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A General Proof of Kripke-completeness for Quantified Modal Logics

In the present paper I aim at providing a general framework to prove Kripke-completeness for normal calculi of Quantified Modal Logic. I shall consider Kripke semantics - as presented in [5] - and extensions of normal propositional modal systems, based both on Kripke's theory of quantification ([11]) and on free and classical first-order logic ([8], [9]).

In [5] Corsi highlights the lack of "a common completeness proof that can cover constant domains, varying domains, and models meeting other conditions"[1], on the other hand the need of such a proof has always been felt[2]. She attempts to provide a *unified completeness theorem for Quantified Modal Logics* - as the title of her paper states - for normal QML calculi based on Kripke's theory of quantification and classical first-order logic. In the end she leaves as open problems the completeness proof for system $Q^{\circ}.K+BF$ (on Kripke's theory of quantification, with BF) and for system $Q^{\circ}.B$ (on Kripke's theory of quantification, with modal base B).

The present paper - corresponding to chapter 1 in [1] - aims at extending and completing Corsi's programme. I present a general proof of Kripke-completeness for QML calculi considered by Corsi, as well as systems based on free logic ([2]), in which quantification is restricted through existence predicate E. In particular I prove Kripke-completeness of $Q^{\circ}.K+BF$ and further original results: every QML calculus based on free logic containing BF is Kripke-incomplete[3].

I start with introducing first-order modal languages $\mathcal{L}$ and $\mathcal{L}^E$, containing only individual variables as terms. In addition $\mathcal{L}^E$ contains existence predicate E as well. Formulas in $\mathcal{L}$ ($\mathcal{L}^E$) are inductively defined as usual.

Then I list ten first-order modal calculi, which are all extensions of propositional modal logic K, thus contain Aristotle's Law $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ and necessitation rule $\phi \Rightarrow \Box\phi$. In addition the presence of either BF or CBF is shown in names of calculi. Systems $Q.K$, $Q.K+BF$ are based on classical first-order logic, thus they both prove CBF. Systems $Q^{\circ}.K$, $Q^{\circ}.K+BF$, $Q^{\circ}.K+CBF$, $Q^{\circ}.K+CBF+BF$ are based on Kripke' theory of quantification as appears in [11]. Systems $Q^E.K$, $Q^E.K+BF$, $Q^E.K+CBF$, $Q^E.K+CBF+BF$ are based on free logic as in [8].

For assigning a meaning to formulas and theorems in these calculi, I make use of Kripke frames as defined in [5]: a Kripke frame $\mathcal{F}$ - K-frame in short - is a 4-

[1] [7], p.132.

[2] "Ideally, we would like to find a completely general completeness proof", [8], p. 273.

[3] I refer to Appendix A in [1] for the formal details

tuple $\langle W, R, D, d\rangle$ s.t. W is the non-empty domain of possible worlds $w, w', \ldots$; R is the accessibility relation on W; each outer domain $D(w)$ is a non-empty set of individuals s.t. wRw' implies $D(w) \subseteq D(w')$; each inner domain $d(w)$ is a subset of $D(w)$. A K-frame $\mathcal{F}$ has *constant* (*increasing*, *decreasing*) inner domains iff wRw' implies $d(w) = d(w')$ (resp. $d(w) \subseteq d(w')$, $d(w) \supseteq d(w')$).

By means of the canonical model method, I show that the following QML calculi are complete w.r.t. the corresponding classes of K-frames:

calculi	*inner domain*	*outer domain*
$Q.K$	increasing	$=$ inner
$Q.K + BF$	constant	$=$ inner
$Q^{\circ}.K$	all	constant
$Q^{\circ}.K + BF$	decreasing	constant
$Q^{\circ}.K + CBF$	increasing	constant
$Q^{\circ}.K + CBF + BF$	constant	constant
$Q^{E}.K$	all	constant
$Q^{E}.K + CBF$	increasing	constant

I underline that the proof is extremely general, as the definition of canonical model for a QML calculus L is the same for each system above. Notice that the fourth result solves the problem left open by Corsi.

Nonetheless Kripke semantics is quite unsatisfactory when applied to QML calculi: the canonical model for system L on Kripke's theory of quantification is not at all based on a K-frame for L. This means that we have to filtrate the canonical model in order to prove Kripke-completeness, but by lemma 1.13 in [1] it is possible to show that this filtration is somewhat unique for the various calculi.

Furthermore calculi $Q^{E}.K+BF$ and $Q^{E}.K+CBF+BF$ are Kripke-incomplete: they both validate the necessity of fictionality $\neg E(x) \rightarrow \Box\neg E(x)$, but this principle is provable in neither of them[4].

As regards QML calculi on normal modal bases stronger than K, for T and $S4$ we have the same completeness results available for K, of course w.r.t. reflexive (resp. transitive and reflexive) K-frame. As to B and $S5$, even systems $Q^{\circ}.B+BF$, $Q^{\circ}.S5+BF$ and $Q^{E}.B+BF$, $Q^{E}.S5B+BF$ are Kripke-incomplete[5], while Kripke-completeness of $Q^{\circ}.B$ and $Q^{\circ}.S5B$ are still open problems.

Finally I introduce counterpart semantics as a solution to the limits of Kripke semantics in Quantified Modal Logic. This semantic approach - developed in [3], [4] and [10] - is able to draw more accurate distinctions on valid formulas: for instance, the necessity of fictionality and BF are no more semantically equivalent, as it is the case in Kripke semantics; as a consequence typed QML calculi on free logic with BF are counterpart-complete. In fact we have completeness results for all our typed QML calculi based on free and classical logic, for every normal modal base[6].

[4]For a formal proof of this fact I refer to Appendix A in [1].

[5]The proof for calculi $Q^{\circ}.B + BF$, $Q^{\circ}.S5 + BF$ appears in [6].

[6]Even these theorems are proved in [1], by referring to [3] and [4].

For all these reasons I maintain that counterpart semantics is more suitable to deal with individuals in modal settings than Kripke semantics.

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