Semantics and Proof Theory of Conditional Logics

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Munich, Nov 18, 2010

Conditional Logics

Syntax. Extension of (classical) propositional logic with a binary operator $\Rightarrow$

$$\mathcal{F} \ni A, B ::= p \mid \bot \mid A \to B \mid A \Rightarrow B$$

where $p \in V$ is a propositional variable.

Reading. Depending on the axiomatisation, various possibilities:

- default implication: if A, then normally B
- counterfactual implication: if A were the case, then so would B
- relevant implication: A is a relevant assumption for B to hold
- ...

Crucially. All readings come with a different axiomatisation and semantics.

Semantics

Basic Conditional Logic. Models are triples $M = (W, \gamma, \pi)$ where

- W is a set of *worlds*
- $\gamma : W \to (\mathcal{P}(W) \to \mathcal{P}(W))$ assigns a *selection function* to each world
- $\pi : V \to \mathcal{P}(W)$ is a *valuation* for the propositional variables

Interpretation. Formulas $A \in \mathcal{F}$ denote subsets $[\![A]\!]_M$ of the carrier of the model:

$$w \models A \Rightarrow B \iff \gamma(w)([\![A]\!]_M) \subseteq [\![B]\!]_M$$

where $[\![C]\!]_M = \{w \in W \mid w \models C\}$.

Reading.

- A is a condition under which B holds
- selection functions return alternative worlds for each antecedent

Non-Monotonicity

Given. Model $M = (W, \gamma : W \to (\mathcal{P}(W) \to \mathcal{P}(W)), \pi : V \to \mathcal{P}(W))$.

Non-Montotonicty. We can have the following situation

$$w \models A \Rightarrow B \text{ but } w \not\models (A \wedge A') \Rightarrow B$$

as $\gamma(w)(\llbracket A \rrbracket_M)$ and $\gamma(w)(\llbracket A \wedge A' \rrbracket_M)$ are unrelated.

Example. Assuming

$$\texttt{rich} \Rightarrow \texttt{happy}$$

we may *not* conclude that also

$$\texttt{rich} \wedge \texttt{cancer} \Rightarrow \texttt{happy}$$

Non-monotonicity is a *distinguishing feature* of conditional logics.

Axiomatisation

Exchange of Equivalents in the antecedent

$$\frac{A \leftrightarrow A'}{A \Rightarrow B \to A' \Rightarrow B}$$

Meet Perservation in the succedent

$$\frac{B}{A \Rightarrow B} \qquad \frac{}{(A \Rightarrow B) \wedge (A \Rightarrow B') \leftrightarrow (A \Rightarrow (B \wedge B'))}$$

Soundness over selection function semantics

$$w \models A \Rightarrow B \iff \gamma(w)([\![A]\!]_M) \subseteq [\![B]\!]_M$$

follows immediately from $X \subseteq Y \wedge X \subseteq Y' \iff X \subseteq Y \cap Y'$.

Completeness similar to completeness proofs of modal logic.

Systems of Conditional Logic

Extension. Additional axioms that give $\Rightarrow$ application-specic flavour.

Extension 1: Default Reasoning. Extension of the basic axioms with

$$(\mathsf{ID}) A \Rightarrow A$$

$$(\mathsf{DE}) (A \Rightarrow C) \wedge (B \Rightarrow C) \rightarrow (A \vee B) \Rightarrow C$$

$$(\mathsf{CM}) (A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \wedge B) \Rightarrow C$$

Example.

If I ordinarily go to work on Mondays, I may choose not to if I'm sick:

$$\not\vdash (\mathtt{monday} \Rightarrow \mathtt{work}) \rightarrow (\mathtt{monday} \wedge \mathtt{sick} \Rightarrow \mathtt{work})$$

But if I'm usually sick on Mondays, then I will go:

$$\vdash (\mathtt{monday} \Rightarrow \mathtt{sick}) \wedge (\mathtt{monday} \Rightarrow \mathtt{work}) \rightarrow (\mathtt{monday} \wedge \mathtt{sick} \Rightarrow \mathtt{work})$$

Modular Buildup of Conditional Logics

Conditional Modus Ponens.

$$(\mathsf{MP}) \quad (A \Rightarrow B) \rightarrow (A \rightarrow B)$$

- accepted under a relevant reading of the conditional
- rejected for default reasoning

Identity Axiom

$$(\mathsf{ID}) A \Rightarrow A$$

- accepted for default reasoning
- rejected for causal interpretation

Conditional Excluded Middle

$$(\mathsf{CEM}) \quad (A \Rightarrow B) \vee (A \Rightarrow \neg B)$$

- rejected in most systems
- adopted for (some) subjunctive conditionals

Semantics and Completeness

Completeness. What class of models provides a complete semantics?

