Research Statement of Domenico Ruoppolo

research team Logique, Calcul et Raisonnement
Laboratoire d'Informatique de Paris Nord
Université Paris 13 - Sorbonne Paris Cité

April 16, 2017

General context

My research fits within the mathematical study of the ideas of *computation* and *program*. In particular, my work is situated at the interface between logic and computer science provided by the *proofs-as-programs correspondence*, also known as *Curry-Howard isomorphism* [How80]. Such correspondence is the realization that the many variants of Church's λ-calculus overlap with a great variety of deductive systems issued from *proof theory*, the branch of mathematical logic giving a prominent role to the dynamics of rewriting of proofs [GLT89, SU06].

The untyped λ-calculus. Despite not being my only interest, most of my focus so far has been on the study of the *untyped λ-calculus* [Bar84], introduced by Alonzo Church in the early 30's within an attempt to give mathematics a logical foundation [Chu32]. It is given by the terms $M, N ::= x \mid \lambda x.M \mid MN$ (where x ranges over countable variables) and the rewriting rule $(\lambda x.M)N \to_\beta M\{N/x\}$. The system provides a model of computation, as it can represent all Turing-computable functions through the notion of λ-definability (Church's thesis). Also, this system is the common core of all functional programming languages. Over the last five decades λ-calculi have played a prominent role in the conception, implementation and analysis of such languages, but also in a number of impressive theoretical insights into the concepts of computation, program and proof.

Equivalences of programs. The study of the untyped λ-calculus is not restricted to the sole β-rule. One is more often interested in *λ-theories*, which are congruences on λ-terms extending β-conversion. All λ-theories form a complete lattice of cardinality $2^{\aleph_0}$, mostly still unexplored. From the point of view of computer science, *observational equivalences* have a certain relevance among λ-theories. Indeed, they provide an answer to a nontrivial question: when two programs are equivalent? The answer is *behavioural*: they are equivalent if they look to behave in the same way in every possible case of execution. Formally, two λ-terms M and N are observationally equivalent with respect to some fixed set $\mathcal{O}$ of *observable terms* when, for every possible context of evaluation $C[-]$, the λ-term $C[M]$ β-reduces to an observable in $\mathcal{O}$ if and only if $C[N]$ β-reduces to an observable in $\mathcal{O}$. The choice of $\mathcal{O}$ is not unique. The most studied instance is the one where the observables are λ-terms in *head normal form.* This λ-theory is denoted by $\mathcal{H}^*$. An alternative choice is to take as $\mathcal{O}$ the set of λ-terms in *β-normal form.* This last is called *Morris's observational equivalence*, and denoted by $\mathcal{H}^+$ hereafter.

Denotational semantics Mostly, my research concerns the *denotational semantics* of the λ-calculus. Dana Scott discovered the first denotational model [Sco72] in the late 60's. Since then, a large number of such models, lying in many different categories, have been studied. In most of them λ-terms are interpreted as structure-preserving functions between some order-theoretic, algebraic or topological structures. A limitation of these traditional models is to abstract away from the execution process and overlook *quantitative* aspects such as the time, space, or energy consumed by a computation. My work fits in a wider research program aiming to overcome these limitations. Such a quantitative approach has its inspiration and technical roots in the semantics of Girard's *Linear Logic* [Gir87]. More specifically, most of the results that I achieved so far concern *relational semantics*, which interprets λ-terms as relations where their inputs are grouped together in multisets. As a result of this usage of multisets, relational models are *resource-sensitive*, in that they represent explicitly the consumption of input resources during the execution of programs. The first concrete examples of relational models of λ-calculus were built in [BEM07, HNPR06].

Relational graph models

In my thesis [Ruo16] and related publications [MR14, BMPR16, BMR17] I studied a proper subclass of relational models, called *relational graph models* (rgm's). On the one hand, the definition of an rgm is the relational analogue of the definition of a graph models *à la* Plotkin-Scott-Engeler [Plo93, Eng81, Lon83], a well-known kind of continuous model. In particular, rgm's can be built by free completion and by forcing like the continuous ones. On the other hand, rgm's can be seen as a resource-sensitive reformulation of filter models [BDS13]. The classical Stone duality between filter models and intersection type systems shows that some interesting classes of domain-based models can be described in logical form. The intuition is that a functional intersection type

$\alpha_1 \wedge \cdots \wedge \alpha_n \to \beta$ can be seen as a continuous step function sending the set $\{\alpha_1, \ldots, \alpha_n\}$ to the element β. Our idea, already present in [dC09, PPRDR15], is that in the absence of idempotency and partial orders the functional type $\alpha_1 \wedge \cdots \wedge \alpha_n \to \beta$ can be seen as a relation associating the multiset $[\alpha_1, \ldots, \alpha_n]$ with the element β. As a consequence, even rgm's can be presented in logical form. Precisely, as *non-idempotent intersection type systems.* This logical representation comes in handy when studying the quantitative features of these models.

