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Quantified Modal Logic and the Ontology of Physical Objects

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Contents

Introduction

The present work deals with logics and ontology. In the first part we analyse three semantic accounts for quantified modal logic: **Kripke semantics**, **the substantial interpretation** and **counterpart semantics**; whereas in the second part we make use of the formal machinery developed thus far to formally compare three ontological theories on persistence conditions of physical objects: **endurantism**, **perdurantism** and **sequentialism**.

Quantified modal logic - QML in short - has always had a strong philosophical appeal, since it has first appeared in papers by Barcan Marcus[1], Hintikka[2], Prior[3] and Kripke[4]. As it is the case for propositional modal logic, QML deals with necessity and possibility, as well as individual knowledge, obligations and permissions, programs and computations: these are major philosophical concepts, which QML has been applied to. In addition quantified modal logic has a special concern for individuals: we can talk about *actual* and *possible objects*, *existence*, *modal properties*, *counterfactual situations*. In the philosophy of QML we find dramatically relevant issues as *actualism/possibilism*[5], *realism about possible worlds*[6], *trans-world identity of individuals*[7]. These subject matters relate to most different fields in the philosophical investigation: *ontology*, *epistemology*, *philosophy of science*, *ethics*, as was first remarked by Leibniz.

The formal development of quantified modal logic would provide a useful tool to philosophical analysis, in order to precisely define the notions listed above.

This is exactly our task in the first part of the present work, in which we thoroughly study the above-mentioned semantic accounts for QML: Kripke semantics, the substantial interpretation and counterpart semantics.

In chapter 1 we consider Kripke semantics as presented by Corsi in [20]. Our aim is the same as Corsi's: to provide a general completeness proof for first-order modal calculi based both on Kripke's theory of quantification and on free and classical logic. In particular we prove Kripke-completeness for system $Q^{\circ}.K + BF$, left as an open problem by Corsi, and state Kripke-incompleteness for calculi $Q^{E}.K + BF$ and $Q^{E}.K + CBF + BF$ on free logic, containing the Barcan formula. We refer to appendix A for a formal proof of the latter fact. Completeness results w.r.t. Kripke semantics reveal a lack of generality and deep limitations, as we point out in section 1.5, thus there is an urging demand for a more satisfactory interpretation of quantified modal logic.

This is why in chapter 2 we introduce counterpart semantics for QML, the key feature of which is the notion of *counterpart*. Counterparts were first introduced by

[1] [7], [6] and [8].
[2] [42], [43].
[3] [71], [72] and [73].
[4] [54], [55] and [56].
[5] For a survey on this topic, see [67].
[6] For instance consider [15].
[7] See [62].

Lewis in [59], as he avoided interpreting the relation of trans-world sameness as strict identity. In the last decade a new interest in counterpart semantics has risen, in part due to Ghilardi's incompleteness results in Kripke semantics in [35], [36]. We start from [10], [19] and present semantics for QML based on counterparts. The main philosophical interest of this proposal lies in a *more accurate treatment of individuals* and in *fresh distinctions among modal principles*, as the Barcan formula and its converse, the necessity of identity and the necessity of difference. In section 2.4 we prove counterpart-completeness for typed QML_t calculi by means of Ghilardi's elegant proof in [10]. Finally we compare the completeness results available in Kripke semantics with those provable w.r.t. classes of counterpart frames. The advantages of the latter account will be clear, once we have shown that every typed QML_t calculus is complete w.r.t. counterpart semantics.

In chapter 3 we introduce identity in our formal language, in order to express relevant ontological issues as persistence conditions for objects in time, trans-world identity and change. We analyse the features of the equality relation in modal settings, in particular we check the meaning of identity statements in our semantics for QML. We first model identity in Kripke semantics, as it unrestrictedly validates modal versions of classical principles as Leibniz's Law and self-identity. Moreover $QML^=$ calculi with identity reveal strong completeness properties w.r.t. Kripke semantics. Nonetheless there are philosophical reasons we consider in section 3.3, to maintain that Kripke semantics is not completely suitable for identity. In order to solve these problems, we develop a further semantic approach to quantified modal logic; we introduce the substantial interpretation of [32], the main feature of which is an *ontology of functions*. This interpretation does not unrestrictedly validate Leibniz's Law, in particular the necessity of identity and the necessity of difference are not sound principles. In par. 3.4 we show that there are QML calculi sound and complete w.r.t. this semantic account, even if their completeness properties are particularly weak. Finally in section 3.5 we present counterpart semantics for identity, which has interesting characteristics: the necessity of identity and the necessity of difference do not hold, as it is the case in the substantial interpretation, but we need not to limit Leibniz's Law. Furthermore both the systems with contingent and classical identity are complete w.r.t. counterpart semantics.

The second part of the present work is devoted to formal ontology. First we compare semantics for quantified modal logic and ontologies for physical objects, and uphold a *correspondence thesis*, then make use of formal results concerning the formers to establish levels of generality and reducibility among the latters[8].

In chapter 4 we introduce three theories on persistence conditions for material objects: endurantism, perdurantism and sequentialism[9]. We claim that there exists a precise correspondence between the semantic accounts in the first part and these ontological theses. We test the alleged correspondence, by referring to three features of semantics: (i) the nature of individuals appearing in the domains of models, (ii) the principles sound with respect to each account, (iii) the representation and

[8]See [69].

[9]For a survey of these theories, see [94].

solution of ontological puzzles within our logical frameworks. I aim at proving that Kripke semantics, the substantial interpretation and counterpart semantics soundly formalize endurantism, perdurantism and sequentialism respectively.

Once we have proved our *correspondence thesis* between logics and ontology, we deal with comparison of ontologies. In chapter 5 we consider the translation functions in [27] and [53], [52], which connect formulas valid in the substantial interpretation and in counterpart semantics. We precisely state the necessary and sufficient conditions by which a formal approach is reducible to another one, then make use of these formal results to clarify the reducibility relationships among our theories on persistence, in virtue of the analysis in chapter 4. We conclude that under determinate constraints we can deal with the endurantist and perdurantist ontology within the sequentialist framework.

Our conclusions are quite strong and someone might not be entirely willing to accept them. Nonetheless we point out that logics has given positive contributions to the philosophy of modality, and we think that it can be useful even on the subject matter of persistence and change for individuals. The present work attempts to be a first step in this direction.

Part I

Logics

Chapter 1

Kripke Semantics

In the present chapter we aim at providing a general framework to prove completeness results w.r.t. Kripke semantics for QML calculi of quantified modal logic. We consider formal systems based both on Kripke's theory of quantification, as presented in [56], and on free and classical first-order logic, as in [32], [47].

In section 1.2 we introduce QML calculi on modal base K, in particular we prove Kripke-completeness for system $Q^{\circ}.K + BF$, left as an open problem by Corsi in [20]. This calculus is obtained by adding the Barcan formula to Kripke's original system. Furthermore we state Kripke-incompleteness for calculi $Q^E.K + BF$ and $Q^E.K + CBF + BF$ based on free logic. Since we need counterpart semantics in order to prove this result, a formal proof will be given only in appendix A.

In sections 1.3 and 1.4 we analyse completeness properties of QML calculi based on modalities stronger than K. We will see how to modify the techniques applied to quantified extensions of K, in order to prove completeness for quantified extensions of B and $S5$. Notice that Kripke-incompleteness for free logic systems with the Barcan formula persists.

We conclude that Kripke semantics reveals a lack of generality and deep limitations when applied to QML calculi, thus there is an urging demand for a more satisfactory account.

We briefly introduce the contents of the following two sections. In par. 1.1.1 we present first-order modal languages $\mathcal{L}$ and $\mathcal{L}^E$, whereas in par. 1.1.2 and 1.1.3 we respectively define ten QML calculi of quantified modal logic and Kripke semantics for $\mathcal{L}$ and $\mathcal{L}^E$.

Section 1.2 is devoted to the completeness proofs for these calculi w.r.t. Kripke semantics. In par. 1.2.1 we propose a general framework for proving completeness by means of the canonical model method: the clue of this technique is lemma 1.13, which generalizes Henkin's construction in modal settings.

Finally in par. 1.2.2 we show how to obtain completeness for a single QML calculus, by filling in the general framework with specific lemmas concerning the envisaged system. In particular Kripke-completeness is proved for $Q^{\circ}.K+BF$, whereas calculi $Q^E.K + BF$ and $Q^E.K + CBF + BF$ turn out to be Kripke-incomplete.

1.1 Syntax and Semantics

In this section we present the syntax and Kripke semantics for quantified modal logic. In par. 1.1.1 we introduce first-order modal languages $\mathcal{L}$ and $\mathcal{L}^E$: they both contain individual variables, predicative constants, propositional connectives, quantifiers and modal operators; moreover the latter contains also existence predicate E. In par. 1.1.2 we define ten QML calculi on $\mathcal{L}$ and $\mathcal{L}^E$, most of which were originally considered by Corsi in [20], but the ones on free logic. Finally in par. 1.1.3 we assign a meaning to formulas in $\mathcal{L}$ and $\mathcal{L}^E$ through Kripke semantics.

1.1.1 Languages $\mathcal{L}$ and $\mathcal{L}^E$

We start with considering the alphabets over which modal formulas are defined. Our alphabet $\mathcal{A}$ contains:

- a denumerable infinite set of individual variables $x_1, x_2, \ldots$;
- a denumerable infinite set of n-ary predicative constants $P_1^n, P_2^n, \ldots$, for every $n \in \mathbb{N}$;
- propositional connectives $\neg$ and $\rightarrow$;
- universal quantifier $\forall$;
- modal operator $\Box$.

Alphabet $\mathcal{A}^E$ includes all the logical and descriptive symbols in $\mathcal{A}$ with in addition the unary predicative constant E. Symbols for constants and functors appear neither in $\mathcal{A}$ nor in $\mathcal{A}^E$, thus the only terms in our alphabets are individual variables. Then we define set $For_{\mathcal{A}}$ of modal formulas on $\mathcal{A}$.

Definition 1.1 (Modal formulas in $For_{\mathcal{A}}$) *Modal formulas $\phi_1, \phi_2, \ldots$ on alphabet $\mathcal{A}$ are inductively defined as follows:*

- *if P^n is an n-ary predicative constant and $\langle x_1, \ldots, x_n \rangle$ is an ordered n-tuple of individual variables, then $P^n(x_1, \ldots, x_n)$ is a (atomic) modal formula;*
- *if ϕ, ψ are modal formulas, then $\neg\phi$, $\phi \rightarrow \psi$ and $\Box\phi$ are modal formulas;*
- *if ϕ is a modal formula and x is a variable, then $\forall x\phi$ is a modal formula;*
- *nothing else is a modal formula.*

Set $For_{\mathcal{A}^E}$ of modal formulas on alphabet $\mathcal{A}^E$ is inductively defined in the same way. First-order modal language $\mathcal{L}$ consists in alphabet $\mathcal{A}$ and set $For_{\mathcal{A}}$; we similarly define language $\mathcal{L}^E$.

Notational conventions:

1. Symbol for falsehood $\bot$, propositional connectives $\wedge, \vee, \leftrightarrow$, existential quantifier $\exists$ and modal operator $\diamond$ are defined in the usual way, by means of the other logical constants.

2. By $\phi[x_1, \ldots, x_n]$ we mean that the free variables in formula ϕ are among $x_1, \ldots, x_n$; whereas by $\phi[x/y]$ we denote the formula obtained by substituting in ϕ free occurrences of x with y, renaming bounded variables if necessary.

3. For referring to the sets of individual variables in $\mathcal{L}$ and $\mathcal{L}^E$, we write $Var(\mathcal{L})$ and $Var(\mathcal{L}^E)$. Metalinguistic variables $\mathcal{L}_0, \mathcal{L}_1, \ldots$ vary over languages.

4. By $\mathcal{L}_0^+$ we denote the language obtained by adding to $\mathcal{L}_0$ an infinite denumerable set of new individual variables. Moreover we define $\mathcal{L}_0$ an *infinitely proper sublanguage* of $\mathcal{L}_1$ - $\mathcal{L}_0 \subset_\infty \mathcal{L}_1$ in short - iff the latter contains an infinite denumerable set of variables that do not appear in $\mathcal{L}_0$.

5. If Y is any set of variables, $\mathcal{L}_0^Y$ is the expansion of $\mathcal{L}_0$ obtained by adding Y.

At this point we have the formal machinery to introduce our QML calculi of quantified modal logic.

1.1.2 QML Calculi

Hereafter we list the schemes of axioms and inference rules, by means of which we define our calculi; we start with the only four postulates appearing in every QML system:

A1.	tautologies of classical propositional calculus,	
A2.	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$	distribution axiom,
R1.	$\frac{\phi \to \psi, \phi}{\psi}$	separation rule,
R2.	$\frac{\phi}{\Box\phi}$	necessitation.

These four postulates give QML calculi with modality K, that is the one we analyse for the moment. For obtaining a different normal modal base - T, $S4$, B or $S5$ - we have to use an appropriate combination of the following schemes of axioms:

A3.	$\Box\phi \to \phi$	axiom T,
A4.	$\Box\phi \to \Box\Box\phi$	axiom 4,
A5.	$\phi \to \Box \diamond \phi$	axiom B.

In order to add quantification we have three distinct choices, which correspond to the tripartition among possibilist quantification, actualist quantification with the existential predicate and actualist quantification *à la* Kripke. The first group of postulates consists in the classical theory of quantification:

A6.	$\forall x\phi \to \phi[x/y]$	exemplification,
R3.	$\frac{\phi \to \psi}{\phi \to \forall x\psi}$, where x is not free in ϕ	universal instantiation.

Whenever our language contains existence predicate E, we can consider a second set of postulates:

A7.	$\forall x\phi \rightarrow (E(y) \rightarrow \phi[x/y])$	E-exemplification,
R4.	$\frac{\phi\rightarrow(E(x)\rightarrow\psi)}{\phi\rightarrow\forall x\psi}$, where x is not free in ϕ	universal E-instantiation.

Finally, Kripke's theory of quantification consists in the following principles:

A8.	$\phi \rightarrow \forall x\phi$, where x is not free in ϕ	vacuous quantification,
A9.	$\forall x(\phi \rightarrow \psi) \rightarrow (\forall x\phi \rightarrow \forall x\psi)$	universal distribution,
A10.	$\forall x\forall y\phi \leftrightarrow \forall y\forall x\phi$	permutation,
A11.	$\forall y(\forall x\phi \rightarrow \phi[x/y])$	$\forall$-exemplification,
R5.	$\frac{\phi}{\forall x\phi}$	generalization.

For regulating the relationship between quantification and modality, we refer to the following well-known schemes of axioms:

A12.	$\forall x\Box\phi \rightarrow \Box\forall x\phi$	Barcan formula,
A13.	$\Box\forall x\phi \rightarrow \forall x\Box\phi$	converse of the Barcan formula.

Remarks:

1. Kripke's postulates for quantification are taken from [20]. In [28] and [47] there appears a slightly different version of A8:

   A8'. $\phi \leftrightarrow \forall x\phi$, where x is not free in ϕ.

   We will see later that assuming $\forall x\phi \rightarrow \phi$, where x is not free in ϕ, tantamounts to ruling out empty quantification domains in Kripke frames for QML calculi.

   In [56] Kripke considered A8 and did not list axiom A10; for a long time it has been uncertain whether A10 could be proved by means of A8, A9, A11, R5 or was independent. This question was settled by K. Fine in [26], where he demonstrated the independence of A10 from the other principles of Kripke's theory of quantification.

2. Once introduced existence predicate E, we have to modify the universal instantiation rule, even if it is valid in varying domain settings. In fact in these models formula $\forall x E(x)$ holds - expressing the trivial truth that all the existing individuals exist - nonetheless it is not possible to prove it, unless we assume the form of universal instantiation codified in R4.

Now we define our QML calculi of quantified modal logic, the names of the systems are the same as in [20].

Definition 1.2 (*QML* **Calculi**) *The following QML calculi consist in schemes of axioms A1, A2 and inference rules R1, R2, with the respective postulates:*

calculi	*schemes of axioms*	*inference rules*
$Q.K$	*A6,*	*R3*
$Q.K + BF$	*A6, A12*	*R3*
$Q^{\circ}.K$	*A8, A9, A10, A11*	*R5*
$Q^{\circ}.K + BF$	*A8, A9, A10, A11, A12*	*R5*
$Q^{\circ}.K + CBF$	*A8, A9, A10, A11, A13*	*R5*
$Q^{\circ}.K + CBF + BF$	*A8, A9, A10, A11, A12, A13*	*R5*
$Q^{E}.K$	*A7*	*R4*
$Q^{E}.K + BF$	*A7, A12*	*R4*
$Q^{E}.K + CBF$	*A7, A13*	*R4*
$Q^{E}.K + CBF + BF$	*A7, A12, A13*	*R4*

We stress the fact that the first six calculi are on language $\mathcal{L}$, whereas the last four ones are on $\mathcal{L}^E$. Hereafter by L we refer to a generic QML calculus.

Proofs, theorems and derivability:

- A *proof* in a system L on language $\mathcal{L}_0$ is a finite sequence of formulas in $\mathcal{L}_0$, s.t. each of them is either an axiom of L or it is obtained by previous formulas in the sequence by means of an application of an inference rule.
- A formula $\phi \in \mathcal{L}_0$ is a *theorem* of L - $\vdash_L \phi$ in short - iff ϕ is the last formula in a proof in L.
- A formula $\phi \in \mathcal{L}_0$ is *derivable* in L from set Δ of formulas in $\mathcal{L}_0$ - $\Delta \vdash_L \phi$ in short - iff there are $\phi_1, \ldots, \phi_n \in \Delta$ s.t. $\vdash_L \phi_1 \wedge \ldots \wedge \phi_n \rightarrow \phi$.

We write $L \subseteq L'$ to express the fact that all the theorems of L are theorems of L'; we summarize the inclusion relationships of the ten systems above in the two following tables.

$$\begin{array}{ccccc} Q^{\circ}.K & \subseteq & Q^{\circ}.K + CBF & \subseteq & Q.K \\ \subseteq & & \subseteq & & \subseteq \\ Q^{\circ}.K + BF & \subseteq & Q^{\circ}.K + CBF + BF & \subseteq & Q.K + BF \end{array}$$

$$\begin{array}{ccc} Q^{E}.K & \subseteq & Q^{E}.K + CBF \\ \subseteq & & \subseteq \\ Q^{E}.K + BF & \subseteq & Q^{E}.K + CBF + BF \end{array}$$

It is important to remark that postulates A8, A9, A10, A11 and R5 of Kripke's theory of quantification are all provable in $L \supseteq Q^{E}.K$. Thus if ϕ is a theorem in $L \supseteq Q^{\circ}.K$, then it can be proved in the corresponding $L' \supseteq Q^{E}.K$. We shall make use of this fact in the completeness proofs for systems based on free logic. In particular the following formulas are theorems and derived rules in any QML calculus.

T1. $\forall z_1, \ldots, z_k, \forall v_{j_1}, \ldots, v_{j_m}, \forall y_1, \ldots, y_n(\forall x_1\phi_1 \to \phi_1[v_1, \ldots, v_h, x_1/y_1] \wedge \ldots \wedge \forall x_n\phi_n \to \phi_n[v_1, \ldots, v_h, x_n/y_n])$, where $z_1, \ldots, z_k$ do not appear in any $\phi_i[v_1, \ldots, v_h, x_i/y_i]$ and $\{v_{j_1}, \ldots, v_{j_m}\} \subseteq \{v_1, \ldots, v_h\}$.

T2. if $\vdash_L \phi_1 \wedge \ldots \wedge \phi_n \to \psi$, then $\vdash_L \forall x_1, \ldots, x_k\phi_1 \wedge \ldots \wedge \forall x_1, \ldots, x_k\phi_n \to \forall x_1, \ldots, x_k\psi$.

T3. if $\vdash_L \phi_1 \wedge \ldots \wedge \phi_n \to \psi$, then $\vdash_L \Box\phi_1 \wedge \ldots \wedge \Box\phi_n \to \Box\psi$.

T4. $\forall x\phi \leftrightarrow \forall y\phi[x/y]$, where y does not occur in ϕ.

T5. $\forall x\phi \wedge \exists y\psi \to \exists y(\phi[x/y] \wedge \psi)$, where y does not occur in ϕ.

Names of calculi:

The various systems of quantified modal logic appear with a number of different names in the literature. In [32] calculi $Q.K + BF$, $Q^E.K$ and $Q^\circ.K$ are considered among others, respectively named $Q1$, $Q1R$ and QK. In [28] there appear systems $Q^\circ.K$ and $Q^\circ.K + CBF + BF$, slightly modified w.r.t. A8 as remarked above; whereas in [47] Kripke-completeness is proved for calculi $Q.K + BF$, $Q.K$, $Q^E.K$ and $Q^\circ.K$, which are respectively called $K + BF$, $LPC + K$, $LPCE$ and $LPCK$.

Let L be a QML calculus on language $\mathcal{L}_0$, L^Y is the same calculus on expansion $\mathcal{L}_0^Y$ of $\mathcal{L}_0$. As remarked above, for the time being we consider only calculi with modal base K.

1.1.3 Kripke Semantics for $\mathcal{L}$ and $\mathcal{L}^E$

Once introduced languages $\mathcal{L}$ and $\mathcal{L}^E$, we have to assign a meaning to modal formulas, so that it is possible to verify which is the actual informative content of theorems in QML calculi. We accomplish this task by means of Kripke frames, widely used in the analysis of modality, as modified in [20].

Definition 1.3 (Kripke frame) *A Kripke frame $\mathcal{F}$ - K-frame in short - is an ordered 4-tuple $\langle W, R, D, d\rangle$ defined as follows:*

- *W is a non-empty set;*
- *R is a binary relation on W;*
- *D is a function assigning to every $w \in W$ a non-empty set $D(w)$ s.t. if wRw' then $D(w) \subseteq D(w')$;*
- *d is a function assigning to every $w \in W$ a set $d(w) \subseteq D(w)$.*

Set W is intuitively interpreted as the domain of possible worlds, whereas R is the accessibility relation among worlds. Each outer domain $D(w)$ contains the objects which it makes sense to talk about in w, on the other hand in each inner domain $d(w)$ there appear individuals actually existing in w.

A K-frame $\mathcal{F}$ has *varying* inner domains when no condition is imposed on it; otherwise $\mathcal{F}$ has *constant* (*increasing*, *decreasing*) inner domains iff for all w, w' in W, wRw' implies $d(w) = d(w')$ ($d(w) \subseteq d(w')$, $d(w) \supseteq d(w')$). Hence we define the notion of interpretation of a language in a K-frame.

Definition 1.4 (Interpretation) *An interpretation I of language $\mathcal{L}_0$ in a K-frame $\mathcal{F}$ is a function from $\mathcal{L}_0$ to $\mathcal{F}$ s.t.*

- *if P^n is an n-ary predicative constant and $w \in W$, then $I(P^n, w)$ is an n-ary relation on $D(w)$;*
- *if our language includes predicative constant E, then $I(E, w) = d(w)$.*

Notice that interpretation I is so defined that in a world w, we can assign to an n-ary predicative constant P^n an ordered n-tuple of elements in $D(w)$, some of which may not belong to $d(w)$. This feature of Kripke semantics is utterly unsatisfactory from an actualist point of view.

Definition 1.5 (Model) *A Kripke model $\mathcal{M}$ - K-model in short - of language $\mathcal{L}_0$, based on K-frame $\mathcal{F}$, is an ordered couple $\langle \mathcal{F}, I \rangle$ s.t. I is an interpretation of $\mathcal{L}_0$ in $\mathcal{F}$.*

To define truth conditions for formulas in $\mathcal{L}_0$, we need the notion of w-*assignment* - i.e. any function from $Var(\mathcal{L}_0)$ to outer domain $D(w)$ - to deal with atomic and quantified formulas.

Since the only terms in our languages are variables, the valuation I^σ of terms, induced by w-assignment σ into interpretation I in world w, is so defined that for every variable x, $I^\sigma(x, w) = \sigma(x)$. Then we consider the variant of a w-assignment.

Definition 1.6 *For $x \in Var(\mathcal{L}_0)$ and $a \in D(w)$, variant $\sigma\binom{x}{a}$ of w-assignment σ is the w-assignment s.t. (i) it does not coincide with σ at most on x and (ii) it assigns element a to x.*

Now we introduce truth conditions for a formula in a world w.r.t. a valuation; in this way we fix the meaning of all the formulas in formal languages $\mathcal{L}$ and $\mathcal{L}^E$.

Definition 1.7 (Satisfaction) *Let I^σ be the valuation induced by w-assignment σ into interpretation I, let w be a world in $W \in \mathcal{F}$. The relation of satisfaction in w for formula $\phi \in \mathcal{L}_0$ w.r.t. I^σ is inductively defined as follows:*

$$
\begin{aligned}
(I^\sigma, w) \models P^n(x_1, \ldots, x_n) \quad & iff \quad \langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in I(P^n, w) \\
(I^\sigma, w) \models \neg\psi \quad & iff \quad not\ (I^\sigma, w) \models \psi \\
(I^\sigma, w) \models \phi \to \psi \quad & iff \quad not\ (I^\sigma, w) \models \phi\ or\ (I^\sigma, w) \models \psi \\
(I^\sigma, w) \models \Box\phi \quad & iff \quad for\ every\ w' \in W,\ wRw'\ implies\ (I^\sigma, w') \models \phi \\
(I^\sigma, w) \models \forall x\phi \quad & iff \quad for\ every\ a \in d(w),\ (I^{\sigma\binom{x}{a}}, w) \models \phi.
\end{aligned}
$$

Truth conditions for modal formulas containing propositional connectives $\wedge$, $\vee$, $\leftrightarrow$, existential quantifier $\exists$ and modal operator $\diamond$ are defined from the ones above in the usual way. In particular $(I^\sigma, w) \models \bot$ never holds. Notice that the increasing outer domain condition on K-frames guarantees that it is always possible to evaluate modalized formulas, in fact if wRw' and σ is a w-assignment then it is also a w'-assignment.

Finally we define the notions of truth and validity.

Definition 1.8 (Truth and validity) *Formula ϕ in $\mathcal{L}_0$ is*

true in world w	*iff*	*it is satisfied by every w-assignment σ*
true in K-model $\mathcal{M}$	*iff*	*it is true in every world in $\mathcal{M}$*
valid in K-frame $\mathcal{F}$	*iff*	*it is true in every K-model based on $\mathcal{F}$*
valid in class $\mathcal{C}$ of K-frames	*iff*	*it is valid in every K-frame belonging to $\mathcal{C}$*

Let Δ be a set of formulas in $\mathcal{L}_0$, $\mathcal{M}$ is a K-model for Δ iff $\mathcal{M}$ verifies every formula in Δ; furthermore $\mathcal{F}$ is a K-frame for Δ iff every K-model based on $\mathcal{F}$ is a K-model for Δ. It is a routine proof to check that $\mathcal{F}$ is a K-frame for a calculus in the first column of the following table - conceived as the set of its theorems - iff it satisfies the conditions on inner and outer domains in the second and third column.

calculi	inner domain	outer domain
$Q.K$	increasing	$=$ inner
$Q.K + BF$	constant	$=$ inner
$Q^\circ.K$	varying	increasing
$Q^\circ.K + BF$	decreasing	increasing
$Q^\circ.K + CBF$	increasing	increasing
$Q^\circ.K + CBF + BF$	constant	increasing
$Q^E.K$	varying	increasing
$Q^E.K + CBF$	increasing	increasing

We conclude this paragraph by stating a well-known lemma, which is needed to prove Kripke-completeness for QML calculi.

Lemma 1.9 (Conversion lemma) *Let I be an interpretation of language $\mathcal{L}_0$, σ a w-assignment, $\phi \in \mathcal{L}_0$:*

$$(I^\sigma, w) \models \phi[x/y] \quad iff \quad (I^{\sigma\binom{x}{\sigma(y)}}, w) \models \phi$$

For a proof of this lemma we refer to [14], pp. 137-138. For the time being we just remark that this lemma relates the syntactic operation of substitution with the semantic notion of variant of an assignment.

1.2 Adequacy Results in Kripke Semantics

In this section we prove adequacy results for QML calculi in par. 1.1.2 w.r.t. specific classes of Kripke frames. This means that each set of theorems correctly describes the corresponding structures, and this description is complete.

All these results but one appear in the literature on quantified modal logic, though scattered in a number of references. In [20] a general approach to completeness proofs for QML calculi is developed, even if systems based on free logic are not considered. Our aim is to review completeness theorems for major systems, based both on Kripke's theory of quantification and on classical and free logic, while pointing out the common features. Moreover we provide an original proof for calculus $Q^{\circ}.K{+}BF$ and state Kripke-incompleteness for $Q^{E}.K{+}BF$ and $Q^{E}.K{+}CBF{+}BF$.

We start with listing the adequacy results to be proved.

Theorem 1.10 (Adequacy) *The following QML calculi are adequate w.r.t. the respective classes of K-frames:*

calculi	*inner domain*	*outer domain*
$Q.K$	*increasing*	$=$ *inner*
$Q.K + BF$	*constant*	$=$ *inner*
$Q^{\circ}.K$	*varying*	*constant*
$Q^{\circ}.K + BF$	*decreasing*	*constant*
$Q^{\circ}.K + CBF$	*increasing*	*constant*
$Q^{\circ}.K + CBF + BF$	*constant*	*constant*
$Q^{E}.K$	*varying*	*constant*
$Q^{E}.K + CBF$	*increasing*	*constant*

The first three results are proved in [20], as well as the fifth and the sixth one; the last two ones can be obtained, with some minor changes, from analogous theorems in [32] and [47]. The proof of the fourth result is original.

It can seem surprising, but calculi $Q^{E}.K + BF$ and $Q^{E}.K + CBF + BF$ turn out to be incomplete w.r.t. Kripke semantics. In fact K-models for $Q^{E}.K + BF$ and $Q^{E}.K + CBF + BF$ validate

A14. $\neg E(x) \to \Box\neg E(x)$ necessity of non-existence ($N\neg E$)

but A14 is a theorem neither in $Q^{E}.K + BF$ nor in $Q^{E}.K + CBF + BF$. In order to prove this fact we need counterpart semantics, which is introduced only in chapter 2, thus we postpone the incompleteness proof to appendix A. For the time being compare the independence of A14 from the other postulates in $Q^{E}.K + BF$, $Q^{E}.K + CBF + BF$ with provability in $Q^{E}.K + CBF$ of

A15. $E(x) \to \Box E(x)$ necessity of existence

This fact determines Kripke-completeness for the latter calculus. All these considerations reveal an asymmetry between BF and CBF: as Corsi points out in [21] the duality between the Barcan formula and its converse, upheld in [28] for instance, is not intrinsic to these formulas but is a by-product of the strong assumptions on individuals underlying Kripke semantics.

As regards the other systems, notice that the following soundness results hold.

Theorem 1.11 (Soundness) *The following QML calculi are sound w.r.t. the respective classes of K-frames:*

calculi	*inner domain*	*outer domain*
$Q.K$	*increasing*	$=$ *inner*
$Q.K + BF$	*constant*	$=$ *inner*
$Q^{\circ}.K$	*varying*	*increasing*
$Q^{\circ}.K + BF$	*decreasing*	*increasing*
$Q^{\circ}.K + CBF$	*increasing*	*increasing*
$Q^{\circ}.K + CBF + BF$	*constant*	*increasing*
$Q^{E}.K$	*varying*	*increasing*
$Q^{E}.K + CBF$	*increasing*	*increasing*

We prove the reverse implication by using the *canonical model method*, which we discuss in the following paragraph, then we show that K-frames with constant outer domains suffice.

1.2.1 The canonical model method

For proving Kripke-completeness of QML calculi, we make use of the canonical model method - as it happens in the propositional case - which basically consists in the following theorem.

Theorem 1.12 *If a QML calculus L does not prove formula $\phi \in \mathcal{L}_0$, then there exists a K-model $\mathcal{M}^L$ for L - i.e. the* canonical model *w.r.t L - s.t. $\mathcal{M}^L$ does not verify ϕ.*

This theorem is proved by means of two lemmas - the *saturation lemma* and the *truth lemma* - then we have to check that the canonical model w.r.t. L is actually based on a K-model for L. In order to state these partial results we need some definitions: let Λ be a set of formulas in language $\mathcal{L}_0$,

Λ is L-*consistent*	iff	$\Lambda \nvdash_L \bot$;
Λ is L-*complete*	iff	for every formula $\phi \in \mathcal{L}_0$, $\phi \in \Lambda$ or $\neg\phi \in \Lambda$;
Λ is L-*maximal*	iff	Λ is L-consistent and L-complete.

Furthermore let Y be a set of variables in $\mathcal{L}_0$,

Λ is *Y-rich*	iff	for $\phi \in \mathcal{L}_0$, if $\exists x\phi \in \Lambda$ then there is $y \in Y$ s.t. $\phi[x/y] \in \Lambda$;
Λ is *Y-universal*	iff	for $\phi \in \mathcal{L}_0$, if $\forall x\phi \in \Lambda$ then for every $y \in Y$, $\phi[x/y] \in \Lambda$;
Λ is *L-saturated*	iff	Λ is L-maximal and Y-rich, Y-universal for some $Y \subseteq Var(\mathcal{L}_0)$.

Since A8' is not a theorem in our QML calculi, a set Λ of formulas can be L-consistent and yet $\Lambda \vdash_L \forall z_1, \ldots, z_h \bot$ for some $h \geq 1$. Finally we adopt the following notational convention: let Y be a set of variables in language $\mathcal{L}_0$,

$$\mathcal{E}(Y) = \{\forall x\phi \to \phi[x/y] | y \in Y \textit{ and } \phi \in \mathcal{L}_0\}$$

For $y \in Var(\mathcal{L}_0)$, we write $\mathcal{E}(y)$ instead of $\mathcal{E}(\{y\})$.

With these definitions we state a comprehensive result, which enables us to prove both the saturation lemma and the truth lemma for a wide range of calculi.

Saturation of theories

The following lemma is intended to provide the kind of generality Corsi was looking for in the introduction to [20].

Lemma 1.13 (modal Henkin) *Let Δ be an L-consistent set of formulas in $\mathcal{L}_0$, let Y be a denumerable set of individual variables and $Y_1, Y_2 \subseteq Y$. Assume that there are enumerations of existential formulas in $\mathcal{L}_0^Y$ and of Y_2, then define by recursion a chain of sets of formulas in $\mathcal{L}_0^Y$ s.t.*

$$\Gamma_0 = \Delta \cup \mathcal{E}(Y_1)$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\theta_n[x/y_n]\} \cup \mathcal{E}(y_n) & \textit{if } \Gamma_n \cup \{\exists x\theta_n\} \textit{ is } L^Y\textit{-consistent,} \\ & \textit{and } y_n \textit{ is the first variable in } Y_2 \textit{ s.t.} \\ & \Gamma_n \cup \{\theta_n[x/y_n]\} \cup \mathcal{E}(y_n) \textit{ is } L^Y\textit{-consistent;} \\ \Gamma_n & \textit{otherwise.} \end{cases}$$

Suppose that (i) Γ_0 is L^Y-consistent and (ii) for every $n \in \mathbb{N}$, if $\Gamma_n \cup \{\exists x\theta_n\}$ is L^Y-consistent, then there exists $y \in Y_2$ s.t. $\Gamma_n \cup \{\theta_n[x/y]\} \cup \mathcal{E}(y)$ is L^Y-consistent. Therefore $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ is an L^Y-consistent set of formulas in $\mathcal{L}_0^Y$ s.t. it extends Δ and is Z-rich and Z-universal, for some $Y_1 \subseteq Z \subseteq Y_1 \cup Y_2$.