Observation.

- all systems arise by extending (CK) by a set of axioms
- axioms are of a specific form: they are *shallow* (nesting depth of $\Rightarrow$ is ≤ 1)

Example

$$(\mathsf{ID}) \quad A \Rightarrow A \qquad (\mathsf{MP}) \quad (A \Rightarrow B) \rightarrow (A \rightarrow B)$$

Definition. A formula is *shallow* if it is a propositional combination of variables and modal atoms $A \Rightarrow B$, where A, B are propositional combinations of variables.

(Here: formula-valued variables $A, B, C, \dots$.)

Goal. General Completeness Result for Conditional Logic, extended with shallow axioms.

More General Framework

Coalgebric Models. triples $(W, \gamma : W \to TW, \pi)$ for $T : \mathsf{Set} \to \mathsf{Set}$

- for Kripke Frames: $W \to \mathcal{P}(W)$
- for Conditional Frames: $W \to (\mathcal{P}(W) \to \mathcal{P}(W))$

Coalgebraic Operators. $[\![\heartsuit]\!] : \mathcal{P}(W)^n \to \mathcal{P}(W)$

- for Kripke Frames: $[\![\Box]\!](A) = \{B \in \mathcal{P}(W) \mid B \subseteq A\}$
- for Conditional Frames: $[\![\Rightarrow]\!](A, B) = \{f : \mathcal{P}(W) \to \mathcal{P}(W) \mid f(A) \subseteq B\}$

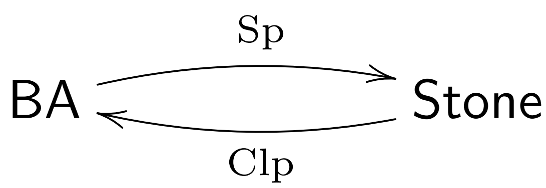
Coalgebraic Interpretation over a model $(W, \gamma : W \to TW, \pi)$

$$w \models \heartsuit(A_1, \dots, A_n) \iff \gamma(w) \in [\![\heartsuit]\!]([\![A_1]\!], \dots, [\![A_n]\!])$$

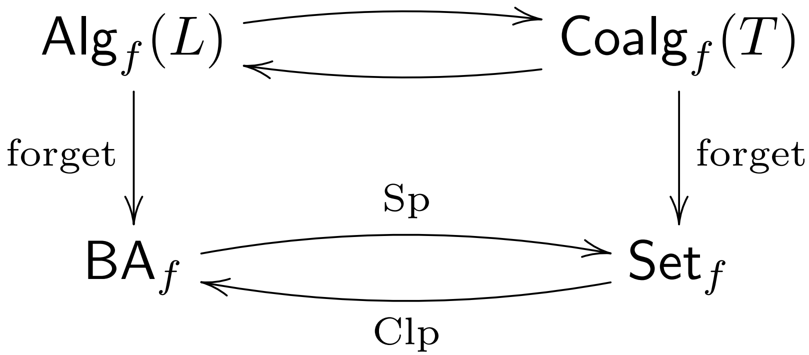
This recovers the standard semantics of both modal and conditional logic.

Completeness via Duality and Finite Models

Finite Models via Finite Algebras: Restricting Stone-duality



to a duality between *finite* modal algebras and *finite* coalgebras (over Set):



Completeness via **finite models**:

- transport a finite *algebraic* modal along duality to a (finite) *coalgebraic* one
- completeness follows from the finite algebra property

Basic Scheme

In general. Extension of a *complete logic* with *extralogical axioms*

Example. Basic Conditional Logics: System (CK)

$$\text{(RCEA)} \quad \frac{A \leftrightarrow A'}{A \Rightarrow B \rightarrow A' \Rightarrow B}$$

$$\text{(RCK)} \quad \frac{B}{A \Rightarrow B} \qquad \frac{}{(A \Rightarrow B) \wedge (A \Rightarrow B') \leftrightarrow (A \Rightarrow (B \wedge B'))}$$

Observation. Axioms / Rules are of a specific form

- *premise* is a propositional combination of *propositional* variables
- *conclusion* is a prop. comb. of *operators*, applied to propositional formulae.

