The Coalgebraic μ-Calculus

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Many Faces of Modal Logic

Modal Logic. Classical Propositional Logic + *Modalities*, e.g.:

Coalition Logic to reason about *Multi-agent systems.*

$\langle a, b \rangle \phi$ $\rightsquigarrow$ Agents a, b can force ϕ

Probabilistic Modal Logic for *Reactive Systems*

$\langle p \rangle \phi$ $\rightsquigarrow$ ϕ holds with probability ≤ 0.2

Graded Modal Logic in *Knowledge Representation*

$\exists \leq 5.\phi$ $\rightsquigarrow$ at most 5 components satisfy ϕ

Historically: **Relational Modal Logic** in *Philosophy*

$\Box \phi$ $\rightsquigarrow$ necessarily ϕ

State of the Art: Different Logics – Different Semantics

Possible World Semantics of standard modal logic

$$W \xrightarrow{\gamma} \mathcal{P}(W) \text{ to interpret } \Box\phi \text{ as "necessarily } \phi\text{"}$$

$$w \models \Box\phi \iff \boxed{\forall w' \in \gamma(w) : w' \in [\![\phi]\!]} \quad \rightsquigarrow \gamma(w) \in \mathbb{P}$$

Distribution Semantics of Probabilistic Logics

$$W \xrightarrow{\gamma} \underbrace{\mathsf{D}(W)}_{\text{prob. dist.}} \text{ to interpret } L_p\phi \text{ as "} \phi \text{ with probability } \geq p\text{"}$$

$$w \models L_p\phi \iff \boxed{\gamma(w)([\![\phi]\!]) \geq p} \quad \rightsquigarrow \gamma(w) \in \mathbb{P}$$

Game Frame Semantics of Coalition Logic

$$W \xrightarrow{\gamma} ((S_a)_{a\in A}, f : \prod_{a\in A} S_a \to W) \text{ to interpret } \langle a, b\rangle\phi \text{ as "a, b can force } \phi\text{"}$$

$$w \models \langle a, b\rangle\phi \iff \boxed{\gamma(w) = ((S_a), f) : f(\ldots, s_a, s_b, \ldots) \in \phi} \quad \rightsquigarrow \gamma(w) \in \mathbb{P}$$

Coalgebras Provide a Semantic Umbrella

Semantic Structures map States to Successors

Coalgebras: $\boxed{W \xrightarrow{\gamma} TW}$

where $T : \mathsf{Set} \to \mathsf{Set}$ is a “construction” (technically: a functor) on sets

