Modal Logics are Coalgebraic

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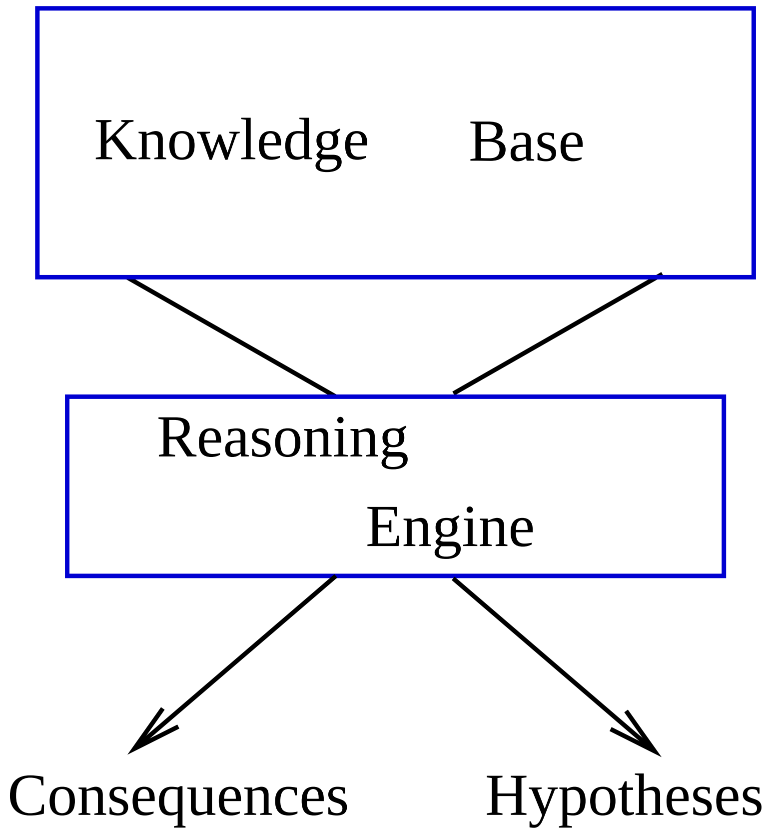
Example: Logics in Knowledge Representation



Knowledge Base

- formulated in domain-specific (logical) language
- example here: *traffic data*

