multimap B_0)}$$

for every $n \in \omega$. As above, we abbreviate $\mathbf{B} = B_1, \dots, B_k$, $\mathbf{A} = A_1, \dots, A_k$ and $\mathbf{A} \multimap$

$\mathbf{B} = A_1 \multimap B_1, \dots, A_k \multimap B_k$. The sequent rules for CK then take the form

$$(C) \frac{\mathbf{B} \Rightarrow B_0 \quad A_0 \Rightarrow A_1 \quad \dots \quad A_0 \Rightarrow A_k \quad A_1 \Rightarrow A_0 \quad \dots \quad A_k \Rightarrow A_0}{\Gamma, \mathbf{A} \multimap \mathbf{B} \Rightarrow A_0 \multimap B_0, \Delta}.$$

The set of one-step rules needed to axiomatise CK_{Id} contains the additional rule

$$\frac{A_0 \leftrightarrow A_1}{A_0 \multimap A_1}$$

which induces the sequent rule

$$(I) \frac{A_0 \Rightarrow A_1 \quad A_1 \Rightarrow A_0}{\Gamma \Rightarrow A_0 \multimap A_1, \Delta}$$

The rules (C) express that the second component obeys normality whereas the first behaves like the modal \Box of neighbourhood frames and (I) formalises an identity law. The calculus extending \mathbf{G} with (C) is later referenced as \mathbf{GCK} and its extension with the rule (I) as \mathbf{GCK}_{Id} .

We note the following immediate corollary of Lemma 3.3.

Corollary 3.6 *Suppose $\phi \rightarrow \psi \in \text{Prop}(W)$ is a propositional tautology. Then for every $\Gamma \Rightarrow \Delta \in \mathcal{S}(\psi)$ there exists $\Gamma' \Rightarrow \Delta' \in \mathcal{S}(\phi)$ such that $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$.*

Informally, if $\phi \rightarrow \psi$ is a propositional tautology and all sequents $E \in \mathcal{S}(\phi)$ are provable, the above lemma entails that we can use weakening (rather than cut, introduced next) to show that also all sequents $E' \in \mathcal{S}(\psi)$ are provable.

Convention 3.7 *If \mathbf{R} is a set of one-step rules, we denote the system that contains the rules of \mathbf{GR} together with the Cut rule*

$$(cut) \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

by \mathbf{GRc} . The notation $\mathbf{GR}[c]$ indicates a statement that applies both to \mathbf{GR} and \mathbf{GRc} ; eg. $\mathbf{GR}[c] \vdash \Gamma \Rightarrow \Delta$ reads as derivability both in the system \mathbf{GR} and \mathbf{GRc} .

As already hinted at above, the system \mathbf{GR} absorbs the structural rules of weakening and contraction. The following two propositions are devoted to a syntactic proof of this fact. We will later return to this point and present an alternative argument via semantic completeness. Weakening comes first:

Proposition 3.8 *The following left and right rules of weakening*

$$(LW) \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad (RW) \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

are admissible in $\mathbf{GR}[c]$.

Proof. By induction on the derivation of $\mathbf{GR}[c] \vdash \phi$. For admissibility of (LW) we have to show that $\mathbf{GR}[c] \vdash \Gamma, A \Rightarrow \Delta$ whenever $\mathbf{GR}[c] \vdash \Gamma \Rightarrow \Delta$ for multisets $\Gamma, \Delta \subseteq \mathcal{F}(\Lambda)$ and $A \in \mathcal{F}(\Lambda)$. The cases dealing with the rules of \mathbf{G} are standard (see [22]). Suppose $\Gamma \Rightarrow \Delta$ has been derived with the help of a sequent rule associated with a one-step rule ϕ/ψ . Then there exists a substitution $\sigma : V \rightarrow \mathcal{F}(\Lambda)$ such that

- $\mathbf{GR}[c] \vdash E\sigma$ for all $E \in \mathcal{S}(\psi)$
- $\mathcal{S}(\psi)\sigma \subseteq \Gamma \Rightarrow \Delta$.

This implies $\mathcal{S}(\psi)\sigma \subseteq \Gamma, A \Rightarrow \Delta$ so that the same rule also derives $\Gamma, A \Rightarrow \Delta$. Applications of the cut rule follow directly from the induction hypothesis. \square

The syntactic proof of admissibility of contraction is slightly involved and requires a few lemmas. Recall that the support $\text{supp}(E)$ of a sequent disregards multiplicities in antecedent and succedent.

Lemma 3.9 *Suppose $\tau : V \rightarrow V$ satisfies $\tau(B) = A$ for some $A, B \in V$ and $\tau(C) = C$ for all $C \neq B$. Then, for all $\phi \in \text{Prop}(V)$ and all $E \in \mathcal{S}(\phi\tau)$ there exists $E^* \in \mathcal{S}(\phi)$ such that $\text{supp}(E^*\tau) \subseteq E$.*

Proof. Let $E \in \mathcal{S}(\phi\tau)$. Then there exists $\rho : V \rightarrow \{\top, \perp\} \in \min(\phi\tau)$ such that $(\phi\tau)\rho = \perp$ and $E = \{C \in \text{dom}(\rho) \mid \rho(C) = \top\} \Rightarrow \{C \in \text{dom}(\rho) \mid \rho(C) = \perp\}$. Note that $B \notin \text{dom}(\rho)$ as B does not occur in $\phi\tau$. If $A \notin \text{dom}(\rho)$ then $\rho \in \min(\phi)$ and we are done. So suppose $A \in \text{dom}(\rho)$. If $\rho' : V \rightarrow \{\perp, \top\}$ extends ρ by mapping $B \mapsto A$, i.e. $\rho'(B) = A$ and $\rho'(C) = \rho(C)$ for all $C \neq B$, we have $\phi\rho' = \perp$. Take a minimal falsification $\rho'' \subseteq \rho'$ of ϕ whence $\phi\rho'' = \perp$ and put $E^* = \{C \mid \rho''(C) = \top\} \Rightarrow \{C \mid \rho''(C) = \perp\}$. Then $E^* \in \mathcal{S}(\phi)$. To see that $\text{supp}(E^*\tau) \subseteq E$ first pick $\tau(C)$ that occurs positively in $E^*\tau$. *Case 1.* $\tau(C) \neq A$. The $C = \tau(C)$ and $\rho''(C) = \top$ whence $\rho(C) = \top$ and C occurs positively in E . *Case 2.* $\tau(C) = A$. Then $C \in \{A, B\}$. In both cases, $\rho''(C) = \top$ whence $\rho(A) = \top$ so A occurs positively in E ; note that A will only occur once in E . The cases where $\tau(C)$ occurs negatively in $E^*\tau$ are symmetric. \square