Proof. Whenever conditions (i) and (ii) are satisfied, it is possible to perform the construction above and each Γ_n is L^Y-consistent by construction. By the chain lemma Γ is L^Y-consistent, moreover Γ is Z-universal for some $Y_1 \subseteq Z \subseteq Y_1 \cup Y_2$, and Z'-rich for some $Z' \subseteq Z$. Therefore Γ is Z-rich as well.

This lemma demands some comment for the extremely general way it is stated. We do not require any constraint on set Y of variables, as we aim at encompassing all the constructions *à la* Henkin needed to prove our completeness results. In order to show that every L-consistent set of formulas can be extended to an L^Y-saturated set, we have to consider as Y a new infinite set of individual variables. On the other hand for proving completeness for system $Q^\circ.K + BF$, we have to set $Y = d(w)$, $Y_1 = \emptyset$ and $Y_2 = d(w)$; as regards $Q^\circ.K + CBF$, $Y_1 = d(w)$ while Y_2 is an infinite set of new variables. Finally when we consider system $Q^{\circ}.K+CBF+BF$, we have to chose $Y=Y_1=Y_2=d(w)$. All these different cases are summarized in lemma 1.13.

By means of lemma 1.13 we even prove the saturation lemma for any QML calculus L, which guarantees that the canonical model w.r.t. L actually exists.

Lemma 1.14 (Saturation) *If Δ is an L-consistent set of formulas in $\mathcal{L}_0$, then there exists an L^Y-saturated set $\Pi \supseteq \Delta$ of formulas in $\mathcal{L}_0^Y$, for Y infinite denumerable set of new individual variables.*

Proof. We consider two different cases:

(a) $\Delta \vdash_{L^Y} \forall z_1, \dots, z_h \bot$ for some $h \geq 1$.

Let Π be an L^Y-maximal extension of Δ in $\mathcal{L}_0^Y$. By induction on h we show that if $\exists x\phi \in \Pi$ then $\bot \in \Pi$, therefore no existential formula is in Π. By hypotheses

$$\Pi \vdash_{L^Y} \exists x\phi \wedge \forall z_1, \dots, z_h \bot$$

and by T5

$$\Pi \vdash_{L^Y} \exists x(\phi \wedge \forall z_2, \dots, z_h \bot)$$

By predicate calculus

$$\Pi \vdash_{L^Y} \exists x \forall z_2, \dots, z_h \bot$$

and by A8

$$\Pi \vdash_{L^Y} \forall z_2, \dots, z_h \bot$$

By applying the induction hypothesis, we obtain that $\Pi \vdash_{L^Y} \bot$ against L^Y-consistency of Π. For $Z=\emptyset$, Π is Z-rich and Z-universal, i.e. is a L^Y-saturated set including Δ.

(b) $\Delta \nvdash_{L^Y} \forall z_1, \dots z_h \bot$ for any $h \geq 1$.

We apply lemma 1.13 for $Y_1, Y_2 \subseteq Y$ s.t. $Y_1=\emptyset$ and $Y_2=Y$, we only have to prove that hypotheses (i) and (ii) hold. First of all we notice that $\Gamma_0=\Delta$ is L^Y-consistent.

As regards (ii) we show that if $\Gamma_n \cup \{\exists x\theta_n\}$ is L^Y-consistent, then it is always possible to find $y \in Y$ s.t. $\Gamma_n \cup \{\theta_n[x/y]\} \cup \mathcal{E}(y)$ is L^Y-consistent as well: just consider the first $y_n \in Y$ that appears neither in $\Gamma_n \setminus \mathcal{E}(\{y_0, \dots, y_{n-1}\})$ nor in θ_n. Since Y is infinite, it is easy to check that such an y_n exists.

Suppose for reduction that the so-defined Γ_{n+1} is not L^Y-consistent, this means that there exists $\phi_1, \dots, \phi_m \in \Gamma_n \setminus \mathcal{E}(\{y_0, \dots, y_{n-1}\})$, $\chi_1, \dots, \chi_k \in \mathcal{L}_0^Y$ s.t.

1. $\forall z_i\chi_i \rightarrow \chi_i[z_i/y_{j_i}] \in \mathcal{E}(\{y_0, \dots, y_n\})$ for $1 \leq i \leq k$;
2. $\vdash_{L^Y} \bigwedge(\forall z_i\chi_i \rightarrow \chi_i[z_i/y_{j_i}]) \rightarrow (\bigwedge \phi_l \rightarrow \neg\theta_n[x/y_n])$.

From 2 we deduce by T2

$$\vdash_{L^Y} \forall y_{j_1}, \ldots, y_{j_k}, y_n \bigwedge(\forall z_i \chi_i \to \chi_i[z_i/y_{j_i}]) \to \forall y_{j_1}, \ldots, y_{j_k}, y_n(\bigwedge \phi_l \to \neg\theta_n[x/y_n])$$

in L^Y we prove T1, thus by R1

$$\vdash_{L^Y} \forall y_{j_1}, \ldots, y_{j_k}, y_n(\bigwedge \phi_l \to \neg\theta_n[x/y_n])$$

Since y_n does not appear in $\bigwedge \phi_l$, by T2 and A8

$$\vdash_{L^Y} \forall y_{j_1}, \ldots, y_{j_k}(\bigwedge \phi_l \to \forall y_n \neg\theta_n[x/y_n])$$

where $\{y_{j_1}, \ldots, y_{j_k}\} \subseteq \{y_0, \ldots, y_{n-1}\}$, therefore $\forall y_{j_1}, \ldots, y_{j_k}(\bigwedge \phi_l \to \forall y_n \neg\theta_n[x/y_n]) \to (\bigwedge \phi_l \to \forall y_n \neg\theta_n[x/y_n]) \in \mathcal{E}(\{y_0, \ldots, y_{n-1}\})$ and

$$\mathcal{E}(\{y_0, \ldots, y_{n-1}\}) \vdash_{L^Y} \bigwedge \phi_l \to \forall y_n \neg\theta_n[x/y_n]$$

but $\bigwedge \phi_l \in \Gamma_n \setminus \mathcal{E}(\{y_0, \ldots, y_{n-1}\})$, then

$$\Gamma_n \vdash_{L^Y} \forall y_n \neg\theta_n[x/y_n]$$

Since y_n does not appear in θ_n, by T4 this contradicts the L^Y-consistency of $\Gamma_n \cup \{\exists x \theta_n\}$.

By lemma 1.13, $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ is an L^Y-consistent set of formulas in $\mathcal{L}_0^Y$ s.t. it extends Δ and is Z-rich and Z-universal for some $Z \subseteq Y$. By Lindenbaum's lemma Γ can be extended to an L^Y-saturated set $\Pi \supseteq \Delta$ in $\mathcal{L}_0^Y$.

Before introducing the canonical model w.r.t. L, notice that the previous lemma corresponds to lemma 1.16 in [20].

Canonical model

In this paragraph we define the canonical model w.r.t. a QML calculus L, then prove that it satisfies the conditions on K-models stated in par. 1.1.3.

Definition 1.15 (Canonical frame) *The canonical frame $\mathcal{F}^L$ for calculus L on language $\mathcal{L}_0$ with an expansion $\mathcal{L}_0^+$ is an ordered 4-tuple $\langle W^L, R^L, D^L, d^L \rangle$, defined as follows:*

- *W^L is the class of L_w-saturated sets w of formulas in $\mathcal{L}_w$, for $\mathcal{L}_0 \subset_\infty \mathcal{L}_w \subset_\infty \mathcal{L}_0^+$;*
- *R^L is the relation on W^L s.t. wR^Lw' iff $\{\phi | \Box\phi \in w\} \subseteq w'$;*
- *for $w \in W^L$, $D^L(w)$ is the set of variables in $\mathcal{L}_w$;*
- *for $w \in W^L$, $d^L(w)$ is the set of $y \in \mathcal{L}_w$ s.t. for every $\phi \in \mathcal{L}_w$, $\forall x\phi \to \phi[x/y] \in w$.*

We prove that the so defined canonical frame actually exists and satisfies conditions on K-frames.

First of all we assumed that $\nvdash_L \phi$, thus set $\{\neg\phi\}$ is L-consistent and by lemma 1.14 can be extended to an L_w-saturated set of formulas on some language $\mathcal{L}_w \supset_\infty \mathcal{L}_0$, infinitely proper sublanguage of $\mathcal{L}_0^+$. Therefore $W^L \neq \emptyset$.

For every $w \in W^L$, $Var(\mathcal{L}_w) \neq \emptyset$ by definition of $w \in W^L$, thus $D^L(w) \neq \emptyset$. Moreover the definition of canonical frame satisfies the increasing outer domain condition: if wR^Lw' and $\phi[x_1, \ldots, x_n] \in w$ is a tautology containing free variables $x_1, \ldots, x_n \in \mathcal{L}_w$, then also $\Box\phi[x_1, \ldots, x_n] \in w$ and thus $\phi[x_1, \ldots, x_n] \in w'$; it follows that $Var(\mathcal{L}_w) \subseteq Var(\mathcal{L}_{w'})$. For every $w \in W^L$, inner domain $d^L(w)$ is a subset of outer domain $D^L(w)$ by definition.

Finally, every $w \in W^L$ is $d^L(w)$-universal and $d^L(w)$-rich: by definition w is $d^L(w)$-universal and $d^L(w)$ is the greatest set w.r.t. which w is universal. Therefore w is Z-universal and Z-rich for some $Z \subseteq d^L(w)$, in particular w is $d^L(w)$-rich.

We conclude that $\mathcal{F}^L$ is a K-frame, but it is still left to prove that it is based on a K-frame for L, for each QML calculus L. The proof of canonicity for QML calculi is postponed to par. 1.2.2. We go on with the notions of canonical interpretation and assignment.

Definition 1.16 (Canonical interpretation) *The canonical interpretation I^L of language $\mathcal{L}_0^+$ into canonical frame $\mathcal{F}^L$ is the function s.t.*

- *for $x_1, \ldots, x_n \in D^L(w)$, $\langle x_1, \ldots, x_n \rangle \in I^L(P^n, w)$ iff $P^n(x_1, \ldots, x_n) \in w$;*
- *if $\mathcal{L}_0$ includes predicative constant E, then $I^L(E, w) = d^L(w)$.*

We easily check that canonical interpretation I^L, as defined above, satisfies the constraints on interpretations in def. 1.4. The ordered couple $\langle \mathcal{F}^L, I^L \rangle$ constitutes the canonical model $\mathcal{M}^L$ w.r.t calculus L.

Definition 1.17 (Canonical w-assignment) *The canonical w-assignment σ^L is the identity function from $Var(\mathcal{L}_w)$ to outer domain $D^L(w)$, that is, $\sigma^L(x) = x$ for every $x \in Var(\mathcal{L}_w)$.*

Even the canonical w-assignment is actually a w-assignment for each $w \in W$. At this point it is left to check that the canonical model w.r.t. L is actually based on a K-frame for L. This result is fundamental in the completeness proof and not so trivial as it may appear at first sight, as there exist many non-canonical modal calculi. Since the proof varies according to the envisaged QML calculus, we defer it to the discussion of single systems in par. 1.2.2.

In order to simplify our notation, in the next paragraphs we write I, $\mathcal{M}$ and σ instead of I^L, $\mathcal{M}^L$, σ^L.

Truth lemma

Up to now we proved that if formula $\phi \in \mathcal{L}_0$ is not provable in L, then the L_w-saturated extension w of L-consistent set $\{\neg\phi\}$, obtained by lemma 1.14, is a world in the canonical model w.r.t. L. Of course $\phi \notin w$. In order to prove completeness we would like to show that if $\phi \notin w$, then ϕ is false in w. This is exactly the content of the next lemma - the *truth lemma* - that links the notions of membership to a set and satisfaction in a world.

Lemma 1.18 (Truth lemma) *For every $w \in W$, for every $\phi \in \mathcal{L}_w$,*

$$(I^\sigma, w) \models \phi \quad iff \quad \phi \in w$$

Proof. The proof is by induction on the length of $\phi \in \mathcal{L}_w$, we have to pay attention to the different languages on which the worlds in our canonical model are constructed.

As to the base of induction, consider atomic formula $P^n(x_1, \ldots, x_n)$. By definition of satisfaction $(I^\sigma, w) \models P^n(x_1, \ldots, x_n)$ iff $\langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in I(P^n, w)$ iff $\langle x_1, \ldots, x_n \rangle \in I(P^n, w)$. According to the definition of canonical interpretation, $\langle x_1, \ldots, x_n \rangle \in I(P^n, w)$ iff $P^n(x_1, \ldots, x_n) \in w$.

As to the inductive step, we separately consider each connective, the universal quantifier and the box operator.

If ϕ has form $\neg\psi$, then notice that $\phi \in \mathcal{L}_w$ iff $\psi \in \mathcal{L}_w$. Thus $(I^\sigma, w) \models \neg\psi$ iff not $(I^\sigma, w) \models \psi$ iff by induction hypothesis $\psi \notin w$. Since w is L_w-maximal, this is the case iff $\neg\psi \in w$.

If ϕ has form $\psi \to \psi'$, then $\phi \in \mathcal{L}_w$ iff $\psi, \psi' \in \mathcal{L}_w$. Moreover $(I^\sigma, w) \models \psi \to \psi'$ iff not $(I^\sigma, w) \models \psi$ or $(I^\sigma, w) \models \psi'$. By induction hypothesis it tantamounts to $\psi \notin w$ or $\psi' \in w$; in both cases we have $\psi \to \psi' \in w$ because w is L_w-maximal.

Suppose that ϕ has form $\forall x\psi$. $\Leftarrow$ Assume that $\forall x\psi \in w$ and y is an individual in $d(w)$. Since w is $d(w)$-universal we have that $\psi[x/y] \in w$, in particular $\psi[x/y] \in \mathcal{L}_w$ and by induction hypothesis $(I^\sigma, w) \models \psi[x/y]$. By the conversion lemma $(I^{\sigma\binom{x}{y}}, w) \models \psi$, and given the arbitrariness of variant $\sigma\binom{x}{y}$, we obtain $(I^\sigma, w) \models \forall x\psi$.

$\Rightarrow$ Assume that $\forall x\psi \notin w$. Since w is L_w-maximal, $\exists x\neg\psi \in w$ and w is $d(w)$-rich, so there exists $y \in d(w)$ s.t. $\neg\psi[x/y] \in w$. Notice that $\neg\psi[x/y] \in \mathcal{L}_w$ and by induction hypothesis not $(I^\sigma, w) \models \psi[x/y]$. By the conversion lemma there exists $y \in d(w)$ s.t. not $(I^{\sigma\binom{z}{y}}, w) \models \psi$, i.e. not $(I^\sigma, w) \models \forall x\psi$.

Suppose that ϕ has form $\Box\psi$. $\Leftarrow$ Assume that $\Box\psi \in w$ and wRw'. By definition of R, we have $\psi \in w'$, thus $\psi \in \mathcal{L}_{w'}$ and by induction hypothesis $(I^\sigma, w') \models \psi$. Therefore $(I^\sigma, w) \models \Box\psi$.

$\Rightarrow$ Assume that $\Box\psi \notin w$ and consider set $\{\phi | \Box\phi \in w\} \cup \{\neg\psi\}$, which is L_w-consistent: if it were not the case there would be $\phi_1, \ldots, \phi_n \in \{\phi | \Box\phi \in w\}$ s.t. $\vdash_{L_w} \bigwedge \phi_i \to \psi$. By T3 $\vdash_{L_w} \bigwedge \Box\phi_i \to \Box\psi$ and since $\bigwedge \Box\phi_i \in w$, also $\Box\psi \in w$ against the L_w-consistency of w. We apply lemma 1.14 for $\Delta = \{\phi | \Box\phi \in w\} \cup \{\neg\psi\}$ and obtain an $L_{w'}$-saturated set w', for $\mathcal{L}_w \subset_\infty \mathcal{L}_{w'} \subset_\infty \mathcal{L}_0^+$, s.t. $\{\phi | \Box\phi \in w\} \cup \{\neg\psi\} \subseteq w'$. This means that wRw' and $(I^\sigma, w') \models \neg\psi$ by induction hypothesis, hence not $(I^\sigma, w) \models \Box\psi$.

By means of lemma 1.18 we prove that the canonical model w.r.t. calculus L is actually a K-model for L.

Theorem 1.19 (Canonical model theorem) *For every $\phi \in \mathcal{L}_0$,*

$$\mathcal{M}^L \models \phi \quad iff \quad \vdash_L \phi$$

Proof. $\Leftarrow$ Suppose that $\vdash_L \phi$ and $x_1, \ldots, x_n$ are all the free variables in ϕ. Consider w-assignment τ s.t. $\tau(x_i) = y_i$, for $1 \leq i \leq n$. By the conversion and coincidence lemma $(I^\tau, w) \models \phi$ iff $(I^\sigma, w) \models \phi[x_1/y_1, \ldots, x_n/y_n]$. By hypothesis $\vdash_L \phi$ and thus $\vdash_L \phi[x_1/y_1, \ldots, x_n/y_n]$, by lemma 1.18 $(I^\sigma, w) \models \phi[x_1/y_1, \ldots, x_n/y_n]$. Finally we have that $(I^\tau, w) \models \phi$ for any τ, therefore $\mathcal{M}^L \models \phi$.

$\Rightarrow$ Whenever $\nvdash_L \phi$, set $\{\neg\phi\}$ is L-consistent. By lemma 1.14 there exists an L_w-saturated extension w of $\{\neg\phi\}$, for some language $\mathcal{L}_0 \subset_\infty \mathcal{L}_w \subset_\infty \mathcal{L}_0^+$, which is a world in the canonical model w.r.t. L. By lemma 1.18, $\neg\phi \in w$ implies $(I^\sigma, w) \models \neg\phi$, i.e. $\mathcal{M}^L$ does not verify ϕ.

Once we have proved theorem 1.19, theorem 1.12 immediately follows: for every formula $\phi \in \mathcal{L}_0$ refutable in L, there exists a K-model for L - i.e. the canonical model w.r.t L - which falsifies ϕ.

In the first part of this section we set out a general framework, into which we can arrange all the completeness results for QML calculi w.r.t. Kripke semantics. Still there are some missing details: for each system L we have to prove that the canonical model w.r.t. L is based in a K-frame for L, i.e. system L is canonical. This is our task in the second part of this section.

1.2.2 Filling in the details

In order to prove Kripke-completeness for a QML calculus L, by using the techniques previously displayed, we have to check that the canonical model w.r.t L - as defined in par. 1.2.1 - is actually based on a K-frame for L. If L is canonical, then we prove the canonical model theorem with $\mathcal{M}^L$ based on a K-frame for L and the converse of theorem 1.11 immediately follows. Hence we devote next paragraphs to check canonicity for each QML calculus in par. 1.1.2, starting with the simplest case: $Q^\circ.K$. But first we have to demonstrate next remark:

Remark 1.20 *Let w be a world in the canonical model, $w \vdash_{L_w} \forall z_1, \ldots, z_h \bot$, for some $h \geq 1$, iff inner domain $d(w)$ is empty.*

Proof. $\Rightarrow$ Suppose for reduction that there exists $y \in d(w)$, this means that for every $\phi \in \mathcal{L}_w$, $\forall x\phi \to \phi[x/y] \in w$, and in particular $\forall z_1, \ldots, z_h \bot \to \forall z_2, \ldots, z_h \bot[z_1/y] \in w$. Since $w \vdash_{L_w} \forall z_1, \ldots, z_h \bot$ and z_1 does not appear in $\bot$, $w \vdash_{L_w} \forall z_2, \ldots, z_h \bot$. By reiterating this argument h times, we obtain that $w \vdash_{L_w} \bot$ against the L_w-consistency of w.

$\Leftarrow$ By contraposition if $w \nvdash_{L_w} \forall z_1, \ldots, z_h \bot$ for any $h \geq 1$, then in particular $w \nvdash_{L_w} \forall x \bot$ and thus $\forall x \bot \notin w$. Since w is L_w-maximal $\exists x \top \in w$, but w is also $d(w)$-rich and thus there exists $y \in d(w)$ s.t. $\top[x/y] \in w$.

$\boxed{Q^\circ.K}$

Since no specific condition is imposed on K-frames for $Q^\circ.K$, the canonical model w.r.t. $Q^\circ.K$ is actually based on a K-frame for $Q^\circ.K$. Kripke-completeness follows.

Corollary 1.21 (Completeness of $Q^\circ.K$) *If formula $\phi \in \mathcal{L}$ is valid in the class of K-frames with varying inner domains and increasing outer domains, then ϕ is a theorem in $Q^\circ.K$.*

$\boxed{Q^\circ.K + BF}$

In order to prove that calculus $Q^\circ.K + BF$ is canonical, we have to check the decreasing inner domain condition on the canonical model. Unfortunately, the definition of canonical frame $\mathcal{F}$ in par. 1.2.1 does not prevent worlds in $\mathcal{F}$ from being such that wRw' but $d(w') \nsubseteq d(w)$. Therefore we modify the canonical model to obtain a K-model $\mathcal{M}'$ for $Q^\circ.K+BF$ s.t. $\mathcal{M}'$ is based on a K-frame for $Q^\circ.K+BF$ and falsifies unprovable ϕ.

Consider $w \in W$ s.t. $\neg\phi \in w$ and sub-model $\mathcal{M}^w$ generated by w. By the generated sub-model theorem $\mathcal{M}^w$ is a K-model for $Q^\circ.K + BF$ falsifying ϕ, but neither in this case the decreasing inner domain condition is guaranteed; thus we define K-model $\mathcal{M}'$ as follows:

- W', D', d' and I' are the same as $\mathcal{M}^w$;
- R' is the relation on W' s.t. $w_1R'w_2$ iff $w_1R^ww_2$ and $d(w_2) \subseteq d(w_1)$.

It is trivial to check that $\mathcal{M}'$ satisfies the decreasing inner domain condition. Now we have to show that in passing from $\mathcal{M}^w$ to $\mathcal{M}'$ we did not eliminate too many worlds and relations. In fact for each world w_1 in the generated sub-model s.t. $\neg\Box\phi \in w_1$, there existed world w_2 s.t. $w_1R^ww_2$ and $\neg\phi \in w_2$ by lemma 1.18. We have to prove that for such a w_1 there exists also w_3 s.t. $w_1R'w_3$ and $\neg\phi \in w_3$, to obtain that lemma 1.18 and theorem 1.19 still hold for $\mathcal{M}'$. This is exactly the content of the next lemma.

Lemma 1.22 *Let w be a world in the generated sub-model s.t. $\neg\Box\phi \in \mathcal{L}_w$ belongs to w. There exists an $L_{w'}$-saturated set w', for $\mathcal{L}_{w'} \supseteq \mathcal{L}_w$, s.t. $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\} \subseteq w'$ and $d(w') \subseteq d(w)$.*

Proof. We distinguish two cases as in lemma 1.14.

(a) $w \vdash_{L_w} \forall z_1, \ldots, z_h \bot$ for some $h \geq 1$.

By *ex falso quodlibet*

$$w \vdash_{L_w} \forall z_1, \ldots, z_h \Box\bot$$

and by several applications of BF

$$w \vdash_{L_w} \Box\forall z_1, \ldots, z_h \bot$$

Thus $\forall z_1, \ldots, z_h \bot$ belongs to L_w-consistent set $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\}$, and we are back to case (a) in lemma 1.14: there exists an $L_{w'}$-saturated set w', for $\mathcal{L}_{w'} \supseteq \mathcal{L}_w$, s.t. $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\} \subseteq w'$ and $d(w') = d(w) = \emptyset$ by remark 1.20.

(b) $w \nvdash_{L_w} \forall z_1, \ldots, z_h \bot$ for any $h \geq 1$.

We apply lemma 1.13 for $Y = d(w)$, $Y_1 = \emptyset$ and $Y_2 = d(w)$. To perform the construction we have to show that hypotheses (i) and (ii) hold, without any expansion of the set of variables.

(i) $\Gamma_0 = \{\psi | \Box\psi \in w\} \cup \{\neg\phi\}$ is L_w-consistent as in lemma 1.18.

(ii) We prove that if $\Gamma_n \cup \{\exists x \theta_n\}$ is L_w-consistent, then there always exists $y \in d(w)$ s.t. $\Gamma_n \cup \{\theta_n[x/y]\} \cup \mathcal{E}(y)$ is L_w-consistent.

First of all notice that by remark 1.20 inner domain $d(w)$ is not empty. Suppose for reduction that there exists $n \in \mathbb{N}$ s.t. $\Gamma_n \cup \{\exists x \theta_n\}$ is L_w-consistent and for every $y \in d(w)$, there are $\Box\psi_1, \ldots, \Box\psi_k$ in w and $\zeta_1, \ldots, \zeta_j \in \mathcal{L}_w$ s.t.

1. $\forall z_i \zeta_i \to \zeta_i[z_i/y_{m_i}] \in \mathcal{E}(\{y_0, \ldots, y_{n-1}, y\})$ for $1 \leq i \leq j$;
2. $\vdash_{L_w} \bigwedge(\forall z_i \zeta_i \to \zeta_i[z_i/y_{m_i}]) \to (\bigwedge \psi_l \to (\theta_0 \wedge \ldots \wedge \theta_{n-1} \to \neg\theta_n[x_n/y]))$.

From 2 we obtain by T3

$$\vdash_{L_w} \bigwedge(\Box\forall z_i \zeta_i \to \Box\zeta_i[z_i/y_{m_i}]) \to (\bigwedge \Box\psi_l \to \Box(\theta_0 \wedge \ldots \wedge \theta_{n-1} \to \neg\theta_n[x_n/y]))$$

and by BF

$$\vdash_{L_w} \bigwedge(\forall z_i \Box\zeta_i \to \Box\zeta_i[z_i/y_{m_i}]) \to (\bigwedge \Box\psi_l \to \Box(\theta_0 \wedge \ldots \wedge \theta_{n-1} \to \neg\theta_n[x_n/y]))$$

Since $y_0, \ldots, y_{n-1}, y$ belong to $d(w)$, we have that $\bigwedge(\forall z_i \Box\zeta_i \to \Box\zeta_i[z_i/y_{m_i}]) \in w$; therefore $\bigwedge \Box\psi_l \to \Box(\theta_0 \wedge \ldots \wedge \theta_{n-1} \to \neg\theta_n[x_n/y]) \in w$. Again $\bigwedge \Box\psi_l \in w$ and thus $\Box(\theta_0 \wedge \ldots \wedge \theta_{n-1} \to \neg\theta_n[x_n/y]) \in w$ for every variable $y \in d(w)$.

Notice that w is $d(w)$-rich and L_w-maximal, so that $\forall z \Box(\theta_0 \wedge \ldots \wedge \theta_{n-1} \to \neg\theta_n[x_n/z]) \in w$ for some variable z not occurring in $\theta_0, \ldots, \theta_n$. By another application of BF, we obtain that $\Box\forall z(\theta_0 \wedge \ldots \wedge \theta_{n-1} \to \neg\theta_n[x_n/z]) \in w$. Since z does not appear in $\theta_0, \ldots, \theta_{n-1}$, by T2 and A8

$$\Box(\theta_0 \wedge \ldots \wedge \theta_{n-1} \to \forall z \neg\theta_n[x_n/z]) \in w$$

that is, $\theta_0 \wedge \ldots \wedge \theta_{n-1} \to \forall z \neg\theta_n[x_n/z] \in \{\psi | \Box\psi \in w\}$. But this means that

$$\Gamma_n \vdash_{L_w} \forall z \neg\theta_n[x_n/z]$$

and this contradicts the L_w-consistency of Γ_n, as z does not appear in θ_n.

In conclusion, if $\Gamma_n \cup \{\exists x \theta_n\}$ is L_w-consistent then it is always possible to find $y \in d(w)$ s.t. $\Gamma_n \cup \{\theta_n[x/y]\} \cup \mathcal{E}(y)$ is L_w-consistent.

By lemma 1.13, $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ is an L_w-consistent set of formulas in $\mathcal{L}_w$ s.t. it extends $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\}$ and is Z-rich and Z-universal, for some $Z \subseteq d(w)$. Lindenbaum's lemma guarantees that Γ can be extended to an L_w-saturated set w' s.t. $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\} \subseteq w'$ and $d(w') \subseteq d(w)$.

By this lemma the coimplication in theorem 1.19 holds even for $\mathcal{M}'$ and Kripke-completeness for $Q^\circ.K + BF$ directly follows.

Corollary 1.23 (Completeness of $Q^\circ.K + BF$) *If formula $\phi \in \mathcal{L}$ is valid in the class of K-frames with decreasing inner domains and increasing outer domains, then ϕ is a theorem in $Q^\circ.K + BF$.*

$\boxed{Q^\circ.K + CBF}$

For proving that the canonical model $\mathcal{M}$ w.r.t. $Q^\circ.K + CBF$ is based on a K-frame for $Q^\circ.K + CBF$, we have to check the increasing inner domain condition on $\mathcal{M}$. As it was the case for calculus $Q^\circ.K + BF$, the definition of canonical frame $\mathcal{F}$ does not prevent worlds in $\mathcal{F}$ from being such that wRw' but $d(w) \not\subseteq d(w')$. Hence we construct K-model $\mathcal{M}'$ as above, but for the accessibility relation that is defined as follows:

- R' is the relation on W' s.t. $w_1 R' w_2$ iff $w_1 R^w w_2$ and $d(w_1) \subseteq d(w_2)$.

Even in the present case we have to show that K-model $\mathcal{M}'$ does not cut off too many worlds, that is, we show that lemma 1.18 and theorem 1.19 still hold for $\mathcal{M}'$ by means of next result.

Lemma 1.24 *Let w be a world in the generated sub-model s.t. $\neg\Box\phi \in \mathcal{L}_w$ belongs to w. There exists an $L_{w'}$-saturated set w', for $\mathcal{L}_{w'} \supseteq \mathcal{L}_w$, s.t. $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\} \subseteq w'$ and $d(w) \subseteq d(w')$.*

Proof. We apply lemma 1.13 for $Y_1, Y_2 \subseteq Y$ s.t. $Y_1 = d(w)$ and Y_2 is a denumerable set of new variables s.t. $\mathcal{L}_w^Y \subset_\infty \mathcal{L}^+$. First of all we have to prove that Γ_0 is L_w^Y-consistent, even this time we distinguish two cases:

(a) $w \vdash_{L_w} \forall z_1, \ldots, z_h \bot$ for some $h \geq 1$.

By remark 1.20 inner domain $d(w)$ is empty, thus $\Gamma_0 = \{\psi | \Box\psi \in w\} \cup \{\neg\phi\}$ is L_w^Y-consistent.

(b) $w \nvdash_{L_w} \forall z_1, \ldots z_h \bot$ for any $h \geq 1$.

Inner domain $d(w)$ is not empty. Suppose for reduction that $\Gamma_0 = \{\psi | \Box\psi \in w\} \cup \{\neg\phi\} \cup \mathcal{E}(d(w))$ is not L_w^Y-consistent, this means that there exist $\Box\psi_1, \ldots, \Box\psi_n \in w$, $\chi_1, \ldots, \chi_k \in \mathcal{L}_w^Y$ s.t.

1. $\forall z_i \chi_i \to \chi_i[z_i/y_i] \in \mathcal{E}(d(w))$ for $1 \leq i \leq k$;
2. $\vdash_{L_w^Y} \bigwedge(\forall z_i \chi_i \to \chi_i[z_i/y_i]) \to (\bigwedge \psi_l \to \phi)$.

From 2 we deduce by T3

$$\vdash_{L_w^Y} \Box \bigwedge(\forall z_i \chi_i \to \chi_i[z_i/y_i]) \to (\bigwedge \Box\psi_l \to \Box\phi)$$

and by T2

$$\vdash_{L_w^Y} \forall y_1, \ldots, y_k \Box \bigwedge(\forall z_i \chi_i \to \chi_i[z_i/y_i]) \to \forall y_1, \ldots, y_k (\bigwedge \Box\psi_l \to \Box\phi)$$

By several applications of CBF

$$\vdash_{L_w^Y} \Box \forall y_1, \ldots, y_k \bigwedge(\forall z_i \chi_i \to \chi_i[z_i/y_i]) \to \forall y_1, \ldots, y_k (\bigwedge \Box\psi_l \to \Box\phi)$$

T1 is a theorem in $Q^\circ.K + CBF$, thus by R2 and R1

$$\vdash_{L_w^Y} \forall y_1, \ldots, y_k (\bigwedge \Box\psi_l \to \Box\phi)$$

Moreover w is $d(w)$-universal and so $\bigwedge \Box\psi_l \to \Box\phi \in w$, but $\bigwedge \Box\psi_l \in w$ and then $\Box\phi \in w$ against the L_w-consistency of w.

(ii) The inductive step is proved as in lemma 1.14.

By lemma 1.13, $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ is an L_w^Y-consistent set of formulas in $\mathcal{L}_w^Y$ s.t. it extends $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\}$ and is Z-rich and Z-universal for some $Z \supseteq d(w)$. Lindenbaum's lemma guarantees that Γ can be extended to an $L_{w'}$-saturated set w', for $\mathcal{L}_{w'} \supseteq \mathcal{L}_w$, s.t. $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\} \subseteq w'$ and $d(w') \supseteq d(w)$.

Finally Kripke-completeness for $Q^\circ.K + CBF$ can be proved by theorem 1.19.

Corollary 1.25 (Completeness of $Q^\circ.K + CBF$) *If formula $\phi \in \mathcal{L}$ is valid in the class of K-frames with increasing inner domains and increasing outer domains, then ϕ is a theorem in $Q^\circ.K + CBF$.*

$\boxed{Q^\circ.K + CBF + BF}$

As it is the case for calculi $Q^\circ.K + BF$ and $Q^\circ.K + CBF$, the definition of canonical frame makes it possible for worlds $w, w' \in W$ to be s.t. wRw' but $d(w) \neq d(w')$. Once more we have to construct K-model $\mathcal{M}'$, where the accessibility relation is defined as follows:

- R' is the relation on W' s.t. $w_1 R' w_2$ iff $w_1 R^w w_2$ and $d(w_1) = d(w_2)$.

Lemmas 1.22 and 1.24 guarantee that we do not eliminate too many worlds in passing from $\mathcal{M}^w$ to $\mathcal{M}'$.

Lemma 1.26 *Let w be a world in the generated sub-model s.t. $\neg\Box\phi \in \mathcal{L}_w$ belongs to w. There exists an $L_{w'}$-saturated set w', for $\mathcal{L}_{w'} \supseteq \mathcal{L}_w$, s.t. $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\} \subseteq w'$ and $d(w') = d(w)$.*

Proof. Once again we consider cases (a) and (b):

(a) $w \vdash_{L_w} \forall z_1, \ldots, z_h \bot$ for some $h \geq 1$.

In $Q^{\circ}.K + CBF + BF$ we have BF as an axiom, thus we prove that $\forall z_1, \ldots, z_h \bot$ belongs to L_w-consistent set $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\}$ as in lemma 1.22, and we are back to case (a) in lemma 1.14: there exists an $L_{w'}$-saturated set w', for $\mathcal{L}_{w'} \supseteq \mathcal{L}_w$, s.t. $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\} \subseteq w'$ and $d(w') = d(w) = \emptyset$.

(b) $w \nvdash_{L_w} \forall z_1, \ldots, z_h \bot$ for any $h \geq 1$.

We apply lemma 1.13 for $Y = Y_1 = Y_2 = d(w)$. As to (i), $\Gamma_0 = \{\psi | \Box\psi \in w\} \cup \{\neg\phi\} \cup \mathcal{E}(d(w))$ is L_w-consistent by lemma 1.24. As to (ii), if $\Gamma_n \cup \{\exists x \theta_n\}$ is L_w-consistent, then by lemma 1.22 there exists $y \in d(w)$ s.t. $\Gamma_n \cup \{\theta_n[x/y]\} \cup \mathcal{E}(y)$ is L_w-consistent.