Reasoning Engine

- automated reasoning

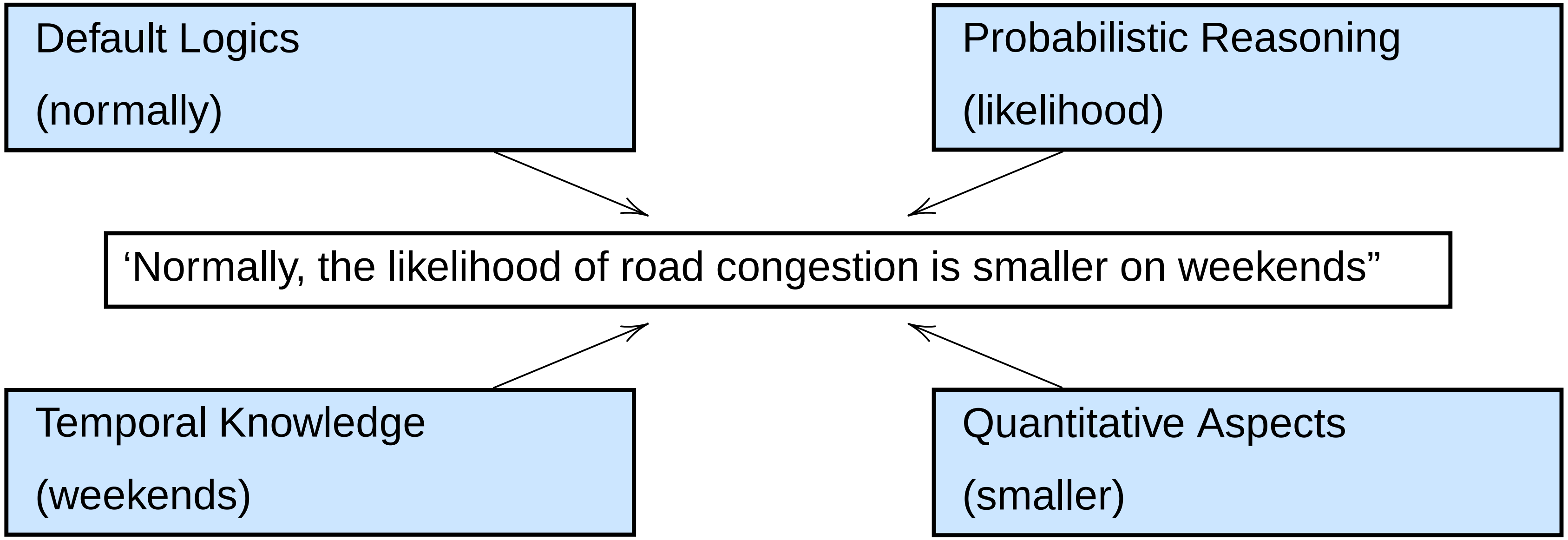
Outcomes

- consistency of hypotheses
- induction of hypotheses

Example. Reasoning about traffic data

“Normally, the likelihood of road congestion is smaller on weekends”

Example: Logics in Knowledge Representation



Reasoning about Knowledge

- a priori: *conjoin* different reasoning principles in a *modular* way
- a fortiori: a *common* “universe” where this is possible

State of the Art: Different Logics – Different Semantics

Possible World Semantics of standard modal logic

$$W \xrightarrow{\gamma} \mathcal{P}(W) \text{ to interpret } \Box\phi \text{ as "necessarily } \phi\text{"}$$

$$w \models \Box\phi \iff \boxed{\forall w' \in \gamma(w) : w' \in [\![\phi]\!]} \quad \leadsto \gamma(w) \in \mathbb{P}$$

Distribution Semantics of Probabilistic Logics

$$W \xrightarrow{\gamma} \underbrace{\mathcal{D}(W)}_{\text{prob. dist.}} \text{ to interpret } L_p\phi \text{ as "}\phi \text{ with probability } \geq p\text{"}$$

$$w \models L_p\phi \iff \boxed{\gamma(w)([\![\phi]\!]) \geq p} \quad \leadsto \gamma(w) \in \mathbb{P}$$

Selection Function Semantics of Conditional Logic

$$W \xrightarrow{\gamma} (\mathcal{P}(W) \to \mathcal{P}(W)) \text{ to interpret } \phi \Rightarrow \psi \text{ as "}\psi \text{ under condition } \phi\text{"}$$

$$w \models \phi \Rightarrow \psi \iff \boxed{\gamma(w)(\phi) \subseteq \psi} \quad \leadsto \gamma(w) \in \mathbb{P}$$

Coalgebras Provide a Semantic Umbrella

Semantic Structures map States to Successors

Coalgebras: $\boxed{W \xrightarrow{\gamma} TW}$

where $T : \mathsf{Set} \to \mathsf{Set}$ is a “construction” (technically: a functor) on sets