Definition. A rule is *one-step* if its premise is a propositional formula, and its premise is a propositional combination of operators applied to propositional formulae.

Algebraic Semantics of Modal Logics

Given. A (presentation of) a logic $\mathcal{L} = (\Lambda, \mathcal{R})$ where Λ are operators and $\mathcal{R}$ are one-step rules.

Defn. The *functorial presentation* of $\mathcal{L}$ is the endofunctor

$$L : \mathsf{BA} \to \mathsf{BA}, \quad A \mapsto F(\{\heartsuit a \mid a \in A, \heartsuit \in \Lambda\})/\sim$$

where F is free construction and $\sim$ the equivalence induced by $\mathcal{R}$ (!).

Remark. L-algebras are the boolean algebras with operators satisfying $\mathcal{R}$.

Synthetic Coalgebraic Semantics by (Stone) duality:

$$L^\star : \mathsf{Set}_f \to \mathsf{Set}_f, \quad X \mapsto \mathrm{Sp} \circ L \circ \mathcal{P}(X)$$

Correspondence Theorem. $\mathsf{Alg}(L)_f \simeq \mathsf{Coalg}(L^\star)_f$ preserving the semantics.

Synthetic vs Coalgebraic Semantics

Coalgebraic Semantics in two guises:

- synthetically (extracted from the logical axiomatisation)
- structurally (given in terms of an interpretation of modal operators)

Reconciliation Theorem:

Structural Semantics	$=$	Synthetic Semantics

from a satisfiability perspective, or more precisely:

For every $L^\star$-coalgebra $\mathbf{C}$ there exists a T-coalgebra $\mathbf{D}$ with the same carrier s.t.

$$c \models_{\mathbf{C}} \phi \iff c \models_{\mathbf{D}} \phi$$

for all modal formulas ϕ, and vice versa (assuming completeness for T).

Corollary.

Finite L-algebras	$\iff$	Finite $L^\star$-Coalgebras	$\iff$	Finite T-Coalgebras

Completeness via Finite Algebras

Given. A (presentation of) a logic $\mathcal{L} = (\Lambda, \mathcal{R})$ where Λ are operators and $\mathcal{R}$ are one-step rules. G

Commmon Signature. propositional symbols + Λ

modal formula	algebraic equation
A	$A = \top$
$A \leftrightarrow B$	$A = B$

Lemma. For every set $\mathcal{R}$ of rank-1 rules there is a set $\mathcal{E}$ of equations such that

$$\mathcal{R} \vdash_{\mathsf{ML}} A \iff \mathcal{E} \vdash_{\mathsf{EL}} A = \top$$

Corollary. If Θ is a set of axioms, then

$$\mathcal{R} + \Theta \vdash_{\mathsf{ML}} A \iff \mathcal{E} + \{A = \top \mid A \in \Theta\} \vdash_{\mathsf{EL}} A = \top$$

Plan of Attack. Completeness from Equational Completeness via finite algebras.

Filtrations for Shallow Axioms

Definition. Suppose Δ is a finite set of formulas, closed under subformulas and (normalised) negation and $\mathcal{R}$, Θ are as above.

Define a boolean algebra with operators $(\mathbb{A}, (f_\heartsuit)_{\heartsuit \in \Lambda})$ by

- $\mathbb{A} = \mathcal{P}(\mathcal{M})$ where $\mathcal{M} = \{\Phi \subseteq \Delta \mid \Phi$ maximally consistent$)$
- $f_\heartsuit : \mathbb{A}^n \to \mathbb{A}$ is n-ary if $\heartsuit$ is n-ary and given by

$$f_\heartsuit(A_1, \ldots, A_n) = \{\Phi \in \mathcal{M} \mid \nu(\Phi) \vdash \heartsuit(\bigvee_{\Psi \in A_1} \bigwedge \Psi, \ldots, \bigvee_{\Psi \in A_n} \bigwedge \Psi)\}$$

where $\nu(\Phi)$ is a fixed maximally consistent extension of Φ.