Full abstractions. Every denotational model induces a λ-theory through the kernel of its interpretation function. In particular, a model is *fully abstract* when the induced λ-theory is an observational equivalence. In exploring the λ-theories induced by rgm's my coauthors and I paid a particular attention to the full abstraction problem. Until recently, researchers were only able to prove full abstractions for individual models [Hyl76, Wad78, CDZ87], or at best to provide sufficient conditions for models living in some class to be fully abstract [Man09]. A substantial advance was made in [Bre14], where Breuvart was able to provide a characterization of all the models fully abstract for $\mathcal{H}^*$ living in a certain class. My coauthors and I achieved equally general full abstraction theorems for the class of rgm's. We proved that an rgm is fully abstract for $\mathcal{H}^+$ iff it is extensional (it models η-conversion) and *λ-König* [BMPR16, BMR17]. Intuitively, a model is λ-König when every recursive tree has an infinite path that is witnessed by some element of the model, in a certain type-theoretical sense. By dualizing the notion of λ-König rgm we also proved a characterization for the other main observational equivalence: an rgm is fully abstract for $\mathcal{H}^*$ iff it is extensional and *hyperimmune* [BMR17].

Other results on Morris's λ-theory

The observational equivalence $\mathcal{H}^+$ is maybe less ubiquitously studied in the literature than $\mathcal{H}^*$, but nevertheless important. For instance, its notion of observables is central in Böhm's Theorem [Böh68] and similar separabilities [CDR78]. This is why I focused on $\mathcal{H}^+$ during my PhD, and, together with my coauthors, proved some other notable results concerning it.

Extensional Taylor expansion. Ehrhard-Regnier's *Taylor expansion* is a translation developing every λ-term as an infinite series of terms living in a resource-sensitive version of the λ-calculus, known as *differential λ-calculus* [ER03, ER08]. In [MR14] we defined the *extensional Taylor expansion*, a version of this notion taking η-reduction into account, and proved that it provides another model of $\mathcal{H}^+$.

The ω-rule. The *ω-rule* is a strong form of extensionality defined as follows: for all M, N $\big(MP = NP$ for all closed $P\big) \Rightarrow M = N$. In [BMPR16] we proved that $\mathcal{H}^+$ satisfies the ω-rule. This solved a long-standing open question [Bar84, §17.4].

Ongoing investigations

A number of more or less precise open questions concerning rgm's remain to investigate.

- Are all λ-theories in the interval $[\mathcal{H}^+, \mathcal{H}^*]$ relational graph theories? If it is not the case, is it possible to provide a characterization of the representable ones?
- Do all extensional rgm's satisfy the ω-rule?
- What is the *extensional collapse* [Ehr12] of the class of rgm's?
- Is there a game semantics reading of rgm's?

Finally, in a much more abstract perspective, a 2-categorical version of the relational semantics was contemplated in [FGHW08, Hyl10], where the categorical notion of *profunctors* takes the role of relations. An ambitious longterm aim is to explore the possibility of studying a profunctorial version of rgm's.

Operads of control

A completely different ongoing investigation, in collaboration with Paul-André Melliès, is rooted in the higher-algebraic counterpart of denotational semantics: *categorical logic*, a branch of category theory interested in the interpretation of proofs [LS86, Mel09].

The context. Starting from [Gri90], researches have extended the proofs-as-programs correspondence beyond the limits of purely *functional* programming on one side, and of purely *constructive* proof systems on the other. In this perspective, the *$\lambda\mu$-calculus* introduced in [Par92] is still a major reference. Its untyped version adds to the syntax of the λ-calculus a GOTO-like mechanism of *control*, by means of *exception raising* terms $[\alpha]M$ and *exception handler* abstractions $\mu\alpha.M$. From a logical point of view, a simply typed fragment of the $\lambda\mu$-calculus is a deduction system for *classical* logic, that is the logic underlying ordinary mathematics.