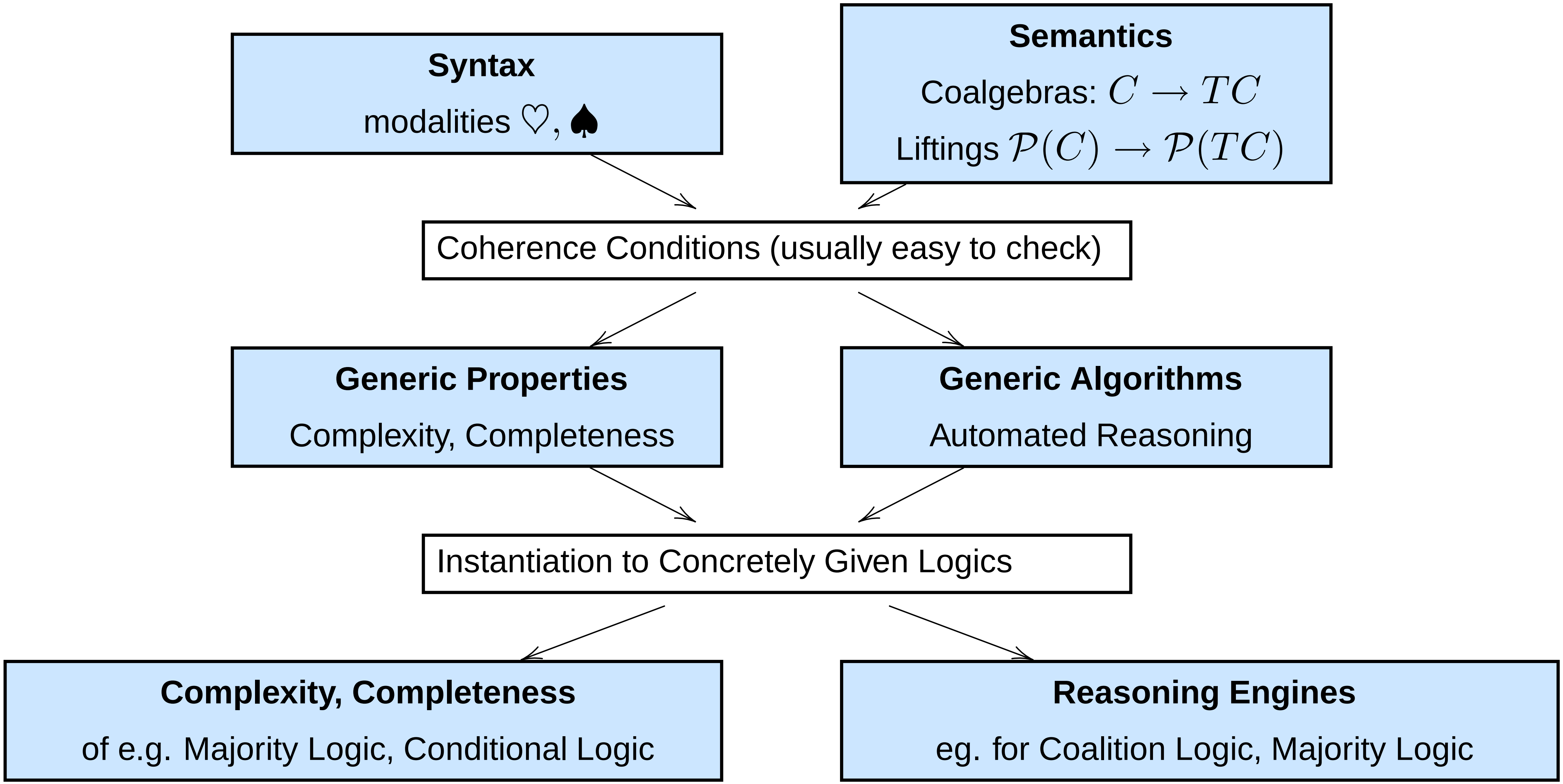
Modalities express properties of Successors in terms of States

$$w \models \heartsuit\phi \iff \gamma(w) \in [\![\heartsuit]\!]([\![\phi]\!])$$

Examples.

- (Standard) modal logic, classical and monotone modal logic
- graded modal logic, probabilistic modal logic
- conditional logic, coalition logic
- ...

Coalgebraic Semantics is UNIFORM



Compositionality By Example: Games and Probabilities

Strategic Games

- *Semantics:* $W \to \mathsf{G}W$

(outcomes of strategic games)

- *Syntax:* $[C]\phi$

(coalition C can force ϕ)

Quantitative Uncertainty

- *Semantics:* $W \to \mathcal{D}W$

(probability distributions)

- *Syntax:* $L_p\phi$

(ϕ with probability $\geq p$)

Taken Together: Games with Uncertain Outcomes

- *Semantics.* $W \longrightarrow \mathcal{D}(\mathsf{G}(W))$

probability distributions over strategic games

- *Syntax:* $L_p[C]\phi$ (and combinations)

Coalition C can bring about ϕ with probability $\geq p$.

Compositionality by Uniformity

Semantics: Defined by Operations (Functors) $T, S : \mathsf{Set} \to \mathsf{Set}$

- **Combination** of Semantical Structures: **Functor Composition**

$$T \circ S, \quad T + S, \quad T \times S : \quad \mathsf{Set} \to \mathsf{Set}$$

- **Synthesis** of Logics, Proof Calculi and Algorithms

$$\heartsuit \rightsquigarrow T\text{-succesors} \qquad \spadesuit \rightsquigarrow S\text{-successors}$$

- **Induced Combinations**
  - $(\heartsuit\spadesuit) \rightsquigarrow T \circ S$-successors describe *sequencing*
  - $(\heartsuit \times \spadesuit) \rightsquigarrow T \times S$-successors describe *fusion*
  - $(\heartsuit + \spadesuit) \rightsquigarrow T + S$-successors describe *choice*