Lemma 3.10 *Suppose $\tau : V \rightarrow V$ satisfies $\tau(B) = A$ for some $A, B \in V$ and $\tau(C) = C$ for $C \neq B$ and let $\phi \in \text{Prop}(V)$, $\sigma : V \rightarrow \mathcal{F}(\Lambda)$. If $\mathbf{GR} \vdash \text{supp}(E\tau)\sigma$ for all $E \in \mathcal{S}(\phi)$ then $\mathbf{GR} \vdash E\sigma$ for all $E \in \mathcal{S}(\phi\tau)$.*

Proof. Pick $E \in \mathcal{S}(\phi\tau)$. By the above lemma, we can find $E^* \in \mathcal{S}(\phi)$ such that $\text{supp}(E^*\tau) \subseteq E$. By assumption, $\mathbf{GR} \vdash \text{supp}(E^*\tau)\sigma$. Since $\text{supp}(E^*\tau) \subseteq E$ we have $\text{supp}(E^*\tau\sigma) \subseteq E\sigma$. Since $\text{supp}(E^*\tau)\sigma$ is derivable the admissibility of weakening implies $\mathbf{GR} \vdash E\sigma$. \square

For admissibility of contraction, we have to rely on an additional property of the set of one-step rules:

Definition 3.11 (Definition 5.1 of [18]) *A clause χ over $\Lambda(V)$ is contracted if all its literals are distinct and a one-step rule ϕ/ψ is contracted if its conclusion ψ is contracted. A set \mathbf{R} of one-step rules is contraction closed, if, for every $\sigma : V \rightarrow V$ and every $\phi/\psi \in \mathbf{R}$ there exists $\sigma' : V \rightarrow V$ and $\phi'/\psi' \in \mathbf{R}$ such that $\psi'\sigma'$ is contracted, $\psi\sigma$ is propositionally implied by $\psi'\sigma'$ and $\phi'\sigma'$ is a propositional consequence of $\phi\sigma$.*

Proposition 3.12 *The following left and right rules of contraction*

$$(LC) \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad (RC) \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

are admissible in \mathbf{GRc} . If moreover \mathbf{R} is contraction closed, then they are also admissible in \mathbf{GR} .

Proof. Admissibility of (LC) and (RC) in \mathbf{GRc} is just an application of the cut rule.

We focus on admissibility in **GR**. For the admissibility of (LC) we have to show that $\mathbf{GR} \vdash \Gamma, A \Rightarrow \Delta$ whenever $\mathbf{GR} \vdash \Gamma, A, A \Rightarrow \Delta$ where again $\Gamma, \Delta \subseteq \mathcal{F}(\Lambda)$ are multisets of formulas and $A \in \mathcal{F}(\Lambda)$. We proceed by induction over the derivation of $\Gamma, A, A \Rightarrow \Delta$ where the applications of rules of **G** is standard. So suppose that a rule $\mathcal{S}(\phi/\psi)$ associated with a one-step rule $\phi/\psi \in \mathbf{R}$ has been applied in the derivation of $\Gamma, A, A \Rightarrow \Delta$; assume that $\mathcal{S}(\psi) = \Gamma_1 \Rightarrow \Delta_1$. Then there exists a substitution $\sigma : V \rightarrow \mathcal{F}(\Lambda)$ and two subsets $\Gamma_0, \Delta_0 \subseteq \mathcal{F}(\Lambda)$ such that

- $\mathbf{GR} \vdash E\sigma$ for all $E \in \mathcal{S}(\phi)$
- $\Gamma, A, A \Rightarrow \Delta = \Gamma_0, \Gamma_1\sigma \Rightarrow \Delta_0, \Delta_1\sigma$.

If $A \in \Gamma_0$ then putting $\Gamma'_0 = \Gamma \setminus \{A\}$ allows to derive $\Gamma, A \Rightarrow \Delta$ using the same rule, so suppose that $A \notin \Gamma_0$ whence $\{A, A\} \subseteq \Gamma_1\sigma$ as multisets.

As ψ is contracted as a conclusion of a one-step rule, there exist two distinct elements $B_0, B_1 \in V$ such that $\sigma(B_0) = \sigma(B_1)$. Consider the substitution $\tau : V \rightarrow V$ with $\tau(B_1) = B_0$ and $\tau(C) = C$ otherwise. By contraction closure, we can find a one-step rule $\phi'/\psi' \in \mathbf{R}$ and a substitution $\tau' : V \rightarrow V$ such that $\psi'\tau'$ is contracted, and both formulas $\psi'\tau' \rightarrow \psi\tau$ and $\phi\sigma \rightarrow \phi'\sigma'$ are tautologies. As $\psi'\tau'$ is contracted, we can assume w.l.o.g. that τ' is a bijection.

By assumption, we have $\mathbf{GR} \vdash E\sigma$ for all $E \in \mathcal{S}(\phi)$. As $E\sigma = E\tau\sigma$ we conclude that $\mathbf{GR} \vdash E\tau\sigma$ for all $E \in \mathcal{S}(\phi)$. By induction hypothesis, we have $\mathbf{GR} \vdash \text{supp}(E\tau)\sigma$ and by Lemma 3.10 $\mathbf{GR} \vdash E\sigma$ for all $E \in \mathcal{S}(\phi\tau)$. Since $\phi\tau \rightarrow \phi'\tau'$ is a tautology, Corollary 3.6 allows us to conclude that $\mathbf{GR} \vdash E\sigma$ for all $E \in \mathcal{S}(\phi'\tau')$. Since τ' is a bijection, this immediately gives derivability of $E\tau'\sigma$ for all $E \in \mathcal{S}(\phi')$ as $\{E\tau' \mid E \in \mathcal{S}(\phi')\} = \mathcal{S}(\phi'\tau)$. If $F = \mathcal{S}(\psi')$, the sequent rule associated with ϕ'/ψ' now implies derivability of $F\tau'\sigma$. Since $\psi'\tau' \rightarrow \psi\tau$ is a tautology we have $\mathcal{S}(\psi'\tau') \subseteq \mathcal{S}(\psi\tau)$. It follows that $\mathbf{GR} \vdash \Gamma, A \Rightarrow \Delta$ by admissibility of weakening as $\psi'\tau'$ is contracted. \square