The *continuation-passing-style* translations (CPS for short), traditionally providing implementation procedures for λ-calculi, extend also to the $\lambda\mu$-calculus. The idea behind CPS's is to provide to (the translation of) a program M a parameter (itself a program, according to the functional paradigm) containing all the information about the environment of execution of M (the *rest* of the code surrounding M). CPS's also have a logical meaning, which stresses the Curry-Howard isomorphism to the level of implementation mechanisms: they correspond to certain versions of *double negation* embeddings of classical logic into intuitionistic logic, whose pioneers in the 20's were Glivenko, Gödel, Kolmogorov. Generalizing to a categorical setting a CPS appeared in [LRS93], Selinger axiomatized the algebraic nature of denotational models of the $\lambda\mu$-calculus in [Sel01]. The corresponding structure is named *control category*. In [MT08] and [Tab08] Melliès and Tabareau investigated what happens when one performs Selinger's axiomatization in a *linear* context. Their analysis relies on *dialogue categories*, which can be considered as the basic categorical notion suitable to study from a *resource-sensitive* perspective the idea of *response*, and therefore the semantics of programs and environments *interacting* by mutual responses. The offspring of their work is a notion of *linear control operads*.

Ongoing work. I am focusing on the reconstruction of the *non-linear* setting from the aforementioned work of Melliès and Tabareau. Intuitively, the question is: can we find a semantic model for the $\lambda\mu$-calculus, or even more generally for any non-linear higher-order calculus with control, by adding some structure on top of a linear control operad? The emerging picture is one in which operads playing the role of linear multi-inputs programs interact with operads representing non-linear programs.

Bibliography

[Bar84] Henk Barendregt. *The lambda-calculus, its syntax and semantics.* Number 103 in Stud. Logic Found. Math. North-Holland, second edition, 1984.

[BDS13] Henk Barendregt, Wil Dekkers, and Richard Statman. *Lambda Calculus with Types.* Perspectives in logic. Cambridge University Press, 2013.

[BEM07] Antonio Bucciarelli, Thomas Ehrhard, and Giulio Manzonetto. Not enough points is enough. In *CSL'07*, volume 4646 of *LNCS*, pages 298–312. Springer, 2007.

[BMPR16] Flavien Breuvart, Giulio Manzonetto, Andrew Polonsky, and Domenico Ruoppolo. New results on Morris's observational theory: the benefits of separating the inseparable. In Delia Kesner and Brigitte Pientka, editors, *FSCD 2016*, volume 52 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 15:1–15:18, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[BMR17] Flavien Breuvart, Giulio Manzonetto, and Domenico Ruoppolo. Relational graph models at work. *submitted to Logical Methods in Computer Science*, 2017.

[Böh68] Corrado Böhm. Alcune proprietà delle forme β-η-normali nel λ-K-calcolo. *Pubbl. INAC*, 696, 1968.

[Bre14] Flavien Breuvart. On the characterization of models of H^*. In *Joint Meeting of CSL-LICS '14, Vienna, Austria,*, CSL-LICS '14, pages 24:1–24:10, New York, NY, USA, 2014. ACM.

[CDR78] M. Coppo, M. Dezani-Ciancaglini, and S. Ronchi Della Rocca. (Semi)-separability of finite sets of terms in Scott's $\mathcal{D}_\infty$-models of the lambda-calculus. In G. Ausiello and C. Böhm, editors, *Automata, Languages and Programming*, volume 62 of *Lecture Notes in Computer Science*, pages 142–164. Springer, 1978.

[CDZ87] Mario Coppo, Mariangiola Dezani, and Maddalena Zacchi. Type theories, normal forms and D_∞-lambda-models. *Inf. Comput.*, 72(2):85–116, 1987.

[Chu32] Alonzo Church. A set of postulates for the foundation of logic. *The Annals of Mathematics*, 32:346–366, 1932.

[dC09] Daniel de Carvalho. Execution time of λ-terms via denotational semantics and intersection types. Draft available on the web page of the author, 2009.

[Ehr12] Thomas Ehrhard. The scott model of linear logic is the extensional collapse of its relational model. *Theor. Comput. Sci.*, 424:20–45, 2012.

[Eng81] Erwin Engeler. Algebras and combinators. *Algebra Universalis*, 13(3):389–392, 1981.

[ER03] Thomas Ehrhard and Laurent Regnier. The differential lambda-calculus. *Theor. Comput. Sci.*, 309(1):1–41, 2003.

[ER08] Thomas Ehrhard and Laurent Regnier. Uniformity and the Taylor expansion of ordinary lambda-terms. *Theor. Comput. Sci.*, 403(2-3):347–372, 2008.

[FGHW08] M. Fiore, N. Gambino, M. Hyland, and G. Winskel. The cartesian closed bicategory of generalised species of structures. *Journal of the London Mathematical Society*, 77(1):203, 2008.