By lemma 1.13, $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ is an L_w-consistent set of formulas in $\mathcal{L}_w$ s.t. it extends $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\}$ and is Z-rich and Z-universal, for some $d(w) \subseteq Z \subseteq d(w)$. Lindenbaum's lemma guarantees that Γ can be extended to an $L_{w'}$-saturated set w', for $\mathcal{L}_{w'} \supseteq \mathcal{L}_w$, s.t. $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\} \subseteq w'$ and $d(w') = d(w)$.

Kripke-completeness for $Q^{\circ}.K + CBF + BF$ follows, as lemma 1.18 and theorem 1.19 can be proved for $\mathcal{M}'$.

Corollary 1.27 (Completeness of $Q^{\circ}.K + CBF + BF$) *If formula $\phi \in \mathcal{L}$ is valid in the class of K-frames with constant inner domains and increasing outer domains, then ϕ is a theorem in $Q^{\circ}.K + CBF + BF$.*

The proofs of lemmas 1.22 and 1.24 are the most interesting part of the present chapter. Together with lemma 1.14, they justify the general form in which lemma 1.13 was stated. Furthermore completeness theorems for systems based on classical and free logic, where available, will be proved by means of these results.

$\boxed{Q.K}$

In order to show that calculus $Q.K$ is canonical, we have to check that the canonical model w.r.t $Q.K$ is based on a K-frame for $Q.K$. This tantamounts to proving that in the canonical model w.r.t $Q.K$:

1. for every $w \in W$, $d(w) = D(w)$;
2. if wRw' then $d(w) \subseteq d(w')$.

The first point follows from next remark concerning each calculus $L \supseteq Q.K$.

Remark 1.28 *If Δ is an L-saturated set of formulas in $\mathcal{L}$, then it is $Var(\mathcal{L})$-rich and $Var(\mathcal{L})$-universal.*

Proof. Any calculus $L \supseteq Q.K$ contains postulate A6 and since Δ is L-maximal, for every $\phi \in \mathcal{L}$, for every $y \in Var(\mathcal{L})$, $\forall x\phi \rightarrow \phi[x/y] \in \Delta$. Therefore Δ is $Var(\mathcal{L})$-rich and $Var(\mathcal{L})$-universal.

Since each world w in the canonical model w.r.t. $Q.K$ is L_w-saturated in $\mathcal{L}_w$, by the previous remark it is $Var(\mathcal{L}_w)$-rich and $Var(\mathcal{L}_w)$-universal as well. Hence $d(w) = D(w)$. Moreover in the canonical model, if wRw' then $D(w) \subseteq D(w')$, thus also $d(w) \subseteq d(w')$ and the increasing inner domain condition is satisfied.

Therefore $Q.K$ is canonical and Kripke-completeness easily follows.

Corollary 1.29 (Completeness of $Q.K$) *If formula $\phi \in \mathcal{L}$ is valid in the class of K-frames with increasing inner domains and outer domains identical to inner ones, then ϕ is a theorem in $Q.K$.*

$\boxed{Q.K + BF}$

For proving that the canonical model w.r.t $Q.K + BF$ is based on a K-frame for $Q.K + BF$, we have to show that:

1. for every $w \in W$, $d(w) = D(w)$;
2. if wRw' then $d(w) = d(w')$.

Since each world w in the canonical frame w.r.t. $Q.K+BF$ is L_w-saturated in $\mathcal{L}_w$, by remark 1.28 it is $Var(\mathcal{L}_w)$-rich and $Var(\mathcal{L}_w)$-universal as well. Thus $d(w) = D(w)$.

For what concerns the second point, as it is the case in the canonical model w.r.t. $Q^{\circ}.K + BF$, the definition of accessibility relation R does not prevent worlds in W from being such that wRw' but $d(w') \nsubseteq d(w)$. Then we construct K-model $\mathcal{M}'$ defined as for calculus $Q^{\circ}.K + BF$, that is

- R' is the relation on W' s.t. $w_1R'w_2$ iff $w_1R^w w_2$ and $d(w_2) \subseteq d(w_1)$.

The present definition and equality of $d(w)$ with $D(w)$ imply together with the increasing outer domain condition that if $w_1R'w_2$ then $d(w_1) = d(w_2)$.

Moreover every world w in the canonical model w.r.t. $Q.K+BF$ can be thought of as constructed on the same language $\mathcal{L}^+$, that is, W' is the class of L^+-saturated sets w of formulas in $\mathcal{L}^+$, for $\mathcal{L}^+ \supset_\infty \mathcal{L}$.

By lemma 1.22 and remark 1.28 we prove that if w is a world in the generated sub-model s.t. $\neg\Box\phi \in \mathcal{L}^+$ belongs to w, then there exists an L^+-saturated set w' s.t. $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\} \subseteq w'$ and $d(w) = d(w') = Var(\mathcal{L}^+)$.

Once more we prove Kripke-completeness through theorem 1.19.

Corollary 1.30 (Completeness of $Q.K + BF$) *If formula $\phi \in \mathcal{L}$ is valid in the class of K-frames with constant inner domains and outer domains identical to inner ones, then ϕ is a theorem in $Q.K + BF$.*

$\boxed{Q^E.K}$

As it was the case for calculus $Q^\circ.K$, no specific condition is imposed on K-frames for $Q^E.K$, thus the canonical model as defined in par. 1.2.1 is trivially based on a K-frame for $Q^E.K$. But in order to prove the truth lemma for $\phi = E(y)$ we need next remark.

Remark 1.31 *Let Λ be a L-saturated set in $\mathcal{L}^E$, $\forall x\phi \to \phi[x/y] \in \Lambda$ for every $\phi \in \mathcal{L}^E$ iff $E(y) \in \Lambda$.*

Proof. $\Rightarrow$ If $\forall x\phi \to \phi[x/y] \in \Lambda$ for every $\phi \in \mathcal{L}^E$, then in particular $\forall x E(x) \to E(y) \in \Lambda$ and since $\vdash_L \forall x E(x)$, $E(y) \in \Lambda$.

$\Leftarrow$ In L there is postulate A7, thus by the L-maximality of Λ, $\forall x\phi \to (E(y) \to \phi[x/y]) \in \Lambda$ for every $\phi \in \mathcal{L}^E$. Since $E(y) \in \Lambda$, also $\forall x\phi \to \phi[x/y] \in \Lambda$ for every $\phi \in \mathcal{L}^E$.

In the truth lemma we show that $(I^\sigma, w) \models E(x)$ iff $E(x) \in w$, i.e. $x \in d(w)$ iff $E(x) \in w$, that immediately follows from the previous remark.

By theorem 1.19 we have the following completeness result.

Corollary 1.32 (Completeness of $Q^E.K$) *If formula $\phi \in \mathcal{L}^E$ is valid in the class of K-frames with varying inner domains and increasing outer domains, then ϕ is a theorem in $Q^E.K$.*

$\boxed{Q^E.K + CBF}$

The main difference between calculus $Q^E.K + CBF$ and its companion $Q^\circ.K + CBF$ consists in the fact that for the latter we were not able to prove that the canonical model w.r.t. $Q^\circ.K + CBF$ is based on a K-frame for $Q^\circ.K + CBF$, whereas for the former is canonical. Consider the following remark.

Remark 1.33 *Let w, w' be worlds in the canonical model w.r.t. $L \supseteq Q^E.K + CBF$, if wRw' then $d(w) \subseteq d(w')$.*

Proof. If $y \in d(w)$ then by remark 1.31 $E(y) \in w$. Since $E(y) \to \Box E(y)$ is a theorem in L, $\Box E(y) \in w$. But $\{\psi | \Box\psi \in w\} \subseteq w'$, thus $E(y) \in w'$ and again by remark 1.31, $y \in d(w')$.

We conclude that the canonical model w.r.t. $Q^E.K + CBF$ - as defined in par. 1.2.1 - is based on a K-frame for $Q^E.K + CBF$, and completeness immediately follows.

Corollary 1.34 (Completeness of $Q^E.K + CBF$) *If formula $\phi \in \mathcal{L}^E$ is valid in the class of K-frames with increasing inner domains and increasing outer domains, then ϕ is a theorem in $Q^E.K + CBF$.*

$\boxed{Q^E.K + BF + N\neg E \textbf{ and } Q^E.K + CBF + BF + N\neg E}$

At the beginning of section 1.2 we noticed that calculi $Q^E.K + BF$, $Q^E.K + CBF + BF$ are Kripke-incomplete. If we are looking for a calculus on language $\mathcal{L}^E$ adequate w.r.t. the class of K-frames with decreasing (resp. constant) inner domains and increasing outer domains, we just have to strengthen calculus $Q^E.K + BF$ ($Q^E.K + CBF + BF$) by adding postulate A14. We show that this extension is all that we need, by proving Kripke-completeness for systems $Q^E.K + BF + N\neg E$ and $Q^E.K + CBF + BF + N\neg E$.

For showing that the canonical model $\mathcal{M}$ w.r.t $Q^E.K + BF + N\neg E$ ($Q^E.K + CBF + BF + N\neg E$) is actually based on a K-frame for this system, we have to check the decreasing (constant) inner domain condition on $\mathcal{M}$, which directly follows from next remark.

Remark 1.35 *Let w, w' be worlds in the canonical frame w.r.t $L \supseteq Q^E.K + BF + N\neg E$, if wRw' then $d(w') \subseteq d(w)$.*

Proof. If $y \notin d(w)$ then by remark 1.31, $E(y) \notin w$ and by the L_w-maximality of w, $\neg E(y) \in w$. Since $\neg E(y) \to \Box\neg E(y)$ is a theorem in L, $\Box\neg E(y) \in w$. But $\{\psi | \Box\psi \in w\} \subseteq w'$, thus $E(y) \notin w'$ and again by remark 1.31, $y \notin d(w')$.

Hence calculus $Q^E.K + BF + N\neg E$ is canonical and the same holds for $Q^E.K + CBF + BF + N\neg E$ by remark 1.33. Kripke-completeness follows by theorem 1.19.

Corollary 1.36 (Completeness of $Q^E.K + BF + N\neg E$) *If formula $\phi \in \mathcal{L}^E$ is valid in the class of K-frames with decreasing inner domains and increasing outer domains, then ϕ is a theorem in $Q^E.K + BF + N\neg E$.*

Corollary 1.37 (Completeness of $Q^E.K + CBF + BF + N\neg E$) *If formula $\phi \in \mathcal{L}^E$ is valid in the class of K-frames with constant inner domains and increasing outer domains, then ϕ is a theorem in $Q^E.K + CBF + BF + N\neg E$.*

1.2.3 Summing up

In the following table we summarize the completeness theorems proved thus far, we consider also the systems introduced in the previous paragraph.

Theorem 1.38 (Completeness) *The following QML calculi are complete w.r.t. the respective classes of K-frames:*

calculi	inner domain	outer domain
$Q.K$	*increasing*	$=$ *inner*
$Q.K + BF$	*constant*	$=$ *inner*
$Q^{\circ}.K$	*varying*	*increasing*
$Q^{\circ}.K + BF$	*decreasing*	*increasing*
$Q^{\circ}.K + CBF$	*increasing*	*increasing*
$Q^{\circ}.K + CBF + BF$	*constant*	*increasing*
$Q^{E}.K$	*varying*	*increasing*
$Q^{E}.K + BF + N\neg E$	*decreasing*	*increasing*
$Q^{E}.K + CBF$	*increasing*	*increasing*
$Q^{E}.K + CBF + BF + N\neg E$	*constant*	*increasing*

All these QML calculi are complete w.r.t. the classes of K-frames for them. We strengthen our last eight results according to the next lemma, which appears as theorem 1.32 in [20].

Lemma 1.39 *If formula $\phi \in \mathcal{L}_0$ is not true in K-model $\mathcal{M}$ with increasing outer domains, then there exists a constant outer domain K-model $\mathcal{M}^*$ s.t. not $\mathcal{M}^* \models \phi$.*

Proof. Starting from K-model $\mathcal{M}$, we build K-model $\mathcal{M}^*$ defined as $\mathcal{M}$ but for the fact that for every $w \in W$, $D^*(w) = \bigcup_{w' \in W} D(w')$. It is easy to check that a world $w \in \mathcal{M}$ satisfies formula $\phi \in \mathcal{L}_0$ for w-assignment σ iff ϕ is satisfied by the same assignment in $w \in \mathcal{M}^*$. Therefore if $\phi \in \mathcal{L}_0$ is not true in K-model $\mathcal{M}$, then neither $\mathcal{M}^* \models \phi$.

In the next table we summarize the completeness theorems matching with the results in theorem 1.10.

Theorem 1.40 (Completeness) *The following QML calculi are complete w.r.t. the respective classes of K-frames:*

calculi	inner domain	outer domain
$Q.K$	*increasing*	$=$ *inner*
$Q.K + BF$	*constant*	$=$ *inner*
$Q^{\circ}.K$	*varying*	*constant*
$Q^{\circ}.K + BF$	*decreasing*	*constant*
$Q^{\circ}.K + CBF$	*increasing*	*constant*
$Q^{\circ}.K + CBF + BF$	*constant*	*constant*
$Q^{E}.K$	*varying*	*constant*
$Q^{E}.K + BF + N\neg E$	*decreasing*	*constant*
$Q^{E}.K + CBF$	*increasing*	*constant*
$Q^{E}.K + CBF + BF + N\neg E$	*constant*	*constant*

We conclude the present section with some remarks on the effectiveness of the canonical model method for QML calculi. In par. 1.2.1 we claimed that in our framework it is possible to accommodate the completeness proofs for a wide range of systems, but in par. 1.2.2 we pointed out that the canonical models w.r.t. $Q^{\circ}.K+BF$, $Q^{\circ}.K+CBF$, $Q^{\circ}.K+CBF+BF$ are not at all based on K-frames for these calculi. Hence we had to modify the canonical model accordingly. This fact reveals a lack of generality in the canonical model method for QML calculi based on Kripke's theory of quantification.

The situation is not nicer for QML calculi based on free logic: systems $Q^{E}.K+BF$ and $Q^{E}.K+CBF+BF$ turned out to be Kripke-incomplete. On the contrary completeness for $Q^{E}.K+CBF$ immediately follows from theorem 1.19, as this calculus is canonical. This result is due to the fact that in language $\mathcal{L}^{E}$ we can express existence of individual y by a single formula $E(y)$; we have not to refer to infinite sets of formulas, written on different languages, as it is the case for systems on Kripke's theory of quantification.

In the following section we consider what happens when we adopt a different modal base for our QML calculi.

1.3 Modal Bases Stronger than K

In this section we consider completeness proofs for QML calculi on modal bases stronger than K, in particular we analyse quantified extensions of normal systems of propositional modal logic, such as T, $S4$, B and $S5$. As we pointed out in par. 1.1.2, for obtaining one of these modalities we have to make use of an appropriate combination of the following schemes of axioms:

A3.	$\Box\phi \rightarrow \phi$	axiom T,
A4.	$\Box\phi \rightarrow \Box\Box\phi$	axiom 4,
A5.	$\phi \rightarrow \Box \diamond \phi$	axiom B.

By *quantified extension* of a propositional modal logic M we refer to any QML calculus in par. 1.1.2, with in addition one or more of the axioms listed above, according to the axiomatization of M.

We begin with quantified extensions of T and $S4$, as the canonical model method applies with no change to these calculi.

1.3.1 Quantified extensions of T and $S4$

Adequacy results for quantified extensions of propositional modal logics T and $S4$ are obtained by means of the theorems proved in the previous section. Soundness results are easy to check, so we skip them and focus on completeness proofs. Once more we make use of the canonical model method explained in par. 1.2.1. Even in these cases we have to prove that the canonical model w.r.t L is actually based on a K-frame for L. In fact, we have to show that the accessibility relation in the canonical model w.r.t. a quantified extension of T (resp. $S4$) is reflexive (resp. reflexive and transitive). These results immediately follow from next lemma.

Lemma 1.41 *Let ϕ be a formula on language $\mathcal{L}_w$,*

(a) If $\vdash_{L_w} \Box\phi \rightarrow \phi$, then canonical relation R is reflexive.

(b) If $\vdash_{L_w} \Box\phi \rightarrow \Box\Box\phi$, then canonical relation R is transitive.

Proof.

(a) We prove that for every $w \in W$, wRw, that is $\{\phi | \Box\phi \in w\} \subseteq w$. This condition tantamounts to: for all $\phi \in \mathcal{L}_w$, $\Box\phi \in w$ implies $\phi \in w$. But $\vdash_{L_w} \Box\phi \rightarrow \phi$ and since w is L_w-maximal, if $\Box\phi \in w$ then actually $\phi \in w$.

(b) We prove that for every $w, w', w'' \in W$, if wRw' and $w'Rw''$ then wRw''. This condition tantamounts to: if $\{\phi | \Box\phi \in w\} \subseteq w'$ and $\{\phi | \Box\phi \in w'\} \subseteq w''$ then $\{\phi | \Box\phi \in w\} \subseteq w''$, i.e. for all $\phi \in \mathcal{L}_w$, if $\Box\phi \in w$ then $\Box\Box\phi \in w$. But $\vdash_{L_w} \Box\phi \rightarrow \Box\Box\phi$ and since w is L_w-maximal, if $\Box\phi \in w$ then actually $\Box\Box\phi \in w$.

From this lemma we infer that the canonical model w.r.t a quantified extension of T (resp. $S4$) is based on a reflexive (resp. reflexive and transitive) K-frame, thus Kripke-completeness follows by theorem 1.19. We summarize in the next table these further adequacy results.

Theorem 1.42 (Adequacy) *The following QML calculi are adequate w.r.t. the respective classes of reflexive K-frames:*

calculi	*inner domain*	*outer domain*
$Q.T$	*increasing*	$=$ *inner*
$Q.T + BF$	*constant*	$=$ *inner*
$Q^{\circ}.T$	*varying*	*constant*
$Q^{\circ}.T + BF$	*decreasing*	*constant*
$Q^{\circ}.T + CBF$	*increasing*	*constant*
$Q^{\circ}.T + CBF + BF$	*constant*	*constant*
$Q^{E}.T$	*varying*	*constant*
$Q^{E}.T + BF + N\neg E$	*decreasing*	*constant*
$Q^{E}.T + CBF$	*increasing*	*constant*
$Q^{E}.T + CBF + BF + N\neg E$	*constant*	*constant*

Theorem 1.43 (Adequacy) *The following QML calculi are adequate w.r.t. the respective classes of reflexive and transitive K-frames:*

calculi	*inner domain*	*outer domain*
$Q.S4$	*increasing*	$=$ *inner*
$Q.S4 + BF$	*constant*	$=$ *inner*
$Q^{\circ}.S4$	*varying*	*constant*
$Q^{\circ}.S4 + BF$	*decreasing*	*constant*
$Q^{\circ}.S4 + CBF$	*increasing*	*constant*
$Q^{\circ}.S4 + CBF + BF$	*constant*	*constant*
$Q^{E}.S4$	*varying*	*constant*
$Q^{E}.S4 + BF + N\neg E$	*decreasing*	*constant*
$Q^{E}.S4 + CBF$	*increasing*	*constant*
$Q^{E}.S4 + CBF + BF + N\neg E$	*constant*	*constant*

Remarks:

1. In order to prove Kripke-completeness for QML calculi based on Kripke's theory of quantification, we have to modify the canonical model as we did in par. 1.2.2 for systems $Q^{\circ}.K + BF$, $Q^{\circ}.K + CBF$ and $Q^{\circ}.K + CBF + BF$.

2. By lemma 1.41 we easily obtain completeness results even for quantified extensions of $K4$.

3. Calculi $Q^{E}.T+BF$, $Q^{E}.T+CBF+BF$ and $Q^{E}.S4+BF$, $Q^{E}.S4+CBF+BF$ are still Kripke-incomplete: of course they all validate A14, but this formula is

not provable in anyone of these systems. We refer to appendix A for a formal proof of this fact. As it was the case for modal base K, Kripke-completeness is attained by adding the unprovable principle to incomplete calculi.

1.3.2 Quantified extensions of B and $S5$

In the definition of the canonical model $\mathcal{M}$ w.r.t. a QML calculus L, all the worlds are set of formulas on different superlanguages of $\mathcal{L}_0$. This feature of $\mathcal{M}$, due to the peculiar form in which lemma 1.18 was to be proved, prevents us from showing that the canonical model w.r.t. a quantified extension L of either B or $S5$ is actually based on a K-frame for L.

In fact, in order to prove canonicity we should show that the accessibility relation in the canonical model w.r.t. a quantified extension of B (resp. $S5$) is reflexive and symmetric (resp. reflexive, transitive and symmetric). By lemma 1.41, this tantamounts to the following fact:

(c) If $\vdash_{L_w} \phi \rightarrow \Box \diamond \phi$, then canonical relation R is symmetric.

But we shall see that (c) does not hold, whenever the worlds in our canonical model are constructed on different languages.

Symmetry of R is equivalent to: for all $w, w' \in W$, if wRw' then $w'Rw$, i.e. if $\{\phi|\Box\phi \in w\} \subseteq w'$ then $\{\phi|\Box\phi \in w'\} \subseteq w$. But assuming that there exists $\psi \in \mathcal{L}_{w'}$ s.t. $\Box\psi \in w'$ and $\psi \notin w$ is consistent with premise $\vdash_{L_w} \psi \rightarrow \Box \diamond \psi$: just consider formula ψ in language $\mathcal{L}_{w'}$ s.t. $\psi \notin \mathcal{L}_w$. It is the case that $\{\phi|\Box\phi \in w\} \subseteq w'$, $\Box\psi \in w'$ but $\psi \notin w$.

This problem can be restated in terms of $\diamond$ rather than $\Box$. In this way we obtain the condition, the invalidity of which Garson points out as the culprit of non-canonicity for quantified extensions of B and $S5$:

$$if\ wRw'\ and\ \phi \in w\ then\ \diamond \phi \in w'$$

Notice that this fact does not imply Kripke-incompleteness for all the quantified extensions of B and $S5$, anyway it represents an important limitation to the canonical model method in comparison with the corresponding propositional systems. We overcome these difficulties by redefining saturation, so that all the worlds in the canonical model are constructed on a unique language. Before turning into this path, we can nonetheless list some completeness and incompleteness results for specific quantified extensions of B and $S5$.

$\boxed{Q^{\circ}.B + CBF}$

First of all notice that $\vdash_{Q^{\circ}.B+CBF} BF$; here's the proof appearing in [28], p. 138.

1. $\forall x(\forall x\Box\phi \rightarrow \Box\phi)$ A11
2. $\Box\forall x(\forall x\Box\phi \rightarrow \Box\phi)$ from 1 by R2
3. $\forall x\Box(\forall x\Box\phi \rightarrow \Box\phi)$ from 2 by A13
4. $\forall x(\diamond\forall x\Box\phi \rightarrow \diamond\Box\phi)$ from 3 by $\Box(\psi \rightarrow \psi') \rightarrow (\diamond\psi \rightarrow \diamond\psi')$
5. $\forall x(\diamond\forall x\Box\phi \rightarrow \phi)$ from 4 by A5
6. $\forall x \diamond \forall x\Box\phi \rightarrow \forall x\phi$ from 5 by A9
7. $\diamond\forall x\Box\phi \rightarrow \forall x\phi$ from 6 by A8
8. $\Box \diamond \forall x\Box\phi \rightarrow \Box\forall x\phi$ from 7 by T3
9. $\forall x\Box\phi \rightarrow \Box\forall x\phi$ from 8 by A5

Therefore $Q^{\circ}.B + CBF$ is equivalent to $Q^{\circ}.B + CBF + BF$. By making use of lemma 1.26, we construct K-model $\mathcal{M}'$ where the accessibility relation is defined as follows:

- R' is the relation on W' s.t. $w_1 R' w_2$ iff $w_1 R^w w_2$, $d(w_1) = d(w_2)$ and $\mathcal{L}_{w_1} = \mathcal{L}_{w_2}$.

All the worlds in $\mathcal{M}'$ are built on the same language $\mathcal{L}' \supset_{\infty} \mathcal{L}_0$ s.t. $\mathcal{L}'$ is an infinite sublanguage of $\mathcal{L}_0^+$. We conclude that we can actually prove condition (c) in par. 1.3.2, and by theorem 1.19 we have the following completeness result.

Corollary 1.44 (Completeness of $Q^{\circ}.B + CBF$) *If formula $\phi \in \mathcal{L}$ is valid in the class of reflexive and symmetric K-frames with constant inner domains and constant outer domains, then ϕ is a theorem in $Q^{\circ}.B + CBF$.*

It is easy to check equivalence between $Q^{\circ}.S5 + CBF$ and $Q^{\circ}.S5 + CBF + BF$, and also in this case we refine the canonical model to obtain a K-model where all the worlds are defined on a unique language. Therefore these calculi are complete w.r.t. the class of reflexive, transitive and symmetric K-frames with constant inner domains and constant outer domains.

$\boxed{Q.B}$

In $Q.B$ we prove BF as above, thus $Q.B$ is equivalent to $Q.B + BF$. We refine the canonical model w.r.t. $Q.B$ by defining

- R' is the relation on W' s.t. $w_1 R' w_2$ iff $w_1 R^w w_2$, $d(w_2) \subseteq d(w_1)$ and $\mathcal{L}_{w_1} = \mathcal{L}_{w_2}$.

and obtain a K-model where all the worlds are defined on the same language $\mathcal{L}_0^+ \supset_{\infty} \mathcal{L}_0$, that verifies condition (c). Again by theorem 1.19, we prove the following result.

Corollary 1.45 (Completeness of $Q.B$) *If formula $\phi \in \mathcal{L}$ is valid in the class of reflexive and symmetric K-frames with constant inner domains and outer domains identical to inner ones, then ϕ is a theorem in $Q.B$.*

Of course calculus $Q.S5$ is equivalent to $Q.S5 + BF$ and complete w.r.t. the class of reflexive, transitive and symmetric K-frames with constant inner domains and outer domains identical to inner ones.

Kripke-incompleteness of $Q^\circ.B + BF$, $Q^E.B + BF$

In [20] Corsi claims that calculus $Q^\circ.B + BF$ is Kripke-incomplete, a detailed proof can be found in [21] as well as an incompleteness proof for $Q^\circ.S5 + BF$. In fact CBF holds in every K-model for $Q^\circ.B + BF$, as it is bound to have constant inner domains, but $\nvdash_{Q^\circ.B+BF} CBF$. By adapting this result we prove that also calculi $Q^E.B + BF$ and $Q^E.S5 + BF$ are Kripke-incomplete, we postpone the proof to appendix A. For the time being we just state the correspondent to theorem 2.22 in [20].

Theorem 1.46 *Calculi $Q^E.B + BF$ and $Q^E.S5 + BF$ are not characterized by any class of K-frames.*

$\boxed{Q^\circ.B \text{ and } Q^E.B}$

We conclude the present section by reminding that the completeness problem for calculi $Q^\circ.B$ and $Q^\circ.S5$ is still open, whereas in order to prove Kripke-completeness for $Q^E.B$ and $Q^E.B + CBF$ we need to redefine saturation, as we shall see in the next section. Thus the canonical model method for quantified extensions of B and $S5$ is extremely unsatisfactory, as all these systems are non-canonical and we have:

(i) completeness results for some calculi: $Q^\circ.B + CBF$, $Q^\circ.S5 + CBF$, $Q.B$ and $Q.S5$;

(ii) incompleteness results for some others: $Q^\circ.B + BF$, $Q^\circ.S5 + BF$, $Q^E.B + BF$, $Q^E.S5 + BF$;

(iii) no result at all for calculi $Q^\circ.B$, $Q^\circ.S5$, but completeness for corresponding $Q^E.B$, $Q^E.S5$ and $Q^E.B + CBF$, $Q^E.S5 + CBF$, at the cost of redefining saturation.

In particular notice the lack of symmetry between systems with the Barcan formula and systems with the converse of BF, the formers have weaker completeness properties in comparison to the latter. We summarize the completeness results attainable thus far in the next tables.

Theorem 1.47 (Adequacy) *The following QML calculi are adequate w.r.t. the respective classes of reflexive, symmetric K-frames:*

calculi	*inner domain*	*outer domain*
$Q.B \equiv Q.B + BF$	*constant*	$=$ *inner*
$Q^\circ.B + CBF \equiv Q^\circ.B + CBF + BF$	*constant*	*constant*

Calculi $Q^\circ.B + BF$, $Q^E.B + BF$ are incomplete and the question is open for $Q^\circ.B$.

Theorem 1.48 (Adequacy) *The following QML calculi are adequate w.r.t. the respective classes of reflexive, transitive and symmetric K-frames:*

calculi	*inner domain*	*outer domain*
$Q.S5 \equiv Q.S5 + BF$	*constant*	$=$ *inner*
$Q^{\circ}.S5 + CBF \equiv Q^{\circ}.S5 + CBF + BF$	*constant*	*constant*

Calculi $Q^{\circ}.S5 + BF$, $Q^{E}.S5 + BF$ are incomplete and the question is open for $Q^{\circ}.S5$.

To solve the completeness problem for $Q^{E}.B$ and $Q^{E}.B + CBF$, we redefine the notion of saturation and introduce canonical models with constant outer domains.

1.4 Calculi with EBR

In [20] Corsi introduces the Extended Barcan Rule - EBR in short - in order to prove completeness for calculi $Q^E.B$ and $Q^E.S5$. We adopt the very same version of EBR; first of all consider the following inference rule:

$$\text{BR}(k{+}1) \quad \frac{\phi_0 \to \Box(\phi_1 \to \ldots \to \Box(\phi_k \to \Box\phi_{k+1}))\ldots)}{\phi_0 \to \Box(\phi_1 \to \ldots \to \Box(\phi_k \to \Box\forall x\phi_{k+1}))\ldots)} \quad \text{where } x \text{ is not free in } \phi_0, \ldots, \phi_k$$

By EBR we denote the set of all the rules BR(k+1), for $k \geq 0$. According to lemma 2.12 in [20], we state the following result:

Lemma 1.49 *EBR is valid on K-frames with constant outer domains, i.e. for any K-model $\mathcal{M}$ with constant outer domains, if the premise of EBR is valid on $\mathcal{M}$, then the conclusion is also valid on $\mathcal{M}$.*

By this lemma and completeness results in theorem 1.10, we list the following equivalence results for QML calculi, which appear as lemma 2.13 in [20].

$$\begin{array}{lcl}
Q.K + EBR & \equiv & Q.K + BF \\
Q.K + BF + EBR & \equiv & Q.K + BF \\
Q^\circ.K + EBR & \equiv & Q^\circ.K \\
Q^\circ.K + BF + EBR & \equiv & Q^\circ.K + BF \\
Q^\circ.K + CBF + EBR & \equiv & Q^\circ.K + CBF \\
Q^\circ.K + CBF + BF + EBR & \equiv & Q^\circ.K + CBF + BF \\
Q^E.K + EBR & \equiv & Q^E.K \\
Q^E.K + BF + EBR & \equiv & Q^E.K + BF \\
Q^E.K + CBF + EBR & \equiv & Q^E.K + CBF \\
Q^E.K + CBF + BF + EBR & \equiv & Q^E.K + CBF + BF
\end{array}$$

Thus EBR is an eliminable - though not derivable - rule in QML calculi based on Kripke's theory of quantification and free logic. Even if EBR does not improve the deductive power of most of our systems, it permits to redefine saturation for calculi on language $\mathcal{L}^E$, thus providing a way of constructing canonical models on a unique language. This feature strengthens completeness property for quantified extensions of B and $S5$.

1.4.1 Redefining saturation

Whenever our language contains predicative constant E, we can redefine saturation in order to prove Kripke-completeness also for systems $Q^E.B$ and $Q^E.S5$, the completeness of which was not possible to prove through the techniques in previous sections. For presenting this new method we need some definitions. Let Λ be a set of formulas in language $\mathcal{L}^E$:

Λ is $\diamond$-*rich*	iff	for every $\phi_0, \ldots, \phi_k \in \mathcal{L}^E$, where x is not free, if $\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond\exists x\phi_{k+1})\ldots) \in \Lambda$, then there exists $y \in Var(\mathcal{L}^E)$ s.t. $\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond\phi_{k+1}[x/y])\ldots) \in \Lambda$.
Λ is $\diamond$-L-*saturated*	iff	Λ is L-saturated and $\diamond$-rich.

We restate lemma 1.14 according to definitions above. Next lemma guarantees the existence of canonical models with constant outer domains.

Lemma 1.50 (Saturation revisited) *If Δ is an L-consistent set of formulas in $\mathcal{L}^E$, then there exist a $\diamond$-L^Y-saturated set $\Pi \supseteq \Delta$ of formulas in $\mathcal{L}^{E\ Y}$, for Y infinite denumerable set of new individual variables.*

Proof. Assume that there are enumerations of existential formulas in $\mathcal{L}^{E\ Y}$, of formulas in $\mathcal{L}^{E\ Y}$ of type $\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond\exists x\phi_{k+1})\ldots)$, where x is not free in $\phi_0, \ldots, \phi_k$, and of Y; then define by recursion a chain of sets of formulas in $\mathcal{L}^{E\ Y}$ s.t.

$$\Gamma_0 = \Delta$$

$$\Gamma_{2n+1} = \begin{cases} \Gamma_{2n} \cup \{E(y_{2n}) \wedge \theta_n[x/y_{2n}]\}, & \text{if } \Gamma_{2n} \cup \{\exists x\theta_n\} \text{ is } L^Y\text{-consistent,} \\ & \text{and } y_{2n} \text{ is the first variable in } Y \\ & \text{appearing neither in } \Gamma_{2n} \text{ nor in } \theta_n. \\ \Gamma_{2n}, & \text{otherwise.} \end{cases}$$

$$\Gamma_{2n+2} = \begin{cases} \Gamma_{2n+1} \cup \{\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond\phi_{k+1}[x/y_{2n+1}])\ldots)\}, \text{ if} \\ \qquad \Gamma_{2n+1} \cup \{\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond\exists x\phi_{k+1})\ldots)\} \text{ is } L^Y\text{-consistent} \\ \qquad \text{and } y_{2n+1} \text{ is the first variable in } Y \text{ appearing} \\ \qquad \text{neither in } \Gamma_{2n+1} \text{ nor in } \phi_1, \ldots, \phi_{k+1}. \\ \Gamma_{2n+1}, \qquad \text{otherwise.} \end{cases}$$

Set Γ_0 is L^Y-consistent by hypothesis. Suppose that for every $n \in \mathbb{N}$, if Γ_n is L^Y-consistent then it is possible to find L^Y-consistent Γ_{n+1}. By the chain lemma $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ is a L^Y-consistent, $\diamond$-rich set of formulas in $\mathcal{L}^{E\ Y}$, that by Lindenbaum's lemma can be extended to an L^Y-maximal, $\diamond$-rich set $\Pi \supseteq \Delta$, which is Z-universal and Z-rich for some $Z \subseteq Y$ by remark 1.31. This means that Π is $\diamond$-L^Y-saturated.