Filtration Lemma. The BAO $\mathbb{A}$ defined above satisfies $\mathbb{A} \models A$ for all $A \in \Theta$.

Corollary. All shallow extensions of coalgebraic logics are complete.

Application. All shallow extensions of conditional Logic are complete wrt. the defined class of conditional frames.

Proof Theory of Conditional Logics

Recall: Syntax given by classical Propositional Logic + *polyadic* modal operators $\heartsuit \in \Lambda$

$$\mathcal{L}(\Lambda) \ni A, B ::= p \mid \neg A \mid A \wedge B \mid \heartsuit(A_1, \ldots, A_n) \qquad (\heartsuit \in \Lambda\ n\text{-ary})$$

- *no fixed* set of operators (but you're invited to thing $\Rightarrow$)
- *no interpretation* (but you're invited to think conditional frames)

Goal. Cut-free sequent calculus and complexity via proof-search. **Sequents.**

Multisets of Λ-formulas, read disjunctively

$$A \equiv \{A\} \qquad \Gamma, \Delta \equiv \Gamma \cup \Delta$$

General Setup

Sequent Calculi. Right-Handed Gentzen-Schütte Systems.

$$\frac{}{\Gamma, A, \neg A} \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \qquad \frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)} \qquad \frac{\Gamma, A}{\Gamma, \neg\neg A}$$

plus **Modal Rules** of the form

$$\frac{\Gamma_1 \quad \dots \quad \Gamma_n}{\Gamma_0}$$

where $\Gamma_0, \dots, \Gamma_n$ are Λ-sequents (in examples: generated by schemas).

Idea.

- Propositional Part has cut-elimination $\leadsto$ scrutinize modal rules!
- Similarly for structural rules (weakening, contraction, inversion)

Cut Elimination by (trivial) Example.

Modal Logic K. $\Lambda = \{\Box\}$ with $\Box$ unary

Rules. Hilbert-Axiomatisation taken from any textbook

$$(N)\frac{p}{\Box p} \qquad (D)\Box(p \to q) \to \Box p \to \Box q$$

As Sequent Rules, i.e. applying inversion

$$\frac{A}{\Box A} \qquad \frac{}{\neg\Box(A \to B), \neg\Box A, \Box B}$$

Find Occurrences of Cut

$$\frac{\dfrac{A \to B}{\Box(A \to B)} \qquad \dfrac{}{\neg\Box(A \to B), \neg\Box A, \Box B}}{\neg\Box A, \Box B}$$

Idea. Add this as a new rule

$$\frac{\neg A, B}{\neg\Box A, \Box B}$$

More Cuts

New Rule Set.

$$\frac{A}{\Box A} \qquad \frac{\neg A, B}{\neg\Box A, \Box B} \qquad \frac{}{\neg\Box(A \to B), \neg\Box A, \Box B}$$

Find more cuts.

$$\frac{\dfrac{\neg A, B \to C}{\neg\Box A, \Box(B \to C)} \qquad \dfrac{}{\neg\Box(B \to C), \neg\Box B, \Box C}}{\neg\Box A, \neg\Box B, \Box C}$$

Add new Rule.

$$\frac{\neg A, \neg B, \neg C}{\neg\Box A, \neg\Box B, \Box C}$$

After finitely many steps . . .

$$(K_n)\,\frac{\neg A_1, \ldots, \neg A_n, A_0}{\neg\Box A_1, \ldots, \neg\Box A_n, \Box A_0}$$

General Idea. Add cuts between modal rules until this process terminates.

Formal Setup

Given modal similarity type Λ and rule set $\mathcal{R}$

Hilbert-Provability. The predicate $\mathsf{H}\mathcal{R} \vdash$ is the least set of formulas that

- contains all propositional tautologies
- is closed under modus ponens and uniform substitution
- contains $\bigvee \Gamma_0$ whenever it contains $\bigvee \Gamma_1, \ldots, \bigvee \Gamma_n$ and $\Gamma_1 \ldots \Gamma_n / \Gamma_0 \in \mathcal{R}$

Gentzen-Provability. The predicate $\mathsf{G}\mathcal{R} \vdash$ is the least set of sequents that

- contains Γ_0 whenever it contains $\Gamma_1, \ldots, \Gamma_n$
- is closed under application of the rules

$$\frac{}{\Gamma, A, \neg A} \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \qquad \frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)} \qquad \frac{\Gamma, A}{\Gamma, \neg\neg A}$$

Easy Theorem. $\mathsf{H}\mathcal{R} \vdash \bigvee \Gamma$ whenever $\mathsf{G}\mathcal{R} \vdash \Gamma$.