Modalities express properties of Successors in terms of States

$$w \models \heartsuit\phi \iff \gamma(w) \in [\![\heartsuit]\!]([\![\phi]\!])$$

Technically. Modal Operators are *natural transformations*

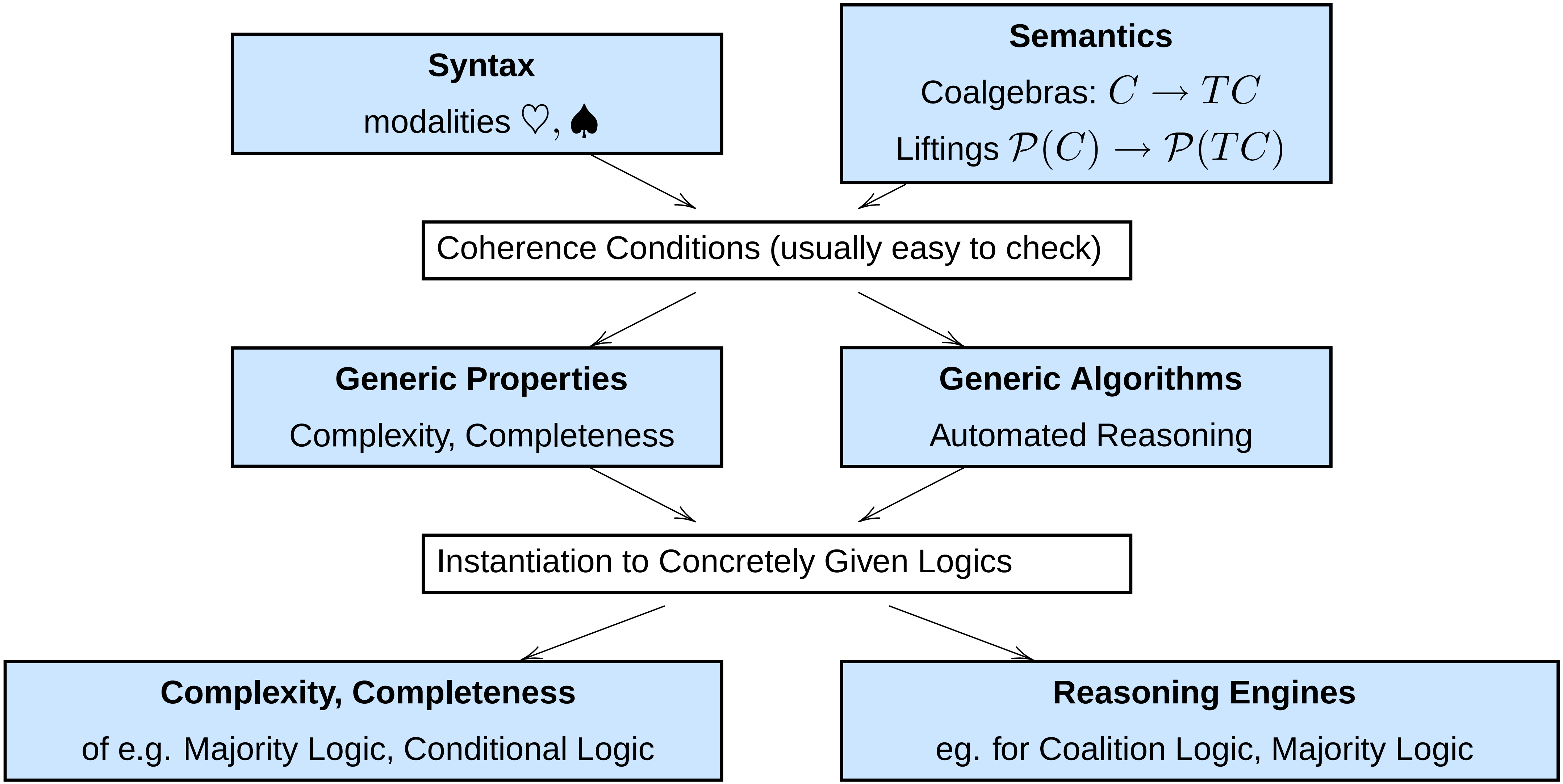
$$[\![\heartsuit]\!]_X : \mathcal{P}(X) \to \mathcal{P}(TX)$$

that we add to (classical) propositional logic.

Examples.

- (Standard) modal logic, classical and monotone modal logic
- graded modal logic, probabilistic modal logic, conditional logic, coalition logic

Coalgebraic Semantics is UNIFORM



Generic Completeness

Approach. Find *coherence conditions* between syntax and semantics

Deduction for Coalgebraic Logics: propositional logic plus a set R of

one-step rules ϕ/ψ: ϕ propositional, ψ clause over $\heartsuit a, a \in V$

Intuition. Rules mimic one-step behaviour

One Step Derivability of χ (propositional over $\{\heartsuit x : x \subseteq X\}$) over a set X

- $TX \models \chi$ defined inductively by $[\![\heartsuit x]\!] = [\![M]\!](x)$
- $\mathcal{R}X \vdash \chi$ iff $\{\psi\sigma : X \models \phi\sigma, \phi/\psi \in \mathcal{R}\} \vdash_{\mathsf{PL}} \chi$

R is one-step sound (complete) if $TX \models \chi$ whenever (only if) $\mathcal{R}X \vdash \chi$

Theorem (P, 2003, Schroeder 2006)

Soundness and weak completeness are implied by their one-step counterparts, decidability in PSPACE.