4 Soundness and Completeness

We now study the relationship between **GR**-derivability and semantic validity. As in previous work, both soundness and completeness will be implied by one-step completeness of the rule set **R**. However, we want to point out two subtle differences: (a) our proof deals with propositional variables directly and (b) it sheds light on the structure of proofs. In particular, we will see that a one-step complete rule set necessitates the use of cut to obtain completeness and eliminability of cut amounts to strict one-step completeness. We recall the definition of one-step soundness and one-step completeness:

Definition 4.1 *A set **R** of one-step rules is one-step sound w.r.t a Λ -structure T if, whenever $\phi/\psi \in \mathbf{R}$, we have $TX, \tau \models \psi$ for each set X and each $\mathcal{P}(X)$ -valuation τ such that $X, \tau \models \phi$. The set **R** is (strictly) one-step complete if, whenever $TX, \tau \models \chi$ for a set X , $\chi \in \text{Cl}(\Lambda(V))$, and a $\mathcal{P}(X)$ -valuation τ , then χ is (strictly) provable over X, τ , i.e. propositionally entailed by clauses (a clause) $\psi\sigma$ where $\phi/\psi \in \mathbf{R}$ and σ is a $\text{Prop}(V)$ -substitution such that $X, \tau \models \phi\sigma$.*

It is an easy exercise to show that $\mathbf{GR}[c]$ is sound provided the rule set **R** is one-step sound. To align the coalgebraic semantics of $\mathcal{F}(\Lambda)$ with the system **GR**, we define the *interpretation* of a sequent $\Gamma \Rightarrow \Delta$ w.r.t. $M = (C, \gamma, \vartheta) \in \text{Mod}(V)$ to be the semantics of

the associated propositional formula, i.e. $\llbracket \Gamma \Rightarrow \Delta \rrbracket_M = \llbracket \bigwedge \Gamma \rightarrow \bigvee \Delta \rrbracket_M$.

Theorem 4.2 (Soundness) *Suppose \mathbf{R} is one-step sound for T . Then $\text{Mod}(T) \models \Gamma \Rightarrow \Delta$ if $\mathbf{GR}[c] \vdash \Gamma \Rightarrow \Delta$.*

Proof. We proceed by induction over the length of the derivation, where the only interesting cases are applications of rules $\mathcal{S}(\phi/\psi)$ for $\phi/\psi \in \mathbf{R}$. So suppose $(C, \gamma, \vartheta) \in \text{Mod}(T)$ and $\Gamma \Rightarrow \Delta$ has been derived via an application of $\mathcal{S}(\phi/\psi)$. That is, there is a substitution $\sigma : V \rightarrow \mathcal{F}(\Lambda)$ such that $\mathbf{GR} \vdash E\sigma$ for all $E \in \mathcal{S}(\phi)$ and $\mathcal{S}(\psi)\sigma \subseteq \Gamma \Rightarrow \Delta$. By induction hypothesis $\llbracket E\sigma \rrbracket_M = \top$ for all $E \in \mathcal{S}(\phi)$ and consequently $\llbracket \phi\sigma \rrbracket_M = \top$ by definition of the associated sequent rule. Consider the $\mathcal{P}(C)$ -valuation $\tau(A) = \llbracket \sigma(A) \rrbracket_M$. We obtain $C, \tau \models \phi$ in the one-step sense and one-step soundness implies $TC, \tau \models \psi$. Consequently, $\llbracket \Gamma \Rightarrow \Delta \rrbracket \supseteq \llbracket \mathcal{S}(\psi)\sigma \rrbracket_M = \llbracket \psi\sigma \rrbracket_M = \top$ which concludes the proof. \square

Completeness can now be established in either of the three following ways:

- (i) syntactically by relating the sequent system \mathbf{GR} to a Hilbert system containing all rules of \mathbf{R} .
- (ii) by applying the shallow proof property [20] of a tableau system induced by the rules of \mathbf{R} and simulating propositional variables by nullary modalities, or
- (iii) by a direct semantical proof that treats propositional variables as first-class citizens.

In this paper, we adopt the latter approach and prove completeness using a terminal sequence argument in the style of [14]. As we are dealing with models, i.e. coalgebras equipped with a valuation, we consider the terminal sequence of the endofunctor $T/\mathcal{P}(V)$ in the category $\text{Set}/\mathcal{P}(V)$. We briefly recapitulate the terminal sequence construction, as used in [14], but phrased in a general categorical setting.

If $F : \mathbb{C} \rightarrow \mathbb{C}$ is an endofunctor on a category \mathbb{C} with terminal object 1 , the finitary part of the terminal sequence of F is the diagram consisting of

- the objects $F^n 1$ for $n \in \omega$ where F^n denotes n -fold application of F , and
- the morphisms $p_j^i : F^i 1 \rightarrow F^j 1$ defined by $p_i^{i+1} = F^i(! : F1 \rightarrow 1)$ and $p_n^{n+k} = p_{n+k-1}^{n+k} \circ \dots \circ p_k^{k+1}$.

Every F -coalgebra (C, γ) gives rise to a canonical cone $(C, (\gamma_n)_{n \in \omega})$ where $\gamma_n : C \rightarrow F^n 1$ over the finitary part of the terminal sequence by stipulating that $\gamma_0 = ! : C \rightarrow F^0 1 = 1$ where $!$ is the unique arrow given by finality of $1 \in \mathbb{C}$ and $\gamma_{n+1} = F\gamma_n \circ \gamma$. We use the terminal sequence construction for the functor $F = T/\mathcal{P}(V)$, the terminal sequence of which is visualised in the following diagram.

$$\begin{array}{ccccccc}
 \overbrace{1 \times \mathcal{P}(V)}^{S_0} & \xleftarrow{p_0^1 = !} & \overbrace{TS_0 \times \mathcal{P}(V)}^{S_1} & \xleftarrow{p_1^2 = Tp_0^1 \times \text{id}} & \overbrace{TS_1 \times \mathcal{P}(V)}^{S_2} & \xleftarrow{p_2^3 = Tp_1^2 \times \text{id}} & \dots \\
 \downarrow \pi_2 & & \downarrow \pi_2 & & \downarrow \pi_2 & & \\
 \mathcal{P}(V) & \xlongequal{\quad} & \mathcal{P}(V) & \xlongequal{\quad} & \mathcal{P}(V) & \xlongequal{\quad} & \dots
 \end{array}$$

The key technique in the proof of completeness via a terminal sequence argument is to associate to every formula of modal rank $\leq n$ an n -step semantics $\llbracket \phi \rrbracket_n$ over the n -th approximant $(T/\mathcal{P}(V))^n 1$ of the terminal sequence. In our case, we take a predicate over