Main Results. Combinations preserve *completeness* and *PSPACE-decidability*

From Logics to Automata and Back

Logical Formulas $\longleftrightarrow$ Automata on Infinite Structures

ϕ satisfiable $\rightsquigarrow$ Logical Formulas

L_ϕ non-empty $\rightsquigarrow$ Automata on Infinite Structures

Example. MSO $\leftrightarrow$ Buchi-Automata and μ-calculus $\leftrightarrow$ alternating parity automata

Automata often work on infinite **coalgebraic** structures

- Words: $W \to \Sigma \times W$
- Trees: $W \to W \times \Sigma \times W$
- LTSs: $W \to \mathcal{P}(A \times W)$

Coalgebra Automata generalise automata on infinite structures

- *Language* consists of states of T*-coalgebras*

Coalgebraic μ-calculus $\longleftrightarrow$ Coalgebra Automata

- *acceptance* is equivalent to *satisfaction* – uniformly in T – and gives *decidability*

Logics for Computational Features

Coalgebras as **State-Based Systems:** $S \to TS$

Computational Features require more structure

- recursion ($\rightsquigarrow$ *fixpoints*)
- local names ($\rightsquigarrow$ *nominal sets*)

Solution. Replace Set with "more structured" sets

- recursion: use *domains*
- local names: use *presheafs*

Example. Logics for the π-calculus, coalgebraically

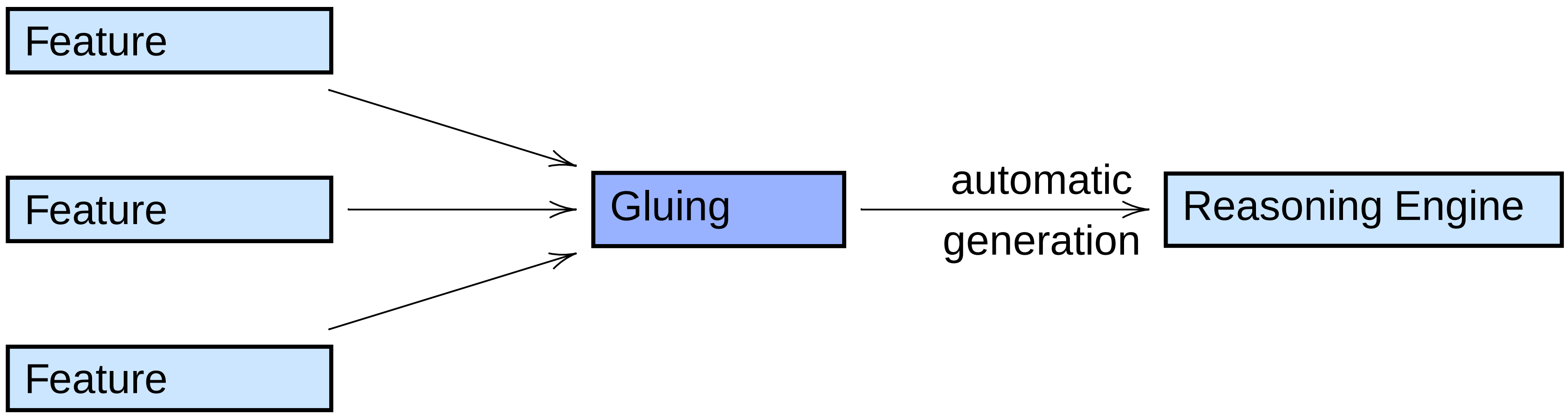
$$S \longrightarrow \mathcal{P}(\underbrace{S}_{\tau\text{-steps}} + \underbrace{N \times S^N}_{\text{name input}} + \underbrace{N \times N \times S}_{\text{name output}} + \underbrace{N \times \delta(S)}_{\text{name creation}})$$

Main Result. Completeness and full abstraction for (strong late) bisimilarity

- *modular combination* of features plus logic of the base category

Conceptually: Uniformity in the base category (sets / domains / presheafs / . . .)

Modular Construction of Coalgebraic Reasoners



Which Reasoning Principles? How Combined?

Example. COLOSS – the Coalgebraic Logic Satisfiability Solver

Conclusions

Coalgebras $W \longrightarrow TW$: *uniform base* for *large class* of modal logics

Parametricity in three different directions:

- semantics of the *particular logic* – choice of endofunctor T
- choice of the *base category* – sets, domains, presheafs etc.
- choice of *logical apparatus* – fixpoints, nominals etc.

Modularity made possible by uniformity

- combinations of features still live in the same framework

Applications in knowledge representation / reactive systems / and logic itself!