Examples of (Cut-Free) Complete Rule Sets

Modal Logic E.

$$\frac{p \leftrightarrow q}{\Box p \rightarrow \Box q}$$

Modal Logic K.

$$\frac{\bigwedge_{i=1,\dots,n} p_i \rightarrow q}{\bigwedge_{i=1,\dots,n} \Box p_i \rightarrow \Box q}$$

Graded Modal Logic.

$$\frac{\sum_{i=1}^{n} p_i \leq \sum_{j=1}^{m} q_j}{\bigwedge_{i=1}^{n} \Diamond_{k_i} p_i \rightarrow \bigvee_{j=1}^{m} \Diamond l_j q_j}$$

Probabilistic Modal Logic.

$$\frac{\sum_{i=1}^{n} p_i + u \leq \sum_{j=1}^{m} q_j}{\bigwedge_{i=1}^{n} L_{u_i} p_i \rightarrow \bigvee_{j=1}^{m} L_{v_j} q_j}$$

Conditional Logic.

$$\frac{\bigwedge_{i=1,\dots,n} q_i \rightarrow q_0 \wedge \bigwedge_{i=1,\dots,n} p_i \leftrightarrow p_0}{\bigwedge_{i=1,\dots,n} (p_i \Rightarrow q_i) \rightarrow (p_0 \Rightarrow q_0)}$$

Coalition Logic.

$$\frac{\bigvee_{i=1}^{n} \neg p_i}{\bigvee_{i=1}^{n} \neg [C_i] p_i} \qquad \frac{\bigwedge_{i=1}^{n} p_i \rightarrow q \vee \bigvee_{j=1}^{m} r_j}{\bigwedge_{i=1}^{n} [C_i] p_i \rightarrow [D] q \vee \bigvee_{j=1}^{m} [N] r_j}$$

This Talk: Extend with Fixpoints

Extend basic logics with least and greatest *fixpoints*: use *negation normal form*

$$A, B ::= p \mid \bar{p} \mid A \vee B \mid A \wedge B \mid \heartsuit(A_1, \ldots, A_n) \mid \bar{\heartsuit}(A_1, \ldots, A_n) \mid \mu p.F \mid \nu p.F$$

where $p \in V$, $\heartsuit$ is n-ary and and $\bar{p}$ does not occur in F.

Dual Operators. For $\heartsuit$ n-ary, $\bar{\heartsuit} = \neg\heartsuit\neg$ (semantically)

Intuition. μ is *finite unfolding* (safety) and ν is *infinite recurrence* (liveness)

Coalgebraic Semantics $[\![A]\!]_M^\pi$ where $M = (C, \gamma)$ and π a valuation

- as before for propositional connectives and modalities
- $[\![\mu p.F]\!]_M = \mathsf{LFP}(F_p^M)$ and $[\![\nu p.F]\!]_M = \mathsf{GFP}(F_p^M)$

where $F_p^M(X) = [\![F]\!]_{M,\sigma'}$ with $\sigma'(q) = \sigma(q)$ for $q \neq p$ and $\sigma'(p) = X$.

Semantically, this is very easy indeed ...

(Dual) Axiomatisation

Here. Easier to use *Tableaux* than *Sequent Calculi*

Tableau Sequents. Finite sets of formulas $\Gamma = \{A_1, \dots, A_n\}$ read *conjunctively*

Tableau Rules. As before, with modal rules dualised

$$(\wedge)\frac{\Gamma; A \wedge B}{\Gamma; A; B} \quad (\vee)\frac{\Gamma; A \vee B}{\Gamma; A \quad \Gamma; B} \quad (\mathsf{f})\frac{\Gamma; \eta p.A}{\Gamma; A[p := \eta p.A]} \quad (\mathsf{m})\frac{\Gamma_0\sigma, \Delta}{\Gamma_1\sigma \dots \Gamma_n\sigma} \quad (\mathsf{Ax})\frac{\Gamma, A, \bar{A}}{}$$

Note. Applying rules starting from Γ only creates a *finite* set of sequents, $\mathsf{Cl}(\Gamma)$

Remarks.