$(T/\mathcal{P}(V))^{n1}$ to be a subset of $S_n = U((T/\mathcal{P}(V))^{n1})$. The formal definition is as follows:

Definition 4.3 *The stratification of $\mathcal{F}(\Lambda)$ is the sequence $\mathcal{F}_n(\Lambda) \subseteq \mathcal{F}(\Lambda)$ of sets of formulas defined inductively by*

$$\mathcal{F}_0(\Lambda) = \text{Prop}(V) \text{ and } \mathcal{F}_n(\Lambda) = \text{Prop}(\Lambda(\mathcal{F}_{n-1}(\Lambda)) \cup V).$$

The n -step semantics of $\phi \in \mathcal{F}_n(\Lambda) \subseteq S_n$ is inductively defined by $S_0 = \mathcal{P}(V)$ and

$$\llbracket p \rrbracket_0 = \{S \in \mathcal{P}(V) \mid p \in S\}$$

for $n = 0$ and $S_n = TS_{n-1} \times \mathcal{P}(V)$ together with

$$\llbracket p \rrbracket_n = \pi_2^{-1}(\{S \in \mathcal{P}(V) \mid p \in S\})$$

and

$$\llbracket \heartsuit(\phi_1, \dots, \phi_k) \rrbracket_n = \pi_1^{-1} \circ \llbracket M \rrbracket_{S_{n-1}}(\llbracket \phi_1 \rrbracket_{n-1}, \dots, \llbracket \phi_k \rrbracket_{n-1})$$

for $\phi_1, \dots, \phi_k \in \mathcal{F}_{n-1}(\Lambda)$ and $\heartsuit \in \Lambda$ an n -ary modality.

Note that $S_n = U((T/\mathcal{P}(V))^{n1})$. We can mediate between the n -step semantics and the semantics w.r.t $\text{Mod}(T)$ as follows:

Lemma 4.4 *Suppose $\phi \in \mathcal{F}_n(\Lambda)$ and $M = (C, \gamma, \vartheta) \in \text{Mod}(T)$. Suppose $(M, (\gamma_n)_{n \in \omega})$ is the canonical cone of M over the terminal sequence of $T/\mathcal{P}(V)$. Then $\llbracket \phi \rrbracket_M = (U\gamma_n)^{-1}(\llbracket \phi \rrbracket_n)$ for all $\phi \in \mathcal{F}_n(\Lambda)$.*

Proof. By induction on n . For $n = 0$ we have $U\gamma_0 = \vartheta$ and $\vartheta^{-1}(\llbracket p \rrbracket_0) = \vartheta^{-1}(\{S \subseteq V \mid p \in S\}) = \{c \in C \mid p \in \vartheta(c)\} = \llbracket p \rrbracket_M$. For $n > 0$, we obtain inductively $U\gamma_n = \langle TU\gamma_{n-1} \circ \gamma, \vartheta \rangle : C \rightarrow TS_{n-1} \times \mathcal{P}(V)$. This gives $(U\gamma_n)^{-1}(\llbracket p \rrbracket_n) = (\pi_2 \circ \langle TU\gamma_n \circ \gamma, \vartheta \rangle)^{-1}(\{S \subseteq V \mid p \in S\}) = \vartheta^{-1}(\{S \subseteq V \mid p \in S\}) = \{c \in C \mid p \in \vartheta(c)\} = \llbracket p \rrbracket_M$ as above. For modal formulas $\heartsuit(\phi_1, \dots, \phi_k)$ with $\phi_1, \dots, \phi_k \in \mathcal{F}_{n-1}(\Lambda)$ we obtain

$$\begin{aligned} & (U\gamma_n)^{-1}(\llbracket \heartsuit(\phi_1, \dots, \phi_k) \rrbracket_n) \\ &= \langle TU\gamma_{n-1} \circ \gamma, \vartheta \rangle^{-1} \circ \pi_1^{-1}(\llbracket \heartsuit \rrbracket_{S_{n-1}}(\llbracket \phi_1 \rrbracket_{n-1}, \dots, \llbracket \phi_k \rrbracket_{n-1})) \\ &= \gamma^{-1} \circ (TU\gamma_{n-1})^{-1} \circ \llbracket \heartsuit \rrbracket_{S_{n-1}}(\llbracket \phi_1 \rrbracket_{n-1}, \dots, \llbracket \phi_k \rrbracket_{n-1}) \\ &= \gamma^{-1} \circ \llbracket \heartsuit \rrbracket_C \circ (U\gamma_{n-1})^{-1} \times \dots \times (U\gamma_{n-1})^{-1}(\llbracket \phi_1 \rrbracket_{n-1}, \dots, \llbracket \phi_k \rrbracket_{n-1}) \\ &= \gamma^{-1} \circ \llbracket \heartsuit \rrbracket(\llbracket \phi_1 \rrbracket_M, \dots, \llbracket \phi_k \rrbracket_M) \\ &= \llbracket \heartsuit(\phi_1, \dots, \phi_k) \rrbracket_M \end{aligned}$$

using the induction hypothesis and naturality of \heartsuit . □

We recall the following lemma, whose proof directly translates to a general categorical setting, from [14]:

Lemma 4.5 *Suppose that $f^0 : 1 \rightarrow F1$ is a morphism of \mathbb{C} and let $f^n = Ff^{n-1}$ inductively. Then $f_n^n = \text{id}_{F^n 1}$ for all $n \in \omega$.*

This immediately implies that semantical validity of a sequent $\Gamma \Rightarrow \Delta$, with $\Gamma, \Delta \subseteq \mathcal{F}_n(\Lambda)$ is equivalent to validity w.r.t the n -step semantics.

Corollary 4.6 *Suppose $\Gamma \Rightarrow \Delta$ is a sequent with $\Gamma, \Delta \subseteq \mathcal{F}_n(\Lambda)$. Then $\text{Mod}(T) \models \Gamma \Rightarrow \Delta$ iff $\llbracket \Gamma \Rightarrow \Delta \rrbracket_n = \top$.*

Proof. The ‘if’-part is a consequence of Lemma 4.4 above.