Absorption of Structural Rules

Goal. $\mathsf{H}\mathcal{R} \vdash \bigvee \Gamma \iff \mathsf{G}\mathcal{R} \vdash \Gamma$

Observation. The following rules of weakening, contraction and inversion

$$\frac{\Gamma}{\Gamma, A} \qquad \frac{\Gamma, A, A}{\Gamma, A} \qquad \frac{\Gamma, \neg\neg A}{\Gamma, A} \qquad \frac{\Gamma, \neg(A_1 \wedge A_2)}{\Gamma, \neg A_1, \neg A_2} \qquad \frac{\Gamma, A_1 \wedge A_2}{\Gamma, A_i}(i = 1, 2)$$

should be admissible, as they are in $\mathsf{H}\mathcal{R}$.

Idea. Isolate the "essence" of e.g. inversion lemma as a property of rule sets

Defn. For a Λ-sequent Δ, let $\mathsf{A}(\Delta)$ denote the closure of Δ under the above rules.

The rule set $\mathcal{R}$ *absorbs structural rules* if

$$\mathsf{G}\mathcal{R} + \mathsf{A}(\Gamma_1) \cup \cdots \cup \mathsf{A}(\Gamma_n) \vdash \Gamma$$

for all $\frac{\Gamma_1 \ldots \Gamma_n}{\Gamma_0} \in \mathcal{R}$ and all $\Gamma \in \mathsf{A}(\Gamma_0)$.

Propn. Admissibility of the structural rules follows from their absorption.

(Counter)Examples

For the modal logic K:

Negative. Absorption of weakening fails for

$$\frac{\neg A_1, \ldots, \neg A_n, A_0}{\neg\Box A_1, \ldots, \neg\Box A_n, \Box A_0}$$

For $n = 0$, have $\Box A_0, p \in \mathsf{A}(\Box A)$ but $\mathsf{G}\mathcal{R} + \mathsf{A}(A) \not\vdash \Box A_0, p$.

Positive. Absorption holds if weakening is built in:

$$\frac{\neg A_1, \ldots, \neg A_n, A_0}{\neg\Box A_1, \ldots, \neg\Box A_n, \Box A_0, \Delta}$$

(Note that inversion is trivial)

More (Counter)Examples

For the modal logic T = K + $\Box A \to A = K + \neg\Box A, A$

Find Cuts between K and T

$$\dfrac{\dfrac{\neg A, B}{\neg\Box A, \Box B} \qquad \dfrac{}{\neg\Box B, B}}{\neg\Box A, B}$$

Negative. Absorption of inversion fails if we adopt this as a new rule

$$\frac{\neg A, B}{\neg\Box A, B} \qquad (\text{take } B = \neg(B_0 \wedge B_1))$$

Negative. Inversion works, but contraction fails for

$$\frac{\neg A, \Gamma}{\neg\Box A, \Gamma} \qquad (\text{take } \Gamma = \neg\Box A)$$

Positive. Absorption of structural rules holds for

$$\frac{\neg A, \neg\Box A, \Gamma}{\neg\Box A, \Gamma}$$

(which is the version of the T-rule that we know and like)

Absorption of the Cut-Rule

(Same) Idea. Distill the "essence" of cut-elimination proofs into rule properties

- think double induction on cut rank and proof size
- allow cuts on smaller proofs and smaller cut formulas

Defn. A ruleset $\mathcal{R}$ *absorbs cut*, if for all rules $(r_1)\frac{\Gamma_1,\ldots,\Gamma_n}{A,\Gamma_0}$, $(r_2)\frac{\Delta_1,\ldots,\Delta_k}{\neg A,\Delta_0} \in \mathcal{R}$

$$\mathsf{G}\mathcal{R} + \mathsf{Cut}(A, r_1, r_2) \vdash \Gamma_0, \Delta_0$$

where $\mathsf{Cut}(A, r_1, r_2)$ contains structural rules, cut on formulas $< A$, the premises $\Gamma_1, \ldots, \Gamma_n, \Delta_1, \ldots, \Delta_k$ and Γ, Δ where, for some formula B,