- *rule application* preserves and reflects *satisfiability*
- *No distinction* between *least* and *greatest* fixpoints
- naive *unfolding* leads to infinite *loops*

Construction of Tableaux

Idea. Tableaux as *finite graphs* that may not unfold *least* fixpoints *infinitely* often

Definition. A *Tableaux* for a sequent Γ is a *finite*, *rooted* and *labelled* graph (N, K, R, ℓ) where

- N is the set of nodes and $K \subseteq N \times N$ is the set of edges,
- R is the root node and $\ell : N \to \mathsf{S}(\Gamma)$ is a labelling with $\ell(R) = \Gamma$
- if a rule can be applied to $\ell(n)$ then n has appropriately labelled successors
- otherwise, n may not have any successors

Closed Tableaux. axioms on all end nodes, infinite unfolding of lfp's on infinite paths

Conceptually. Closed tableaux are *finitely represented* witnesses for unsatisfiability

Example: Coalition Logic with Fixpoints

Recall. $[C]\phi \rightsquigarrow$ "Coalition C can force ϕ"

Closed Tableau for $[C]\,\underbrace{\nu X.(p \wedge \overline{[N]}X))}_{B} \wedge [D]\,\underbrace{\mu Y.(\bar{p} \vee [D]Y)}_{A}$:

$$\begin{array}{c}
[C]B \wedge [D]A \\ \hline
[C]B \, ; \, [D]A \\ \hline
B \, ; \, A \\ \hline
p \wedge \overline{[N]}B \, ; \, A \\ \hline
p \wedge \overline{[N]}B \, ; \, \overline{p} \vee [D]A \\ \hline
p \, ; \, \overline{[N]}B \, ; \, \overline{p} \vee [D]A \\ \hline
\begin{array}{c} p \, ; \, \overline{[N]}B \, ; \, \overline{p} \\ \hline \end{array} \qquad p \, ; \, \overline{[N]}B \, ; \, [D]A
\end{array}$$

"C can perpetuate p indefinitely whereas D can achieve $\bar{p}$ in finitely many steps"

Games for the coalgebraic μ-calculus

Main Technical Tool. Two-Player Parity Games

- every board position b has a priority $\Omega(b)$
- $\exists$ wins (and $\forall$ looses) a play if largest infinitely occurring priority is even
- unfolding of *least fixpoints* gives *odd* priorities

Model Checking Game

- modal satisfiability game
- played on state/formula pairs
- unfolding of fixpoints

Tableaux Game

- played on sequents and rules
- $\forall$ chooses rule
- $\exists$ chooses conclusion

Crucially. *formulas* in the model checking game vs *formula sets* in tableaux

Goal. Γ has closed tableau $\iff$ $\forall$ wins tableau game $\iff$ Γ unsatisfiable

Traces: The Good, The Bad And The Ugly

Traces, or: how to align the two games

- in the tableau game: plays are sequences Γ_1, Γ_2 of *formula sets*
- in the tableau game: plays are sequences $A_1, A_2, \ldots$ of *formulas*

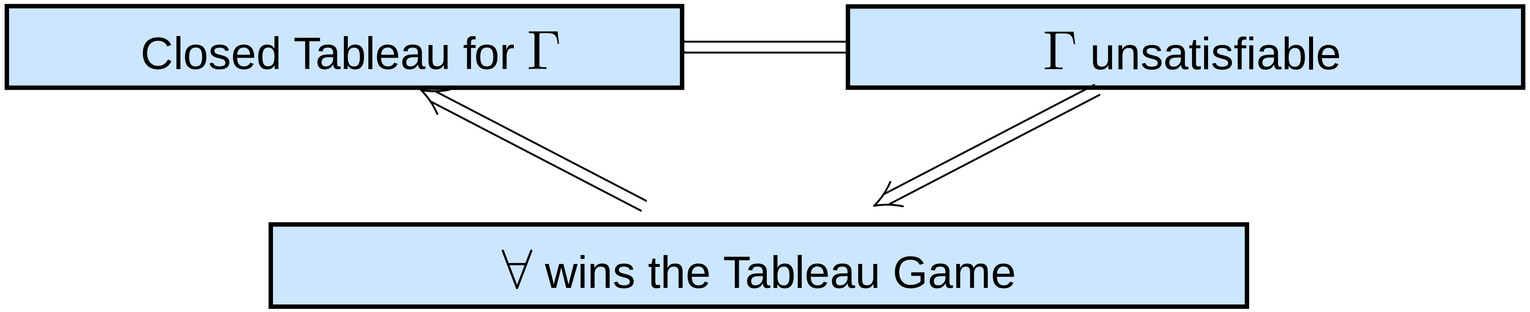
Intuition. $\forall$ looses play (Γ_i) if it carries (A_i) which looses model checking game.