For the ‘only if’-part assume that $\text{Mod}(T) \models \Gamma \Rightarrow \Delta$ and pick $f^0 : 1 \rightarrow (T/\mathcal{P}(V)1) \in \text{Set}/\mathcal{P}(V)$ where 1 is a terminal object of $\text{Set}/\mathcal{P}(V)$. Consider $M = (C, \gamma) \in \text{Coalg}(T/\mathcal{P}(V))$ where $C = (T/\mathcal{P}(V))^n$ and $\gamma = (T/\mathcal{P}(V))^n(f^0)$. As $\text{Mod}(T) \models \Gamma \Rightarrow \Delta$ we have that $M \models \Gamma \Rightarrow \Delta$ and Lemma 4.4 above implies that $\llbracket \Gamma \Rightarrow \Delta \rrbracket_n = \top$. \square

The proof of completeness (and later cut-free completeness) relies on a stratification of the provability predicate of $\mathbf{GR}[c]$. We introduce an auxiliary provability judgement $\mathbf{GR}[c] \vdash_n \Gamma \Rightarrow \Delta$, indexed by modal depth. Both \mathbf{GR} and \mathbf{GRc} contain the rules

$$(P_n) \frac{\mathbf{GR}[c] \vdash_n E_1 \quad \dots \quad \mathbf{GR}[c] \vdash_n E_k}{\mathbf{GR}[c] \vdash_{n+1} E_0}$$

for all rules $P = E_1, \dots, E_k/E_0$ associated with one-step rules $\phi/\psi \in \mathbf{R}$ together with the rules

$$(R_n) \frac{\mathbf{GR}[c] \vdash_n E_1 \quad \dots \quad \mathbf{GR}[c] \vdash_n E_k}{\mathbf{GR}[c] \vdash_n E_0}$$

for all rules $S = E_1, \dots, E_k/E_0$ of \mathbf{G} . Application of the rules (P_n) , (R_n) and (Cut_n) is restricted to substitutions $\sigma : V \rightarrow \mathcal{F}_n(\Lambda)$.

For \mathbf{GRc} we have the additional rule

$$(Cut_n) \frac{\mathbf{GRc} \vdash_n \Gamma \Rightarrow \Delta, A \quad \mathbf{GRc} \vdash_n A, \Gamma \Rightarrow \Delta}{\mathbf{GRc} \vdash_n \Gamma \Rightarrow \Delta}$$

whose application is also substitutions to valuations $\sigma : V \rightarrow \mathcal{F}_n(\Lambda)$.

The following proposition is the key stepping stone in the completeness proof and relates validity in the n -step semantics to derivability in rank n .

Proposition 4.7 *Let $\Gamma \Rightarrow \Delta$ be a sequent where $\Gamma, \Delta \subseteq \mathcal{F}_n(\Lambda)$. Then $\llbracket \Gamma \Rightarrow \Delta \rrbracket_n = \top$ implies that $\mathbf{GRc} \vdash_n \Gamma \Rightarrow \Delta$. If moreover \mathbf{R} is strictly one-step complete and reduction closed, then $\mathbf{GR} \vdash_n \Gamma \Rightarrow \Delta$.*

Proof. By induction on n . If $n = 0$ the statement follows from semantic completeness of \mathbf{G} . For $n > 0$, it suffices to consider $\llbracket \Gamma \tau \Rightarrow \Delta \tau \rrbracket_n = \top$ for $\tau : V \rightarrow \mathcal{F}_{n-1}(\Lambda)$ and

$$\Gamma = \heartsuit_1 \mathbf{q}_1, \dots, \heartsuit_i \mathbf{q}_i, p_1, \dots, p_j \text{ and } \Delta = \heartsuit'_1 \mathbf{q}'_1, \dots, \heartsuit'_{i'} \mathbf{q}'_{i'}, p'_1, \dots, p'_{j'}$$

for $\mathbf{q}_k \in V^n$ if \heartsuit_i is n -ary and analogously for \mathbf{q}'_k . By definition of $\llbracket \cdot \rrbracket_n$ and elementary boolean algebra, we deduce that either

$$\llbracket \bigwedge_{k=1}^i \heartsuit_k \mathbf{q}_k \tau \rightarrow \bigvee_{k=1}^{i'} \heartsuit'_k \mathbf{q}'_k \tau \rrbracket_n = \top \text{ or } \llbracket \bigwedge_{k=1}^j p_k \rightarrow \bigvee_{k=1}^{j'} p'_k \rrbracket_n = \top$$

holds. In the latter case, $\bigwedge_{k=1}^j p_k \rightarrow \bigvee_{k=1}^{j'} p'_k$ is a propositional tautology and the result follows from semantical completeness of \mathbf{G} . So assume that the left-hand identity above holds. Consider the clause

$$\chi = \bigwedge \Gamma \rightarrow \bigvee \Delta$$

and write τ_{n-1} for the $\mathcal{P}(S_{n-1})$ valuation $p \mapsto \llbracket \tau(p) \rrbracket_{n-1}$. Then $T S_{n-1}, \tau \models \chi$. By one-step completeness, there exist one-step rules $\phi_1/\psi_1, \dots, \phi_r/\psi_r$ and substitutions $\sigma_1, \dots, \sigma_r : V \rightarrow \text{Prop}(V)$ such that

- $S_{n-1}, \tau_{n-1} \models \phi_k \sigma_k$ for $k = 1, \dots, r$, and
- $\bigwedge_{k=1}^r \psi_k \sigma_k \rightarrow \chi$ is a propositional tautology.

Consequently, for $k = 1, \dots, r$ we have that $\llbracket \phi_k \sigma_k \tau \rrbracket_{n-1} = \top$. Suppose that $\mathcal{S}(\psi_k) = \Gamma_k \Rightarrow \Delta_k$. By induction hypothesis, we have that $\mathbf{GR}[\mathbf{c}] \vdash_{n-1} E$ for all $E \in \mathcal{S}(\phi_k)$ and all $k = 1, \dots, r$. Applying the sequent rule associated with ϕ_k/ψ_k , we obtain $\mathbf{GR}[\mathbf{c}] \vdash_n \Gamma\tau, \Gamma_k \sigma_k \tau \Rightarrow \Delta\tau, \Delta_k \sigma_k \tau$ for all $k = 1, \dots, r$ which entails that $\mathbf{GR}[\mathbf{c}] \vdash_n \Gamma\tau \Rightarrow \psi_k \sigma_k \tau, \Delta\tau$ again for all $k = 1, \dots, r$ and therefore $\vdash_n \Gamma\tau \Rightarrow (\bigwedge_{k=1}^r \psi_k \sigma_k), \Delta\tau$. As $\bigwedge_{k=1}^r \psi_k \sigma_k \rightarrow \chi$ is a propositional tautology we have that

$$\mathbf{GR} \vdash_n \Gamma\tau, \left(\bigwedge_{k=1}^r \psi_k \sigma_k \right) \Rightarrow \Delta\tau$$

and an application of the cut rule now gives $\mathbf{GR} \vdash_n \Gamma\tau \Rightarrow \Delta\tau$ as required.