All we have to prove are the inductive cases. We start with the step from $2n$ to $2n+1$ and show that if $\Gamma_{2n} \cup \{\exists x\theta_n\}$ is L^Y-consistent, then the same holds for $\Gamma_{2n} \cup \{E(y_{2n}) \wedge \theta_n[x/y_{2n}])\}$. Suppose for reduction that so-defined Γ_{2n+1} is not L^Y-consistent, this means that there exist $\phi_1, \ldots, \phi_m \in \Gamma_{2n}$ s.t.

$$\vdash_{L^Y} \bigwedge \phi_i \rightarrow (E(y_{2n}) \rightarrow \neg\theta_n[x/y_{2n}])$$

By R4 we deduce

$$\vdash_{L^Y} \bigwedge \phi_i \rightarrow \forall y_{2n}\neg\theta_n[x/y_{2n}]$$

thus $\Gamma_{2n} \vdash_{L^Y} \forall y_{2n} \neg \theta_n[x/y_{2n}]$ and by T4, $\Gamma_{2n} \cup \{\exists x \theta_n\}$ is not L^Y-consistent against hypothesis.

We go on showing the inductive step from $2n+1$ to $2n+2$, i.e. if $\Gamma_{2n+1} \cup \{\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond \exists x \phi_{k+1}) \ldots)\}$ is L^Y-consistent, then the same holds for $\Gamma_{2n+1} \cup \{\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond \phi_{k+1}[x/y_{2n+1}]) \ldots)\}$. Suppose for reduction that so-defined Γ_{2n+2} is not L^Y-consistent, this means that there exist $\psi_1, \ldots, \psi_m \in \Gamma_{2n+1}$ s.t.

$$\vdash_{L^Y} \bigwedge \psi_i \rightarrow \neg(\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond \phi_{k+1}[x/y_{2n+1}]) \ldots))$$

By propositional modal calculus we deduce

$$\vdash_{L^Y} \bigwedge \psi_i \wedge \phi_0 \rightarrow \Box(\phi_1 \rightarrow \ldots \rightarrow \Box(\phi_k \rightarrow \Box \neg \phi_{k+1}[x/y_{2n+1}]) \ldots)$$

and by an application of EBR

$$\vdash_{L^Y} \bigwedge \psi_i \wedge \phi_0 \rightarrow \Box(\phi_1 \rightarrow \ldots \rightarrow \Box(\phi_k \rightarrow \Box \forall y_{2n+1} \neg \phi_{k+1}[x/y_{2n+1}]) \ldots)$$

that is equivalent to

$$\vdash_{L^Y} \bigwedge \psi_i \rightarrow \neg(\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond \exists y_{2n+1} \phi_{k+1}[x/y_{2n+1}]) \ldots))$$

But $\bigwedge \psi_i \in \Gamma_{2n+1}$, thus

$$\Gamma_{2n+1} \vdash_{L^Y} \neg(\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond \exists x \phi_{k+1}[x/y_{2n+1}]) \ldots))$$

and by T4, $\Gamma_{2n+1} \cup \{\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond \exists x \phi_{k+1}) \ldots)\}$ is not L^Y-consistent, against hypothesis.

Before defining the canonical model with constant outer domains w.r.t. a calculus L on language $\mathcal{L}^E$, notice that the previous lemma is analogous to lemma 2.14 in [20].

1.4.2 Canonical models with constant outer domains

In the present paragraph we give a new definition of canonical model w.r.t. a QML calculus L, and prove that also this version satisfies the constraints on K-models in par. 1.1.3. Most important, it has constant outer domains.

Definition 1.51 (Constant outer domain canonical frame) *Constant outer domain canonical frame $\mathcal{F}^L$ for calculus L on language $\mathcal{L}^E$, with an expansion $\mathcal{L}^{E+}$, is an ordered 4-tuple $\langle W^L, R^L, D^L, d^L \rangle$ s.t.*

- *W^L is the class of $\diamond$-L^+-saturated sets of formulas in $\mathcal{L}^{E+}$;*
- *accessibility relation R^L and functions D^L, d^L are defined as in def. 1.15.*

We show that the so-defined canonical frame with constant outer domains actually exists and satisfies constraints on K-frames as in par. 1.2.1, by making use of lemma 1.50 instead of lemma 1.14. Notice that all the worlds in this frame are defined on the same language $\mathcal{L}^{E+}$; moreover outer domains are constant, as for every $w \in W^L$, $D^L(w)$ is set $Var(\mathcal{L}^{E+})$.

We conclude that $\mathcal{F}^L$ is a K-frame with constant outer domains, but it is left to prove that it is based on a K-frame for $L = Q^E.B, Q^E.B + CBF$. The notions of canonical interpretation and assignment are the same as in par. 1.2.1, and there we checked that they satisfy the conditions on interpretations and assignments for language $\mathcal{L}^E$. The ordered couple $\langle \mathcal{F}^L, I^L \rangle$ constitutes the canonical model $\mathcal{M}^L$ with constant outer domains w.r.t QML calculus L, we have to verify that $\mathcal{M}^L$ is a K-model for L.

In order to simplify our notation, in next paragraphs we write I, $\mathcal{M}$ and σ instead of I^L, $\mathcal{M}^L$, σ^L.

For what concerns the proof of the truth lemma, since formulas belong to an unique language, we have to restate it as follows:

Lemma 1.52 (Truth lemma) *For every $w \in W$, for every $\phi \in \mathcal{L}^{E+}$,*

$$(I^\sigma, w) \models \phi \quad iff \quad \phi \in w$$

The proof is by induction on the length of $\phi \in \mathcal{L}^{E+}$ and completely identical to the one of lemma 1.18, but for the case of the modal operator. In fact we refer only to language $\mathcal{L}^{E+}$.

Lemma 1.53 *Let w be a world in the canonical model s.t. $\neg\Box\phi \in \mathcal{L}^{E+}$ belongs to w. There exists a $\diamond$-L^+-saturated set w', s.t. wRw' and $\neg\phi \in w'$.*

Proof. Set $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\}$ is L^+-consistent by remark 1.20, but we cannot apply lemma 1.50 - as we did for calculus $Q^E.K$ - as we would need an expansion of language $\mathcal{L}^{E+}$. On the contrary we show that is possible to extend $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\}$ to a $\diamond$-L^+-saturated set w', by using variables already in $Var(\mathcal{L}^{E+})$.

Assume that there are enumerations of existential formulas in $\mathcal{L}^{E+}$, of formulas in $\mathcal{L}^{E+}$ of type $\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond\exists x\phi_{k+1})\ldots)$, where x is not free in $\phi_0, \ldots, \phi_k$, and of $Var(\mathcal{L}^{E+})$; then define by recursion a chain of sets of formulas in $\mathcal{L}^{E+}$ s.t.

$$\Gamma_0 = \{\psi | \Box\psi \in w\} \cup \{\neg\phi\}$$

$$\Gamma_{2n+1} = \begin{cases} \Gamma_{2n} \cup \{E(y_{2n}) \wedge \theta_n[x/y_{2n}]\}, & \text{if } \Gamma_{2n} \cup \{\exists x\theta_n\} \text{ is } L^+\text{-consistent and} \\ & y_{2n} \text{ is the first variable in } Var(\mathcal{L}^{E+}) \text{ s.t.} \\ & \Gamma_{2n} \cup \{E(y_{2n}) \wedge \theta_n[x/y_{2n}]\} \text{ is } L^+\text{-consistent.} \\ \Gamma_{2n}, & \text{otherwise.} \end{cases}$$

$$\Gamma_{2n+2} = \begin{cases} \Gamma_{2n+1} \cup \{\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond\phi_{k+1}[x/y_{2n+1}])\ldots)\}, \text{ if} \\ \quad \Gamma_{2n+1} \cup \{\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond\exists x\phi_{k+1})\ldots)\} \text{ is } L^+\text{-consistent} \\ \quad \text{and } y_{2n+1} \text{ is the first variable in } Var(\mathcal{L}^{E+}) \text{ s.t.} \\ \quad \Gamma_{2n+1} \cup \{\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond\phi_{k+1}[x/y_{2n+1}])\ldots)\} \text{ is } L^+\text{-consistent} \\ \Gamma_{2n+1}, \quad \text{otherwise.} \end{cases}$$

Set Γ_0 is L^+-consistent by hypothesis, and if for every $n \in \mathbb{N}$, from L^+-consistent Γ_n it is possible to construct L^+-consistent Γ_{n+1}, then by the chain lemma $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ is a L^+-consistent, $\diamond$-rich set of formulas in $\mathcal{L}^{E+}$. By Lindenbaum's lemma it can be extended to an L^+-maximal, $\diamond$-rich set $w' \supseteq \{\psi | \Box\psi \in w\} \cup \{\neg\phi\}$, that by remark 1.31 is Z-universal and Z-rich for some $Z \subseteq Var(\mathcal{L}^{E+})$. Therefore w' is $\diamond$-L^+-saturated.

We prove the inductive steps, beginning with the case from Γ_{2n} to Γ_{2n+1}. We show that if $\Gamma_{2n} \cup \{\exists x \theta_n\}$ is L^+-consistent, then there exists $y \in Var(\mathcal{L}^{E+})$ s.t. the same holds for $\Gamma_{2n} \cup \{E(y) \wedge \theta_n[x/y]\}$. Suppose for reduction that there exists $n \in \mathbb{N}$ s.t. for every $y \in Var(\mathcal{L}^{E+})$, there exist $\Box\psi_1, \dots, \Box\psi_m \in w$ and

$$\vdash_{L^+} \bigwedge \psi_i \to (\bigwedge \gamma_j \to (E(y) \to \neg\theta_n[x/y]))$$

where $\gamma_1, \dots, \gamma_l$ are all the formulas in $\Gamma_{2n} \setminus \{\psi | \Box\psi \in w\}$. By T3 we infer

$$\vdash_{L^+} \Box \bigwedge \psi_i \to \Box(\bigwedge \gamma_j \to (E(y) \to \neg\theta_n[x/y]))$$

and since $\Box \bigwedge \psi_i \in w$, for every $y \in Var(\mathcal{L}^{E+})$ we have $\Box(\bigwedge \gamma_j \to (E(y) \to \neg\theta_n[x/y])) \in w$. By hypothesis w is $\diamond$-rich and L^+-maximal, thus $\Box\forall z(\bigwedge \gamma_j \to (E(z) \to \neg\theta_n[x/z])) \in w$, for some variable z occurring neither in $\bigwedge \gamma_j$ nor in θ_n. By T3 and A8 we deduce

$$\Box(\bigwedge \gamma_j \to (\forall z E(z) \to \forall z \neg\theta_n[x/z])) \in w$$

and since $\forall z E(z)$ is a theorem in L^+

$$\bigwedge \gamma_j \to \forall z \neg\theta_n[x/z] \in \{\psi | \Box\psi \in w\}$$

Therefore $\Gamma_{2n} \vdash_{L^+} \forall z \neg\theta_n[x/z]$, and by T4 this contradicts the L^+-consistency of $\Gamma_{2n} \cup \{\exists x \theta_n\}$.

Consider the step from Γ_{2n+1} to Γ_{2n+2}. We show that if $\Gamma_{2n+1} \cup \{\psi_0 \wedge \diamond(\psi_1 \wedge \dots \wedge \diamond(\psi_k \wedge \diamond\exists x \phi_{k+1}) \dots)\}$ is L^+-consistent, then there exists $y \in Var(\mathcal{L}^{E+})$ s.t. the same holds for $\Gamma_{2n+1} \cup \{\psi_0 \wedge \diamond(\psi_1 \wedge \dots \wedge \diamond(\psi_k \wedge \diamond\psi_{k+1}[x/y]) \dots)\}$. Suppose for reduction that there exists $n \in \mathbb{N}$ s.t. for every $y \in Var(\mathcal{L}^{E+})$, there exist $\Box\psi_1, \dots, \Box\psi_m \in w$ and

$$\vdash_{L^+} \bigwedge \psi_i \to (\bigwedge \gamma_j \to \neg(\phi_0 \wedge \diamond(\phi_1 \wedge \dots \wedge \diamond(\phi_k \wedge \diamond\phi_{k+1}[x/y]) \dots)))$$

where $\gamma_0, \dots, \gamma_l$ are all the formulas in $\Gamma_{2n+1} \setminus \{\psi | \Box\psi \in w\}$. By T3 we deduce

$$\vdash_{L^+} \Box \bigwedge \psi_i \to \Box(\bigwedge \gamma_j \to \neg(\phi_0 \wedge \diamond(\phi_1 \wedge \dots \wedge \diamond(\phi_k \wedge \diamond\phi_{k+1}[x/y]) \dots)))$$

and since $\Box \bigwedge \psi_i \in w$, we have $\Box(\bigwedge \gamma_j \to \neg(\phi_0 \wedge \diamond(\phi_1 \wedge \dots \wedge \diamond(\phi_k \wedge \diamond\phi_{k+1}[x/y]) \dots))) \in w$ for every $y \in Var(\mathcal{L}^{E+})$. By hypothesis w is $\diamond$-rich and L^+-maximal, thus $\neg \diamond$

$(\bigwedge \gamma_j \wedge \diamond(\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond\exists z\phi_{k+1}[x/z])\ldots))) \in w$ for some variable z occurring neither in $\bigwedge \gamma_j$ nor in $\phi_0, \ldots \phi_{k+1}$. In turn this tantamounts to

$$\bigwedge \gamma_j \to \neg(\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond\exists z\phi_{k+1}[x/z])\ldots)) \in \{\psi|\Box\psi \in w\}$$

Hence $\Gamma_{2n+1} \vdash_{L^+} \neg(\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond\exists z\psi_{k+1}[x/z])\ldots))$, and by T4 this contradicts the L^+-consistency of $\Gamma_{2n+1} \cup \{\phi_0 \wedge \diamond(\phi_1 \wedge \ldots \wedge \diamond(\phi_k \wedge \diamond\exists x\phi_{k+1})\ldots)\}$.

The proof of lemma 1.53 is complete, the truth lemma holds also for canonical models with constant outer domains. As a consequence the canonical model theorem is valid, that is, formula $\phi \in \mathcal{L}^E$ holds in $\mathcal{M}^L$ iff it is provable in L. Therefore if L does not prove formula $\phi \in \mathcal{L}^E$, then the canonical model with constant outer domains, which is a K-model for L, does not verify ϕ.

There are some details missing in the present completeness proof: we show that calculi $Q^E.B$, $Q^E.B + CBF$ as well as $Q^E.S5$, $Q^E.S5 + CBF$ are canonical.

1.4.3 Filling in the details again

In order to prove completeness for a QML calculus L on language $\mathcal{L}^E$ by means of canonical models with constant outer domains, we have to check that $\mathcal{M}^L$, as defined in par. 1.4.2, is actually based on a K-frame for L. Canonicity for the envisaged quantified extensions of B and $S5$ follows by next remark:

(c) If $\vdash_{L^+} \Box\phi \to \Box\diamond\phi$, then canonical relation R is symmetric.

Symmetry of R tantamounts to: for every $w, w' \in W$, if $\{\phi|\Box\phi \in w\} \subseteq w'$ then $\{\phi|\Box\phi \in w'\} \subseteq w$. But now assuming that there exists $\psi \in \mathcal{L}^{E+}$ s.t. $\Box\psi \in w'$ and $\psi \notin w$ does imply a contradiction: by the L-maximality of w, $\neg\psi \in w$ and by A5, $\Box\diamond\neg\psi \in w$ i.e. $\diamond\neg\psi \in w'$. Therefore $\neg\Box\psi \in w'$ against the L-consistency of w'.

Hereafter we state the completeness results for calculi $Q^E.B$ and $Q^E.S5$ on language $\mathcal{L}^E$.

Corollary 1.54 (Completeness of $Q^E.B$) *If formula $\phi \in \mathcal{L}^E$ is valid in the class of reflexive and symmetric K-frames with varying inner domains and constant outer domains, then ϕ is a theorem in $Q^E.B$.*

Corollary 1.55 (Completeness of $Q^E.S5$) *If formula $\phi \in \mathcal{L}^E$ is valid in the class of reflexive, transitive and symmetric K-frames with varying inner domains and constant outer domains, then ϕ is a theorem in $Q^E.S5$.*

Finally we have to show that the canonical models with constant outer domains w.r.t. calculi $Q^E.B + CBF$ and $Q^E.S5 + CBF$ have constant inner domains too. In par. 1.1.2 we remarked that all the postulates of $Q^\circ.K$ are theorems in $Q^E.K$, thus $\vdash_{Q^E.B+CBF} BF$ as well. Furthermore A14 is a theorem in $Q^E.B + CBF$:

1.	$E(x) \to \Box E(x)$	by CBF
2.	$\neg\Box E(x) \to \neg E(x)$	from 1 by contraposition
3.	$\Box\neg\Box E(x) \to \Box\neg E(x)$	from 2 by T3
4.	$\Box\Diamond\neg E(x) \to \Box\neg E(x)$	from 3
5.	$\neg E(x) \to \Box\neg E(x)$	from 4 by A5

Hence $Q^E.B+CBF$ is equivalent to $Q^E.B+CBF+BF+N\neg E$, and by remarks 1.33 and 1.35 the canonical model with constant outer domains has also constant inner domains. Kripke-completeness immediately follows.

Corollary 1.56 (Completeness of $Q^E.B + CBF$) *If formula $\phi \in \mathcal{L}^E$ is valid in the class of reflexive and symmetric K-frames with constant inner domains and constant outer domains, then ϕ is a theorem in $Q^E.B + CBF$.*

Corollary 1.57 (Completeness of $Q^E.S5 + CBF$) *If formula $\phi \in \mathcal{L}^E$ is valid in the class of reflexive, transitive and symmetric K-frames with constant inner domains and constant outer domains, then ϕ is a theorem in $Q^E.S5 + CBF$.*

We conclude by noting that EBR is a derivable rule in quantified extensions of B and $S5$, for a proof we refer to [20].

1.5 Conclusions

In these conclusive remarks we test the effectiveness of the canonical model method in proving Kripke-completeness for QML calculi. By means of this technique we demonstrated Kripke-completeness for a certain number of quantified extensions of K, T and $S4$, based both on Kripke's theory of quantification and on free and classical logic. Nonetheless we had to make some minor adjustments for systems based on Kripke's theory of quantification - as $Q^{\circ}.K + BF$, $Q^{\circ}.K + CBF$ and $Q^{\circ}.K + CBF + BF$ - which are non-canonical. Moreover calculi based on free logic, containing BF, turned out to be Kripke-incomplete.

As to quantified extensions of B, they are all non-canonical and we had completeness proofs only for calculi $Q^{\circ}.B + CBF$, $Q^{\circ}.B + CBF + BF$ and $Q.B$, $Q.B + BF$, which are pairwise equivalent. Systems $Q^{\circ}.B + BF$ and $Q^{E}.B + BF$ are Kripke-incomplete, whereas the completeness problem for $Q^{\circ}.B$ is still open. Analogous results hold for the corresponding extensions of $S5$.

By means of canonical models with constant outer domains we have been able to prove Kripke-completeness for systems $Q^{E}.B$, $Q^{E}.B + CBF$ and $Q^{E}.S5$, $Q^{E}.S5 + CBF$; these results are unavailable through the simple canonical model method. Unfortunately this technique is at our disposal only for calculi on language $\mathcal{L}^{E}$. Thus the question for $Q^{\circ}.B$ and $Q^{\circ}.S5$ is once more left unanswered.

We summarize the results obtained thus far by saying that Kripke semantics reveals deep limitations when applied to QML calculi. In particular notice the discrepancies between systems based on Kripke's theory of quantification and free logic:

1. The canonical model is based on a K-frame for $Q^{E}.K + CBF$, but not for $Q^{\circ}.K + CBF$; the canonical model method need some adjustments to prove completeness for calculi $Q^{\circ}.K + BF$ and $Q^{\circ}.K + CBF + BF$, whereas corresponding $Q^{E}.K + BF$ and $Q^{E}.K + CBF + BF$ are Kripke-incomplete. The same holds for stronger modalities T and $S4$.

2. By canonical models with constant outer domains we proved completeness for $Q^{E}.B$, $Q^{E}.S5$ but not for $Q^{\circ}.B$, $Q^{\circ}.S5$.

These remarks - not to talk about Ghilardi's incompleteness results in [35] - seem to point out an irremediable flaw in Kripke semantics and a demand for a more satisfactory account. In what follows we attempt to answer this question.

Chapter 2

The Semantics of Counterparts

In chapter 1 we discussed Kripke semantics for quantified modal logic, stressed some utterly unsatisfactory features of that formal account - in particular w.r.t. completeness properties of QML calculi - and demanded a more refined framework to handle individuals in modal settings. In this chapter we focus on the semantics of counterpart.

Counterpart theory was introduced by D. Lewis in [59] as a first-order calculus. The key points of his proposal are the *notion of counterpart*, which is a consequence of Lewis' refusal to interpret the relation of trans-world sameness as strict identity, and the *extensionalization of modal discourse*, obtained by translating modal operators into quantification over possible worlds and counterparts.

Lewis' counterpart theory - CT in short - was developed by A. Hazen in [40] to provide semantics for first-order modal logic, but during the eighties there have not been significant contributions to this topic. A new interest in applying counterpart semantics to QML has risen in the past decade, in part due to Ghilardi's incompleteness results in Kripke semantics[1].

In [10] Ghilardi, Corsi in [19] and Kracht Kutz in [53], [52] presented semantics for quantified modal logic based on counterparts, the main philosophical relevance of which lies in *dealing with individuals more accurately*, and in *drawing fresh distinctions on valid formulas.*

In section 2.1 we analyse some unsatisfactory features of Kripke semantics from an actualist point of view; we consider problems due to assuming that the same individual appears and is identified across different worlds. These remarks yield to counterpart semantics. In section 2.2 we introduce counterpart frames, then explain why infinitary assignments of Kripke semantics are to be substituted by finitary ones. In section 2.3 we present typed languages for quantified modal logic, counterpart semantics and typed QML_t calculi. Finally in section 2.4 we prove counterpart-completeness for typed QML_t calculi.

As regards this last point, in the literature many completeness proofs are available. In [34] and [38] counterpart-completeness for typed QML_t calculi is proved by

[1] See [35], [36].

means of category theory, thus only for modalities at least as strong as $S4$. In [19] the notion of graph is introduced, in order to deal also with quantified extensions of K, T and B. In [10] Ghilardi provides an elegant and comprehensive completeness proof w.r.t. counterpart semantics, that we apply even to systems based on free logic.

In the conclusion we compare the completeness results available in Kripke semantics with those provable w.r.t. classes of counterpart frames. The advantages of the latter account will be clear, once we have proved that each typed QML_t calculus is complete w.r.t. counterpart semantics.

2.1 From Kripke to Counterpart Semantics

In Kripke frames we assumed the increasing outer domain condition: for all $w, w' \in W$, if wRw' then $D(w) \subseteq D(w')$. We need this constraint for evaluating $\Box$-formulas - otherwise variable x s.t. $\sigma(x) \in D(w)$ could have no denotation in $D(w')$ - but is it philosophically motivated? In this section we negatively answer this question, on the grounds of problems related to existence and trans-identity of individuals, thus laying the foundation for a counterpart-theoretic account of quantified modal logic.

2.1.1 Increasing outer domains

In chapter 1 we presented Kripke semantics for first-order modal languages $\mathcal{L}$ and $\mathcal{L}^E$. We briefly recall the evaluation clause for formulas, in which $\Box$ is the main operator:

Definition 2.1 *The relation of satisfaction in w for formula $\Box\phi \in \mathcal{L}_0$ w.r.t. valuation I^σ is defined as follows:*

$$(I^\sigma, w) \models \Box\phi \quad iff \quad for\ every\ w', wRw'\ implies\ (I^\sigma, w') \models \phi$$

The same assignment σ for individual variables in $\mathcal{L}_0$ appears in evaluating both $\Box\phi$ and ϕ. This means that statement $\Box\phi$ is true in world w of elements $a_1, \ldots, a_n$ in the outer domain of w, iff in all the worlds accessible from w statement ϕ is true of the same $a_1, \ldots, a_n$. The present definition lays down a problem of trans-world existence: in order to evaluate $\Box$-formulas in a K-model, we have to assume that objects $a_1, \ldots a_n$ in world w exist in all the worlds accessible from w. Kripke semantics requires the increasing outer domain condition, which is assumed in def. 1.3, i.e. for every world $w, w' \in W$, if w' is accessible from w then the outer domain of w is included in the outer domain of w'.

Nonetheless there is a number of contexts in which the increasing outer domain condition is not intuitive at all, just consider temporal logic: things now existing probably will not exist in some future time[2]. Even in epistemic and modal logic, we may be willing to think of epistemic states and possible worlds containing fewer individuals than the present one. After all, when we adopt an actualist point of view, we deny existence to all the possible individuals but the actual ones. Therefore we are eventually forced to dropping the increasing outer domain condition.

2.1.2 Varying domain K-models

In Kripke semantics we have at our disposal a way to reconcile increasing outer domains and actualism. It consists in distinguishing for each possible world w an outer domain $D(w)$ of objects, for which it makes sense to ascribe properties and relationships, from an inner domain $d(w)$ of existing individuals, over which quantifiers

[2] As a roman epigraph states: *Fui non sum, es non estis, nemo immortalis.* This ontological account is known as *presentism*, for a survey of the eternalism/presentism issue we refer to [63] and [65].

range. In this way we obtain the varying domain K-models in par. 1.1.3, that first appeared in [56] as a formal representation of the actualist account in the author's intent. This approach has some point, as varying domain K-models formalize the idea of diverse individuals existing in different instants. Moreover possibilist principles - as the Barcan formula and its converse, the necessity of existence $\forall x \Box E(x)$, that are all rejected by actualists - are no more valid. In conclusion, can actualists be content with varying domain settings in Kripke semantics?

In [67] Menzel lists two actualist issues which are not completely satisfied by varying domain K-models:

1. In the object-language quantifiers range only over individuals existing in the inner domain, as it is expressed by the evaluation clause for $\forall$-formulas:

$$(I^{\sigma}, w) \models \forall x \phi \quad iff \quad for\ every\ a \in d(w),\ (I^{\sigma\binom{x}{a}}, w) \models \phi$$

   but in the meta-language of K-frames we deal with two distinct types of set, i.e. $D(w)$ and $d(w)$, for each $w \in W$. Thus mere *possibilia*, swept out by the door, come back inside through the window. Furthermore, in virtue of this feature of Kripke semantics we are forced to introducing existence predicate E and free logic for recovering a sound first-order calculus. This is a quite ironic consequence for a philosophical account, that does not want to discriminate between actual and possible existence.

2. In varying domain K-models it can be the case that some individual a belongs to $D(w)$ but not to $d(w)$, for some $w \in W$, nonetheless properties and relationships are usually ascribed to a even in w. From a certain perspective this is quite intuitive: think about Plato who is considered, at the present time, a great philosopher even if he died in 347 BC. But this characteristic of Kripke semantics conflicts with the fundamental thesis of *Strong Actualism*[3]: if object a does not exist in world w, then nothing can be said about a in w. If we accept Strong Actualism, then we must admit truth-value gaps in Kripke semantics, even for modal formulas evaluated on existing objects $a_1, \ldots, a_n$, whenever any a_i does not appear in some accessible world.

We conclude that Kripke models with varying inner domains does not seem a satisfactory proposal for accommodating increasing outer domains and the actualist account, in particular w.r.t. Strong Actualism. These last remarks seem to deny the possibility itself of a formal representation for actualism in Kripke semantics.

2.1.3 Trans-world identity

There is a further question, related to the application of the increasing outer domain condition in evaluating $\Box$-formulas, which deserves more insight. Definition 2.1 is an *a priori* construction, the well-definiteness of which is guaranteed by the inductive process of defining. When *a posteriori* we want to check whether modal statement

[3]For a brief exposition of Strong Actualism, see [73].

$\Box\phi$ is true of individual a, we need a method to recognize the same a across possible worlds. This tantamounts to the well-known problem of *trans-world identity*, the bibliography of which has been enlarging during last half-century[4]. This issue is not of our concern for the time being, we consider only the (negative) solution to this problem given by D. Lewis in [59].

Since there is no agreed way out the puzzles of trans-world identity, Lewis denies the possibility of identifying the same individual across possible worlds. He even rejects that an individual may exist in different worlds, by axiom P2 of counterpart theory:

$$\forall xyz(I(x,y) \wedge I(x,z) \rightarrow y = z)$$

where $I(x,y)$ stands for *object x is in world y*. Lewis substitutes the notion of trans-world identity with a not further explained counterpart relation C, that - he claims - need to be neither transitive, nor symmetric, nor functional, nor injective, nor surjective, nor defined everywhere; but it is only reflexive (axiom P6: anything in a world is counterpart to itself). By denying the characters of equivalence relations to C, Lewis is able to maintain that his proposal does not differ from Kripke semantics just verbally, but it enjoys a wider generality.

Moreover Lewis extensionalizes modal language. The translation into the first-order language of CT of formula $\Box\phi$ with free variables $x_1, \ldots, x_n$, w.r.t. world w, goes as follows:

$$(\Box\phi)^w := \forall y \forall z_1, \ldots, z_n(W(y) \wedge R(w,y) \wedge \bigwedge_{1 \leq i \leq n} (I(z_i, y) \wedge C(x_i, z_i)) \rightarrow \phi[x_1/z_1, \ldots, x_n/z_n])$$

Truth conditions for $\Box$-formulas assert that statement $\Box\phi$ is true in world w of individuals $a_1, \ldots, a_n$, iff in every world y accessible from w statement ϕ is true not of $a_1, \ldots, a_n$, which do not exist in y, but of all the counterparts $b_1, \ldots, b_n$ in y of $a_1, \ldots, a_n$. In the next section we will see how to formally define the notion of counterpart and how to modify assignments, in order to apply Lewis' ideas to modal languages and develop a counterpart-theoretic semantics for quantified modal logic.

In conclusion we notice that Lewis' proposal is far from being free from criticisms. Rejecting trans-worlds identity is as open to questions as accepting it: truth conditions for the counterpart relation between individuals living in different worlds are as obscure as those for identity. Even Lewis is aware of these difficulties:

> The counterpart relation is a relation of similarity. So it is problematic in the way all relations of similarity are: it is the resultant of similarities and dissimilarities in a multitude of respects, weighted by the importances of the various respects and by the degree of the similarities. ([59], p. 112)

We do not pursue this matter any further, we are content with a negative characterization of the counterpart relation as not necessarily identity. This is our starting point to develop a fresh semantics for QML, in which assumptions on individuals, that are usually hidden in Kripke frames, are stated crystal-clear instead.

[4] We refer to [62], which contains relevant papers on this subject.

2.2 Counterpart Frames and Finitary Assignments

In this section we first present counterpart frames, then try to define truth conditions for $\Box$-formulas reflecting those in Lewis' Counterpart Theory. This is no trivial task. In par 2.2.2 we discuss several proposals, due to Fitting ([27]), Kracht and Kutz ([53], [52]), Corsi ([19]), and finally opt for the last one. In par. 2.2.3 we introduce finitary assignments and typed formulas, which constitute two major technical novelties of the counterpart-theoretic approach to QML.

2.2.1 Counterpart frames

In order to assign a meaning to terms *necessary* and *possible* according to Lewis' CT, we refer to K-frames in par. 1.1.3, enriched by a function C s.t. for all $w, w' \in W$, $C_{w,w'}$ is a binary relation on $D(w) \times D(w')$, intuitively interpreted as the counterpart relation.

Definition 2.2 (Counterpart frame) *A counterpart frame $\mathcal{F}$ - c-frame in short - is an ordered 5-tupla $\langle W, R, D, d, C\rangle$ defined as follows:*

- *W, R, D, d are defined as for K-frames, but D need not to satisfy the increasing outer domain condition;*
- *C is a function assigning to every 2-tuple $\langle w, w'\rangle$ a subset of $D(w) \times D(w')$.*

As usual W is meant to be the set of possible worlds, while R is the accessibility relation between worlds. Each outer domain $D(w)$ is the set of individuals, whom talking about makes sense in w; whereas each inner domain $d(w)$ is the set of individuals actually existing in w. As anticipated in section 2.1, D need not to satisfy the increasing outer domain condition. Finally C assigns to every couple $\langle w, w'\rangle$, counterpart relation $C_{w,w'}$ between members of $D(w)$ and $D(w')$. We remark that neither individual a nor its counterpart b need to belong either to $d(w)$ or to $d(w')$; furthermore counterpart relation $C_{w,w'}$ can be empty.

Before interpreting modal languages $\mathcal{L}$ and $\mathcal{L}^E$ on counterpart frames, we underline some differences between the present definition of c-frame and those in [10], [19] and [53]. Kracht and Kutz consider only counterpart relations having the *Counterpart-Existence Property*, i.e. for every $w, w' \in W$, for every $a \in D(w)$, there exists $b \in D(w')$ s.t. $C_{w,w'}(a, b)$. This is a very strong assumption, for which there seems to be no deep philosophical justification, but its need to validate a basic modal principle - distribution axiom A2 - as we will see in a little while. We have already remarked that Lewis explicitly rejects the Counterpart-Existence Property, as he maintains that "[i]t would not have been plausible to postulate that, for any two worlds, anything in one had some counterpart in the other"[5]. Definition 2.2 is more general and faithful to Lewis' original account; we keep working with it, even if it causes some problems with which we deal in next paragraphs.

[5] [59], p.113.

On the other hand, both Corsi and Ghilardi set $d(w) = D(w)$ for every $w \in W$. This assumption is a consequence of using typed languages, that we will analyse in par. 2.3.5, and it simplifies truth conditions for quantified formulas and the proof-theory. On the other hand it automatically validates some modal principle - namely the converse of the Barcan formula - the soundness of which we do not want to take for granted; moreover c-frames with varying domains are to be used in Appendix A. Later on we will see how to provide a notion of satisfaction and adequate calculi within this context.

2.2.2 Infinitary assignments

If we aim at developing a lewisian treatment of quantified modal logic, in a c-frame we have to give to formula $\Box\phi$ in world w truth conditions reflecting those for its translation $(\Box\phi)^w$ in CT. We run into problems if we try to provide a formal account of this idea, by means of infinitary assignments from variables to individuals. Consider for instance the definition of satisfaction appearing in [27]:

$$\begin{aligned}(I^\sigma, w) \models \Box\phi \quad iff \quad & for\ every\ w' \in W, for\ every\ \tau\ C_{w,w'} - counterpart\ to\ \sigma,\\ & wRw'\ implies\ (I^\tau, w') \models \phi\end{aligned}$$

where w'-assignment τ is a $C_{w,w'}$-counterpart to σ iff for every variable x, $\tau(x)$ is a counterpart in $D(w')$ of $\sigma(x) \in D(w)$. This clause has some counterintuitive consequence, just consider the following c-frame $\mathcal{F}$:

- $W = \{w, w'\}$;
- $R = \{\langle w, w'\rangle\}$;
- $D(w) = \{a, b\}$, $D(w') = \{a\}$;
- $C_{w,w'} = \{\langle a, a\rangle\}$;
- $d = D$.

Interpretation I assigns the empty set to predicative constant P^1 in w', and σ is a w-assignment s.t. $\sigma(x_1) = a$, $\sigma(x_2) = b$. Everyone can see that there is no w'-assignment $\tau : Var_{\mathcal{L}_0} \to D(w')$ that is a $C_{w,w'}$-counterpart to σ, as individual b has no counterpart in w'. Therefore valuation I^σ satisfies formula $\Box P^1(x_1)$ in w, even if there is a counterpart of a - a itself - which lacks property $I(P^1, w')$.

The present problem is due to two facts: (i) the definition of $C_{w,w'}$-counterpart and (ii) the lack of the Counterpart-Existence Property. In fact by (i), for evaluating formula $\Box P^1(x_1)$ in w w.r.t. individual a, we have to consider also what happens to b; but by (ii), b may not have a counterpart in w', thus w' does not count as a meaningful context to evaluate $\Box P^1(x_1)$.