Fortunately. There is a (det.) parity word A automaton that rejects all such (Γ_i).

The Tableau Game.

- board positions: $(\Gamma, a) \cup (\{\Gamma_1, \ldots, \Gamma_n\}, a)$ where $a \in A$
- $\forall$ chooses tableau rule, $\exists$ chooses conclusions
- $(\Gamma, a) \to_\forall (\{\Gamma_1, \ldots, \Gamma_n\}, a) \to_\exists (\Delta, a')$ if $a \xrightarrow{\Delta} a'$ in the automaton
- priorities given by automaton

Soundness, Completeness and Decidability



Proof Ideas.

- closed tableaux "guide" $\forall$ in the model checking game
- tableau-winning strategies for $\forall$ yield closed tableaux
- tableau-winnning strategies for $\exists$ code satisfying models

Crucial Step. use of coalgebraic *coherence conditions* in model construction

EXPTIME Complexity Bounds

Idea. Decidability via parity games

Parity Word Automaton (via Safra construction)

- exponentially many states, *polynomial* index (in $|\Gamma|$)

Rules and Conclusions. Require *exponentially tractable* rules (cf NPMV earlier):

- polynomial rule codes, premise and conclusions in EXPTIME

Game Board of the parity game

- exponential in $|\Gamma|$ (word automaton $\times$ (rule codes $\cup$ sequents))

Bean Counting Theorem. Assuming exponential tractability, satisfiability is in EXPTIME.

Examples and New Results

Observation. All naturally occuring rule sets are tractable (even in NP)

The modal and graded μ-calculus. $TX = \mathcal{P}(X)$ / $TX = \mathsf{B}(X)$

- known EXPTIME bounds by automata-theoretic methods (Emerson/Jutla and Kupferman/Sattler/Vardi)
- alternative proof via tableaux (Niwinsiki/Walukiewicz)

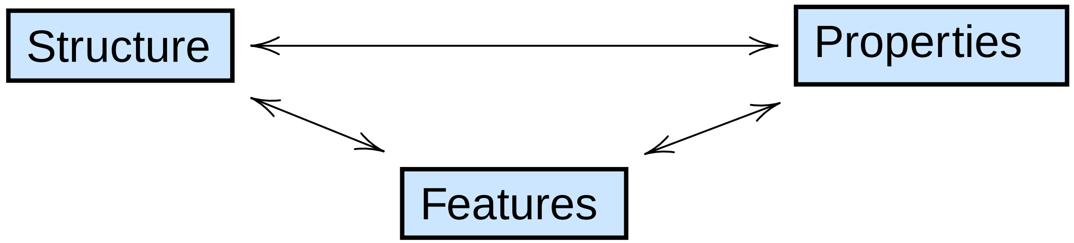
New EXPTIME bounds

- the probabilistic μ-calculus:$TX = \mathsf{D}X$
- the coalitional μ-calculus: $TX = \mathsf{G}X$
- the monotone μ-calculus: TX = $\mathsf{M}X$

Proof. Using (known) exponential tractability of rule sets.

Via Compositionality. E.g. EXPTIME for probabilistic games: $TX = \mathsf{D} \circ \mathsf{G}X$