- Γ, B and $\Delta, \neg B \in \{\Gamma_1, \ldots, \Gamma_n, \Delta_1, \ldots, \Delta_k\}$ (cut premise/premise), or
- $\Gamma, B = \Gamma_0, A$ and $\Delta, \neg B \in \{\Delta_1, \ldots, \Delta_k\}$ (cut conclusion/premise), or
- $\Gamma, B = \Delta_0, \neg A$ and $\Delta, \neg B \in \{\Gamma_1, \ldots, \Gamma_n\}$ (cut premise/conclusion).

A rule set that absorbs structural rules and the cut rule is called *absorbing*.

Cut Absorption implies Cut Elimination

Thm. If $\mathcal{R}$ absorbs structural rules and cut, then cut is admissible.

Proof. (Sketch) Double induction on cut rank and proof size, case modal / prop. rule:

$$\dfrac{\dfrac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \qquad \dfrac{\Delta_1 \qquad \dots \qquad \Delta_n}{\neg(A \wedge B), \Delta}}{\Gamma, \Delta}$$

- closure under inversion gives proof of $\neg A, \neg B, \Delta$
- cut on (smaller) formulas A, B gives

$$\dfrac{\dfrac{\Gamma, A \qquad \Delta, \neg A, \neg B}{\Gamma, \Delta, \neg B} \qquad \Gamma, B}{\Gamma, \Gamma, \Delta}$$

- closure under contraction gives proof of Γ, Δ

Cor. If $\mathcal{R}$ is closed under uniform substitution and absobrs cut and structural rules, then $\mathsf{G}\mathcal{R} \vdash \Gamma \iff \mathsf{H}\mathcal{R} \vdash \bigvee \Gamma$.

Construction of Cut-Free Rule Sets

Conceptually. Absorption by addition of cuts + equivalence transformations

Admissibility. A rule $(r)\Gamma_1 \dots \Gamma_n/\Gamma_0$ is *admissible* in $\mathsf{H}\mathcal{R}$ if, for all formulas A,

$$\mathsf{H}\mathcal{R} \vdash A \iff \mathsf{H}(\mathcal{R} \cup \{r\}) \vdash A.$$

Simple Lemma 1. (Propositional weakening / strengthening is sound in $\mathsf{H}\mathcal{R}$)

The rule $\Delta_1 \dots \Delta_k/\Delta_0$ is admissible, if there is $\Gamma_1, \dots, \Gamma_n/\Gamma_0 \in \mathcal{R}$ s.t.

$$\{\bigvee \Delta_1, \dots, \bigvee \Delta_k\} \vdash_{\mathsf{PL}} \bigvee \Gamma_i (1 \le i \le n) \text{ and } \bigvee \Gamma_0 \vdash_{\mathsf{PL}} \bigvee \Delta_0.$$

Simple Lemma 2. (Cut is sound in $\mathsf{H}\mathcal{R}$)

If $\Gamma_1 \dots \Gamma_n/A, \Gamma_0$ and $\Delta_1 \dots \Delta_k\ \neg A, \Delta_0 \in \mathcal{R}$ then

$$\frac{\Gamma_1 \dots \Gamma_n \quad \Delta_1 \dots \Delta_k}{\Gamma_0, \Delta_0}$$

is admissible in $\mathsf{H}\mathcal{R}$.

Applications: Conditional Logics

Similarity Type. $\Lambda = \{\Rightarrow\}$ with $\Rightarrow$ binary (nonmonotonic conditional)

Basic Rules.

$$(\mathsf{RCEA})\frac{A \leftrightarrow A'}{(A \Rightarrow B) \leftrightarrow (A' \Rightarrow B)} \qquad (\mathsf{RCK})\frac{B_1 \wedge \cdots \wedge B_n \to B}{(A \Rightarrow B_1) \wedge \cdots \wedge (A \Rightarrow B_n) \to (A \Rightarrow B)}$$

Additional Axioms.

$$(\mathsf{ID}) A \Rightarrow A \qquad (\mathsf{MP})(A \Rightarrow B) \to (A \to B) \qquad (\mathsf{CEM})(A \Rightarrow B) \vee (A \Rightarrow \neg B)$$

Terminology. $\mathsf{CK} = (\mathsf{RCEA}) + (\mathsf{RCK})$, additional axioms juxtaposed, e.g. $\mathsf{CKCEMMP} = \mathsf{CK} + (\mathsf{CEM}) + (\mathsf{MP})$.