To avoid such problems, while respecting Lewis' idea that only counterparts of a are to be considered in evaluating formula $\Box P^1(x_1)$ in w, Kracht and Kutz provide a different notion of $C_{w,w'}$-counterpart, depending only on the free variables appearing in a formula. In [53] we read the same truth conditions for formula $\Box\phi$ with free variables $x_1, \ldots, x_n$ as above, but now w'-assignment τ is a $C_{w,w'}$-counterpart to σ only if for every x_i, $\tau(x_i)$ is a counterpart in $D(w')$ of $\sigma(x_i) \in D(w)$, for $1 \leq i \leq n$. In this way we eliminate the pathological case previously considered. In w formula $\Box P^1(x_1)$ is falsified by interpretation I^σ, as τ is a $C_{w,w'}$-counterpart to σ by Kracht and Kutz's new definition.

The two authors do not solve every problem, as they tackle only fact (i). Their proposal conflicts with the validity of a universally accepted principle, i.e. distribution axiom A2. Consider the following example from [19]. Assume that our language contains two predicative constants Q^2 and D^1, which are interpreted as relation *to quarrel with* and property *to get angry*. If we adopt Kracht and Kutz's clause for evaluating modal formulas, then $\Box(Q^2(x_1, x_2) \to D^1(x_2))$, $\Box Q^2(x_1, x_2)$ can be true in a world w w.r.t. a valuation I^σ, even if $\Box D^1(x_2)$ is false in the same world w.r.t. the same valuation. In fact,

> [s]uppose it is true that 'a always quarrels with b' and that 'every time that a quarrels with b, then b gets angry'. From this it doesn't follow that 'b is always angry', for b may not be angry in those worlds where a is absent. ([19], p. 13)

We formally represent this situation through the following counterpart model $\mathcal{M}$:

- $W = \{w, w'\}$;
- $R = \{\langle w, w' \rangle\}$;
- $D(w) = \{a, b\}$, $D(w') = \{b\}$;
- $C_{w,w'} = \{\langle b, b \rangle\}$;
- $d = D$.

where $I(D^1, w') = \emptyset$. If $\sigma(x_1) = a$ and $\sigma(x_2) = b$, then there is no $C_{w,w'}$-counterpart τ to σ w.r.t. variables x_1, x_2; hence valuation I^σ vacuously satisfies both $\Box(Q^2(x_1, x_2) \to D^1(x_2))$ and $\Box Q^2(x_1, x_2)$ in w. On the contrary, there is a $C_{w,w'}$-counterpart τ' to σ w.r.t. x_2, i.e. $\tau'(x_2) = b$, thus not $(I^\sigma, w) \models \Box D^1(x_2)$.

Failure of A2 is due to fact (ii): in evaluating formulas $\Box(Q^2(x_1, x_2) \to D^1(x_2))$, $\Box Q^2(x_1, x_2)$ we must consider set Φ of $C_{w,w'}$-counterparts to σ w.r.t. variables x_1, x_2; whereas for formula $\Box D^1(x_1)$ we think of set Ψ of $C_{w,w'}$-counterparts to σ w.r.t. variable x_2 only. Since our c-frames lack the Counterpart-Existence Property, it can be the case that Φ is strictly included in Ψ. The whole problem stems from the possibility for a subformula to contain less free variables than the formula itself, as it happens with $\Box(Q^2(x_1, x_2) \to D^1(x_2)) \to (\Box Q^2(x_1, x_2) \to \Box D^1(x_2))$ and $\Box D^1(x_2)$. We feel the need to make explicit and homogeneous the occurrences of variables in a formula and its subformulas.

At this point there are three practicable paths for reconciling validity in counterpart frames and axiom A2. Either we assume Kracht and Kutz's Counterpart-Existence Property, or we adopt *finitary assignments* as Corsi and Ghilardi do, or we give up box-distribution over implication. By the first strategy we cut away the odd model previously considered: $a \in D(w)$ has a counterpart in $D(w')$ and b cannot be happy in those worlds where a is absent, just because in every world there exists a counterpart of a. We have already stressed the strength of this assumption and its unfaithfulness to Lewis' original proposal. The third choice forces to a non-normal modal logic and I am not aware of anyone who has developed such an account. We explore the second possibility, follow closely the ideas in [19] and start with the notion of finitary assignment.

2.2.3 Finitary assignments and types

Even if Kracht and Kutz's proposal does not solve all our problems, it has some point in addressing the definition of truth conditions for $\Box$-formulas, namely the distinction among assignments based on free variables appearing in formulas. By this device we maintain that in the first c-model considered formula $\Box P^1(x_1)$ is not satisfied by valuation I^σ, by specifying that we are looking for $C_{w,w'}$-counterparts to σ only w.r.t. variable x_1. This constraint parallels a well-known result in classical first-order logic, i.e. the coincidence lemma: the latter says that in deciding whether an assignment satisfies a formula, we *can* consider *just* free variables appearing therein; whereas by the former we *must* think of *only* free variables.

The first step towards a lewisian definition of satisfaction for $\Box$-formulas consists in making clear which are the elements w.r.t. which a formula is evaluated. As Corsi remarks:

> [...] the notion of satisfaction with respect to an assignment function σ should be replaced by the notion of satisfaction with respect to a finite list $\sigma(x_1), \ldots \sigma(x_m)$ of elements of the domain under consideration, thus: $^\sigma \models_w A(x_1, \ldots, x_m)$ should be replaced by $\langle \sigma(x_1), \ldots \sigma(x_m) \rangle \models_w A(x_1, \ldots, x_m)$. ([19], p.11)

Thus far we agree with Kracht and Kutz, but if we do not accept Kracht and Kutz's unbearable assumption, we have to be careful in dealing with free variables in a formula, as we learnt from A2's failure. We have already remarked that the culprit of that failure is the possibility for a subformula to contain less free variables than the formula itself, therefore:

> each formula must contain information about the length of the list of elements with respect to which it has to be evaluated, or, which is the same thing, the variables that occur in it (either explicitly or implicitly). ([19], p.13)

This task is performed by a natural number - called the *type* of the formula - that intuitively expresses the number of free variables implicitly appearing in it. A formula ϕ of type n is or is not satisfied by n-tuples of individuals in a world w, hence it stands for a subset of $D(w)^n$: the set of n-tuples satisfying ϕ in w. According to this set-theoretic interpretation, it is legitimate only to combine formulas with the same type. For instance consider the following formula:

$$\Box(Q^2(x_1, x_2) \to D^1(x_2)) \to (\Box Q^2(x_1, x_2) \to \Box D^1(x_2)) \tag{2.1}$$

Though D^1 is an unary predicative constant and only one variable appears in it, the whole formula makes sense only w.r.t. couples of elements, thus we need a way for making clear the presence of the second variable in $D^1(x_2)$. To solve this problem, Corsi introduces projection functions $\pi^m_k : Var^m_{\mathcal{L}_0} \rightarrow Var_{\mathcal{L}_0}$, for $1 \leq k \leq m$. A formula like 2.1 is nothing but a shorthand for

$$\Box(Q^2(\pi^2_1(x_1, x_2), \pi^2_2(x_1, x_2)) \rightarrow D^1(\pi^2_2(x_1, x_2))) \rightarrow \\ (\Box Q^2(\pi^2_1(x_1, x_2), \pi^2_2(x_1, x_2)) \rightarrow \quad \Box D^1(\pi^2_2(x_1, x_2)))$$

where both variable x_1 and x_2 explicitly appear in all the subformulas. This formula is evaluated w.r.t. couples of individuals, and A2 recovers validity as finitary assignment $\langle a, b \rangle$ vacuously satisfies $\Box D^1(\pi^2_2(x_1, x_2))$ in w.

In conclusion, in order to give a meaning to $\Box$-formulas in counterpart frames according to Lewis' CT, we have to adopt *finitary assignments*, so that we are concerned only with objects actually appearing in each formula, and *typed formulas*, to keep track of every individual w.r.t. which a formula is evaluated. Here we have a first justification to the claim that counterpart semantics offers a finer treatment of individuals in modal settings: a formula is evaluated w.r.t all and only the objects actually appearing in it. In the next section we modify the definition of terms and formulas in chapter 1, so that we can define satisfaction w.r.t finitary assignments and develop a counterpart-theoretic account of QML.

2.3 A Counterpart-theoretic account of QML

In this section we thoroughly develop a counterpart-theoretic account of quantified modal logic. While following the considerations in the previous section, in par. 2.3.1 we introduce typed languages $\mathcal{L}_t$ and $\mathcal{L}_t^E$, then in par. 2.3.2 we define the notions of satisfaction, truth and validity for these languages w.r.t. finitary assignments. In par. 2.3.3 we prove some equivalences between features of c-frames and validity of formulas, which throw light on the new meaning well-known modal principles acquire in counterpart semantics. Par. 2.3.4 is devoted to discuss whether counterpart semantics is a sound formalization for actualism, we will see in which sense the answer is positive. Finally in par. 2.3.5 we present typed QML_t calculi for quantified modal logic, which correspond to QML calculi on classical and free logic in chapter 1.

2.3.1 Typed languages $\mathcal{L}_t$ and $\mathcal{L}_t^E$

Projection functions are a handy tool to represent the context w.r.t. which a formula is meaningful, nonetheless this notation is rather cumbersome. In fact there is no common formalism for typed modal languages, in the present work we follow [10] with some minor changes. Even in this case we consider alphabets $\mathcal{A}$ and $\mathcal{A}^E$ introduced in chapter 1, containing only individual variables, no constant nor functors, and define typed terms.

Definition 2.3 (Typed terms in $tTer_{\mathcal{A}}$ ($tTer_{\mathcal{A}^E}$)) *Terms $t_1, t_2, \ldots$ of type n, for $n \in \mathbb{N}$ - $t_1 : n, t_2 : n, \ldots$ in short - are inductively defined as follows:*

- *for every variable x_i, for every $n \geq i$, x_i is a term of type n;*
- *nothing else is a typed term.*

Intuitively, for every $n \geq i$ typed term $x_i : n$ stands for projection function π_i^n applied to $x_1, \ldots, x_n$; this explains why we require that $n \geq i$. Notice that the only terms in our languages are typed variables, and an n-term is just a term of type n.

Whenever $t = x_i$ is an n-term and $s_1, \ldots, s_n$ are m-terms, substituted m-term $t[s_1, \ldots, s_n]$ tantamounts to $s_i : m$.

We have to modify also the inductive definition of modal formula on alphabet $\mathcal{A}$ ($\mathcal{A}^E$), in order to insert types.

Definition 2.4 (Typed modal formulas in $tFor_{\mathcal{A}}$ ($tFor_{\mathcal{A}^E}$)) *Modal formulas $\phi_1, \phi_2, \ldots$ of type n, for $n \in \mathbb{N}$ - $\phi_1 : n, \phi_2 : n, \ldots$ in short - are inductively defined as follows:*

- *if P^m is an m-ary predicative constant and $\langle t_1, \ldots, t_m \rangle$ is an ordered m-tuple of n-terms, then $P^m(t_1, \ldots, t_m)$ is a (atomic) formula of type n;*
- *if ϕ, ψ are n-formulas, then $\neg\phi$ and $\phi \rightarrow \psi$ are formulas of type n;*
- *if ϕ is an m-formula and $\langle t_1, \ldots, t_m \rangle$ is an ordered m-tuple of n-terms, then $\Box\phi(t_1, \ldots, t_m)$ is a formula of type n;*

- *if ϕ is an $n+1$-formula and x_{n+1} is an individual variable, then $\forall x_{n+1}\phi$ is a formula of type n;*
- *nothing else is a formula.*

First-order typed modal language $\mathcal{L}_t$ consists in sets $tTer_{\mathcal{A}}$ and $tFor_{\mathcal{A}}$; we similarly define language $\mathcal{L}_t^E$. Now let ϕ be an n-formula and $\vec{s}$ an n-tuple of m-terms, we inductively define substituted m-formula $\phi[\vec{s}]$ as follows:

- if ϕ is atomic formula $P^m(t_1, \ldots, t_m)$, then $\phi[\vec{s}]$ is $P^m(t_1[\vec{s}], \ldots, t_m[\vec{s}])$;
- if $\phi = \neg\psi$, then $\neg\psi[\vec{s}] = \neg(\psi[\vec{s}])$;
- if $\phi = \psi \to \psi'$, then $(\psi \to \psi')[\vec{s}] = \psi[\vec{s}] \to \psi'[\vec{s}]$;
- if $\phi = \Box\psi(t_1, \ldots, t_m)$, then $\Box\psi(t_1, \ldots, t_m)[\vec{s}] = \Box\psi(t_1[\vec{s}], \ldots, t_m[\vec{s}])$;
- if $\phi = \forall x_{n+1}\psi$, then $\forall x_{n+1}\psi[\vec{s}] = \forall x_{m+1}(\psi[\vec{s}, x_{m+1}])$.

In the present section $\mathcal{L}_0, \mathcal{L}_1, \ldots$ are used as variables on typed languages.

Remarks

1. According to def. 2.4 formulas $\Box Q^2(x_1, x_3) : 3$ and $\Box Q^2(x_1, x_3) : 5$ are different, as the former stands for $\Box Q^2(\pi_1^3(x_1, x_2, x_3), \pi_3^3(x_1, x_2, x_3))$, whereas the latter is an abbreviation of $\Box Q^2(\pi_1^5(x_1, x_2, x_3, x_4, x_5), \pi_3^5(x_1, x_2, x_3, x_4, x_5))$. The first one is evaluated w.r.t. 3-tuples of individuals, on the contrary the second one will be satisfied or not by ordered 5-tuples. In general formula $\Box Q^2(x_1, x_3) : 3$ turns out to be true for more finitary assignments than $\Box Q^2(x_1, x_3) : 5$. This is why binary connectives can be applied only to formulas of the same type. Finally we remark that sentences have type 0, they will be satisfied or not by the empty assignment $\langle\rangle$.

2. By the fourth clause in def. 2.4, in typed formula $\phi : n+1$ we can bind only the variable, the position of which in the enumeration corresponds to the type of ϕ, i.e. x_{n+1}, which does occur in ϕ. Hence formulas $\forall x_1 Q^2(x_1, x_2)$, $\forall x_3 Q^2(x_1, x_2)$ are not well-formed. As a consequence, Corsi in [19] states the following lemma:

   Lemma 2.5 *If $\phi : n$ is a typed modal formula in $\mathcal{L}_0$, then the free variables occurring in ϕ have index at most n, and any quantifier occurring in ϕ binds variables with index greater than n.*

   Moreover we have neither vacuous quantification, nor quantified formulas differentiating w.r.t. bound variables.

3. By the definition of substitution, we show that the following formula holds:

   $$\Box\phi[t_1, \ldots, t_m] \leftrightarrow \Box\phi(t_1, \ldots, t_m)$$

   where $\Box\phi$ stands for $\Box\phi(x_1, \ldots, x_n)$. Though we cannot prove that substitution commutes with the modal operator.

2.3.2 Counterpart semantics

In this paragraph we present the conclusive notions for interpreting typed modal formulas in counterpart frames. An interpretation I for language $\mathcal{L}_0$ in a c-frame $\mathcal{F}$ is defined as for K-frames, that is, I is a function from $\mathcal{L}_0$ to $\mathcal{F}$ s.t.

- if P^n is an n-ary predicative constant and $w \in W$, then $I(P^n, w)$ is an n-ary relation on $D(w)$;
- if our language includes predicative constant E, then $I(E, w) = d(w)$.

The definition of counterpart model is straightforward: a c-model $\mathcal{M}$ for $\mathcal{L}_0$, based on c-frame $\mathcal{F}$, is an ordered 2-tuple $\langle \mathcal{F}, I \rangle$ s.t. I is an interpretation of $\mathcal{L}_0$ in $\mathcal{F}$.

In order to apply remarks in par. 2.2.3, we have to substitute infinitary assignments with finitary ones, where a finitary assignment of type n - or n-assignment - for language $\mathcal{L}_0$ in a world w, is an ordered n-tuple of elements in $D(w)$.

Now let I be an interpretation of $\mathcal{L}_0$ in c-frame $\mathcal{F}$, let $\vec{a} \in D(w)^n$ be an n-assignment, valuation $I^{\vec{a}}$ of type n - n-valuation in short - assigns as a meaning to n-term t individual $a_i = \vec{a}(t)$, whenever $t = x_i$.

Well-definiteness of valuations is guaranteed by the fact that if term t is variable $x_i : n$, then $i \leq n$. Finally we state truth conditions for typed modal formulas w.r.t. finitary valuations. Since there is no risk of confusion, we simply write $\vec{a}$ instead of $I^{\vec{a}}$.

Definition 2.6 (Satisfaction) *Let $\vec{a}$ be an n-valuation and $w \in W$. The relation of satisfaction in w for typed modal formula $\phi : n \in \mathcal{L}_0$ w.r.t. $\vec{a}$ is inductively defined as follows:*

$$
\begin{array}{rcl}
(\vec{a}, w) \models P^m(t_1, \dots, t_m) & iff & \langle \vec{a}(t_1), \dots, \vec{a}(t_m) \rangle \in I(P^m, w) \\
(\vec{a}, w) \models \neg\psi & iff & not\ (\vec{a}, w) \models \psi \\
(\vec{a}, w) \models \psi \to \psi' & iff & not\ (\vec{a}, w) \models \psi\ or\ (\vec{a}, w) \models \psi' \\
(\vec{a}, w) \models \Box\psi(t_1, \dots, t_m) & iff & for\ every\ w' \in W, for\ every\ b_1, \dots, b_m \in D(w'), \\
 & & wRw',\ C_{w,w'}(\vec{a}(t_i), b_i)\ imply\ (\vec{b}, w') \models \psi \\
(\vec{a}, w) \models \forall x_{n+1}\psi & iff & for\ every\ a^* \in d(w), (\vec{a} \cdot a^*, w) \models \psi
\end{array}
$$

The clauses for atomic formulas, as well as propositional connectives, are almost trivial. The clause for the universal quantifier makes it clear why in def. 2.4 we required that if $\forall x_{n+1}\phi : n$ is well-formed, then ϕ must have type $n+1$. The fourth clause reflects truth conditions for the translation of $\Box$-formulas in Lewis' CT.

As it was the case for Kripke semantics, truth conditions for formulas containing propositional connectives $\wedge$, $\vee$, $\leftrightarrow$, existential quantifier $\exists$ and modal operator $\diamond$, are defined from the ones above. We conclude by defining truth and validity in counterpart semantics.

Definition 2.7 (Truth and Validity) *Typed modal formula $\phi : n$ in $\mathcal{L}_0$ is*

true in world w	*iff*	*it is satisfied by every n-assignment*
true in c-model $\mathcal{M}$	*iff*	*it is true in every world in $\mathcal{M}$*
valid in c-frame $\mathcal{F}$	*iff*	*it is true in every c-model based on $\mathcal{F}$*
valid in class $\mathcal{C}$ of c-frames	*iff*	*it is valid in every c-frame belonging to $\mathcal{C}$*

Let Δ be a set of typed formulas in $\mathcal{L}_0$, $\mathcal{M}$ is a c-model for Δ iff $\mathcal{M}$ verifies every formula in Δ; furthermore $\mathcal{F}$ is a c-frame for Δ iff every c-model based on $\mathcal{F}$ is a c-model for Δ. The equivalence results in the next paragraph will be useful in stating which c-frames are c-frames for our typed QML_t calculi.

We have thus provided a counterpart-theoretic semantics for quantified modal logic, which respects Lewis's intuitions in CT. Notice that the conversion lemma holds in counterpart semantics too.

Lemma 2.8 (Conversion lemma) *Let $\phi : m$ be a formula in $\mathcal{L}_0$ and $\vec{a}$ an n-assignment, it is the case that:*

$$(\vec{a}, w) \models \phi[t_1, \ldots, t_m] \quad iff \quad (\langle \vec{a}(t_1), \ldots, \vec{a}(t_m) \rangle, w) \models \phi$$

Our next step consists in comparing the analytical approach of semantics with the synthetic perspective of proof-theory, that is, we define QML calculi on typed languages and prove that they are sound and complete w.r.t. counterpart semantics. But before we state some equivalences between validity of formulas and features of c-frames; then in par. 2.3.4 we test counterpart semantics from an actualist point of view.

2.3.3 Formulas and features of c-frames

Our present aim is to establish some correspondence between validity of a formula ϕ in a c-frame $\mathcal{F}$ and characteristics of relations R and $C_{w,w'}$ in $\mathcal{F}$. The proofs of the following lemmas are not particularly complex, nonetheless they are useful to understand the new meaning that some well-known principle - as the Barcan formula and its converse - acquires in counterpart semantics. This is a further justification to our claim that counterpart semantics draws fresh distinctions on individuals in modal settings. We start with listing modal principles, which have already appeared in untyped languages in chapter 1, and some definitions necessary to state these correspondences. Let $\phi : n$, $\psi : n+1$ be formulas in $\mathcal{L}_0$:

A3.	$\Box\phi \to \phi$	axiom T,
A4.	$\Box\phi \to \Box\Box\phi$	axiom 4,
A5.	$\phi \to \Box \Diamond \phi$	axiom B,
A6.	$\forall x_{n+1}\psi[x_1, \ldots, x_n] \to \psi$	exemplification,
A12.	$\forall x_{n+1}\Box\psi \to \Box\forall x_{n+1}\psi$	Barcan formula,
A13.	$\Box\forall x_{n+1}\psi \to \forall x_{n+1}\Box\psi$	converse of Barcan formula,
A14.	$\neg E(x_1) \to \Box\neg E(x_1)$,	necessity of non-existence $N\neg E$,
A15	$\exists x_{n+1}\Box\psi \to \Box\exists x_{n+1}\psi$	Ghilardi formula GF,

Even if these formulas are different from the ones in chapter 1, we keep the same names and enumeration. Now consider the following conditions on c-frames.

Definition 2.9 *A c-frame $\mathcal{F}$ is*

existentially faithful	*iff*	*if wRw', $a \in d(w)$ and $C_{w,w'}(a,b)$, then $b \in d(w')$;*
fictionally faithful	*iff*	*if wRw', $b \in d(w')$ and $C_{w,w'}(a,b)$, then $a \in d(w)$;*
everywhere-defined	*iff*	*if wRw' and $a \in D(w)$, then there is $b \in D(w')$ s.t. $C_{w,w'}(a,b)$;*
total	*iff*	*if wRw' and $a \in d(w)$, then there is $b \in d(w')$ s.t. $C_{w,w'}(a,b)$;*
surjective	*iff*	*if wRw' and $b \in d(w')$, then there is $a \in d(w)$ s.t. $C_{w,w'}(a,b)$;*
classical	*iff*	*for every $w \in W$, $d(w) = D(w)$;*
reflexive	*iff*	*R is reflexive and for $w \in W$, $C_{w,w} \supseteq id$;*
transitive	*iff*	*R is transitive and for $w, w', w'' \in W$, $C_{w,w'} \circ C_{w',w''} \subseteq C_{w,w''}$;*
symmetric	*iff*	*R is symmetric and for $w, w' \in W$, $\breve{C}_{w,w'} \subseteq C_{w',w}$.*

All these constraints are quite clear. Notice that everywhere-definiteness tantamounts to the Counterpart-Existence Property in [53]. Moreover existential faithfulness and totality on the one hand, fictional faithfulness and surjectivity on the other one, are strictly linked one to another. In fact existential faithfulness affirms that every counterpart of something existing is an existent, whereas totality says that there is an existing counterpart for every existent. Again, by fictional faithfulness every existent is possibly counterpart to an existent, whereas by surjectivity every existent is actually counterpart to some existent. In both cases the interplay between the existential and universal quantifier does matter. These are quite different constraints, that Kripke semantics can not discriminate: as we shall see the former two collapse into the increasing inner domain condition, whereas the latter are flattened to decreasing inner domains.

There is no agreed nomenclature on the conditions listed above, but for reflexivity, transitivity and symmetry. *Existential faithfulness* and *fictional faithfulness* are taken from [53]; *surjectivity* appears in [19], whereas Kracht and Kutz name it *existential friendliness*. Finally in [10] and [19] there is no distinction between *totality* and *everywhere-definiteness*, as both authors consider classical c-frames.

Now it is possible to prove the following results, most of which appear in [19], [38].

Lemma 2.10 *A c-frame $\mathcal{F}$ is surjective iff $\forall x_{n+1}\Box\psi \rightarrow \Box\forall x_{n+1}\psi$ is valid in $\mathcal{F}$.*

Proof. $\Leftarrow$ Consider c-frame $\mathcal{F}$ validating BF, a world $w \in W$ and for all $w' \in W$ s.t. wRw' define $I(P^1, w') = \{b \in D(w') |$ *there exists* $a \in d(w)$ *s.t.* $C_{w,w'}(a,b)\}$. It is the case that $(\langle\rangle, w) \models \forall x_1 \Box P^1(x_1)$, as for all $a \in d(w)$, $(a, w) \models \Box P^1(x_1)$, i.e. for $w' \in W$, $b \in D(w')$, wRw' and $C_{w,w'}(a,b)$ imply $(b, w') \models P^1(x_1)$ by definition of $I(P^1, w')$. By BF we have that $(\langle\rangle, w) \models \Box\forall x_1 P^1(x_1)$, hence for $w' \in W$, $b \in d(w')$, wRw' implies $(b, w') \models P^1(x_1)$. It follows that $d(w') = I(P^1, w')$, i.e. for all $b \in d(w')$ there is $a \in d(w)$ s.t. $C_{w,w'}(a,b)$.

$\Rightarrow$ Suppose that $\mathcal{F}$ is surjective and $\langle a_1, \ldots, a_n\rangle$ satisfies $\forall x_{n+1}\Box\psi$ in w. This means that for every $a_{n+1} \in d(w)$, $w' \in W$, $b_1, \ldots, b_{n+1} \in D(w')$, if wRw' and

$C_{w,w'}(a_i, b_i)$ then $(\langle b_1, \dots, b_{n+1}\rangle, w') \models \psi$. Since $\mathcal{F}$ is surjective, for $b \in d(w')$ there exists $a \in d(w)$ s.t. $C_{w,w'}(a, b)$, thus $(\langle b_1, \dots, b_n\rangle, w') \models \forall x_{n+1}\psi$, that is, $\langle a_1, \dots, a_n\rangle$ satisfies $\Box\forall x_{n+1}\psi$ in w.

Lemma 2.11 *A c-frame $\mathcal{F}$ is existentially faithful iff $\Box\forall x_{n+1}\psi \to \forall x_{n+1}\Box\psi$ is valid in $\mathcal{F}$.*

Proof. $\Leftarrow$ Consider c-frame $\mathcal{F}$ validating CBF, a world $w \in W$ and for all $w' \in W$ s.t. wRw' define $I(P^1, w') = d(w')$. It is the case that $(\langle\rangle, w) \models \Box\forall x_1 P^1(x_1)$, as for all $w' \in W$, wRw' imply $(\langle\rangle, w') \models \forall x_1 P^1(x_1)$, i.e. for all $b \in d(w')$, $(b, w') \models P^1(x_1)$ by definition of $I(P^1, w')$. By CBF we have that $(\langle\rangle, w) \models \forall x_1\Box P^1(x_1)$, hence for all $a \in d(w)$, $w' \in W$, wRw' and $C_{w,w'}(a, b)$ imply $(b, w') \models P^1(x_1)$, i.e. $b \in d(w')$.

$\Rightarrow$ Suppose that $\mathcal{F}$ is existentially faithful and $\langle a_1, \dots, a_n\rangle$ satisfies $\Box\forall x_{n+1}\psi$ in w. This means that for every $w' \in W$, $b_1, \dots, b_n \in D(w')$, $b_{n+1} \in d(w')$ if wRw' and $C_{w,w'}(a_i, b_i)$, then $(\langle b_1, \dots, b_{n+1}\rangle, w') \models \phi$. Since $\mathcal{F}$ is existentially faithful, for every $a \in d(w)$, $C_{w,w'}(a, b)$ implies $b \in d(w')$, thus $(\langle a_1, \dots, a_{n+1}\rangle, w') \models \Box\psi$, i.e. $\langle a_1, \dots, a_n\rangle$ satisfies $\forall x_{n+1}\Box\psi$ in w.

Lemma 2.12 *Let $\mathcal{F}$ be an everywhere-defined c-frame, $\mathcal{F}$ is total iff $\exists x_{n+1}\Box\psi \to \Box\exists x_{n+1}\psi$ is valid in $\mathcal{F}$.*

Proof. $\Leftarrow$ Consider everywhere-defined c-frame $\mathcal{F}$ validating GF, a world $w \in W$, $a \in d(w)$ and for all $w' \in W$ s.t. wRw' define $I(P^1, w') = \{b \in D(w') | C_{w,w'}(a, b)\}$. It is the case that $(\langle\rangle, w) \models \exists x_1\Box P^1(x_1)$, as $(a, w') \models \Box P^1(x_1)$, that is for all $w' \in W$, $b \in D(w')$, wRw' and $C_{w,w'}(a, b)$ imply $(b, w') \models P^1(x_1)$ by definition. By GF we have that $(\langle\rangle, w) \models \Box\exists x_1 P^1(x_1)$, hence for all $w' \in W$, wRw' implies that there exists $b \in d(w')$ s.t. $C_{w,w'}(a, b)$.

$\Rightarrow$ Suppose that $\mathcal{F}$ is everywhere-defined and total and $\langle a_1, \dots, a_n\rangle$ satisfies $\exists x_{n+1}\Box\psi$ in w. This means that there exists $a_{n+1} \in d(w)$ s.t. for every $w' \in W$, for every $b_1, \dots, b_{n+1} \in D(w')$, if wRw' and $C_{w,w'}(a_i, b_i)$ then $(\langle b_1, \dots, b_{n+1}\rangle, w') \models \psi$. Since $\mathcal{F}$ is total, we assume that $b_{n+1} \in d(w')$, thus $(\langle b_1, \dots, b_n\rangle, w') \models \exists x_{n+1}\psi$, that is $\langle a_1, \dots, a_n\rangle$ satisfies $\Box\exists x_{n+1}\psi$ in w.

Lemma 2.13 *A c-frame $\mathcal{F}$ is fictionally faithful iff $\neg E(x_1) \to \Box\neg E(x_1)$ is valid in $\mathcal{F}$.*

Proof. $\Leftarrow$ Consider c-frame $\mathcal{F}$ validating A14, and $w, w' \in W$ s.t. $b \in d(w')$ and $C_{w,w'}(a, b)$. By hypotheses $(a, w) \models \diamond E(x_1)$ and by A14 $(a, w) \models E(x_1)$, therefore $a \in d(w)$.

$\Rightarrow$ Suppose that $\mathcal{F}$ is fictionally faithful and 1-assignment $\langle a\rangle$ satisfies $\neg E(x_1)$ in w. For reduction suppose that $(a, w) \models \diamond E(x_1)$, this means that there exist $w' \in W$ and counterpart $b \in D(w')$ of a s.t. wRw' and $b \in d(w')$. Since $\mathcal{F}$ is fictionally faithful, we should have $a \in d(w)$ against hypothesis.

Lemma 2.14 *A c-frame $\mathcal{F}$ is classical iff $\forall x_{n+1}\psi[x_1, \dots, x_n] \to \psi$ is valid in $\mathcal{F}$.*

Proof. $\Leftarrow$ Consider c-frame $\mathcal{F}$ validating A6, a world $w \in W$ and define $I(P^1, w) = d(w)$. It follows that $(\langle\rangle, w) \models \forall x_1 P^1(x_1) : 0$ and by the conversion lemma for every $a \in D(w)$, $(a, w) \models \forall x_1 P^1(x_1) : 1$. By A6 $(a, w) \models P^1(x_1)$ and $a \in d(w)$.

$\Rightarrow$ Suppose that $\mathcal{F}$ is classical and assignment $\langle a_1, \dots, a_{n+1}\rangle$ satisfies $\forall x_{n+1}\psi[x_1, \dots, x_n]$ in w. This means that for every $a^* \in d(w)$, $(\langle a_1, \dots, a_n, a^*\rangle, w) \models \psi$. By hypothesis a_{n+1} belongs to $d(w) = D(w)$, thus $(\langle a_1, \dots, a_{n+1}\rangle, w) \models \psi$.

Lemma 2.15 *A c-frame $\mathcal{F}$ is reflexive iff $\Box\phi \to \phi$ is valid in $\mathcal{F}$.*

Proof. $\Leftarrow$ Consider c-frame $\mathcal{F}$ validating A3, a world $w \in W$, $a \in D(w)$ and define $I(P^1, w') = \{b \in D(w') | C_{w,w'}(a, b)\}$ whenever wRw'. By definition $(a, w) \models \Box P^1(x_1)$ and by A3 $(a, w) \models P^1(x_1)$. It follows that $C_{w,w'}(a, a)$.

$\Rightarrow$ Suppose that $\mathcal{F}$ is reflexive and $\vec{a}$ satisfies $\Box\phi$ in w. Since R is reflexive and for $w \in W$, $C_{w,w} \supseteq id$, $\vec{a}$ is bound to satisfy ϕ in w.

Lemma 2.16 *A c-frame $\mathcal{F}$ is transitive iff $\Box\phi \to \Box\Box\phi$ is valid in $\mathcal{F}$.*

Proof. $\Leftarrow$ Consider c-frame $\mathcal{F}$ validating A4, a world $w \in W$, $a \in D(w)$ and define $I(P^1, w')$ as in the previous lemma. By definition $(a, w) \models \Box P^1(x_1)$ and by A4 $(a, w) \models \Box\Box P^1(x_1)$. It follows that for every w', w'', $b \in D(w')$ and $c \in D(w'')$, if wRw', $w'Rw''$, $C_{w,w'}(a, b)$ and $C_{w',w''}(b, c)$, then $(c, w'') \models P^1(x_1)$. Therefore $C_{w,w''}(a, c)$

$\Rightarrow$ Suppose that $\mathcal{F}$ is transitive and $\vec{a}$ satisfies $\Box\phi$ in w. Since R is transitive and for $w, w', w'' \in W$, $C_{w,w'} \circ C_{w',w''} \subseteq C_{w,w''}$, $\vec{a}$ satisfies also $\Box\Box\phi$ in w.

Lemma 2.17 *A c-frame $\mathcal{F}$ is symmetric iff $\phi \to \Box \diamond \phi$ is valid in $\mathcal{F}$.*

Proof. $\Leftarrow$ Consider c-frame $\mathcal{F}$ validating A5, a world $w \in W$, $a \in D(w)$ and define $I(P^1, w) = \{a\}$, the empty set in all the other cases. By definition $(a, w) \models P^1(x_1)$ and thus $(a, w) \models \Box \diamond P^1(x_1)$. It follows that for $w' \in W$, $b \in D(w')$, if wRw' implies $C_{w,w'}(a, b)$, then there exists w'' and $c \in D(w'')$ s.t. $(c, w'') \models P^1(x_1)$. Therefore $w'' = w$, $c = a$, $w'Rw$ and $C_{w',w}(b, a)$.

$\Rightarrow$ Suppose that $\mathcal{F}$ is symmetric and $\vec{a}$ satisfies ϕ in w. Since R is symmetric and for every $w, w' \in W$, $\breve{C}_{w,w'} \subseteq C_{w',w}$, $\vec{a}$ satisfies $\Box \diamond \phi$ in w.

Lemmas 2.10 and 2.12 were originally proved in [59]. In [10], [19] and [59] CBF is expected to be valid, as Corsi and Ghilardi consider only classical c-frames, which are in particular existentially and fictionally faithful.