Long-Term Goal: Bespoke Logics

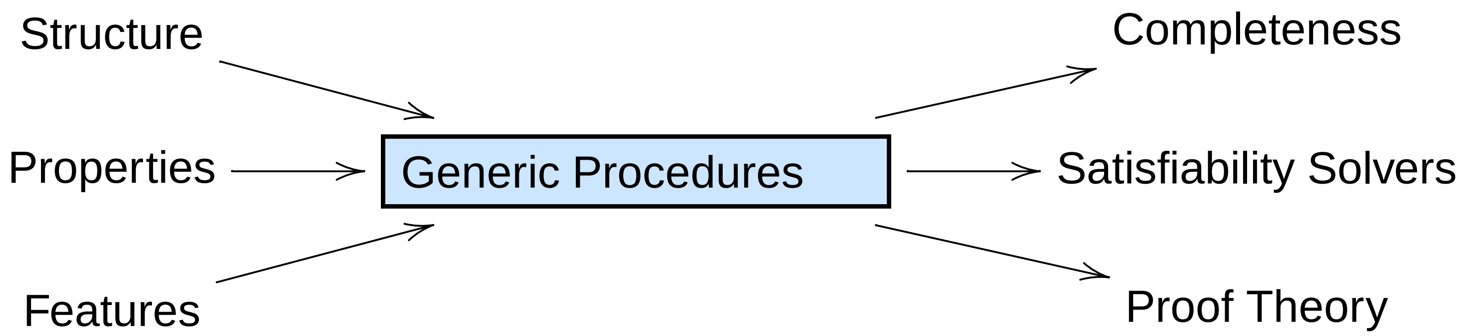


Structures. Basic semantic model – e.g. probabilistic systems or game frames

Properties. Additional frame conditions, e.g. generalised transitivity

Features. Logical means of expressivity, e.g. fixpoints, nominals

Long term goal: Pick and Choose Aproach to Modal Logics



Application Pull: Logic-Based Knowledge Representation

Complexity and Decidability: Sequent Calculi

Applications.

- *Proof Search*: logical complexity decreases from conclusion to premise
- *Subformula Property*: every proof of ϕ only mentions subformulas of ϕ
- *Interpolation:* Craig Interpolation by induction on proofs

Complexity and Decidability: Sequent Calculi

Applications.

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Our Enemy: The Cut Rule

$$(cut) \quad \frac{\Gamma, A \qquad \Delta, \neg A}{\Gamma, \Delta}$$

Sequent Calculi for Coalgebraic Logics

Sequents are multisets of formulas. Write Γ, Δ for $\Gamma \cup \Delta$ and Γ, A for $\Gamma, \{A\}$

Propositional Rules (for a right-handed Gentzen-Schuette System)

$$\frac{}{\Gamma, A, \neg A} \qquad \frac{\Gamma, A}{\Gamma, \neg\neg A} \qquad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)}$$

Modal Rules from a one-step rule ϕ/ψ where σ ranges over substitutions

$$\frac{\mathsf{Lit}(\phi_1)\sigma \quad \ldots \mathsf{Lit}(\phi_n)\sigma}{\mathsf{Lit}(\psi)\sigma, \Delta}$$

and $\mathsf{cnf}(\phi) = \phi_1 \wedge \cdots \wedge \phi_n$ and $\mathsf{Lit}(\cdot)$ is the set of literals occuring in a clause.

Notation. $\mathsf{GenR} \vdash \Gamma$ if Γ can be derived using the propositional rules and the "imported" modal rules.

Sequent Proofs vs Hilbert Proofs

Easy Lemma. $\mathsf{R} \vdash \bigvee \Gamma$ whenever $\mathsf{GenR} \vdash \Gamma$.

Cut-Free Complete Rule Sets: Two equivalent definitions

Semantically.	**Syntactically.**
Clauses over successors are derivable using a *single* rule	Closure under cuts between conclusions of modal rules

Lemma. Suppose R is cut-free complete and contraction closed.

- contraction, cut and weakening are admissible
- the inversion lemma holds for propositional connectives

Thm. Suppose R is strictly complete and contraction closed. Then

$$\mathsf{Coalg}(T) \models \bigvee \Gamma \text{ iff } \mathsf{GenR} \vdash \Gamma.$$

Complexity

PSPACE Bounds via proof search:

- polynomial bound on the height of the proof tree
- for every sequent Γ, the (codes of) rules that entails Γ can be found in polytime
- for every (code of a) rule, its premises can be found in polytime.

