Vector Algebra Revision Notes

A vector is a one dimensional array of numbers, for example

$$\begin{pmatrix} 1 \\ 55 \\ 79.3 \\ -25 \end{pmatrix}$$

is a vector. For the analysis of three dimensional geometry we will consider only vectors of dimension two or three.

A vector

$$\mathbf{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

can be written within a line using the transpose notation as $\mathbf{p} = (p_x, p_y, p_z)^T$. The magnitude of $\mathbf{p}$ is computed by $|\mathbf{p}| = \sqrt{p_x^2 + p_y^2 + p_z^2}$. The direction of $\mathbf{p}$ is defined by the angles with the Cartesian axes: $\theta_x, \theta_y, \theta_z$ where

$$\cos\theta_x = \frac{p_x}{|\mathbf{p}|} \quad \cos\theta_y = \frac{p_y}{|\mathbf{p}|} \quad \cos\theta_z = \frac{p_z}{|\mathbf{p}|}$$

A unit vector is one where $|\mathbf{p}| = 1$. By convention, the unit vectors in the directions of the Cartesian axes are labelled $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ with $\boldsymbol{i} = (1, 0, 0)^T$, $\boldsymbol{j} = (0, 1, 0)^T$ and $\boldsymbol{k} = (0, 0, 1)^T$. Unit vectors are often used for specifying directions. Lower case bold italic letters are used for unit vectors in these notes.

Vectors may be treated *position* vectors which start at the origin and describe a particular position in space. Alternatively, they may be treated as *direction* vectors which simply have direction and magnitude and are not associated with any particular position in the Cartesian coordinate system.

A position vector is simply a coordinate in Cartesian space. It is a particular instance of a vector which starts from the origin. Upper case bold letters are used for position vectors in these notes.

A direction vector is the same as a vector. The name is used particularly when a vector defines a direction and its magnitude is not relevant. Lower case bold letters are used for direction vectors in these notes.

Vector Addition:

There is only one way to add vectors, and that is to add the individual ordinates, so:

$$\mathbf{p} + \mathbf{q} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} + \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} = \begin{pmatrix} p_x + q_x \\ p_y + q_y \\ p_z + q_z \end{pmatrix}$$

Scalar multiplication:

Multiplication of a vector by a scalar means that each element of the vector is multiplied by the scalar, i.e.

$$\mu\mathbf{d} = \mu \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} = \begin{pmatrix} \mu d_x \\ \mu d_y \\ \mu d_z \end{pmatrix}$$

Multiplication by -1 reverses the direction of a vector.

Scalar multiplication and vector addition can be combined to define a line in Cartesian space. The equation: $\mathbf{p} = \mathbf{B} + \mu\mathbf{d}$ specifies a line through point (position vector) $\mathbf{B}$ in direction $\mathbf{d}$. Choosing any value of μ will identify one point on the line.

The scalar product[1]:

There are two ways of 'multiplying' two vectors together. The first we describe is the scalar product which returns a scalar value defined as

$$\mathbf{p} \cdot \mathbf{q} = p_x q_x + p_y q_y + p_z q_z$$

Note that we can write the magnitude of a vector as $|\mathbf{p}| = \sqrt{\mathbf{p} \cdot \mathbf{p}}$. The dot product can be written equivalently as

$$\mathbf{p} \cdot \mathbf{q} = |\mathbf{p}||\mathbf{q}| \cos(\theta)$$

where θ is the angle between the directions of $\mathbf{p}$ and $\mathbf{q}$. The dot product has some useful properties:

1. If two vectors are at right angles, the dot product is zero.
2. If the angle between two vectors is acute the dot product is positive.
3. If one of the vectors is a unit vector, the dot product gives the projection of the other vector along its direction.

The cross product:

This is the second way of 'multiplying' two vectors and is also known as the 'vector product'.

The cross product of two vectors results in a third vector defined by:

$$\mathbf{p} \times \mathbf{q} = \begin{pmatrix} p_y q_z - p_z q_y \\ p_z q_x - p_x q_z \\ p_x q_y - p_y q_x \end{pmatrix}$$

This formula may be easily remembered by considering the evaluation of the determinant:

$$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{vmatrix}$$

The vector product is a vector whose magnitude is given by

$$||\mathbf{p} \times \mathbf{q}|| = ||\mathbf{p}|| \, ||\mathbf{q}|| \, |\sin\theta|$$

where we distinguish between vector magnitudes by using the notation $|| \cdot ||$ and the absolute value the scalar $\sin\theta$ with the notation $| \cdot |$.

It is important to remember that the direction of $\mathbf{p} \times \mathbf{q}$ is at right angles to both $\mathbf{p}$ and $\mathbf{q}$.

[1] This is also known as the 'inner' product or the 'dot' product.

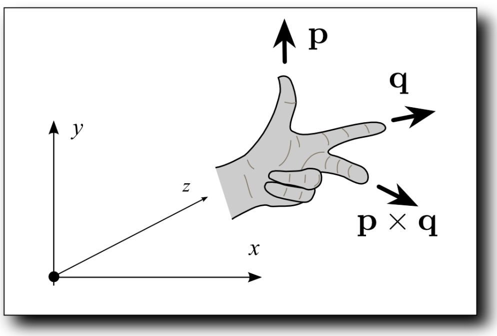

Figure 1: The way vectors and their cross product are oriented depends on the coordinate system.
Left: A left handed coordinate system. Right: A right handed coordinate system.

For a given pair of vectors $\mathbf{p}$ and $\mathbf{q}$, there are in fact *two* possible vectors with the above magnitude and which are at right angles to both $\mathbf{p}$ and $\mathbf{q}$.

This means we need to fix which one is to be defined as the cross product $\mathbf{p} \times \mathbf{q}$. We do this by taking the handedness of the coordinate system into account. If the coordinate xyz frame is left-handed (see Figure left), then the vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{p} \times \mathbf{q}$, taken in order follow the left hand rule. Similarly, they will follow the right hand rule if the coordinate frame is right-handed (See Figure, right).

If we assume that the coordinate system is left-handed, and assume that $\mathbf{p}$ and $\mathbf{q}$ are unit vectors and at right angles to each other, then the *ordered* set of three vectors $\{\mathbf{p},\ \mathbf{q},\ \mathbf{p} \times \mathbf{q}\}$ can be viewed as a left-handed frame in their own right. Additionally, every cyclic permutation of this ordered set will also be a left-handed frame, i.e. $\{\mathbf{p} \times \mathbf{q},\ \mathbf{p},\ \mathbf{q}\}$ and $\{\mathbf{q},\ \mathbf{p} \times \mathbf{q},\ \mathbf{p}\}$.

Note that $\mathbf{p} \times \mathbf{p} = (0, 0, 0)^T$ and has no direction. The same is also true for any two vectors with the same direction or with exactly opposite directions.

Note that, for any pair of vectors, $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ since, if the angle from $\mathbf{u}$ to $\mathbf{v}$ is θ, then the angle from $\mathbf{v}$ to $\mathbf{u}$ will be $\sin(360 - \theta)$, and $\sin(360 - \theta) = \sin(-\theta) = -\sin(\theta)$. The useful property of the cross product is that it defines a direction at right angles to two vectors. Thus we can use it to find the normal vector to a plane simply by taking the cross product of two vectors on the plane. For most graphics applications we need to test which direction the normal vector goes.