If we assume that the counterpart relation is everywhere-defined and it is identity, then we are back to the validity conditions in Kripke semantics for the previously listed formulas: for instance, BF and CBF correspond to nested inner domain conditions. Hence it is clear in which way counterpart-theoretic approach is more general than Kripke's one. We can provide even more explicit examples of differences in the meaning of formulas, which are completely obliterated in K-frames. Consider formulas CBF and GF, which are provable in $L \supseteq Q.K$ as follows:

$$
\begin{array}{ccc}
\forall x\phi \to \phi & \text{A6} & \phi \to \exists x\phi \\
\hline
\Box\forall x\phi \to \Box\phi & \text{T3} & \Box\phi \to \Box\exists x\phi \\
\hline
\Box\forall x\phi \to \forall x\Box\phi & \text{R3} & \exists x\Box\phi \to \Box\exists x\phi
\end{array}
$$

On a proof-theoretic level these two principles go together, as their proofs show the same axioms and inference rules. Furthermore in Kripke semantics each of these formulas is valid iff the increasing inner domain condition holds. On the contrary, counterpart semantics discriminates between two distinct constraints: *existential faithfulness* and *totality* respectively, which say quite different things, and collapse into one only in K-frames. We prove this fact.

Consider existential faithfulness and assume identity as counterpart relation. We obtain that for every $w, w' \in W$, if wRw', $a \in d(w)$ and $a = b$ then $b \in d(w')$; since counterpart relation is defined everywhere in K-frames, for every $w, w' \in W$, wRw' implies $d(w) \subseteq d(w')$. As regards totality, for every $w, w' \in W$, wRw' and $a \in d(w)$ imply that a exists also in $d(w')$. Thus it turns out that if we assume identity as everywhere-defined counterpart relation, a c-frame is existentially faithful iff it is total iff it satisfies the increasing inner domain condition.

There are even more astonishing examples of the flattening power of Kripke semantics. In K-frames BF and $N\neg E$ on the one hand, and CBF and the necessity of existence $E(x) \to \Box E(x)$ on the other one, are equivalent formulas, as the formers correspond to increasing inner domain condition, whereas the latter refer to decreasing inner domains[6]. In c-frames formulas CBF and $E(x) \to \Box E(x)$ are still equivalent, both correspond to existential faithfulness; whereas BF and $N\neg E$ have no more the same meaning: the Barcan formula is valid in all and only the surjective c-frames, $N\neg E$ is valid iff a c-frame is fictionally faithful. These two constraints collapse into one if we take identity as counterpart relation and we deem it everywhere-defined. As to fictional faithfulness, suppose that for every $w, w' \in W$, if wRw', $b \in d(w')$ and $a = b$, then $a \in d(w)$. This tantamounts to: wRw' implies $d(w') \subseteq d(w)$ for every $w, w' \in W$. As to surjectivity, assume that for every $w, w' \in W$, if wRw' and $b \in d(w')$, then there exists $a \in d(w)$ s.t. $a = b$, that is, for every $w, w' \in W$, if wRw' and $b \in d(w')$, then b exists in $d(w)$. Both conditions collapse into decreasing inner domains.

These are probably the most striking distinctions counterpart semantics can draw, which are completely removed by Kripke semantics. Notice that the different validity conditions for BF and $N\neg E$ constitute the reason for Kripke-incompleteness of systems $Q^E.BF$ and $Q^E.CBF + BF$ stated in chapter 1; this is an anticipation of the formal proof given in appendix A. In K-frames BF and $N\neg E$ are equivalent, but they are not at all the same from a proof-theoretic point of view, as it is disclosed by counterpart semantics. All these results agree with Corsi's claim that "duality is not intrinsic to the meaning of CBF and BF but rather depends on general features of Kripke semantics"[7]. Maybe BF and CBF are one the reverse implication of the other from a graphical or Kripke-theoretic point of view, but surely not from a proof- or counterpart-theoretic perspective.

[6]See [28], pp. 181-182, for a formal proof of these facts.

[7][21], pg. 1.

Finally we remark that validity of axioms T, 4, B does not depend only on characteristics of accessibility relation R, but also on features of counterpart relation $C_{w,w'}$. In c-frames the interaction of these two relations determines validity for those modal principles. It would be interesting to study which formulas would be valid, given only conditions on R or $C_{w,w'}$. It is plausible to state that accessibility relation R in c-frame $\mathcal{F}$ is reflexive, transitive or symmetric iff respectively $\Box\phi \to \phi$, $\Box\phi \to \Box\Box\phi$, $\phi \to \Box \diamond \phi$ are valid, where ϕ has type 0, i.e. it is a sentence. I have no idea if similar conditions can be given also for the counterpart relation, but it would not be only an oddity, as axiom P6 in Lewis' CT requires that "anything in a world is a counterpart of itself". If we aim at formalizing Lewis' account in a modal language, we need some formula expressing reflexivity of $C_{w,w'}$.

2.3.4 Actualism revisited

In par. 2.1.2 we focused on three features of varying domain K-models, which are not completely satisfactory from an actualist point of view:

1. the presence of *possibilia* at least in the meta-language of Kripke semantics;
2. the recourse to existence predicate E and free logic to recover quantification;
3. the violation of the principle of Strong Actualism, according to which something not existing in a world w cannot have properties in w.

We show that counterpart semantics can deal with all these problems and solve them, while giving actualism the first adequate formal presentation probably. As regards the presence of *possibilia* in the meta-language of semantics, we assume that for every $w \in W$, $D(w) = d(w)$, that is the individuals, which it makes sense to talk about in w, are all and only the objects existing in w. By this choice we recover classical quantification, thus we need neither existence predicate E nor free logic.

Pay attention to the different consequences of assuming $D(w) = d(w)$ in Kripke and counterpart semantics. In the former this constraint validates classical quantification, but classical quantification implies some principle the kripkean reading of which is rejected by actualism, i.e. the converse of the Barcan formula. Hence Kripke semantics seems to force actualists towards varying domain K-models and free logic. In counterpart semantics we have none of this, we can set $D(w) = d(w)$ for every $w \in W$ and refuse possibilism and free logic at once. Clearly CBF is still valid in this framework, but its interpretation clashes no more with the actualist account, as it only corresponds to the existential faithfulness condition, which is actualistically acceptable.

As to the third point, if individual a does not belong to $D(w')$, we need not to ascribe properties or relationships to a in w' to avoid truth-value gaps. In evaluating modal formulas w.r.t. a, we consider features of a only in the actual world, and of its counterpart in the other accessible worlds. Thus counterpart semantics seems to provide a nice formalization to actualism, as it is free from all the three faults we listed at the beginning.

2.3.5 Typed QML_t calculi

In this paragraph we introduce typed QML_t calculi for quantified modal logic on languages $\mathcal{L}_t$ and $\mathcal{L}_t^E$, but before we have to discuss the role of free logic in typed contexts, as such languages already make existential presuppositions explicit. Consider the following example drawn from [37], in classical logic we have the following proof:

$$\begin{array}{cc} \forall x P(x) \to P(x) & \text{A6} \\ \hline P(x) \to \exists x P(x) & \text{A6} \\ \hline \forall x P(x) \to \exists x P(x) & \text{by transitivity} \end{array}$$

In this proof non-emptiness of quantification domain is presupposed, but not explicitly stated. Free logic is an attempt to solve this problem, by paraphrasing the proof above as:

$$\begin{array}{cc} \forall x P(x) \to (E(x) \to P(x)) & \text{A7} \\ \hline P(x) \wedge E(x) \to \exists x P(x) & \text{A7} \\ \hline \forall x P(x) \to (E(x) \to \exists x P(x)) & \text{by transitivity} \end{array}$$

where the existential presupposition is clearly stated by means of formula $E(x)$. On the contrary we need not predicative constant E in typed languages, as formulas carry in their types the context w.r.t. which they are meaningful:

$$\begin{array}{cc} \forall x P(x) \to P(x) : 1 & \text{A6} \\ \hline P(x) \to \exists x P(x) : 1 & \text{A6} \\ \hline \forall x P(x) \to \exists x P(x) : 1 & \text{by transitivity} \end{array}$$

The axioms in this proof have type 1, thus are meaningful w.r.t. strings of single objects in the domain of quantification, which is meant to be non-empty. However there is still room for existential predicate E in counterpart semantics.

In par. 2.2.1 we distinguished inner and outer domains in c-frames, so that we can model a possibilist account of counterparts as well, but this fact entails the failure of classical principles of quantification. We have to reconsider predicative constant E in case that we aim at axiomatizing the set of validities in general c-frames. Therefore in this section we consider classical and free typed calculi, while Kripke's theory of quantification makes no sense in the present framework.

We start with listing schemes of axioms and inference rules. We keep the same names and enumeration in par. 1.1.2, as there is a close link with principles on untyped languages and there is no risk of confusion. We begin with the only four postulates appearing in each QML_t system, where $\phi : n+1$ and $\psi : n$, $\theta : n$ are typed formulas in $\mathcal{L}_0$:

A1.	tautologies of classical propositional calculus,	
A2.	$\Box(\theta \to \psi) \to (\Box\theta \to \Box\psi)$	distribution axiom,
R1.	$\frac{\theta \to \psi, \theta}{\psi}$	separation rule,
R2.	$\frac{\psi}{\Box\psi}$	necessitation.

As it is the case for QML calculi, these four postulates give typed QML_t calculi with modal base K. Of course, for obtaining a different modal base - such as T, $S4$, B or $S5$ - we have to use an appropriate combination of schemes of axioms A3/A5.

To add quantification, we first consider the classical theory consisting in the following two principles:

A6.	$\forall x_{n+1}\phi[x_1,\dots,x_n] \to \phi$	exemplification,
R3.	$\frac{\psi[x_1,\dots,x_n]\to\phi}{\psi\to\forall x_{n+1}\phi}$	universal instantiation.

In the case that our language contains existence predicate E, we can consider a typed version of free logic:

A7.	$\forall x_{n+1}\phi[x_1,\dots,x_n] \to (E(x_{n+1}) \to \phi)$	E-exemplification,
R4.	$\frac{\psi[x_1,\dots,x_n]\to(E(x_{n+1})\to\phi)}{\psi\to\forall x_{n+1}\phi}$	universal E-instantiation.

We checked that substitution commutes with all the propositional connectives and quantifiers, and applies to atomic and modal formulas; for the modal operator we have only the following one-way implication:

A16. $\Box\psi[t_1,\dots,t_m] \to \Box(\psi[t_1,\dots,t_m])$ continuity axiom.

This is a major difference in comparison to untyped languages for quantified modal logic. Validity of axiom A16's converse implication

A16'. $\Box(\psi[t_1,\dots,t_m]) \to \Box\psi[t_1,\dots,t_m]$

depends on the form of terms $t_1,\dots,t_m$. In the case that these are $n+1$-terms $x_1,\dots,x_n$, A16' holds by assuming that the counterpart relation is everywhere-defined[8]. We refer to [10] for mathematically-motivated models in which 16' does not hold.

In the following table we summarize typed QML_t calculi for quantified modal logic, with the respective schemes of axioms and inference rules. We choose to preserve names of QML calculi, with a subscript t, in order to make clear the relationship with systems in par. 1.1.2.

Definition 2.18 (Typed QML_t calculi) *The following typed QML_t calculi consist in schemes of axioms A1, A2, A16 and inference rules R1, R2, with in addition the respective postulates:*

[8]For a formal proof of this fact we refer to [19], p. 23.

calculi	*schemes of axioms*	*inference rules*
$Q.K_t$	*A6,*	*R3*
$Q.K + BF_t$	*A6, A12*	*R3*
$Q^E.K_t$	*A7*	*R4*
$Q^E.K + BF_t$	*A7, A12*	*R4*
$Q^E.K + CBF_t$	*A7, A13*	*R4*
$Q^E.K + CBF + BF_t$	*A7, A12, A13*	*R4*

We recall that the first two calculi are on language $\mathcal{L}_t$, whereas the last four ones are on $\mathcal{L}_t^E$. Hereafter by L we refer to a generic typed QML_t calculus, while the notions of proof and theorem are defined as in par. 1.1.2.

On the other hand, formula $\phi : n \in \mathcal{L}_0$ is *derivable* in L from set Δ of formulas in $\mathcal{L}_0$ - $\Delta \vdash_L \phi$ in short - iff there are $\phi_1, \ldots, \phi_n \in \Delta$ and substitutions $[\vec{x}_1], \ldots, [\vec{x}_n], [\vec{x}]$ s.t. $\vdash_L \phi_1[\vec{x}_1] \wedge \ldots \wedge \phi_n[\vec{x}_n] \to \phi[\vec{x}]$.

We summarize inclusion relationships of free logic systems in the following table, which parallels the analogous table for QML calculi.

$$\begin{array}{|ccc|}
\hline
Q^E.K_t & \subseteq & Q^E.K + CBF_t \\
\subseteq & & \subseteq \\
Q^E.K + BF_t & \subseteq & Q^E.K + CBF + BF_t \\
\hline
\end{array}$$

By the equivalence results in par. 2.3.3, it is easy to check that $\mathcal{F}$ is a c-frame for a calculus in the first column of the following table iff it satisfies the constraints in the second column.

calculi	c-frames
$Q.K_t$	classical
$Q.K + BF_t$	classical, surjective
$Q^E.K_t$	all
$Q^E.K + BF_t$	surjective
$Q^E.K + CBF_t$	existentially faithful
$Q^E.K + CBF + BF_t$	existentially faithful, surjective

We conclude this section by discussing the translation of Lewis' Counterpart Theory in modal language.

Lewis' quantified modal logic

In this last paragraph we describe a research program concerning Lewis' CT, and the development of a first-order modal calculus corresponding to that theory. The idea behind Lewis' translation schemes is that modal formula ϕ is true in a world w iff translation ϕ^w is plainly true: it would be nice to work out a quantified modal calculus that proves all and only the formulas, the translations of which are provable in CT.

This project is unlewisian in a very deep sense. In the introduction we remarked that one of the aims of CT was *extensionalizing the modal discourse*, therefore a quantified modal calculus equivalent to CT, would be of no interest to Lewis. On the contrary, a proof of the impossibility of such a formal system would show that modal logic is not the appropriate instrument to talk about possibility and necessity for individuals, at least as far as we are concerned with the intuitions on which CT is based.

For the time being we state some partial results on the way to modal CT. Since CT is a consistent first-order theory, by completeness there is a class $\mathcal{C}$ of first-order models w.r.t. which CT is adequate. Consider now the class $\mathcal{C}'$ of counterpart frames s.t.

1. counterpart relation is reflexive,
2. for all $w \in W$, $d(w) = D(w)$,
3. for all $w, w' \in W$, $D(w) \cap D(w') = \emptyset$.

If formula ϕ is not true in a world w, in a c-model $\mathcal{M}'$ belonging to $\mathcal{C}'$, then translation ϕ^w can be falsified in a model $\mathcal{M}$ for CT. On the other hand if ϕ^w is false in a model $\mathcal{M}$ for CT, then we can construct a c-model $\mathcal{M}' \in \mathcal{C}'$ s.t. ϕ is false in $w \in \mathcal{M}'$. Therefore it is possible to prove the following lemma:

Lemma 2.19 *A formula ϕ is valid in the class $\mathcal{C}'$ of counterpart frames iff $\forall w \phi^w$ is valid in $\mathcal{C}$.*

By adequacy formula $\forall w \phi^w$ is valid in $\mathcal{C}$ iff CT proves it, then we only have to find a quantified modal logic adequate w.r.t. $\mathcal{C}'$, in order to show that there is a modal Counterpart Theory s.t. $modalCT \vdash \phi$ iff $CT \vdash \forall w \phi^w$. To fulfil this aim it is clear that we need a formula expressing reflexivity of the counterpart relation, thus problems in par. 2.3.3 become even more pressing.

2.4 Adequacy Results in Counterpart Semantics

This section is the most important part of the present chapter, as we prove counterpart-completeness for typed QML_t calculi introduced in par. 2.3.5. The whole proof appears in [10], and it is a major improvement in comparison to analogous proofs in [19], [34] and [38]. Our contribution consists in extending Ghilardi's method to typed systems based on free logic. We show that typed QML_t calculi have stronger completeness properties than their untyped companions, for instance free logic typed calculi $Q^E.K + BF_t$ and $Q^E.K + CBF + BF_t$ are complete w.r.t. the classes of c-frames for them, while the corresponding QML calculi are Kripke-incomplete, as we stated in chapter 1. Thus counterpart semantics enjoys a wider generality than Kripke's one.

Hereafter we list the adequacy results to be proved.

Theorem 2.20 (Adequacy) *The following typed QML_t calculi are adequate w.r.t. the respective classes of c-frames:*

calculi	*c-frames*
$Q.K_t$	*classical*
$Q.K + BF_t$	*classical, surjective*
$Q^E.K_t$	*all*
$Q^E.K + BF_t$	*surjective*
$Q^E.K + CBF_t$	*existentially faithful*
$Q^E.K + CBF + BF_t$	*existentially faithful, surjective*

In [10] Ghilardi proves counterpart-completeness for system $Q.S4_t$, and explains how his method can be extended to weaker modalities. In [19] Corsi considers also system $Q.K + BF_t$ with the Barcan formula, but they both stick to typed QML_t calculi based on the classical theory of quantification.

We skip soundness results, they are not trivial as in the untyped case, nonetheless they are not particularly complex either. In the next paragraph we directly tackle counterpart-completeness.

2.4.1 Ghilardi's method

As it was the case for the canonical model method, also Ghilardi proves counterpart-completeness by contraposition; he shows that if QML_t calculus L does not prove formula $\phi : n \in \mathcal{L}_t$, then there exists a c-model $\mathcal{M}^L$ for L falsifying ϕ. Moreover $\mathcal{M}^L$ is based on a c-frame for L.

The main difference in comparison to the completeness proof w.r.t. Kripke semantics lies in the worlds appearing in counterexample c-model $\mathcal{M}^L$. This time they are not L-saturated sets of formulas, but models for certain first-order typed theories to be specified.

We start with introducing these theories, the models of which constitute the worlds in our c-model falsifying ϕ. First of all we need some definitions.

Let L be a typed QML_t calculus on language $\mathcal{L}_0$, first-order typed language $\mathcal{L}_c$ is obtained by adding to a first-order typed language a new n-ary predicative constant $P_{\Box\phi}$ for every formula $\Box\phi : n$ in $\mathcal{L}_0$. Language $\mathcal{L}_c$ is not to be confused with $\mathcal{L}_0$: the latter is a first-order typed modal language for L, whereas the former is a first-order typed (non-modal) language. We define translation function c from modal formulas in $\mathcal{L}_0$ to formulas in $\mathcal{L}_c$.

Definition 2.21 (Translation) *Let $\psi : n$ be a formula in $\mathcal{L}_0$, translation $\psi_c : n \in \mathcal{L}_c$ is inductively defined as follows:*

- *if ψ is an atomic formula, then $(P^m(t_1, \ldots, t_m))_c = P^m(t_1, \ldots, t_m)$;*
- *if ψ has form $\neg\theta$, then $\psi_c = \neg(\theta_c)$;*
- *if ψ has form $\theta \to \theta'$, then $\psi_c = \theta_c \to \theta'_c$;*
- *if ψ has form $\forall x_{n+1}\theta$, then $\psi_c = \forall x_{n+1}(\theta_c)$;*
- *if ψ has form $\Box\theta(t_1, \ldots, t_m)$, then $\psi_c = P_{\Box\theta}(t_1, \ldots, t_m)$.*

For each typed QML_t calculus L we consider first-order typed theory T_c, containing as logical axioms and inference rules postulates A6, R3 for $L = Q.K_t, Q.K{+}BF_t$, and postulates A7, R4 whenever L is based on free logic. In addition T_c contains as proper axioms any formula ϕ_c s.t. $\phi : n$ is provable in L. Ghilardi proves the following lemma for L based on classical logic, we show that it holds also for systems on free logic.

Lemma 2.22 *A formula $\phi : n \in \mathcal{L}_0$ is provable in L iff $\phi_c : n \in \mathcal{L}_c$ is provable in T_c.*

Proof. $\Rightarrow$ If $\phi : n$ is provable in L then $\phi_c : n$ is an axiom of T_c, thus ϕ_c is provable in T_c.

$\Leftarrow$ Suppose that $\phi_c : n$ is provable in T_c. We show that $\vdash_L \phi$ by induction on the proof of $\phi_c : n$ in T_c. If $\phi_c : n$ is a logical axiom of T_c, then also $\vdash_L \phi$; in case that $\phi_c : n$ is a proper axiom of T_c, we have $\vdash_L \phi$ as well by definition of T_c. As to the inductive step, notice that T_c as no deductive machinery which is not already available in L.

First-order models

The worlds in c-model $\mathcal{M}^L$ falsifying ϕ are models for first-order typed theory T_c, hence we have to introduce the notion of first-order model.

Definition 2.23 (First-order model) *A model $\mathcal{M}$ for first order typed language $\mathcal{L}_c$ - $\mathcal{L}_c$-model for short - is an ordered 3-tuple $\langle D, d, I \rangle$, defined as follows:*

- *D is a non-empty set, the outer domain;*
- *d is a subset of D, the inner domain;*

- *interpretation I assigns an n-ary relation $I(P^n)$ on D^n to every n-ary predicative constant P^n, and if our language includes predicative constant E, then $I(E) = d$.*

We say that an $\mathcal{L}_c$-model is *classical* whenever $D = d$. The definitions of n-assignment and n-valuation are similar to the ones for the modal case, with the obvious changes. Hereafter we present truth conditions for a first-order typed formula $\phi : n$ in $\mathcal{L}_c$-model $\mathcal{M}$ w.r.t. n-assignment $\vec{a}$.

Definition 2.24 (Satisfaction) *The relation of satisfaction in $\mathcal{M}$ for formula $\phi : n \in \mathcal{L}_c$ w.r.t. n-assignment $\vec{a}$ is inductively defined as follows:*

$$\begin{array}{rcl} \mathcal{M} \models_{\vec{a}} P^m(t_1, \dots, t_m) & iff & \langle \vec{a}(t_1), \dots, \vec{a}(t_m) \rangle \in I(P^n) \\ \mathcal{M} \models_{\vec{a}} \neg\psi & iff & not\ \mathcal{M} \models_{\vec{a}} \psi \\ \mathcal{M} \models_{\vec{a}} \phi \to \psi & iff & not\ \mathcal{M} \models_{\vec{a}} \phi\ or\ \mathcal{M} \models_{\vec{a}} \psi \\ \mathcal{M} \models_{\vec{a}} \forall x_{n+1}\phi & iff & for\ every\ a^* \in d,\ \mathcal{M} \models_{\vec{a} \cdot a^*} \phi \end{array}$$

Formula $\phi : n$ is true in $\mathcal{M}$ iff it is satisfied by every n-assignment, and ϕ is valid iff it is true in every $\mathcal{L}_c$-model. The following definitions are taken from [10]: an n-type is a set Δ of n-formulas in $\mathcal{L}_c$, and Δ is T_c-*consistent* iff no finite conjunction of formulas in Δ is refutable in T_c. An n-type Δ is *realized* in an $\mathcal{L}_c$-model $\mathcal{M}$ iff for some $\vec{a} \in D^n$, $\mathcal{M} \models_{\vec{a}} \Delta$, and $\mathcal{M}$ is an $\mathcal{L}_c$-model for Δ whenever it verifies all the formulas in Δ.

In what follows we simply write T_c-model $\mathcal{M}$, whenever $\mathcal{M}$ is an $\mathcal{L}_c$-model for T_c.

Ghilardi proves completeness of first-order typed theories based on the classical theory of quantification w.r.t. classical $\mathcal{L}_c$-model, we extend this result to first-order typed theories on free logic:

Theorem 2.25 (First-order completeness) *If Δ is a T_c-consistent n-type in first-order typed language $\mathcal{L}_c$, then there exists a T_c-model $\mathcal{M}$ realizing Δ.*

For a formal proof of this theorem in the case that T_c is based on free logic, we refer to Appendix B.

The subordination frame method

In this paragraph we define the counterpart model $\mathcal{M}^L$ falsifying $\phi : n$, for $\phi \in \mathcal{L}_0$ not provable in typed QML_t calculus L. The main difference in comparison to the canonical model of QML calculi consists in the counterpart relation: we need a suitable way of defining function C on couples of T_c-models. This is the content of the next definition, where $X(\vec{a}, \vec{b})$ is an abbreviation for $X(a_i, b_i)$ for $1 \leq i \leq n$.

Definition 2.26 (Admissibility) *Let w, w' be T_c-models, relation $X \subseteq D_w \times D_{w'}$ is* admissible *iff every $n \geq 1$, for every $\vec{a} \in D^n_w$, $\vec{b} \in D^n_{w'}$, if $R(\vec{a}, \vec{b})$ then*

$$w \models_{\vec{a}} P_{\Box\psi}(x_1, \dots, x_n) \quad implies \quad w' \models_{\vec{b}} \psi_c$$

for every formula $\psi : n$.

Ghilardi makes use of the *subordination frame* technique in [46] and inductively defines counterpart frame $\mathcal{F}^L$, which underlies c-model $\mathcal{M}^L$ falsifying $\phi : n$. The root of $\mathcal{F}^L$ is any T_c-model realizing T_c-consistent n-type $\{\neg\phi_c : n\}$. Now assume that node w has already been defined, for every couple $\langle \vec{a}, \Box\psi : n\rangle$ s.t. not $w \models_{\vec{a}} P_{\Box\psi}(x_1, \ldots, x_n)$, we consider T_c-model w', admissible relation $C_{w,w'} \subseteq D_w \times D_{w'}$, and n-tuple $\vec{b} \in D^n_{w'}$ s.t. $C_{w,w'}(\vec{a}, \vec{b})$ and not $w' \models_{\vec{b}} \psi_c$. In order to have such a T_c-model, we need the following lemma.

Lemma 2.27 *Let w be a T_c-model s.t. not $w \models_{\vec{a}} P_{\Box\psi}(x_1, \ldots, x_n)$, then n-type $\{\theta_c : n | w \models_{\vec{a}} P_{\Box\theta}(x_1, \ldots, x_n)\} \cup \{\neg\psi_c\}$ is T_c-consistent.*

Proof. For reduction suppose that there are $\theta_{c1}, \ldots, \theta_{ck} \in \{\theta_c : n | w \models_{\vec{a}} P_{\Box\theta}(x_1, \ldots, x_n)\}$ s.t.

$$\vdash_{T_c} \bigwedge \theta_c \to \psi_c$$

By lemma 2.22

$$\vdash_L \bigwedge \theta \to \psi$$

and by T3

$$\vdash_L \bigwedge \Box\theta \to \Box\psi$$

By lemma 2.22 again,

$$\vdash_{T_c} \bigwedge P_{\Box\theta}(x_1, \ldots, x_n) \to P_{\Box\psi}(x_1, \ldots, x_n)$$

and thus $w \models_{\vec{a}} P_{\Box\psi}(x_1, \ldots, x_n)$ against hypothesis.

Since n-type $\{\theta_c : n | w \models_{\vec{a}} P_{\Box\theta}(x_1, \ldots, x_n)\} \cup \{\neg\psi_c\}$ is T_c-consistent, there is a T_c model w' realizing it; i.e. there exist $\vec{b} \in D^n_{w'}$ s.t. $w' \models_{\vec{b}} \{\theta_c : n | w \models_{\vec{a}} P_{\Box\theta}(x_1, \ldots, x_n)\} \cup \{\neg\psi_c\}$. In the next lemma we define an admissible relation between individuals in w and w'.

Lemma 2.28 *Let w, w' be T_c-models s.t. $w' \models_{\vec{b}} \{\theta_c : n | w \models_{\vec{a}} P_{\Box\theta}(x_1, \ldots, x_n)\}$. There exists an admissible relation $C_{w,w'} \subseteq D_w \times D_{w'}$ s.t. $C_{w,w'}(\vec{a}, \vec{b})$.*

Proof. We define an admissible relation $C_{w,w'}$ on $D_w \times D_{w'}$. For $a \in D_w$, $b \in D_{w'}$, we set $C_{w,w'}(a, b)$ iff there is n-term t s.t. $\vec{a}(t) = a$ and $\vec{b}(t) = b$. It is trivial to prove that $C_{w,w'}(\vec{a}, \vec{b})$ holds, then we check the constraint on admissible relations. Suppose that $\vec{c} \in D^k_w$, $\vec{d} \in D^k_{w'}$, $C_{w,w'}(\vec{c}, \vec{d})$ and $w \models_{\vec{c}} P_{\Box\psi}(x_1, \ldots, x_k)$. By definition of $C_{w,w'}$, for each c_i there is n-term t_i s.t. $\vec{a}(t_i) = c_i$, for $1 \leq i \leq k$; hence by hypothesis

$$w \models_{\vec{a}(t_1), \ldots, \vec{a}(t_k)} P_{\Box\psi}(x_1, \ldots, x_k)$$

and by the conversion lemma, $w \models_{\vec{a}} P_{\Box\psi}(t_1, \ldots, t_k)$. Since in L we have continuity axiom A16 and w is a T_c-model

$$w \models_{\vec{a}} P_{\Box(\psi[t_1, \ldots, t_k])}(x_1, \ldots, x_n)$$

but $w' \models_{\vec{b}} \{\theta_c : n | w \models_{\vec{a}} P_{\Box\theta}(x_1, \dots, x_n)\}$, therefore

$$w' \models_{\vec{b}} (\phi[t_1, \dots, t_k])_c = \phi_c[t_1, \dots, t_k]$$

where $\vec{b}(t_i) = d_i$ for $1 \leq i \leq k$, by definition of $C_{w,w'}$. By the conversion lemma $w' \models_{\vec{d}} \phi_c$ as desired.

The worlds $w, w', \dots$ in counterpart frame $\mathcal{F}^L$ are generated as above, as well as the various counterpart relations $C_{w,w'}, \dots$. We set wRw' iff there exists a non-empty admissible relation $C_{w,w'}$ between w and w'; in addition for every $w \in W$, $D(w) = D_w$ and $d(w) = d_w$. If we are considering $L = Q.K_t, Q.K + BF_t$, then $D(w) = d(w) = D_w$ and our counterpart frame $\mathcal{F}^L$ is classical. We define interpretation I^L on c-frame $\mathcal{F}^L$ by gluing together the various interpretations I_w, for each T_c-model w in W^L.

Definition 2.29 *Interpretation I^L of language $\mathcal{L}_0$ into c-frame $\mathcal{F}^L$ is s.t.*

- *for every $a_1, \dots, a_n \in D(w)$, $\vec{a} \in I^L(P^n, w)$ iff $\vec{a} \in I_w(P^n)$;*
- *if $\mathcal{L}_0$ includes predicative constant E, then $I^L(E, w) = d^L(w)$.*

We easily check that interpretation I^L satisfies constraints on interpretations in c-frames. The ordered couple $\langle \mathcal{F}^L, I^L \rangle$ constitutes our counterpart model $\mathcal{M}^L$ w.r.t typed QML_t calculus L. In order to simplify our notation, in next paragraphs we eliminate superscript L.

Truth lemma

Thus far we proved that if formula $\phi : n \in \mathcal{L}_0$ is not provable in L, then by theorem 2.25 there exists T_c-model w s.t. not $w \models \phi_c$. But w is the root in c-model $\mathcal{M}$ w.r.t. L, thus we aim at showing that if not $w \models_{\vec{a}} \phi_c$ then formula ϕ is falsified in $\mathcal{M}$. This is exactly the content of the next result, which corresponds to lemma 1.18 in chapter 1.

Lemma 2.30 (Truth lemma) *For every $w \in W$, $\phi : n \in \mathcal{L}_0$ and $\vec{a} \in D(w)^n$,*

$$(\vec{a}, w) \models \phi \quad iff \quad w \models_{\vec{a}} \phi_c$$

Proof. By induction on the length of formula ϕ. As to the base of induction, consider atomic formula $P^m(t_1, \dots, t_m)$. By definition of satisfaction $(\vec{a}, w) \models P^m(t_1, \dots, t_m)$ iff $\langle \vec{a}(t_1), \dots, \vec{a}(t_m) \rangle \in I(P^m, w)$ iff $\langle \vec{a}(t_1), \dots, \vec{a}(t_m) \rangle \in I_w(P^m)$. It follows that $\langle \vec{a}(t_1), \dots, \vec{a}(t_m) \rangle \in I_w(P^m)$ iff $w \models_{\vec{a}} P^m(t_1, \dots, t_m)$.

As to the inductive step, we separately consider propositional connectives, the universal quantifier and the box operator.

If ϕ has form $\neg\psi$, then $(\vec{a}, w) \models \phi$ iff not $(\vec{a}, w) \models \psi$ iff by induction hypothesis not $w \models_{\vec{a}} \psi_c$, i.e. $w \models_{\vec{a}} \neg\psi_c$.

If ϕ has form $\psi \to \psi'$, then $(\vec{a}, w) \models \psi \to \psi'$ iff not $(\vec{a}, w) \models \psi$ or $(\vec{a}, w) \models \psi'$. By induction hypothesis it tantamounts to: not $w \models_{\vec{a}} \psi_c$ or $w \models_{\vec{a}} \psi'_c$; in both cases we have $w \models_{\vec{a}} \psi_c \to \psi'_c$.

Suppose that ϕ has form $\forall x_{n+1}\psi$, then $(\vec{a}, w) \models \phi$ iff for all $a^* \in d(w)$, $(\langle \vec{a} \cdot a^* \rangle, w) \models \psi$. By induction hypothesis it tantamounts to: $w \models_{\vec{a} \cdot a^*} \psi_c$ for each $a^* \in d_w = d(w)$, which is equivalent to $w \models_{\vec{a}} \forall x_{n+1}\psi_c$.

Suppose that ϕ has form $\Box\psi(t_1, \ldots, t_m)$. $\Leftarrow$ Assume that $w \models_{\vec{a}} P_{\Box\psi}(t_1, \ldots, t_m)$ and $C_{w,w'}(\vec{a}(t_i), b_i)$. By the conversion lemma $w \models_{\langle \vec{a}(t_1), \ldots, \vec{a}(t_m) \rangle} P_{\Box\psi}(x_1, \ldots, x_m)$, and since $C_{w,w'}$ is admissible, $w' \models_{\vec{b}} \psi_c$. By applying the induction hypothesis $(\vec{b}, w') \models \psi$ and thus $(\vec{a}, w) \models \phi$.

$\Rightarrow$ Assume that not $w \models_{\vec{a}} P_{\Box\psi}(t_1, \ldots, t_m)$, that is not $w \models_{\langle \vec{a}(t_1), \ldots, \vec{a}(t_m) \rangle} P_{\Box\psi}(x_1, \ldots, x_m)$ By lemmas 2.27 and 2.28, there exist T_c-model w' and $\vec{b} \in D(w')^m$ s.t. $C_{w,w'}(\vec{a}(t_i), b_i)$ and not $w' \models_{\vec{b}} \psi_c$. By induction hypothesis not $(\vec{b}, w') \models \psi$ and thus not $(\vec{a}, w) \models \phi$.

Once we have proved lemma 2.30, we show that $\mathcal{M}^L$ is a c-model for L and that unprovable formula ϕ does not hold in $\mathcal{M}^L$ by adapting the proof of theorem 1.19. But in order to prove counterpart-completeness w.r.t. specific classes of c-frames, we have to check that $\mathcal{M}^L$ is based on a c-frame for L. In the second part of the present section we consider this point for each typed QML_t calculus.

2.4.2 Filling in the details

In the previous paragraph we explained Ghilardi's method to prove counterpart-completeness for typed QML_t calculi. In order to apply these techniques to a specific system L, we have to check that the subordination c-frame w.r.t L - as defined in par. 2.4.1 - is actually a c-frame for L. We devote next pages to the proof of this fact for every calculus in par. 2.3.5, beginning with $Q^E.K_t$.