If moreover \mathbf{R} is strictly one-step complete and contraction closed, we can assume $r = 1$ in the above so that $\psi_1 \sigma_1 \rightarrow \chi$ is a propositional tautology and $\chi\sigma$ contracted. By Corollary 3.6 $\mathcal{S}(\chi) \supseteq \mathcal{S}(\psi_1 \sigma_1)$ so that $(\Gamma \Rightarrow \Delta)\tau = \mathcal{S}(\chi\tau) \supseteq \mathcal{S}(\psi_1) \sigma_1 \tau$ proving that $\mathbf{GR} \vdash_n (\Gamma \Rightarrow \Delta)\tau$. \square

Note that we needed the power of the cut rule at precisely one point in the above proof: To conclude $\chi\tau$ from the sequent $\psi_1 \sigma_1 \tau, \dots, \psi_r \sigma_r \tau$ – the need for cut is eliminated if we use strictly complete and contracted rule sets. Moreover, closure under contraction is needed as we can only assert that χ is a propositional consequence of $\psi_1 \sigma_1$ which makes no assertion about the multiplicity of literals in either $\psi_1 \sigma_1$ or χ .

Completeness is now an easy corollary.

Corollary 4.8 (Completeness and cut free completeness) *Suppose \mathbf{R} is one-step complete for T and $\text{Mod}(T) \models \Gamma \Rightarrow \Delta$ for a sequent $\Gamma \Rightarrow \Delta$ over $\mathcal{F}(\Lambda)$. Then $\mathbf{GRc} \vdash \Gamma \Rightarrow \Delta$. If moreover \mathbf{R} is contraction closed and strictly complete, then $\mathbf{GR} \vdash \Gamma \Rightarrow \Delta$.*

As a corollary, we have admissibility of cut.

Corollary 4.9 *Suppose \mathbf{R} is contraction closed and strictly complete. Then the cut rule is admissible in \mathbf{GR} .*

Note that we have not used admissibility of weakening and contraction in the above completeness proof. Therefore (cut-free) completeness yields an alternative, semantic proof.

Corollary 4.10 *Suppose \mathbf{R} is one-step sound and one-step complete. Then weakening and contraction are admissible in \mathbf{GRc} . If moreover \mathbf{R} is strictly one-step complete and contraction closed, then weakening and contraction are admissible in \mathbf{GR} .*

Proof. Via soundness and completeness w.r.t. $\text{Mod}(T)$. \square

Note that the above semantical proof yields a slightly weaker result than the syntactic proofs of Section 3 as we pre-suppose soundness and completeness w.r.t. a given Λ -structure. However, for every rank-1 logic we can always construct a Λ -structure for which the given rule set is one-step sound and strictly one-step complete [19]. We conclude the section by re-visiting our two running examples.

Example 4.11 (i) *It has been shown in [18] that the set of one-step rules \mathbf{R}_C is both*

strictly one-step complete and resolution closed. As a consequence, the cut rule is admissible in \mathbf{GC} and \mathbf{GC} is complete for $\text{Mod}(\mathcal{C})$.

(ii) We give a direct proof of the fact that the rule set $\mathbf{R}_{\mathbf{CK}}$ is strictly one-step complete. This then implies completeness and admissibility of cut also for \mathbf{GCK} . Suppose $\chi = \bigwedge_{i \in I} (A_i \multimap B_i) \rightarrow \bigvee_{j \in J} (A'_j \multimap B'_j)$ and let τ be a $\mathcal{P}(X)$ -valuation such that $\tau \models \chi$. We claim that there exists $j \in J$ and $I_0 \subseteq I$ such that

(i) $\tau(A_i) = \tau(A'_j)$ for all $i \in I_0$

(ii) $\bigcap_{i \in I_0} \tau(B_i) \subseteq \tau(B'_j)$.

Assume, for a contradiction, that this is not the case. Then, for every $j \in J$, defining $I_j = \{i \in I \mid \tau(A_i) = \tau(A'_j)\}$ yields $\bigcap_{i \in I_j} \tau(B_i) \not\subseteq \tau(B'_j)$. Define the function $f : 2(X) \rightarrow \mathcal{P}(X)$ by

$$f(S) = \begin{cases} \bigcap_{i \in I_j} \tau(B_i) & S = \tau(A'_j) \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $f(\tau(A_i)) \subseteq \tau(B_j)$ but for all $j \in J$ we have that $f(\tau(A'_j)) \not\subseteq \tau(B'_j)$ by construction, contradicting $\mathbf{CK}, \tau \models \chi$. We can therefore assume the claim and pick $j \in J$ and $I_0 \subseteq I$ such that (i), (ii) above hold. We obtain $X, \tau \models \bigwedge_{i \in I_0} B_i \rightarrow B_j$. The claim follows as $\bigwedge_{i \in I_0} B_i \rightarrow B'_j \wedge \bigwedge_{i \in I_0} A_i \leftrightarrow A'_j / \bigwedge_{i \in I_0} (A_i \multimap B_i) \rightarrow (A_j \multimap B_j)$ is substitution instance of a one-step rule above the premise of which is valid under τ .

It is easy to see that also the rule set $\mathbf{R}_{\mathbf{CK}_{\text{Id}}}$ is strictly one-step complete for \mathbf{CK}_{Id} : if χ is as above, one shows that either $A'_j = B'_j$ for some $j \in J$ or there exists $j \in J$ and $I_0 \subseteq I$ as above; in the latter case we proceed as above. If the former is the case, then the result follows by instantiating the rule $A_0 \leftrightarrow A_1 / A_0 \multimap A_1$. It is easy to see that both $\mathbf{R}_{\mathbf{CK}}$ and $\mathbf{R}_{\mathbf{CK}_{\text{Id}}}$ are one-step sound and closed under reduction, hence \mathbf{GCK} is complete w.r.t $\text{Mod}(\mathbf{CK})$ and cut is admissible in \mathbf{GCK} ; the analogous statement holds for \mathbf{GCK}_{Id} .