[Gir87] Jean-Yves Girard. Linear logic. *Th. Comp. Sc.*, 50:1–102, 1987.

[GLT89] Jean-Yves Girard, Yves Lafont, and Paul Taylor. *Proofs and Types.* Number 7 in Cambridge tracts in theoretical computer science. Cambridge University Press, 1989.

[Gri90] Timothy Griffin. A formulae-as-types notion of control. In *Proceedings of the 1990 Principles of Programming Languages Conference*, pages 47–58. IEEE Comp. Soc. Press, 1990.

[HNPR06] Martin Hyland, Misao Nagayama, John Power, and Giuseppe Rosolini. A category theoretic formulation for Engeler-style models of the untyped λ-calculus. *Electronic Notes in Theor. Comp. Sci.*, 161:43–57, 2006.

[How80] William A. Howard. The formulas-as-types notion of construction. In J. P. Seldin and J. R. Hindley, editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus, and Formalism*, pages 479–490. Academic Press, 1980. Reprint of 1969 article.

[Hyl10] Martin Hyland. Some reasons for generalising domain theory. *Mathematical Structures in Computer Science*, 20(2):239–265, 2010.

[Hyl76] Martin Hyland. A syntactic characterization of the equality in some models for the λ-calculus. *J. London Math. Soc. (2)*, 12(3):361–370, 1975/76.

[Lon83] Giuseppe Longo. Set-theoretical models of λ-calculus: theories, expansions, isomorphisms. *Ann. Pure Appl. Logic*, 24:153–188, 1983.

[LRS93] Yves Lafont, Bernhard Reus, and Thomas Streicher. Continuation semantics or expressing implication by negation. Technical Report 93-21, Ludwig-Maximilians-Universität, München, 1993.

[LS86] J. Lambek and P. J. Scott. *Introduction to higher order categorical logic.* Cambridge University Press, New York, NY, USA, 1986.

[Man09] Giulio Manzonetto. A general class of models of $\mathcal{H}^\star$. In *MFCS 2009*, volume 5734 of *LNCS*, pages 574–586. Springer, 2009.

[Mel09] Paul-André Melliès. Categorical semantics of linear logic. In Jean-Louis Krivine Paul-Andr Mellis Pierre-Louis Curien, Hugo Herbelin, editor, *Interactive models of computation and program behaviour*, number 27 in Panoramas et synthses. Socit Mathmatique de France, 2009.

[MR14] Giulio Manzonetto and Domenico Ruoppolo. Relational graph models, Taylor expansion and extensionality. *Electr. Notes Theor. Comput. Sci.*, 308:245–272, 2014.

[MT08] Paul-André Melliès and Nicolas Tabareau. Linear continuations and duality. draft, 2008.

[Par92] Michel Parigot. $\lambda\mu$-calculus: an algorithmic interpretation of classical natural deduction. In *Proceedings of International Conference on Logic Programming and Automated Reasoning*, volume 624 of *Lecture Notes in Comput. Sci.*, pages 190–201. Springer, 1992.

[Plo93] Gordon D. Plotkin. Set-theoretical and other elementary models of the lambda-calculus. *Theor. Comput. Sci.*, 121(1&2):351–409, 1993.

[PPRDR15] L. Paolini, M. Piccolo, and S. Ronchi Della Rocca. Essential and relational models. *Mathematical Structures in Computer Science*, 2015. Doi:10.1017/S0960129515000316.

[Ruo16] D. Ruoppolo. *Relational Graph Models and Morris's Observability: resource-sensitive semantic investigations on the untyped λ-calculus.* Thése de doctorat, Université Paris 13, 2016.

[Sco72] Dana Scott. Continuous lattices. In Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, volume 274 of *Lecture Notes in Math.*, pages 97–136. Springer, 1972.

[Sel01] Peter Selinger. Control categories and duality: on the categorical semantics of the lambda-mu calculus. *Mathematical Structures in Computer Science*, 11(2):207–260, April 2001.

[SU06] Morten Heine B. Sorensen and Pawel Urzyczyn. *Lectures on the Curry-Howard Isomorphism.* Number 149 in Stud. Logic Found. Math. North-Holland, second edition, 2006.

[Tab08] Nicolas Tabareau. *Modalités de ressources et contrôle en logique tensorielle.* Thèse de doctorat, Université Paris VII, 2008.

[Wad78] Christopher P. Wadsworth. Approximate reduction and lambda calculus models. *SIAM J. Comput.*, 7(3):337–356, 1978.