$\boxed{Q^E.K_t}$

Since no specific condition is defined on c-frames for $Q^E.K_t$, the subordination c-frame w.r.t. $Q^E.K_t$ is trivially a c-frame for $Q^E.K_t$. Thus we state the completeness result for $Q^E.K_t$.

Corollary 2.31 (Completeness of $Q^E.K_t$) *If formula $\phi \in \mathcal{L}_t^E$ is valid in the class of all c-frames, then ϕ is a theorem in $Q^E.K_t$.*

$\boxed{Q^E.K + BF_t}$

We have to modify the construction of the subordination c-frame, in order to prove that for every couple $\langle \vec{a}, \Box\phi : n \rangle$ s.t. not $w \models_{\vec{a}} P_{\Box\phi}(x_1, \ldots, x_n)$, there exist T_c-model w', admissible surjective relation $C_{w,w'} \subseteq D_w \times D_{w'}$ and n-tuple $\vec{b} \in D^n_{w'}$ s.t. $C_{w,w'}(\vec{a}, \vec{b})$ and $w' \models_{\vec{b}} \neg\phi_c$. The present proof is an adaptation from [19].

By lemma 2.27, n-type $\Delta = \{\psi_c | w \models_{\vec{a}} P_{\Box\psi}(x_1, \ldots, x_n)\} \cup \{\neg\phi_c\}$ is $T_{Q^E.K+BF_t}$-consistent; hence we enumerate the existential formulas in $\mathcal{L}_t^E$ and inductively define a sequence $\Gamma_0, \Gamma_1, \ldots$ of types where $\gamma_0 = \neg\phi_c$ and $\vec{a}_{\gamma_0} = \vec{a}$,

$$\Gamma_0 = \Delta$$

$$\Gamma_{k+1} = \begin{cases} \{\theta_c : m | w \models_{\vec{a}_{\gamma_k}} P_{\Box\theta}(x_1, \ldots, x_m)\} \cup \{\gamma_{k+1}\}, \\ \qquad \text{if } w \models_{\vec{a}_{\gamma_k}} P_{\Diamond(\gamma_k \wedge \neg\exists x_{i_k} \psi_k[x_1,\ldots,x_{i_k}])}(\vec{x}) \\ \qquad \text{and } \gamma_{k+1} := \gamma_k \wedge \neg\exists x_{i_k}(\psi_k)_c[x_1, \ldots, x_{i_k}]; \\ \\ \{\theta_c : m+1 | w \models_{\vec{a}_{\gamma_k}\cdot a_k} P_{\Box\theta}(x_1, \ldots, x_{m+1})\} \cup \{\gamma_{k+1}\}, \\ \qquad \text{if } w \models_{\vec{a}_{\gamma_k}} P_{\Diamond(\gamma_k \wedge \exists x_{i_k} \psi_k[x_1,\ldots,x_{i_k}])}(\vec{x}) \text{ and} \\ \qquad \gamma_{k+1} := \gamma_k[x_1, \ldots, x_m] \wedge (\psi_k)_c[x_1, \ldots, x_{i_k}, x_{m+1}] \wedge E(x_{m+1}), \end{cases}$$

where $\vec{a}_{\gamma_{k+1}} := \vec{a}_{\gamma_k} \cdot a_k$ is defined as follows: suppose that $w \models_{\vec{a}_{\gamma_k}} P_{\Diamond(\gamma_k \wedge \exists x_{i_k} \psi_k[x_1,\ldots,x_{i_k}])}(\vec{x})$, then

$$w \models_{\vec{a}_{\gamma_k}} P_{\Diamond(\gamma_k \wedge \exists x_{m+1}(\psi_k[x_1,\ldots,x_{i_k},x_{m+1}]))}(\vec{x})$$

and thus

$$w \models_{\vec{a}_{\gamma_k}} P_{\Diamond\exists x_{m+1}(\gamma_k[x_1,\ldots,x_m] \wedge \psi_k[x_1,\ldots,x_{i_k},x_{m+1}])}(\vec{x})$$

By BF we obtain

$$w \models_{\vec{a}_{\gamma_k}} \exists x_{m+1} P_{\Diamond(\gamma_k[x_1,\ldots,x_m] \wedge \psi_k[x_1,\ldots,x_{i_k},x_{m+1}])}(\vec{x})$$

therefore there exists $a_k \in d_w$ s.t. $w \models_{\vec{a}_{\gamma_k}\cdot a_k} P_{\Diamond(\gamma_k[x_1,\ldots,x_m] \wedge \psi_k[x_1,\ldots,x_{i_k},x_{m+1}])}(\vec{x})$.

Finally notice that, since not $w \models_{\vec{a}} P_{\Box\phi}(x_1, \ldots, x_n)$, for every k either $w \models_{\vec{a}_{\gamma_k}} P_{\Diamond(\gamma_k \wedge \neg\exists x_{i_k} \psi_k[x_1,\ldots,x_{i_k}])}(\vec{x})$ or $w \models_{\vec{a}_{\gamma_k}} P_{\Diamond(\gamma_k \wedge \exists x_{i_k} \psi_k[x_{i_1},\ldots,x_{i_k}])}(\vec{x})$; our cases exhaust all the possibilities.

Each Γ_k is T_c-consistent by construction, so the same holds for $\Gamma = \bigcup_{k\in\mathbb{N}} \Gamma_k$. By lemma B.2 there exists a T_c-maximal set $\Pi \supseteq \Gamma$, moreover Π is Y-rich for $Y = \{y | E(y) \in \Pi\}$. By lemma 2.30, T_c-model Π realizes Δ, in particular there are $\vec{x} \in D_\Pi^n$ s.t. $\Pi \models_{\vec{x}} \Delta$.

Now we show how to define an admissible surjective relation $C_{w,\Pi}$ on $D_w \times D_\Pi$. For each T_c-consistent m-type Γ_k, there exist $\vec{x}_{\gamma_k} \in D_\Pi^m$ s.t. $\Pi \models_{\vec{x}_{\gamma_k}} \Gamma_k$; we set $C_{w,\Pi}(a_{\gamma_k}, x_{\gamma_k})$. It is easy to see that $C_{w,\Pi}(\vec{a}, \vec{x})$ holds for $k = 0$. The so defined relation $C_{w,\Pi}$ is surjective: if $x_j \in d(\Pi)$ then $E(x_j) \in \Pi$, and this means that $w \models_{\vec{a}_{\gamma_j}} P_{\Diamond(\gamma_j \wedge \exists x_{i_j} \psi_j[x_1,\ldots,x_{i_j}])}(\vec{x})$. Find $a_j \in d_w$ as above, it is easy to see that $C_{w,\Pi}(a_j, x_j)$ holds by definition of $C_{w,\Pi}$. We show that $C_{w,\Pi}$ is admissible as well.

Suppose that $w \models_{\vec{c}} P_{\Box\psi}(x_1, \ldots, x_n)$ and $C_{w,\Pi}(\vec{c}, \vec{d})$, by definition of $C_{w,\Pi}$ there exist $\vec{a}_{\gamma_k}$ and an n-tuple $\vec{t}$ of terms s.t. $\vec{a}_{\gamma_k}[\vec{t}] = \vec{c}$, hence $w \models_{\vec{a}_{\gamma_k}[\vec{t}]} P_{\Box\psi}(x_1, \ldots, x_n)$. By the conversion lemma $w \models_{\vec{a}_{\gamma_k}} P_{\Box\psi}(t_1, \ldots, t_n)$ and by A16 $w \models_{\vec{a}_{\gamma_k}} P_{\Box(\psi[t_1,\ldots,t_n])}(x_1, \ldots, x_n)$. This means that there are $\vec{x}_{\gamma_k}$ in T_c-model Π s.t. $\Pi \models_{\vec{x}_{\gamma_k}} (\psi[t_1, \ldots, t_n])_c = \psi_c[t_1, \ldots, t_n]$ and $\vec{x}_{\gamma_k}[\vec{t}] = \vec{d}$. Again by the conversion lemma $w' \models_{\vec{d}} \psi_c$, thus $C_{w,\Pi}$ is admissible.

Completeness immediately follows.

Corollary 2.32 (Completeness of $Q^E.K + BF_t$) *If formula $\phi \in \mathcal{L}_t^E$ is valid in the class of surjective c-frames, then ϕ is a theorem in $Q^E.K + BF_t$.*

$\boxed{Q^E.K + CBF_t}$

We prove that the subordination c-frame w.r.t. $Q^E.K + CBF_t$ is existentially faithful. First notice that $E(x_1) \to \Box E(x_1)$ is a theorem in $Q^E.K+CBF_t$, actually it is equivalent to CBF. Suppose that $a \in d(w)$ and $C_{w,w'}(a, b)$, by the first hypothesis $(a, w) \models E(x_1)$ and thus $\Box E(x_1)$ is satisfied by a in w. By the truth lemma $w \models_a P_{\Box E(x_1)}(x_1)$ and since $C_{w,w'}$ is admissible, $w' \models_b E(x_1)$ as well. Again by the truth lemma $(b, w') \models E(x_1)$ and thus $b \in d(w')$.

Therefore we have counterpart-completeness for calculus $Q^E.K + CBF_t$.

Corollary 2.33 (Completeness of $Q^E.K + CBF_t$) *If formula $\phi \in \mathcal{L}_t^E$ is valid in the class of existentially faithful c-frames, then ϕ is a theorem in $Q^E.K + CBF_t$.*

$\boxed{Q^E.K + CBF + BF_t}$

The proof that the subordination c-frame w.r.t. $Q^E.K + CBF + BF_t$ is existentially faithful and surjective is obtained by means of analogous proofs for calculi $Q^E.K + BF_t$ and $Q^E.K + CBF_t$. First we show that counterpart relation $C_{w,w'}$ can be defined so that it is admissible and surjective, then $C_{w,w'}$ is provably existentially faithful by CBF. Finally we state the completeness result for calculus $Q^E.K + CBF + BF_t$.

Corollary 2.34 (Completeness of $Q^E.K + CBF + BF_t$) *If formula $\phi \in \mathcal{L}_t^E$ is valid in the class of existentially faithful, surjective c-frames, then ϕ is a theorem in $Q^E.K + CBF + BF_t$.*

$\boxed{Q.K_t}$

In order to prove counterpart-completeness for calculus $Q.K_t$, we have to check that the subordination c-frame w.r.t $Q.K_t$ is classical, which tantamounts to proving that for every world $w \in W$, $d(w) = D(w)$. This follows immediately by definition of subordination c-frame for $Q.K_t$, as it is made of classical T_c-models in which $d(w) = D_w = D(w)$ for every $w \in W$.

Corollary 2.35 (Completeness of $Q.K_t$) *If formula $\phi \in \mathcal{L}_t$ is valid in the class of classical c-frames, then ϕ is a theorem in $Q.K_t$.*

$\boxed{Q.K + BF_t}$

The subordination c-frame w.r.t. $Q.K + BF_t$ is classical as it is constructed on classical T_c-models as well, it can be proved to be even surjective by the same argument applied to calculus $Q^E.K + BF_t$. Therefore we list our last completeness result.

Corollary 2.36 (Completeness of $Q.K + BF_t$) *If formula $\phi \in \mathcal{L}_t$ is valid in the class of classical, surjective c-frames, then ϕ is a theorem in $Q.K + BF_t$.*

2.4.3 Summing up

In the following table we summarize the completeness theorems for typed QML_t calculi proved thus far, which match with the results stated in theorem 2.20.

Theorem 2.37 (Completeness) *The following typed QML_t calculi are complete w.r.t. the respective classes of c-frames:*

calculi	*c-frame*
$Q.K_t$	*classical*
$Q.K + BF_t$	*classical, surjective*
$Q^E.K_t$	*all*
$Q^E.K + BF_t$	*surjective*
$Q^E.K + CBF_t$	*existentially faithful*
$Q^E.K + CBF + BF_t$	*existentially faithful, surjective*

We conclude this paragraph with some remarks on the effectiveness of Ghilardi's method for typed QML_t calculi, and a comparison with the results available w.r.t. Kripke semantics. We recall from chapter 1 that QML systems $Q^E.K + BF$ and $Q^E.K+CBF+BF$ turned out to be Kripke-incomplete, because of Kripke semantics' incapability of expressing the subtle distinctions explained in par. 2.3.3. On the contrary, we have counterpart-completeness for every typed QML_t calculus based on free logic. These facts reveal a major advantage of counterpart semantics in comparison to Kripke's one.

In the following paragraph we analyse what happens when we adopt a different modal base.

2.4.4 Modal bases stronger than K

In this paragraph we consider counterpart-completeness for typed QML_t calculi with modal bases stronger than K, in particular we analyse quantified extensions of normal systems of propositional modal logic, such as T, $S4$, B and $S5$. As pointed out in par. 2.3.5, for obtaining one of these modalities we have to make use of an appropriate combination of the following schemes of axioms:

A3.	$\Box\phi \rightarrow \phi$	axiom T,
A4.	$\Box\phi \rightarrow \Box\Box\phi$	axiom 4,
A5.	$\phi \rightarrow \Box\Diamond\phi$	axiom B.

We show that Ghilardi's method applies with no change to these calculi. Also in the present cases the only thing to prove, for each typed QML_t calculus L, is that the counterpart model w.r.t L - as defined in par. 2.4.1 - is actually based on a c-frame for L. This is equivalent to proving that the reflexive (resp. reflexive and transitive, reflexive and symmetric, reflexive and transitive and symmetric) closure of an admissible relation is still admissible in the subordination c-frame w.r.t. a quantified extension of T (resp. $S4$, B, $S5$). These facts immediately follow from the next lemma.

Lemma 2.38 *Let $\phi : n$ be a formula on language $\mathcal{L}_0$,*

(a) If $\vdash_L \Box\phi \to \phi$, then the reflexive closure of an admissible relation is admissible.

(b) If $\vdash_L \Box\phi \to \Box\Box\phi$, then the transitive closure of an admissible relation is admissible.

(a) If $\vdash_L \phi \to \Box\Diamond\phi$, then the symmetric closure of an admissible relation is admissible.

Proof.

(a) We show that for every $w \in W$, identity on $D(w)$ is an admissible relation. Suppose that $w \models_{\vec{a}} P_{\Box\phi}(x_1, \ldots, x_n)$, by axiom A3 $w \models_{\vec{a}} \phi_c$ as desired.

(b) We demonstrate that for every $w, w', w'' \in W$, if $C_{w,w'}$ and $C_{w',w''}$ are admissible, then the same holds for $C_{w,w''} = C_{w,w'} \circ C_{w',w''}$. Suppose that $w' \models_{\vec{b}} \{\phi_c | w \models_{\vec{a}} P_{\Box\phi}(\vec{x})\}$ and $w'' \models_{\vec{c}} \{\phi_c | w' \models_{\vec{b}} P_{\Box\phi}(\vec{x})\}$. By axiom A4 if $w \models_{\vec{a}} P_{\Box\phi}(\vec{x})$ then $w \models_{\vec{a}} P_{\Box\Box\phi}(\vec{x})$, hence $w'' \models_{\vec{c}} \{\phi_c | w \models_{\vec{a}} P_{\Box\phi}(\vec{x})\}$, i.e. $C_{w,w''}$ is admissible.

(c) We prove that for every $w, w' \in W$, if $C_{w,w'}$ is admissible then $C_{w',w} = \breve{C}_{w,w'}$ is admissible as well. By hypothesis $w' \models_{\vec{b}} \{\phi_c | w \models_{\vec{a}} P_{\Box\phi}(\vec{x})\}$. Assume for reduction that $w' \models_{\vec{b}} P_{\Box\psi}(\vec{x})$ and $w \models_{\vec{a}} \neg\psi_c$. By axiom A5 $w \models_{\vec{a}} P_{\Box\Diamond\neg\psi}(\vec{x})$, i.e. $w' \models_{\vec{b}} \neg P_{\Box\psi}(\vec{x})$ against hypothesis.

From lemma 2.38 we infer for each calculus L on a modal base stronger than K, that the subordination c-frame w.r.t L is actually a c-frame for L. Completeness immediately follows. We summarize in the next theorems these new adequacy results, where by a typed QML_t calculus L' on modal base M stronger than K, we mean the system obtained by adding to system L on modal base K in theorem 2.20, the characteristic axioms of M.

Theorem 2.39 (Quantified extensions of T) *If L' is a typed QML_t calculus on modal base T, then L' is complete w.r.t. the class of reflexive c-frames for L.*

Theorem 2.40 (Quantified extensions of $S4$) *If L' is a typed QML_t calculus on modal base $S4$, then L' is complete w.r.t. the class of reflexive and transitive c-frames for L.*

Theorem 2.41 (Quantified extensions of B) *If L' is a typed QML_t calculus on modal base B, then L' is complete w.r.t. the class of reflexive and symmetric c-frames for L.*

Theorem 2.42 (Quantified extensions of $S5$) *If L' is a typed QML_t calculus on modal base $S5$, then L' is complete w.r.t. the class of reflexive, transitive and symmetric c-frames for L.*

We conclude the present paragraph by further comparing counterpart and Kripke semantics. Ghilardi's method applies with no modification even to quantified extensions of T, $S4$, B and $S5$; whereas the canonical model method had deep limitations w.r.t. QML calculi on modal base B and $S5$. First of all we prove counterpart-completeness for systems $Q^E.B + BF_t$, $Q^E.S5 + BF_t$, the corresponding QML calculi of which are Kripke-incomplete. This result is a consequence of counterpart-completeness for $Q^E.K + BF_t$. Furthermore we do not have to introduce canonical models with constant outer domains in order to prove completeness for $Q^E.B_t$ and $Q^E.S5_t$. In the conclusion we sum up the advantages of counterpart semantics in comparison to Kripke's one.

2.5 Conclusions

In this chapter we gave some reasons for preferring counterpart semantics to Kripke's one, in particular we stressed its advantages in dealing with individuals in modal settings. First of all by using finitary assignments and types, we evaluate modal formulas w.r.t. all and only the objects actually appearing therein: modal statement $\Box\phi$ is true of individuals $a_1, \ldots, a_n$ in world w, iff it is true of all and only the counterparts of $a_1, \ldots, a_n$, in every world accessible from w. Moreover on the base of validity conditions counterpart semantics discriminates formulas, which are deemed equivalent in Kripke semantics only in virtue of the strong assumptions of that formal account, as it is the case for BF and $N\neg E$. We even showed that the alleged symmetry between BF and CBF is nothing but the by-product of these assumptions on individuals. Furthermore counterparts seem to be the best available option to formalize actualism.

Finally typed QML_t calculi for quantified modal logic have much stronger completeness properties w.r.t. counterpart semantics, in comparison to QML calculi and Kripke semantics. Systems $Q^E.K + BF$ and $Q^E.K + CBF + BF$ are Kripke-incomplete, but we proved counterpart-completeness for corresponding $Q^E.K + BF_t$ and $Q^E.K + CBF + BF_t$. As regards modalities stronger than K, the advantages of counterpart semantics over Kripke's one are even more striking. Ghilardi's method does not exhibit the deep limitations of the canonical model method, when applied to quantified extensions of B and $S5$. We list below the new results provable in counterpart semantics:

1. We proved counterpart-completeness for systems $Q^E.B + BF_t$, $Q^E.S5 + BF_t$; the corresponding QML calculi are Kripke-incomplete.

2. We did not have to introduce canonical models with constant outer domains to prove completeness for systems $Q^E.B_t$ and $Q^E.S5_t$.

By considering all these results, we affirm that counterpart semantics represents a major improvement in comparison to the kripkean framework: it makes clear the true meaning of modal formulas, by revealing the hidden assumptions on individuals lying behind the thick cover of Kripke semantics.

Chapter 3

Identity

In chapters 1 and 2 we presented Kripke and counterpart semantics for first-order modal languages containing as logical constants propositional connectives, quantifiers and modal operators. In order to express relevant ontological issues, as persistence conditions for objects in time, trans-world identities and change, we need to introduce identity symbol '=' in our alphabets and to extend our languages accordingly. By this extension we will be able to discuss equality and its characteristics in modal settings[1]. In this chapter we verify which is the meaning of identity statements in the various semantics for quantified modal logic, and check which are the formal consequences of our pre-theoretic assumptions on equality.

In section 3.1 we take first-order logic as starting point. We analyse formal features of the equality relation in first-order structures, then consider the following postulates for identity

$$x = x \tag{3.1}$$

$$(x = y) \rightarrow (\phi \rightarrow \phi[x/y]) \tag{3.2}$$

They are necessary and sufficient for axiomatizing sound and complete first-order calculi with identity. Formulas 3.1 and 3.2 - also known as *self-identity* and *Leibniz's Law* - reflects deep-rooted intuitions on equality, the strength of which justifies the extension of these principles to modal languages. Hence we begin by interpreting first-order modal languages with identity in Kripke semantics, as it unrestrictedly validates modal versions of 3.1 and 3.2. Moreover QML calculi with identity reveal particularly strong completeness properties w.r.t. Kripke semantics, even if this fact is due to a violation of the *Methodenreinheit*.

There are further philosophical reasons to consider Kripke semantics not completely suitable for identity. In section 3.3 we wonder whether in modal settings postulate 3.2 still correctly represents the intuitions, which motivate it in first-order structures. It could be the case that this formula has only a limited range of application. In par. 3.3.1 we consider the analyses of Leibniz's Law's failure in modal contexts made by Frege in [30] and by Russell in [79]. They both felt the need to

[1] We use term 'identity' to mean the syntactic notion, whereas 'equality' refers to the relation on the domain of objects.

provide reasons for this fact, while preserving the metatheoretic principle of substitution of identicals *salva veritate*. We list further remarks supporting limitations to Leibniz's Law in par. 3.3.2: (i) by 3.2 identity statements are necessary, but we can think of contingent identities; (ii) universal exemplification A6 is no more unrestrictedly valid; (iii) for logics of time an ontology of intensional objects is perhaps more suitable.

In order to solve all these questions, we develop a fresh semantic approach to quantified modal logic with identity. In sections 3.4.1 and 3.4.2 we introduce the conceptual and substantial interpretations in [32], the main features of which are restricted validity for 3.2 and an ontology of functions. The former admits as meaningful each possible individual concept, but this liberality has some drawbacks: there are counterintuitive formulas valid in the conceptual interpretation and the proof-theory is unaxiomatisable. The substantial interpretation considers only a subclass of all the possible individual concepts, thus we have QML calculi adequate w.r.t. this semantic account, even if their completeness properties are particularly weak.

Finally in section 3.5 we present counterpart semantics for identity, which has interesting characteristics. First of all Leibniz's Law holds for every typed formula, even if identities and differences are not necessary. Furthermore, both systems with contingent identity and with classical identity are complete w.r.t. classes of c-frames. At last we have at our disposal a nice framework to talk about temporal and modal properties of objects, persistence and change.

3.1 Identity in First-order Logic

In this section we consider the formal features of the equality relation on a domain of objects, these characteristics are axiomatized by postulates 3.1 and 3.2. The latter formalizes the metatheoretic principle of substitution of identicals *salva veritate*, which seems to be a feature of equality that cannot be given up without losing identification criteria for objects. In par. 3.1.2 we consider some logical consequence of Leibniz's Law, and their connection to the problem of trans-world identity.

3.1.1 Postulates for identity

We start with the formal characteristics of equality[2], which is a particular equivalence relation, where the latter is defined as a relation $\sim$ satisfying for every individual a, b, c in domain D the following conditions:

- $a \sim a$ (reflexivity),
- $a \sim b$ implies $b \sim a$ (symmetry),
- $a \sim b$ and $b \sim c$ imply $a \sim c$ (transitivity).

Equivalence relation $\sim$ determines a partition of set D into disjoint equivalences classes, each object $a \in D$ is related by $\sim$ to all and only the objects belonging to the equivalence class of a. Equality is defined as the smallest equivalence relation s.t. for every $a \in D$ equivalence class $a_\sim$ is singleton $\{a\}$.

For axiomatizing identity in a first-order (non-modal) calculus, we need postulates by means of which we prove at least reflexivity, symmetry and transitivity for symbol '='. Principles 3.1 and 3.2 are the most commonly used:

3.1	$x = x$	self-identity
3.2	$(x = y) \to (\phi \to \phi[x/y])$	Leibniz's Law

Self-identity tantamounts to reflexivity; whereas from Leibniz's Law, by substituting ϕ with $x = x$ and $z = x$ respectively, we obtain symmetry and transitivity for symbol '='.

Nonetheless this is not enough for principles 3.1 and 3.2 to adequately formalize equality in first-order structures. We have to prove that they are necessary and sufficient conditions for symbol '=' to mimic equality in the domain of the model, that is, we need checking that a first-order model $\mathcal{M}$ satisfies 3.1 and 3.2 iff symbol '=' is interpreted as equality in the domain of $\mathcal{M}$.

Notice that reflexivity, transitivity and symmetry are properties shared by all the equivalence relations, thus they cannot discriminate between equality and some other relation $\sim$. In fact, if our language is not expressive enough, a first-order model $\mathcal{M}$ interprets symbol '=' as the equivalence relation among individuals enjoying the same properties definable in our language. It can be the case that individuals a and b satisfy Leibniz's Law, though they are two distinct objects. In order to keep things

[2] In this paragraph we closely follow [28], in particular as regards names for postulates.

straight, in our semantics for languages with identity we consider only a particular class of interpretations - *normal* interpretations - assigning as meaning to symbol '=' the equality relation.

We have seen that self-identity and Leibniz's Law are not sufficient for expressing equality, now we check whether they are sound principle when identity symbol '=' is interpreted as equality. Postulate 3.1 mirrors a basic intuition on equality - each object is equal to itself - which deserves no further concern. On the contrary, soundness for 3.2 is due to the metatheoretic principle of substitution of identicals *salva veritate*:

(SI) If individuals a and b are one and the same object, then every statement ϕ true of a is true also of b.

Substitution of identicals has been thoroughly investigated in the history of logic. Leibniz considered SI as the definition of equality: *Eadem sunt, quae sibi mutuo substitui possunt salva veritare*[3]; but it has often been questioned, especially in relation with modal contexts. On the one hand, SI expresses a basic intuition on equality: two identical objects are not discernible by anyone of their properties; if there were some true statement about the former which is falsified by the latter, then they would differ under some respect and would not be a unique individual. On the other hand, the contexts called *indirect* by Quine offer several counterexamples to Leibniz's Law, which we analyse in section 3.3.

Thus we have to face a threefold question: does SI is unrestrictedly valid? does SI implies Leibniz's Law? does even Leibniz's Law is unrestrictedly valid? We shall see that all the authors affirmatively answer the first question; some of them - as Frege does - negatively reply to the third one, by denying the implication from SI to 3.2; some others - as Russell does - maintain that 3.2 always holds as well.

For the time being we show that Leibniz's Law is related to some important principles, which describe the behaviour of identity symbol '=' and originate a number of ontological issues.

3.1.2 Indiscernibility of identicals and trans-world identity

Consider the following principle - known as *indiscernibility of identicals* - which is equivalent to Leibniz's Law:

$$(x = y) \rightarrow (\phi \leftrightarrow \phi[x/y]) \tag{3.3}$$

From 3.3 we trivially prove 3.2, and the reverse implication follows by symmetry of '='. Indiscernibility of identicals states that two identical objects share all the properties definable in the language, it has always been considered problematic with respect to trans-world identity. In fact, by contraposition it affirms that if two objects differ in a single aspect, then they are numerically different. One of the best-known example of the counterintuitive consequences of 3.3, when it is applied to modal settings, dates back at least to Leibniz and concerns Julius Caesar. We quote from [28]:

[3]This statement is actually stronger than SI, as it implies also the identity of indiscernibles.

> In the actual world, Julius Caesar crossed the Rubicon and marched on Rome. This is, however, a contingent property of the man. It is possible that he didn't cross the Rubicon and march on Rome, and this is to say that there is another possible world in which he didn't. But how can the Julius Caesar in this world be the very same Julius Caesar as the one in the other world, when a property the Julius Caesar in this world has, the Julius Caesar in another lacks? ([28], pg. 146)

Validity of 3.3 seems to deny individual change through instants or in counterfactual situations. Here we confront with one of the most relevant ontological issue concerning quantified modal logic: how is it possible to reconcile a basic intuition about equality, expressed by 3.2, with change in time?

Furthermore 3.3 and its converse, the identity of indiscernibles:

$$(\phi \rightarrow \phi[x/y]) \rightarrow x = y$$

are the two directions of $x = y \leftrightarrow (\phi \rightarrow \phi[x/y])$, which is the definition of identity in its second-order logic formulation:

$$x = y \leftrightarrow \forall\phi(\phi \rightarrow \phi[x/y])$$

Counterexamples to 3.3 imply the lack of identity criteria for individuals in intensional contexts, and this turns out to tantamount to the problem of trans-world identity once more. In fact a number of puzzles about identity across worlds and time and about persistence conditions for material objects, can be seen as violations of the indiscernibility of identicals[4]. We shall further pursue this subject matter in chapter 4, for the time being we only wish to clearly state that a logical answer to the invalidity of 3.3 constitutes a response to these ontological problems too.

In the next section we check validity for principles 3.1 and 3.2 in Kripke semantics. Then we investigate the meaning identity acquires in K-frames, by verifying which formulas with identity hold in these structures. Are all these validities acceptable and philosophically motivated? Or does Kripke semantics make some counterintuitive formula true? We answer these questions and hint at possible solutions to be developed in later sections. Finally we consider $QML^{=}$ calculi with identity and examine their completeness properties w.r.t. K-frames.

[4] Just to recall some of them, think about the statue/lump of clay argument, the Tibbles/Tib case or Theseus' ship paradox.

3.2 Identity in QML

To deal with equality in quantified modal logic, we extend our formal languages with symbol '=' for identity, then we assign a meaning to formulas in this language by means of Kripke semantics. We show that K-frames validate both 3.1 and 3.2, thus they faithfully extend formal properties of identity in first-order logic to modal settings. The necessity of identity and the necessity of difference follow. Finally we consider the proof-theory for identity in QML calculi.

3.2.1 Language $\mathcal{L}^=$

From alphabet $\mathcal{A}$ in chapter 1 we obtain alphabet $\mathcal{A}^=$ with identity just by adding symbol '='. Set $For_{\mathcal{A}^=}$ of first-order modal formulas on $\mathcal{A}^=$ is defined by modifying as follows the base of induction in def. 1.1:

- if x and y are individual variables, then $x = y$ is a (atomic) formula.

We write $x \neq y$ as a shorthand for $\neg(x = y)$. Language $\mathcal{L}^=$ for quantified modal logic consists in alphabet $\mathcal{A}^=$ and set $For_{\mathcal{A}^=}$.

Notice that we do not introduce language $\mathcal{L}^{E=}$, containing also predicative constant E. This is no omission: if our language contains the symbol for identity, then for any term t in $\mathcal{L}^=$ we define $E(t)$ as $\exists x(x = t)$. Once we have the definition of satisfaction for identities in the next paragraph, it will be easy to check that truth conditions for these two formulas are equivalent. Hereafter we consider only language $\mathcal{L}^=$ and $E(t)$ is an abbreviation for $\exists x(x = t)$.

3.2.2 Kripke semantics for identity

Our present aim is to provide a meaning to formulas in language $\mathcal{L}^=$, so that principles 3.1 and 3.2 hold. We show that Kripke semantics is the right framework to achieve this goal. In this section we refer to notions defined in chapter 1 and - as anticipated - we consider only normal interpretations for K-frames, assigning as meaning to symbol '=' in each $w \in W$ equality on $D(w)$. We make use of character '=' for referring to identity both in the object-language and in the metalanguage of semantics, the difference is made clear by the context.

Let $\mathcal{F}$ be a K-frame and let I^σ be the valuation induced by assignment σ into normal interpretation I; the relation of satisfaction in w for formula $x = y$ in $\mathcal{L}^=$ w.r.t. I^σ is defined as follows:

$$(I^\sigma, w) \models x = y \quad iff \quad \sigma(x) = \sigma(y)$$

that is, identity statement $x = y$ is satisfied by assignment σ if and only if the objects assigned by σ to x and y are identical.

It is easy to check that self-identity and Leibniz's Law are sound principles w.r.t. every K-frame. As to the former, $(I^\sigma, w) \models x = x$ iff $\sigma(x) = \sigma(x)$ which is trivially true. As to 3.2, suppose that $\sigma(x) = \sigma(y)$ and $(I^\sigma, w) \models \phi$; the second premise tantamounts to $(I^{\sigma\binom{x}{\sigma(x)}}, w) \models \phi$, by the first premise and substitution of identicals *salva veritate*, $(I^{\sigma\binom{x}{\sigma(y)}}, w) \models \phi$. By the conversion lemma this is equivalent to $(I^\sigma, w) \models \phi[x/y]$.

Notice that in these soundness proofs for 3.1, 3.2 we use at a metalinguistic level the very same principles we have to prove: for self-identity we assume $\sigma(x) = \sigma(x)$, while in order to prove 3.2 we substitute $\sigma(y)$ for $\sigma(x)$ in $(I^{\sigma\binom{x}{\sigma(x)}}, w) \models \phi$ by SI. Notice also the essential role of the conversion lemma. We maintain that the formal properties of identity come from the corresponding pre-theoretical features of equality, we lack an independent justification for self-identity and Leibniz's Law.

Besides formulas 3.1 and 3.2, Kripke semantics validates two relevant principles:

A22.	$x = y \to \Box(x = y)$	necessity of identity,
A23.	$x \neq y \to \Box(x \neq y)$	necessity of difference.

The former says that if two objects are identical, then they are necessarily identical, whereas by the latter two distinct objects are necessarily distinct. We provide soundness proofs for both formulas in K-frames. As to A22 suppose that $(I^\sigma, w) \models x = y$ and wRw'. By the first premise $\sigma(x) = \sigma(y)$, thus also $(I^\sigma, w') \models x = y$, i.e. $(I^\sigma, w) \models \Box(x = y)$. As to A23 assume that $(I^\sigma, w) \models x \neq y$ and wRw'. By the first premise $\sigma(x) \neq \sigma(y)$, thus $(I^\sigma, w') \models x \neq y$ and $(I^\sigma, w) \models \Box(x \neq y)$.

These are quite controversial principles, the validity of which is due to the particular notion of assignment we adopted in par. 1.1.3. In fact, our assignments are *world-independent*: if σ assigns element a to variable x in world w and wRw', then $\sigma(x) = a$ also in w'. It is important to remark that soundness of 3.2, A22 and A23 is due to world-independence - or *rigidity* - of assignments in K-frames. By rigidity we prove that if $\sigma(x)$ is equal to $\sigma(y)$ in w, then they are the same object, thus substitution in $(I^{\sigma\binom{x}{\sigma(x)}}, w) \models \phi$ is permitted. Again by rigidity we show that if $\sigma(y)$ and $\sigma(x)$ are equal (resp. different) in w, then the same holds in w'.

Postulates 3.1, 3.2, A22 and A23 are the most relevant formulas in language $\mathcal{L}^=$ with identity, hence no surprise that they are sufficient for an adequate axiomatization of validities in Kripke semantics.

3.2.3 $QML^=$ calculi with identity

We obtain calculi with identity adequate w.r.t. Kripke semantics, by adding the following schemes of axioms to QML calculi on languages $\mathcal{L}$ and $\mathcal{L}^E$ in par. 1.1.2.

A22.	$x = y \to \Box(x = y)$	necessity of identity,
A23.	$x \neq y \to \Box(x \neq y)$	necessity of difference,
A24.	$x = x$	self-identity,
A25.	$x = y \to (\phi \to \phi[x/y])$	where ϕ is an atomic formula.