Formally. R is *NPMV* if there exists a finite alphabet Σ such that all sequents can be represented in Σ and a pair

$$f : \Sigma \to \mathcal{P}(\Sigma) \quad g : \Sigma \to \mathcal{P}(\Sigma)$$

of nondeterministic polytime functions such that

$$\{\{\Gamma_1, \ldots, \Gamma_n\} \mid \frac{\Gamma_1, \ldots, \Gamma_n}{\Gamma} \in \mathsf{GenR}\} = \{g(x) \mid x \in f(\Gamma)\}$$

for all sequents Γ.

Thm. If R is NPMV, sound and strictlh complete, then R-satisfiability is in PSPACE.

Implementation of Satisfiability / Provability

Parametric Formulas.

```
data L a
  =  F | T | Atom Int
    | Neg (L a) | And (L a) (L a) | M a (L a)
```

Example. The logic K and graded modal logic

```
data K = K deriving (Eq,Show)
data G = G Int
```

Logic. Type-class that supports matching

```
class (Eq a,Show a) => Logic a where
  match :: Clause a -> [[L a]]
```

(double lists as rule premises are generally in cnf)

Matching and Provability

Example. Syntax of K (again)

```
data K = K
```

Proof Rule.

$$\frac{\neg A_1, \ldots, \neg A_n, A_0}{\neg \Box A_1, \ldots, \neg \Box A_n, \Box A_0}$$

Matching: representation of resolution closed rule sets

```
instance Logic K where
  match (Clause (pl,nl)) =
    let (nls,pls) = (map neg (stripany nl), stripany pl)
    in map disjlst (map (\x -> x:nls) pls)
```

Generic Provability Predicate.

```
pbl :: (Logic a) => L a -> Bool
pbl phi = all (\c -> any (all pbl) ( match c)) (cnf phi)
```

(lazyness of Haskell guarantees polynomial space)

Compositionality By Example: Games and Probabilities

Strategic Games

- *Semantics:* $W \to \mathsf{G}W$

(outcomes of strategic games)

- *Syntax:* $[C]\phi$

(coalition C can force ϕ)

Quantitative Uncertainty

- *Semantics:* $W \to \mathsf{D}W$

(probability distributions)

- *Syntax:* $L_p\phi$

(ϕ with probability $\geq p$)

Taken Together: Games with Uncertain Outcomes

- *Semantics.* $W \longrightarrow \mathsf{D}(\mathsf{G}(W))$

probability distributions over strategic games

- *Syntax:* $L_p[C]\phi$ (and combinations)

Coalition C can bring about ϕ with probability $\geq p$.

Compositionality by Uniformity

Semantics: Defined by Operations (Functors) $T, S : \mathsf{Set} \to \mathsf{Set}$

- **Combination** of Semantical Structures: **Functor Composition**

$$T \circ S, \quad T + S, \quad T \times S : \quad \mathsf{Set} \to \mathsf{Set}$$

- **Synthesis** of Logics, Proof Calculi and Algorithms

$$\heartsuit \rightsquigarrow T\text{-succesors} \qquad \spadesuit \rightsquigarrow S\text{-successors}$$

- **Induced Combinations**
  - $(\heartsuit\spadesuit) \rightsquigarrow T \circ S$-successors describe *sequencing*
  - $(\heartsuit \times \spadesuit) \rightsquigarrow T \times S$-successors describe *fusion*
  - $(\heartsuit + \spadesuit) \rightsquigarrow T + S$-successors describe *choice*

Main Results. Combinations preserve *completeness* and *PSPACE-decidability*

Extensions: Hybrid Coalgebraic Logic

Extend modal logics with *nominals* $i \in N$ and *satisfaction operators* $@_i$

$$\mathcal{L} \ni \phi, \psi ::= a \mid \bot \mid \phi \to \psi \mid \heartsuit(\phi_1, \ldots, \phi_n) \mid @_i \phi$$

for $a \in N \cup V$, $\heartsuit$ n-ary and V a set of propositional variables.