Some remarks are in order. Strictly complete rule sets can always be systematically constructed by closing arbitrary one-step complete rule sets under rule resolution. In a nutshell, if ϕ_1/ψ_1 and ϕ_2/ψ_2 are one-step rules where $\heartsuit a$ occurs positively in ϕ_1 and negatively in ϕ_2 , one adds the rule $\phi_1 \wedge \phi_2/\psi$ where ψ is the resolvent of ψ_1 and ψ_2 at $\heartsuit a$ assuming that the rules share no other propositional variables. This is detailed in Theorem 3.15 of [18] where it is also shown that resolution closed rule sets are automatically strictly one-step complete. In this light, the proof system itself absorbs all the cuts involving rule conclusions, which suffices to prove cut-free completeness. Indeed, the rule sets in our examples have been obtained by systematically resolving the rules of given, one-step complete systems.

5 Applications

This section presents, from a syntactic viewpoint, some applications of cut-free completeness of \mathbf{GR} for a strictly complete and resolution closed set \mathbf{R} of one-step rules. The first application, the subformula property, is immediate:

Theorem 5.1 *Suppose \mathbf{R} is strictly one-step complete and contraction closed. Then \mathbf{GR} has the subformula property, i.e. every deduction $\mathbf{GR} \vdash \Gamma \Rightarrow \Delta$ only mentions subformulas of formulas occurring either in Γ or Δ .*

Proof. By induction on the derivation of $\mathbf{GR} \vdash \Gamma \Rightarrow \Delta$ where both the case of propositional connectives and the application of a sequent rule associated with $\phi, \psi \in \mathbf{R}$ is immediate by the rule format. \square

As a consequence, we obtain alternative proofs of two results of [18] regarding conservativity and complexity of coalgebraic logics.

Corollary 5.2 (Conservativity) *Suppose $\Lambda_0 \subseteq \Lambda$ is a sub-similarity type and \mathbf{R} is strictly one-step complete and contraction closed for a Λ -structure T . If \mathbf{R}_0 only contains those $(\phi, \psi) \in \mathbf{R}$ for which ψ is a clause over $\Lambda_0(V)$ then \mathbf{GR}_0 is complete for T .*

Proof. Suppose $\Gamma \Rightarrow \Delta$ is a valid sequent over $\mathcal{F}(\Lambda_0)$. Then $\mathbf{GR} \vdash \Gamma \Rightarrow \Delta$. By the subformula property, any derivation of $\Gamma \Rightarrow \Delta$ only contains subformulas of $\Gamma \Rightarrow \Delta$ and hence no rules of $\mathbf{R} \setminus \mathbf{R}_0$ are ever applied. \square

As the design of the system \mathbf{GR} is such that the logical complexity of the formula strictly decreases when passing from conclusion to premise, these systems can be used to establish both decidability and complexity of the satisfiability problem. Simply put, proof search in \mathbf{GR} terminates if for every sequent E there are only finitely many substitution instances of rule conclusions equal to E with different premises.

Polynomial bounds on the size of such rules imply decidability in polynomial space using depth-first search.

This allows us to re-prove the main theorem of [18] (to which we refer for the definition of *PSPACE*-tractable) in the setting of sequent calculi:

Theorem 5.3 *Suppose \mathbf{R} is strictly one-step complete, resolution closed and *PSPACE*-tractable. Then the satisfiability problem for $\mathcal{F}(\Lambda)$ w.r.t. T is decidable in polynomial space.*

Proof. As \mathbf{R} is *PSPACE*-tractable, there are only finitely many (rule, substitution)-pairs of polynomial size that allow to derive any given sequent and the depth of the search tree is polynomial in the size of the input formulas. \square

Cut-free proof calculi also provide all the necessary scaffolding to prove Craig interpolation by induction on cut-free proofs. To aid the formulation of the interpolation property, we write $\text{FV}(\phi)$ for the set of propositional variables occurring in $\phi \in \mathcal{F}(\Lambda)$ and extend to multisets and sequents by $\text{FV}(\Gamma) = \cup\{\text{FV}(\phi) \mid \phi \in \Gamma\}$ and $\text{FV}(\Gamma \Rightarrow \Delta) = \text{FV}(\Gamma) \cup \text{FV}(\Delta)$. Interpolation then takes the following form:

Definition 5.4 $\mathcal{F}(\Lambda)$ has the Craig Interpolation Property (CIP) with respect to $\text{Mod}(T)$ if $\text{Mod}(T) \models \phi \rightarrow \psi$ for $\phi, \psi \in \mathcal{F}(\Lambda)$, then there exists an interpolant $\theta \in \mathcal{F}(\Lambda)$ such that $\text{Mod}(T) \models \phi \rightarrow \theta$, $\text{Mod}(T) \models \theta \rightarrow \psi$ and $\text{FV}(\theta) \subseteq \text{FV}(\phi) \cap \text{FV}(\psi)$.

Syntactic proofs of the CIP proceed by induction on cut-free proofs. The following definition introduces the necessary terminology.

Definition 5.5 A split sequent is a quadruple $(\Gamma_0, \Gamma_1, \Delta_0, \Delta_1)$, written as $\Gamma_0 \mid \Gamma_1 \Rightarrow \Delta_0 \mid \Delta_1$. We say that $\Gamma_0 \mid \Gamma_1 \Rightarrow \Delta_0 \mid \Delta_1$ is a splitting of $\Gamma \Rightarrow \Delta$ if $\Gamma = \Gamma_0, \Gamma_1$ and $\Delta = \Delta_0, \Delta_1$.

A formula F is an interpolant of a split sequent $\Gamma_0 \mid \Gamma_1 \Rightarrow \Delta_0 \mid \Delta_1$ if $\mathbf{GR} \vdash \Gamma_0 \Rightarrow F, \Delta_0$ and $\mathbf{GR} \vdash F, \Gamma_1 \Rightarrow \Delta_1$ and $\text{FV}(F) \subseteq \text{FV}(\Gamma) \cap \text{FV}(\Delta)$. We say that a sequent $\Gamma \Rightarrow \Delta$ admits interpolation if every splitting of $\Gamma \Rightarrow \Delta$ has an interpolant.

The system **GR** has the Craig interpolation property (CIP) if every derivable sequent admits interpolation.

The idea of the syntactic proof of Craig interpolation [22, Chapter 4], in contrast to the semantic proofs via amalgamation (see [11] for the case of normal modal logics and [8] for monotone modal logic) is to construct interpolants inductively – clearly this fails in the presence of the cut-rule. Completeness gives the link between both the syntactic and the semantic versions of the CIP.