Logics without (CEM)

Notation. $A_0 = \cdots = A_n$ contains $\neg A_i, A_0$ and $A_i, \neg A_0$ for $i = 1, \ldots, n$.

Basic Conditional Logic CK.

$$(\mathsf{CK}_g)\,\frac{A_0 = \cdots = A_n \quad \neg B_1, \ldots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma}$$

CK + (ID). Axiom schema (CK_g) plus

$$(\mathsf{ID}_g)\,\frac{A = B}{A \Rightarrow B, \Gamma}$$

CK + (MP). Axiom schema (CK_g) plus

$$(\mathsf{MP}_g)\,\frac{A, \neg(A \Rightarrow B), \Gamma \quad \neg B, \neg(A \Rightarrow B), \Gamma}{\neg(A \Rightarrow B), \Gamma}$$

In all cases. Cut elimination holds, and provability is unchanged.

Logics with (CEM)

CK + (CEM). New rule schema

$$(\mathsf{CKCEM}_g)\,\frac{A_0 = \cdots = A_n \quad B_0, \ldots, B_j, \neg B_{j+1}, \neg B_n}{(A_0 \Rightarrow B_0), \ldots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \ldots, \neg(A_n \Rightarrow B_n), \Gamma}$$

for $1 \leq j \leq n$.

CK + CEM + MP. Rule schemas $(\mathsf{CKCEM}_g) + (\mathsf{MP}_g)$ and new schema

$$(\mathsf{MPEM}_g)\,\frac{A, (A \Rightarrow B), \Gamma \qquad B, (A \Rightarrow B), \Gamma}{(A \Rightarrow B), \Gamma}.$$

In all cases. Cut elimination holds, and provability is unchanged.

Complexity.

System. CK plus an arbitrary subset of $\mathsf{CEM}, \mathsf{MP}, \mathsf{ID}$.

- Polynomial bound on height and branching of proof tree

Thm. Provability for extensions of CK with one or more of $\mathsf{CEM}, \mathsf{MP}, \mathsf{ID}$ is in $PSPACE$.

Other Axioms.

- similar techniques work for (DE) and (CM)
- (but the rules are more involved . . .)
- PSPACE-bound for default reasoning

Complexity.

State-of-the-art and related work

Related Work: Conditional Sequent Calculi.

- Olivetti *et.al.*: dedicated (labelled) sequent calculi for many flavours of conditional logics
- here: ‘pure’ calculi and new complexity bounds

Related Work: Complexity

- Halpern et. al.: semantical complexity bounds for many flavours of conditional logic
- here: purely syntactical approach.

Current State-of-the-Art

Basic Idea.

- compute the cut-closure of a set of sequent rules 'by hand'
- check that rule application is decidable in polynomial time.

Better Idea.

- we don't need to pre-compute the cut-closure by hand
- it suffices to be able to compute premises from conclusions 'fast enough'

Good news Lemma. Given a finite set of sequent rules and a sequent Γ, the set of rules $\frac{\Gamma_1 \ldots \quad \ldots \quad \Gamma_n}{\Gamma}$ can be represented in space polynomial in Γ.

(Main Challege: Non-monotonicity of sequent size given repeated cuts)

Problem still. It is still open how to close the ensuing ruleset under contraction.

Summary and Conclusions

Generic Cut Elimination.

- absorbing rule sets: inductive steps for cut elimination
- construction of absorbing sets: add cuts until saturated

Conditional Logics.

- New (internalised, cut-free) sequent calcul for extensions of CK
- Cut-elimination for $\mathsf{CKCEMMP}$.

Questions and Further Work.

- Fine tuning: preservation of proof height for structural rules
- Other base logics (e.g. FOL)
- Other flavours (e.g. intuitionistic)