Tutorial 2: Transformations of Graphics Scenes

1. In a computer graphics animation scene an object is defined as a planar polyhedron. The object centre is located at position P = (0, 0, 10), and the scene is drawn, as normal, in perspective projection with the viewpoint at the origin and the view direction along the z-axis. Calculate the transformation matrix that will shrink the object in size by a factor of 0.8 towards its centre point.

2. Use your matrix of part 1 to check what happens to the points (0, 0, 10) and (0,0,5). Is your result what you expect?

3. In a different animation, the object, defined above is required to rotate clockwise, looking from the origin, while shrinking. In each successive frame it is to rotate by 15^o while shrinking to 0.8 of its original size. The rotation axis is to be the z axis, and the shrinkage is, as before, towards the object's centre. Given that $Cos(15^o) = .97$ and $Sin(15^o) = .26$, what is the transformation matrix that will achieve this animation?

4. The scene above is to be drawn in perspective projection with the plane of projection being $z = 2$. Find the combined transformation that will do animation of part 1 followed by the perspective projection. Is your matrix singular?

5. Use your matrix to find the transformation and perspective projection of the points (0, 0, 10) and (0, 0, 5) in homogenous coordinates and then in Cartesian coordinates.

6. The scene is to be viewed from a moving viewpoint specified by its position **C** and a left-handed viewing coordinate system $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$. At one point in the animation the view direction is $\boldsymbol{w} = (-1, 0, 0)^T$, and the viewpoint is given by **C** = (50, 10, -10). Given that the view is in the horizontal plane ($\boldsymbol{v} = (0, 1, 0)^T$) find the value of $\boldsymbol{u}$.

7. Hence, or otherwise, find the viewing transformation matrix.

Tutorial 2: Solution

Q1.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & -8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So the required transformation with a single matrix is:

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Q2.

The point (0, 0, 10) 'moves' to (0, 0, 10) i.e. it stays in the same place. This is expected because it is the centre of the object.

The point (0, 0, 5) goes to (0, 0, 6). This is a move towards the centre as expected.

Q3.

We want the rotation to be clockwise when viewed from the origin, i.e. when viewed from the negative side of the axis (because the object centre is at $z = 10$). So we need a value of $\theta = -15°$ in the rotation matrix R_z

$$\begin{matrix} \cos(15^\circ) \approx 0.97 \\ \sin(15^\circ) \approx 0.26 \end{matrix} \quad \text{so} \quad \begin{matrix} \cos(-15^\circ) \approx 0.97 \\ \sin(-15^\circ) \approx -0.26 \end{matrix}$$

Q3. (Contd.)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(-15^\circ) & -\sin(-15^\circ) & 0 & 0 \\ \sin(-15^\circ) & \cos(-15^\circ) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 0.77 & 0.21 & 0 & 0 \\ -0.21 & 0.77 & 0 & 0 \\ 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Q4.

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0.5 & 0 \end{pmatrix} \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0.4 & 1 \end{pmatrix}$$

It is singular. The last two rows are multiples of each other.

Q5.

$$\begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0.4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 10 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10 \\ 5 \end{pmatrix}$$

in homogeneous coordinates which normalises into Cartesian coordinate (0, 0, 2).

$$\begin{pmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0.4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 3 \end{pmatrix}$$

in homogeneous coordinates which normalises into Cartesian coordinate (0, 0, 2).

So both points project to the origin in the plane of projection.

Q6.

<u>Method 1:</u> Brute force equation solving

We have the following identity for the left-handed system $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$

$$\mathbf{u} \times \mathbf{v} = \mathbf{w}$$

We know that

$$\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

and would like to solve for $\mathbf{u}$. Writing $\mathbf{u} = (u_1, u_2, u_3)^T$, we can obtain the equation

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} -u_3 \\ 0 \\ u_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

which gives $u_3 = 1$ and $u_1 = 0$. We must also have $u_2 = 0$, because $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (perpendicular), i.e. $\mathbf{u} \cdot \mathbf{v} = 0$.

Putting this all together gives $\mathbf{u} = (0, 0, 1)^T$.

<u>Method 2:</u> Using the cross product identities (see revision notes on vector algebra).

For $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ to form a left-handed system, the following cross product formula holds

$$\mathbf{w} = \mathbf{u} \times \mathbf{v}$$

Cyclic permutations of this formula also hold, i.e.

$$\mathbf{v} = \mathbf{w} \times \mathbf{u} \qquad \text{and} \qquad \mathbf{u} = \mathbf{v} \times \mathbf{w}$$

We can use the last of these formulas to find $\mathbf{u}$:

$$\mathbf{u} = \mathbf{v} \times \mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

which agrees with method 1.

Q7.

The transformation matrix can be written in terms of ***u***, ***v***, ***w*** and **C**

$$\begin{pmatrix} u_x & u_y & u_z & -\mathbf{C}\cdot u \\ v_x & v_y & v_z & -\mathbf{C}\cdot v \\ w_x & w_y & w_z & -\mathbf{C}\cdot w \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And **C** = (50, 10, -10) so

$$\mathbf{C}\cdot u = -10$$
$$\mathbf{C}\cdot v = 10$$
$$\mathbf{C}\cdot w = -50$$

hence we write down the transformation matrix as:

$$\begin{pmatrix} 0 & 0 & 1 & 10 \\ 0 & 1 & 0 & -10 \\ -1 & 0 & 0 & 50 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$