Proposition 5.6 *Suppose that \mathbf{R} is one-step sound and strictly one-step complete w.r.t the Λ -structure T . If moreover \mathbf{R} is contraction closed, then **GR** has the CIP iff $\text{Mod}(T)$ has the CIP.*

We are not yet in the position of presenting a generic proof of the CIP for coalgebraic logics in general; this is left for future work. Instead, we show that the systems used in our running examples, coalition logic and conditional logic have the CIP. For coalition logic, this is not a new result [7] but our proof is shorter due to the smaller number of modal proof rules. For the conditional logics CK and CK_{Id} the CIP is – to the best of our knowledge – a new result which was left as future work in [13] where a substantially different proof calculus is presented.

The proof of the CIP in both examples benefits from the following notions.

Definition 5.7 *A sequent rule E_1, \dots, E_n, E_0 supports interpolation if, for every substitution $\sigma : V \rightarrow \mathcal{F}(\Lambda)$, $E_0\sigma$ admits interpolation provided all of $E_1\sigma, \dots, E_n\sigma$ admit interpolation.*

It is well known (and shown e.g. in [22]) that all rules of **G** support interpolation. The special format of the sequent rules associated with one-step rules allows us to restrict ourselves to those instances without context formulas.

Lemma 5.8 *A sequent rule $E_1, \dots, E_n/E_0$ associated with a one-step rule ϕ/ψ of \mathbf{R} supports interpolation if and only if the rule $E_1, \dots, E_n/\mathcal{S}(\psi)$ supports interpolation.*

Proof. Suppose $E_1, \dots, E_n/E_0$ is a sequent rule associated with the one-step rule ϕ/ψ and suppose that $\mathcal{S}(\psi) = F \Rightarrow G$ whence $E_0 = \Gamma, F \rightarrow \Delta, G$. Pick a substitution $\sigma : V \rightarrow \mathcal{F}(\Lambda)$ and assume that $E_i\sigma$ admits interpolation for all $i = 1, \dots, n$; we have to show that $E_0\sigma$ admits interpolation. So suppose that $S = \Gamma_0\sigma, F_0\sigma \mid \Gamma_1, \sigma F_1\sigma \Rightarrow \Delta_0\sigma, G_0\sigma \mid \Delta_1\sigma, G_1\sigma$ is a splitting of $E_0\sigma$ for some $\sigma : V \rightarrow \mathcal{F}(\Lambda)$ where $\Gamma = \Gamma_0, \Gamma_1$ and similarly for Δ, F and G . Then $S^\sharp = F_0\sigma \mid F_1\sigma \Rightarrow G_0\sigma, G_1\sigma$ is a splitting of $\mathcal{S}(\psi)\sigma$. By assumption, S^\sharp has an interpolant I , i.e. $F_0\sigma \rightarrow I, G_0\sigma$ and $F_1\sigma, I \Rightarrow G_1\sigma$ are derivable. By admissibility of weakening, both $F_0\sigma, \Gamma_0\sigma \Rightarrow I, G_0\sigma, \Delta_0\sigma$ and $I, F_1\sigma, \Gamma_1\sigma \rightarrow G_1\sigma, \Delta_1\sigma$ are **GR**-derivable. It is obvious that I satisfies the restriction on variable occurrences. \square

The above lemma simplifies the proof of the CIP as we can ignore context formulas for sequent rules that are induced by one-step rules. We turn to our main running examples:

Theorem 5.9 *Coalition logic, i.e. the system **GC**, has the CIP.*

Proof. By induction on the derivation of $\Gamma \Rightarrow \Delta$. As all the rules of **G** support interpolation, we focus on applications of sequent rules induced by one-step rules where we can assume w.l.o.g. that the set of context formulas is empty.

Rule (A). If $S = [C]A \mid [C']A' \Rightarrow \emptyset \mid \emptyset$ is a splitting of the (substituted) rule conclusion and I is an interpolant of $A \mid A' \rightarrow \emptyset \mid \emptyset$, then $\neg[\cup C']\neg I$ is an interpolant of S .

Rule (B). First consider the splitting

$$S = [C]A \mid [C']A' \Rightarrow [D]B, [N]B \mid [N]B'$$

of the rule conclusion. If I is an interpolant of $A \mid A' \Rightarrow B, B \mid B'$ then $\neg[\cup C']\neg I$ is an interpolant of S .

Now consider the splitting

$$S = [C]A \mid [C']A' \Rightarrow [N]B \mid [D]B, [N]B'$$

of the rule conclusion. In this case, if I is an interpolant of $A \mid A' \Rightarrow B \mid B, B' \mid B'$ then $\neg[\cup C]\neg I$ is an interpolant of S . \square

By a similar argument we establish the CIP for the conditional logics CK and CK_{Id} .

Theorem 5.10 *The conditional logics CK and CK_{Id} have the CIP.*

Proof. First consider **GCK**; again we have to show that every rule supports interpolation. *Rule (K).* We first show that the rule (C) support interpolation. First suppose that $S = A \multimap B \mid A' \multimap B' \Rightarrow A \multimap B \mid \emptyset$ is a splitting of the rule conclusion. If I is an interpolant of $B \mid B' \Rightarrow B \mid \emptyset$ then $\neg(A \multimap \neg I)$ interpolates S . Now consider the splitting $S = A \multimap B \mid A' \multimap B' \multimap \emptyset \mid A \multimap B \mid \emptyset$. If I interpolates $B \mid B' \Rightarrow \emptyset \mid B$ then $A \multimap I$ interpolates S . Applications of the rule (*Id*) are straightforward. \square

6 Conclusions

We have argued that strict one-step completeness of a system of one-step rules automatically results in a sequent system that is cut free and complete. The interpolation property still provides a challenge, as we were unable to present a proof that simultaneously applies to a large class of logics. Instead, we have established the CIP by separate inductions for our two running examples. It is worthwhile to point out that for coalition logic, the inductive step is not entirely straight forward as the newly constructed interpolant uses a modality that does not necessarily appear in one of the rules. We leave this question as

Open Problem 6.1 *Does the interpolation property hold in general for all coalgebraic modal logics?*

Our second observation pertains to our proof of cut-free completeness, which is heavily based on semantical notions. While we strongly believe that this theorem could also have been obtained purely syntactically, i.e. by comparison of different proof systems, we are as of yet unsure whether these methods extend beyond rank 1. In particular, can cut always be absorbed into the modal proof rules, for example via resolution closure as described at the end of section 4? We formulate this as

Open Problem 6.2 *To what extent can resolution closure be used to absorb the cut rule into a system of modal proof rules?*

We conjecture that this is possible at least for all logics axiomatised by formulas of modal rank ≤ 1 but presently lack intuition concerning arbitrary rank.

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