Hybrid Valuations $\pi : N \cup V \to \mathcal{P}(C)$ assign singleton sets to nominals

Intuition. Nominals are *names* of individual entities in models (like Henry VIII)

Coalgebraic Semantics $[\![\phi]\!]^{\pi}_{(C,\gamma)}$ of $\phi \in \mathcal{L}$ over (C, γ) wrt hybrid valuation π:

- as before for modal operators and nominals (given structure for the modalities)
- and $[\![@_i \phi]\!]^{\pi}_{(C,\gamma)} = \{c \in C \mid \pi(i) \models \phi\}$

NB. The semantics of $@$-formulas is either empty or the carrier of the model.

Hybrid Completeness

Axioms for Nominals

$$
\begin{array}{rlrl}
(K@) & @_i(\phi \to \psi) \to (@_i\phi \to @_i\psi) & (\mathrm{sd}) & @_i\phi \leftrightarrow \neg @_i \neg \phi \\
(\mathrm{in}) & i \wedge \phi \to @_i\phi & (@\bot) & @_i\bot \to \bot \\
(\mathrm{ref}) & @_i i & (\mathrm{sym}) & @_i j \leftrightarrow @_j i \\
(\mathrm{nom}) & @_i j \wedge @_j p \to @_i p & (\mathrm{ag}) & @_j @_i p \to @_i p
\end{array}
$$

Interaction Axiom

$$(mob) \quad @_i a \to (\heartsuit b \leftrightarrow \heartsuit(b \wedge @_i a)) \qquad (\heartsuit \in \Lambda)$$

Thm. (Myers/P/Schröder 2008) Hybrid coalebraic modal logic, i.e. (mob) + nominal axioms + a set of complete one-step rules is complete up to the finite model property and decidable in PSPACE.

Sequent Calculi for Hybrid Logics

Sequents. Multisets of formulas of the form $@_t A$ for formulas A

Static Rules. $@$-prefixed versions of propositional and nominal rules

$$(\mathrm{Ax})\quad @_t\neg A, @_t A, \Gamma \qquad (\mathrm{Ref})\quad @_t t, \Gamma \qquad (@\top)\quad @_t\top, \Gamma$$

$$(\neg\neg)\quad \frac{@_t A, \Gamma}{@_t\neg\neg A, \Gamma} \qquad (\wedge)\quad \frac{@_t A, \Gamma \qquad @_t B, \Gamma}{@_t(A \wedge B), \Gamma}$$

$$(\neg\wedge)\quad \frac{@_t\neg A, @_t\neg B, \Gamma}{@_t\neg(A \wedge B), \Gamma} \qquad (\mathrm{At})\quad \frac{@_t A, \Gamma}{@_s @_t A, \Gamma}$$

$$(\mathrm{Sd})\quad \frac{@_s\neg A, \Gamma}{@_t\neg @_s A, \Gamma} \qquad (\mathrm{Eq})\quad \frac{\Gamma[t := i]}{@_t\neg i, \Gamma}$$

Modal Rules.

$$(R)\quad \frac{@_n\Gamma_1\sigma, @_t\Gamma_0\sigma, \Delta \quad \dots \quad @_n\Gamma_k\sigma, @_t\Gamma_0\sigma, \Delta}{@_t\Gamma_0\sigma, \Delta}(n \text{ fresh})$$

for all sequent rules(!) $\Gamma_1 \dots \Gamma_n/\Gamma_0$

Cut-Elimination in Coalgebraic Hybrid Logic

Thm. The ($@$-prefixed version of the) cut rule

$$\frac{@_t A, \Gamma \qquad @_t \neg A, \Delta}{\Gamma, \Delta}$$

is admissible.

Proof. Triple(!) induction over modal rank, size of cut formula and proof size.

Complexity. Proof search is not neccessarily terminating:

$$(R) \quad \frac{@_n \Gamma_1 \sigma, @_t \Gamma_0 \sigma, \Delta \quad \dots \quad @_n \Gamma_k \sigma, @_t \Gamma_0 \sigma, \Delta}{@_t \Gamma_0 \sigma, \Delta} (n \text{ fresh})$$

but polynomially many applications of (R) suffice on every branch.

Corollary. Given that the original rule set is NPMV, cut-free complete and contraction closed, satisfiability in Coalgebraic Hybrid Logic is PSPACE-decidable.