These principles are sufficient for completeness, that is, all the validities concerning identity are provable by A22/A25. Which is the reason behind A25 version of Leibniz's Law? The point is that A22 and A25 are enough to prove $x = y \to (\phi \to \phi[x/y])$ for any formula ϕ, i.e. principle 3.2. The proof is by induction on the length of ϕ, by A25 we prove the base of induction and the cases for propositional connectives and quantifiers. If ϕ has form $\Box\psi$, then:

1.	$x = y \rightarrow (\psi \rightarrow \psi[x/y])$	by induction hypothesis,
2.	$\Box(x = y) \rightarrow (\Box\psi \rightarrow \Box\psi[x/y])$	from 1 by T3,
3.	$x = y \rightarrow (\phi \rightarrow \phi[x/y])$	from 2 by A22.

On the other hand, both A22 and A25 can be deduced from 3.2: the former is provable by taking as ϕ formula $\Box(x = x)$, the latter is just a special case of 3.2. We choose this axiomatization, as A22 is a relevant principle on its own, then we list postulate A25 to avoid redundancy.

Axiom A23 is provable from A22 only if our modal base is at least B, in general it has to be assumed for completeness. We list our new systems along with their schemes of axioms and inference rules.

Definition 3.1 (**$QML^=$ calculi**) *The following $QML^=$ calculi with identity consists in schemes of axioms A1, A2, A22/A25, and inference rules R1, R2, with in addition the respective postulates:*

calculi	*schemes of axioms*	*inference rules*
$Q.K^=$	*A6,*	*R3*
$Q.K + BF^=$	*A6, A12*	*R3*
$Q^\circ.K^=$	*A8, A9, A10, A11*	*R5*
$Q^\circ.K + BF^=$	*A8, A9, A10, A11, A12*	*R5*
$Q^\circ.K + CBF^=$	*A8, A9, A10, A11, A13*	*R5*
$Q^\circ.K + CBF + BF^=$	*A8, A9, A10, A11, A12, A13*	*R5*
$Q^E.K^=$	*A7*	*R4*
$Q^E.K + BF^=$	*A7, A12*	*R4*
$Q^E.K + CBF^=$	*A7, A13*	*R4*
$Q^E.K + CBF + BF^=$	*A7, A12, A13*	*R4*

Notice that postulates A7 and R4 of free logic are provable in systems based on Kripke's theory of quantification, by defining $E(t)$ as $\exists x(x = t)$. The proof for A7 is taken from [20], where z does not appear in ϕ.

1.	$\exists z(z = y) \rightarrow (\exists z(z = y) \wedge \forall z(\forall x\phi \rightarrow \phi[x/z]))$	by propositional calculus
2.	$\exists z(z = y) \rightarrow \exists z((z = y) \wedge (\forall x\phi \rightarrow \phi[x/z]))$	from 1 by T5
3.	$z = y \rightarrow ((\forall x\phi \rightarrow \phi)[x/z] \rightarrow (\forall x\phi \rightarrow \phi)[x/y])$	by A25
4.	$z = y \wedge (\forall x\phi \rightarrow \phi[x/z]) \rightarrow (\forall x\phi \rightarrow \phi[x/y])$	from 3 by propositional calculus
5.	$\exists z((z = y) \wedge (\forall x\phi \rightarrow \phi[x/z])) \rightarrow \exists z(\forall x\phi \rightarrow \phi[x/y])$	from 4 by T2
6.	$\exists z((z = y) \wedge (\forall x\phi \rightarrow \phi[x/z])) \rightarrow (\forall x\phi \rightarrow \phi[x/y])$	from 5 by A8
7.	$\exists z(z = y) \rightarrow (\forall x\phi \rightarrow \phi[x/y])$	from 2, 6

Here is a proof of R4, where x does not appear free in ϕ.

1.	$\phi \to (\exists z(z = x) \to \psi)$	premise of R4
2.	$\forall x\phi \to (\forall x\exists z(z = x) \to \forall x\psi)$	from 1 by T2
3.	$\phi \to (\forall x\exists z(z = x) \to \forall x\psi)$	from 2 by A8
4.	$\forall x\exists z(z = x)$	theorem in $L \supseteq Q^\circ.K$
5.	$\phi \to \forall x\psi$	from 3,4 by R1

In chapter 1 we remarked that the postulates of Kripke's theory of quantification are provable in free logic. Therefore each $QML^=$ calculus L on Kripke's theory of quantification is equivalent to its free logic companion. This fact means that from time to time we use the axiomatization of L, which suits our aims the most, and it has relevant consequences on completeness properties of $QML^=$ calculi.

3.2.4 Kripke-completeness for $QML^=$ calculi

In this paragraph we sketch a completeness proof for $QML^=$ calculi w.r.t. Kripke semantics, the main interest of which lies in the stronger completeness properties of systems with identity, in comparison to their QML companions. We start with listing the adequacy results to be proved.

Theorem 3.2 (Adequacy) *The following $QML^=$ calculi with identity are adequate w.r.t. the respective classes of K-frames:*

calculi			*inner domain*	*outer domain*
$Q.K^=$			*increasing*	$=$ *inner*
$Q.K + BF^=$			*constant*	$=$ *inner*
$Q^\circ.K^=$	$\equiv$	$Q^E.K^=$	*varying*	*constant*
$Q^\circ.K + BF^=$	$\equiv$	$Q^E.K + BF^=$	*decreasing*	*constant*
$Q^\circ.K + CBF^=$	$\equiv$	$Q^E.K + CBF^=$	*increasing*	*constant*
$Q^\circ.K + CBF + BF^=$	$\equiv$	$Q^E.K + CBF + BF^=$	*constant*	*constant*

The major difference in comparison to QML systems consists in Kripke-completeness for calculi $Q^E.K + BF^=$ and $Q^E.K + CBF + BF^=$. In fact postulate A14, which is the culprit of Kripke-incompleteness for $Q^E.K + BF$ and $Q^E.K + CBF + BF$, is now provable in $L \supseteq Q^E.K + BF^=$. Consider the next proof from [20].

1.	$\diamond(x = y) \to (x = y)$	by A23
2.	$\exists y \diamond (x = y) \to \exists y(x = y)$	from 1 by T2
3.	$\diamond\exists y(x = y) \to \exists y \diamond (x = y)$	by BF
4.	$\diamond\exists y(x = y) \to \exists y(x = y)$	from 2, 3 by transitivity

In order to prove Kripke-completeness for $QML^=$ calculi of quantified modal logic, we make use of the canonical model method in chapter 1, properly modified to fit a framework with identity. In next paragraphs we largely refer to definitions and theorems in section 1.2.

The canonical model

We are considering only normal K-models, thus also the canonical model $\mathcal{M}^L$ w.r.t. $QML^=$ calculus L has to be normal, that is, symbol '=' has to be interpreted as identity in the outer domain of $\mathcal{M}^L$. Therefore individuals in each world w of canonical frame $\mathcal{F}^L$ can no more be variables $x_1, x_2, \ldots$, in language $\mathcal{L}_w$, as it can be the case that $x_1 = x_2 \in w$ for two different variables x_1, x_2. Instead we consider their equivalence classes $x_{1\sim_w}, x_{2\sim_w}, \ldots$ according to equivalence relation $\sim_w$, which is defined on variables in $\mathcal{L}_w$ as: $x \sim_w y$ iff $x = y \in w$. It is easy to check that $\sim_w$ is actually an equivalence relation, as $x = x$, $x = y \rightarrow y = x$ and $x = y \wedge y = z \rightarrow x = z$ are theorems in every L.

By considering equivalence classes of variables, we prove the base of induction in the truth lemma, i.e. $(I^\sigma, w) \models x = y$ iff $x = y \in w$. The left member of this coimplication tantamounts to $x_{\sim_w} = y_{\sim_w}$, that holds iff $x = y \in w$. But this definition of individuals in the canonical model determines a major problem, we have to check the increasing outer domain condition on $\mathcal{F}^L$: if wRw' and $x_{\sim_w} \in D(w)$ then $x_{\sim_w} \in D(w')$, which is equivalent to $x_{\sim_w} = x_{\sim_{w'}}$. Axioms A22, A23 guarantee that for every $x, y \in \mathcal{L}_w$, $x = y \in w$ iff $x = y \in w'$; but worlds in $\mathcal{F}^L$ are all built on different languages, in particular if wRw' then $Var(\mathcal{L}_w) \subseteq Var(\mathcal{L}_{w'})$. Thus we cannot rule out that for some $y \in \mathcal{L}_{w'}$, $x = y \in w'$ but not $x = y \in w$, making $x_{\sim_w}$ and $x_{\sim_{w'}}$ different. Hence we have to modify the definition of accessibility relation R^L in the canonical model.

Definition 3.3 (Canonical frame) *The canonical frame $\mathcal{F}^L$ for calculus L on language $\mathcal{L}^=$, with an expansion $\mathcal{L}^{=+}$, is an ordered 4-tuple $\langle W^L, R^L, D^L, d^L \rangle$ defined as follows:*

- *W^L is the class of L_w-saturated sets w of formulas in $\mathcal{L}_w$, for $\mathcal{L}^= \subset_\infty \mathcal{L}_w \subset_\infty \mathcal{L}^{=+}$;*
- *R^L is the relation on W^L s.t. wR^Lw' iff $\{\phi | \Box\phi \in w\} \subseteq w'$ and $y_{\sim_w} = y_{\sim_{w'}}$, for every $y \in Var(\mathcal{L}_w)$;*
- *for every $w \in W^L$, $D^L(w)$ is the set of equivalence classes of variables in $\mathcal{L}_w$;*
- *for every $w \in W^L$, $d^L(w)$ is the set of equivalence classes $y_{\sim_w}$ s.t. $E(y) \in w$.*

Even in the present case we have to prove that the canonical frame $\mathcal{F}^L$, as defined above, is actually a K-frame. The proof is almost the same as the one in par. 1.2.1, in particular by saturation lemma 1.14 we prove that set W^L is non-empty, whenever there exists an L-consistent set of formulas. Furthermore the increasing outer domain condition is guaranteed by the way accessibility relation R^L is defined. We conclude that $\mathcal{F}^L$ is a K-frame, but notice that we did not prove that it is a K-frame for L. We go on with the notions of canonical interpretation and assignment.

This time canonical interpretation I^L of language $\mathcal{L}^{=+}$ is so defined that for $x_{1\sim_w}, \ldots, x_{n\sim_w} \in D^L(w)$, $\langle x_{1\sim_w}, \ldots, x_{n\sim_w} \rangle \in I^L(P^n, w)$ iff $P^n(x_1, \ldots, x_n) \in w$. By postulate A25 we show that interpretation I^L is well-defined, as it is independent from the choice of representative x_i for equivalence class $x_{i\sim_w}$. Finally canonical assignment σ^L assigns equivalence class $x_{\sim_w}$ to every $x \in Var(\mathcal{L}_w)$.

For each $QML^=$ calculus L, it is left to check that the canonical model w.r.t. L is actually based on a K-frame for L. The proof is postponed to the discussion of single calculi, in what follows we eliminate superscript L.

Truth lemma

In order to prove Kripke-completeness for $QML^=$ calculi by the canonical model method, we have to show that the truth lemma holds, that is, formula $\phi \in \mathcal{L}_w$ is satisfied in w by canonical assignment σ iff ϕ belongs to w. This proof goes as in lemma 1.18, by induction on the length of $\phi \in \mathcal{L}_w$. The base of induction for identity statements holds, as individuals in $\mathcal{M}^L$ are equivalence classes and interpretation I is normal; in fact $(I^\sigma, w) \models x = y$ iff by normality of I, $x_{\sim_w} = y_{\sim_w}$, iff $x = y \in w$. Inductive steps for propositional connectives and quantifiers are straightforward. The only difficult case concerns $\Box$-formulas, in particular we have to prove that if $\Box\phi \notin w$, then there exists a $L_{w'}$-saturated set w', for $\mathcal{L}_{w'} \supseteq \mathcal{L}_w$, s.t. wRw' and $\neg\phi \in w'$. This is exactly the content of the next lemma appearing in [20].

Lemma 3.4 (auxiliary lemma) *Let w be a world in the canonical model s.t. $\neg\Box\phi \in \mathcal{L}_w$ belongs to w. There exists a $L_{w'}$-saturated set w', for $\mathcal{L}_{w'} \supseteq \mathcal{L}_w$, s.t. $\{\psi | \Box\psi \in w\} \cup \{\neg\phi\} \subseteq w'$ and $y_{\sim_w} = y_{\sim_{w'}}$ for every $y \in Var(\mathcal{L}_w)$.*

Proof. This proof is similar to the one for lemma 1.14, but this time we have to respect the condition on equivalence classes of variables in $\mathcal{L}_w$. Let Y be an infinite denumerable set of individual variables not contained in $\mathcal{L}_w$ s.t. $\mathcal{L}_w^Y \subset_\infty \mathcal{L}^{=+}$. Assume that there are enumerations of existential formulas in $\mathcal{L}_w^Y$ and of Y, then define by recursion a chain of sets of formulas in $\mathcal{L}_w^Y$ s.t.:

$$\Gamma_0 = \{\psi | \Box\psi \in w\} \cup \{\neg\phi\}$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{E(y_n) \wedge \theta_n[x/y_n]\} & \text{if } \Gamma_n \cup \{\exists x\theta_n\} \text{ is } L_w^Y\text{-consistent and} \\ & y_n \in \Gamma_n \text{ is s.t. } \Gamma_n \cup \{E(y_n) \wedge \theta_n[x/y_n]\} \\ & \text{is } L_w^Y\text{-consistent;} \\ \Gamma_n \cup \{E(y_n) \wedge \theta_n[x/y_n]\} \cup \{y_n \neq z | z \in \Gamma_n\} & \text{if } \Gamma_n \cup \{\exists x\theta_n\} \text{ is } L_w^Y\text{-consistent and} \\ & \text{for all } z \in \Gamma_n,\ \Gamma_n \cup \{\mathcal{E}(z) \wedge \theta_n[x/z]\} \\ & \text{is not } L_w^Y\text{-consistent,} \\ & \text{nor } y_n \in Y \text{ appears in } \Gamma_n; \\ \Gamma_n & \text{otherwise.} \end{cases}$$

The base of induction Γ_0 is L_w^Y-consistent by remark 1.20. Then we prove that for every $n \in \mathbb{N}$, if Γ_n is L_w^Y-consistent then Γ_{n+1} is L_w^Y-consistent too.

We consider only the case in which $\Gamma_n \cup \{\exists x\theta_n\}$ is L_w^Y-consistent and for all $z \in \Gamma_n$, set $\Gamma_n \cup \{E(z) \wedge \theta_n[x/z]\}$ is not L_w^Y-consistent; we show that so-defined

Γ_{n+1} is L_w^Y-consistent. First of all notice that for $y_n \in Y$ not appearing in Γ_n, set $\Gamma_n \cup \{E(y_n) \wedge \theta_n[x/y_n]\}$ is L_w^Y-consistent. Then suppose for reduction that there exist formulas $\psi_1, \ldots, \psi_m \in \Gamma_n$ and variables $z_1, \ldots, z_k \in \Gamma_n$ s.t.

$$\vdash_{L_w^Y} \bigwedge \psi_l \rightarrow (\theta_n[x/y_n] \rightarrow \neg \bigwedge z_j \neq y_n)$$

that is

$$\vdash_{L_w^Y} \bigwedge \psi_l \rightarrow (\theta_n[x/y_n] \rightarrow \bigvee z_j = y_n)$$

Since $\Gamma_n \cup \{E(y_n) \wedge \theta_n[x/y_n]\}$ is L_w^Y-consistent, we deduce that also $\Gamma_n \cup \{E(y_n) \wedge \theta_n[x/y_n]\} \cup \{z_j = y_n\}$ is L_w^Y-consistent for some $1 \leq j \leq k$. By 3.2, $\Gamma_n \cup \{E(z_j) \wedge \theta_n[x/z_j]\}$ is L_w^Y-consistent too, contrarily to hypothesis.

Therefore $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ is a L_w^Y-consistent set of formulas in $\mathcal{L}_w^Y$, that by Lindenbaum's lemma can be extended to a $L_{w'}$-saturated set w', for $\mathcal{L}_{w'} = \mathcal{L}_w^Y \supseteq_\infty \mathcal{L}_w$. Finally wRw' and $\neg\phi \in w'$ by construction.

Notice that, differently from QML calculi, here we directly proved the auxiliary lemma, as the proof is unique and does not vary according to the envisaged $QML^=$ calculus. Once proved the truth lemma, we can demonstrate the version of theorem 1.19 stated below.

Theorem 3.5 (Canonical model theorem) *For every $\phi \in \mathcal{L}^=$,*

$$\mathcal{M}^L \models \phi \quad iff \quad \vdash_L \phi$$

The only thing left to prove for each $QML^=$ calculus L is that the canonical model w.r.t. L is actually based on a K-frame for L. This is our task in the following paragraph.

Final details

For proving Kripke-completeness for $QML^=$ calculus L, by using the techniques displayed in the previous paragraph, we have only to check that the canonical model w.r.t L is actually based on a K-frame for L.

Here we find the major differences between $QML^=$ calculi and their QML companions. Systems $Q^\circ.K^=$ and $Q^E.K^=$ are complete w.r.t. the class of all K-frames, as no condition is imposed on K-frames for these calculi. Most important is completeness of $Q^E.K + BF^=$ and $Q^E.K + CBF + BF^=$ w.r.t. the class of K-frames with decreasing, resp. constant, inner domains and constant outer domains: these calculi both prove A14 and by remark 1.35 the canonical model w.r.t. $Q^E.K + BF^=$ ($Q^E.K + CBF + BF^=$) satisfies the decreasing (constant) inner domain condition.

We have remarkable improvements also for $QML^=$ calculi based on Kripke's theory of quantification, as we do not need anymore to modify the canonical model for systems containing BF or CBF, to satisfy the decreasing, resp. increasing, inner domain condition. Systems $Q^\circ.K + BF^=$ and $Q^\circ.K + CBF^=$ are equivalent to $Q^E.K + BF^=$ and $Q^E.K + CBF^=$, and they respectively prove A14 and the necessity of existence $E(x) \rightarrow \Box E(x)$. By the canonical model method we prove at once

Kripke-completeness for all the ten $QML^=$ calculi with identity listed in par. 3.2.3.

Now we consider completeness properties of $QML^=$ calculi on modal bases stronger than K. As regards quantified extensions of T and $S4$, by lemma 1.41 we have the same completeness results as for quantified extensions of K. Thus all the $QML^=$ calculi on modal bases T and $S4$ are complete w.r.t. the corresponding classes of reflexive, resp. reflexive and transitive, K-frames.

On the other hand, for quantified extensions of B and $S5$ we have the most significant improvements. First of all for $L \supseteq Q^\circ.B + BF^=$, system L proves the necessity of existence $E(x) \to \Box E(x)$, which is equivalent to A13 (also this proof appears in [20]):

1.	$\Diamond(x = y) \to x = y$	by A23
2.	$\exists y \Diamond (x = y) \to \exists y(x = y)$	from 1 by T2
3.	$\Diamond \exists y(x = y) \to \exists y(x = y)$	from 2 by BF
4.	$\Box \Diamond \exists y(x = y) \to \Box \exists y(x = y)$	from 3 by T3
5.	$\exists y(x = y) \to \Box \exists y(x = y)$	from 4 by A5

By other results in chapter 1, calculi $Q^\circ.B + BF^=$, $Q^\circ.B + CBF^=$ and $Q^\circ.B + CBF + BF^=$ are all equivalent, and thus complete w.r.t. the class of reflexive and symmetric K-frames with constant inner domains and constant outer domains. Even system $Q^E.B + BF^=$ is Kripke-complete, differently from its QML companion, as it is equivalent to $Q^\circ.B + BF^=$ and proves A13, A14 as well. Finally $Q^\circ.B^=$ is equivalent to $Q^E.B^=$, and completeness w.r.t. the class of reflexive and symmetric K-frames can be proved by means of canonical models with constant outer domains.

In the following table we summarize adequacy results for quantified extensions of B, the analogous theorems for $QML^=$ calculi on modal base $S5$ are easily attainable.

Theorem 3.6 (Adequacy) *The following $QML^=$ calculi with identity are adequate w.r.t. the respective classes of reflexive and symmetric K-frames:*

calculi					*inner domain*	*outer domain*
$Q.B^=$	$\equiv$	$Q.B + BF^=$			*constant*	*= inner*
$Q^\circ.B^=$	$\equiv$	$Q^E.B^=$			*varying*	*constant*
$Q^\circ.B + BF^=$	$\equiv$	$Q^\circ.B + CBF^=$	$\equiv$	$Q^\circ.B + CBF + BF^=$	*constant*	*constant*
$Q^E.B + BF^=$	$\equiv$	$Q^E.B + CBF^=$	$\equiv$	$Q^E.B + CBF + BF^=$	*constant*	*constant*

Compare these results with those listed in par. 1.4.3. The introduction of identity significantly improves the deductive power of QML calculi, so that we demonstrate completeness for *all* the $QML^=$ calculi with identity. But this is possible only at some cost. The previously displayed proof of CBF in $Q^\circ.B + BF^=$ violates "a deep-rooted aspiration which assumed the most varied forms over the centuries - for instance Aristotle's *rejection* of the *μετάβασις εις άλλο γένος* (shift to a different kind) or the 19$^{\text{th}}$ century search for *Methodenreinheit* (purity of methods) in number theory - that is, showing that in the proof of a true statement only properties concerning the (specific or logical) concepts occurring in that statement are involved, no further conceptual material"[5]. For proving postulate A13 in which identity does not appear, we have to make a detour through formulas in language $\mathcal{L}^=$ with identity. Once more Kripke semantics is far from being completely satisfactory.

[5] [14], pg. 265.

3.3 Problems with Leibniz's Law

In par. 3.2.2 we deduced soundness of Leibniz's Law w.r.t. Kripke semantics by substituting $\sigma(x)$ with $\sigma(y)$ in $(I^{\sigma\binom{x}{\sigma(x)}}, w) \models \phi$, and the conversion lemma. We made use at a metalinguistic level of the principle we had to prove - substitution of identicals - by supposing that $\sigma(x)$ is equal to $\sigma(y)$. But the equality between $\sigma(x)$ and $\sigma(y)$ can be inferred from the equality of $\sigma(x)$ and $\sigma(y)$ in w, only by rigidity of assignments in Kripke semantics: if $\sigma(x) = a \in D(w)$ and wRw', then $\sigma(x) = a \in D(w')$. It is clear that we can doubt the soundness of 3.2, if we either hold SI as not universally valid, or that equality between $\sigma(x)$ and $\sigma(y)$ in w is not enough to infer that $\sigma(x)$ is equal to $\sigma(y)$. In par. 3.3.1 we test the implication from substitution of identicals to Leibniz's Law, by analysing Frege's and Russell's solutions to some counterintuitive consequence of 3.2 in modal settings; whereas in par. 3.3.2 we check whether rigid assignments of Kripke semantics are philosophically motivated.

3.3.1 Does SI imply Leibniz's Law?

We introduced Leibniz's Law in par. 3.1.1 and justified it on the base of substitution of identicals *salva veritate*. These considerations were developed within a first-order language, then we just restated 3.2 as A22 and A25 in modal language $\mathcal{L}^=$. But in intensional contexts, even if we accept SI, it is not so plain that validity of 3.2 unconditionally follows from SI. In fact Leibniz 's Law does not seem to be unrestrictedly valid in quantified modal logic, contingent identities being the most common counterexamples. Logician and philosophers have always felt uneasy with the failure of Leibniz 's Law[6], reasons for this feeling are to be found in the role played by SI in establishing identity conditions for objects. If we accept that SI implies 3.2, and this latter is not valid, then by modus tollendo tollens even substitution of identicals fails. The failure of 3.2 eventually implies the lack of identity conditions for individuals in intensional contexts, and this turns out to tantamount to the problem of trans-world identity.

In the history of philosophy there have been two main solutions to the questions illustrated so far: since everyone holds SI as true, we can either keep 3.2 unrestricted or deny that SI implies 3.2 and drop the latter. In the former case we are eventually forced to eliminate non-rigid terms and assignments - as Russell does in [79] - and thus give good reasons for considering all the individuals and names as rigid. In the latter case, we discriminate different notions of identity - as Frege does in [30] - and maintain - as Heller does in [41] - that identities of physical objects are determined not only by their material features, but also by our 'conventions' concerning them.

In the next two paragraphs we deal with the problem of substituting identicals in intensional contexts. We analyse Frege's and Russell's replies to Leibniz's Law's failure, which are respectively instances of the second and first strategy described above.

[6]The only counterexample of which I am aware is N. Belnap, who in the introduction to [11] renames Leibniz's Law as Leibniz's Lie.

Frege's *Sinn/Bedeutung* distinction

At the very beginning of [30] Frege remarks that:

> Equality gives rise to challenging questions which are not altogether easy to answer. [...] $a = a$ and $a = b$ are obviously statements of differing cognitive value; $a = a$ holds a priori and, according to Kant, is to be labelled analytic, while statements of the form $a = b$ often contain very valuable extensions of our knowledge and cannot always be established a priori. The discovery that the rising sun is not new every morning, but always the same, was one of the most fertile astronomical discoveries. Even to-day the identification of a small planet or a comet is not always a matter of course. Now if we were to regard equality as a relation between that which the names 'a' and 'b' designate, it would seem that $a = b$ could not differ from $a = a$ (i.e. provided $a = b$ is true).

Frege's example amounts to the well-known morning star-evening star puzzle. We hold the following statements as true:

(1) the morning star is identical to the evening star,

(2) the ancients knew that the morning star is identical to the morning star.

By Leibniz's Law we should be able to infer

(3) the ancients knew that the morning star is identical to the evening star,

which is obviously false. Therefore it seems that we have to limit the validity of 3.2 in intensional contexts; moreover we demand a philosophical justification to this limitation, which preserves validity for SI.

Frege's account consists in distinguishing between a *primary* reference of a term - its *Bedeutung* - and a *secondary* one - its *Sinn*[7] - and he thinks of terms appearing in modal contexts (*oratio obliqua*, as he calls them) not as having their primary reference, but as standing for their *Sinn*. In this way substitution of identicals *salva veritate* is vindicated: since terms 'the morning star' and 'the evening star' have their primary references in identity statement (1) - whereas in (2) and (3) they refer to their *Sinn* - this is no more a case of SI:

> If now $a = b$, then indeed the reference of 'b' is the same as that of 'a', and hence the truth value of '$a = b$' is the same as that of '$a = a$'. In spite of this, the sense of 'b' may differ from that of 'a', and thereby the sense expressed in '$a = b$' differs from that of '$a = a$'. In that case the two sentences do not have the same cognitive value. If we understand by 'judgement' the advance from the thought to its truth value, as in the above paper, we can also say that the judgements are different.

Then SI does not unconditionally imply 3.2 and the latter has to be limited to non-modalized formulas, so that (3) cannot be deduced from (1) and (2).

[7] "It is natural now, to think of there being connected with a sign (name, combination of words, letter) besides that to which the sign refers, which may be called the reference of the sign, also what I should like to call the *sense* of the sign, wherein the mode of presentation is contained." [30].

Russell's term theory

Differently from Frege, Russell deems Leibniz's Law unrestrictedly valid. In [79] he criticizes Frege's *Sinn/Bedeutung* distinction, by considering the case of non-denoting terms. The denoting phrase 'the present King of France' has surely a meaning but no denotation, if we try to formally translate it as a term, then we are compelled to introduce a fictitious entity as denotation - against Russell's actualism - or to ascribe to it a conventional denotation, as Frege does[8]. Russell refuses to treat denoting phrases as terms and through this choice he solves the morning star/evening star puzzle. We consider his example in [79]: from

(4) Scott was the author of Waverly,

(5) George IV wished to know whether Scott was the author of Waverley,

it is possible to infer - via Leibniz's Law - that

(6) George IV wished to know whether Scott was Scott.

But - as Russell says - "an interest in the law of identity can hardly be attributed to the first gentleman in Europe"[9]. To block this inference Russell simply denies termhood to denoting phrase 'the author of Waverly', expressions like this one are to be formally translated into *definite expressions*:

- there existed an x, which was the author of Waverly and was unique.

Since 'the author of Waverly' is to be rendered as a definite expression, (4) is no more an identity, rather it tantamounts to the following existential statement:

(4') There existed an x, which was the author of Waverly and was unique, and this x was Scott.

Moreover Russell discriminates between a *primary* and a *secondary*[10] occurrence of denoting phrase 'the author of Waverly' in (5). If the denoting phrase has a primary occurrence, we have to translate it into *de re*

(5') there existed an x, which was the author of Waverly and was unique, and George IV wished to know whether this x was Scott.

On the contrary if 'the author of Waverley' has a secondary occurrence, then we must render it as *de dicto*

(5") George IV wished to know whether there existed an x, which was the author of Waverly and was unique, and this x was Scott.

[8] "Frege [...] provides by definition some purely conventional denotation for cases in which otherwise there would be none." [79].

[9] [79].

[10] Not to be confused with Frege's homonymous concepts. Later Russell and Whitehead distinguished between a *small* or *narrow* scope of a term and a *large* or *wide* or *broad* scope.

Leibniz's Law can be applied only when terms have primary occurrences, in fact from (4') and (5') we at most conclude that:

(6') There existed an x, which was Scott and George IV wished to know whether this x was Scott,

which causes no argument to Russell[11]. Therefore the elimination of denoting phrases via definite descriptions, and the distinction between primary and secondary occurrences of terms should keep 3.2 unrestricted.

We have seen that both Frege and Russell deem SI unrestrictedly valid, but in order to find a way out the morning star/evening star and George IV's puzzles, they are compelled to amend either proof-theory or language. Frege denies that identity of meanings is enough for substitution in intensional contexts, rather identity of senses is needed, thus Leibniz's Law has to be limited accordingly. Russell denies that 'Scott was the author of Waverley' is an identity at all, as 'the author of Waverley' is not a name. Russell deems 3.2 unrestrictedly valid, as he rejects that contingent identity statements have form $a = b$. Both these accounts have drawbacks. Frege's *Sinn/Bedeutung* distinction has been criticized ever since it has appeared, and if we uphold that in intensional contexts linguistic expressions do not stand for their meanings, but for their senses, it seems that we are begging the question. On the other hand, Russell's term theory eventually eliminates contingently synonymous terms and names for non-existing objects, our language would contain only terms as 'π' and 'the ratio between the circumference and the radius in a circle', and identities as 'π is equal to the ratio between the circumference and the radius in a circle'.

3.3.2 Interpreting variables

In Kripke semantics we think of assignments of variables to individuals as independent from the world, in which the assignment takes place, in the following sense: if assignment σ assigns to variable x individual a in world w and wRw', then $\sigma(x) = a$ also in w'. Since we can consider the worlds in a K-frame all related by R, we have that for every $w, w' \in W$, $\sigma(x)$ in w is equal to $\sigma(x)$ in w'. The present notion of assignment reflects a specific ontological perspective, called *objectual* by Garson in [32], according to which the individuals that it makes sense to talk about and over which quantifiers range, are the objects appearing in the domain of each possible world. Therefore variables, as descriptive symbols for denoting individuals, are interpreted in the same way in all the worlds of a K-model, that is, assignment σ assigns individual a to variable x and this assignment does not vary in passing from a world to another. At most x denote an actually existing object, rather than a mere *possibile*.

The objectual interpretation of quantified modal logic has some relevant consequence, as validity for Leibniz's Law. In fact the antecedent $x = y$ of 3.2 is true in a world w iff $\sigma(x) \in D(w)$ is equal to $\sigma(y) \in D(w)$. By this fact and rigidity

[11] "This would be true, for example, if George IV had seen Scott at a distance, and had asked 'Is that Scott?' " [79].

of assignments, we infer that $\sigma(x) = \sigma(y)$ *simpliciter* and then make use of SI to deduce $(I^{\sigma\binom{x}{\sigma(y)}}, w) \models \phi$ from $(I^{\sigma}, w) \models \phi$. Therefore rigid assignments of Kripke semantics guarantee soundness for Leibniz's Law, by assuming substitution of identicals. But there is a number of contexts in which rigidity is not motivated at all. In [32] and [47] the authors give three different reasons for considering world-dependent assignments to variables.

Contingent identity systems

In [47] Hughes and Cresswell test the acceptability of postulates A22, A23, by analysing the following statement:

(a) the person next door is the major.

Statement (a) is true in the actual world w if there exists an individual a, who is both the major and the person living next door. Since this fact is undoubtedly contingent, there has to be another world w' where the person next door and the major are not the same guy; that is, there exists two distinct individuals b and b' in w' s.t. in w' the person next door is b, whereas b' is the major, and at most one of them, maybe neither of them, is equal to a.

By applying an analogous argument to the necessity of difference A23, we have that the antecedent can be satisfied by non-rigid assignment σ s.t. $\sigma(x, w) = a \neq a' = \sigma(y, w)$, even if the consequent is falsified in world w' accessible from w s.t. $\sigma(x, w') = b = \sigma(y, w')$. These counterexamples to A22, A23 imply a redefinition of the semantic notion of assignment, which is no more a function from the set of variables in language $\mathcal{L}^{=}$ to outer domain $D(w)$, but a correspondence from couples in $Var(\mathcal{L}^{=}) \times W$ to elements in the various outer domains. Even the notion of satisfaction in w for formula ϕ w.r.t. a valuation I^{σ} has to be modified according to this new framework. This peculiar version of Kripke semantics does not validate A22 nor A23. As a consequence Leibniz's Law either is no more unrestrictedly valid, we have to limit 3.2 to formulas not containing modal operators. These characteristics of the present semantics justify the name of *contingent identity systems*, given by Hughes and Cresswell to calculi sound w.r.t. this interpretation.

Exemplifying the universal quantifier

In [32] Garson introduces two further arguments supporting world-dependent assignments to variables, the former has specifically to do with logic. Languages considered thus far contain only individual variables as terms, but we can expand them by adding individual constants. Then we have to choose how to assign a meaning to these new symbols in Kripke semantics, there are two available options: either we rigidly interpret constants, by setting that if wRw' then $I(c, w) = I(c, w')$, or we adopt non-rigid interpretations. Both options have drawbacks. On the one hand, it is unlikely that one and the same individual is the tallest man in the world in every possible world. On the other one, if we non-rigidly interpret constants then exemplification axiom A6 is no more valid, whenever term y is a constant and ϕ has form $\Box\psi$.

To understand why A6 fails, define the *intension* $I^\sigma(t)$ of term t w.r.t. valuation I^σ, as the function that for each $w \in W$, gives as output element $I^\sigma(t, w) \in D(w)$. If term t contains constants, then a non-rigid interpretation assigns any intension to t; whereas quantifiers bind individual variables, the intensions of which are constant functions in Kripke semantics: for all x, for all $w, w' \in W$, $I^\sigma(x)(w) = \sigma(x) = I^\sigma(x)(w')$. This 'difference in treatment' is the culprit of A6's failure. Finally Garson remarks that: "[p]erhaps allowing nonrigid intensions in our *domain* might result in a better match between the quantifiers and the terms, and so yield simpler rules"[12], that is, we have to consider non-rigid assignments also for variables.

Ontologies for logics of time

Besides the technical advantages of non-rigid assignments to variables, Garson maintains that there are philosophically motivated intuitions, supporting ontologies of intensional objects, in particular with respect to temporal logics:

> [I]magine that our possible worlds are now states of our universe at a given time. The extension of a term at a given time will turn out to be a temporal slice of some thing, 'frozen' as it is at that instant. Notice that things, since they change, cannot be identified with term extensions. Instead, things are world-lines, or functions from times into time slices, and so they correspond to term intensions or individual concepts. Since our ontology takes thing, not their slices as ontologically basic, it is natural to quantify over term intensions in temporal logic. [...] The so called 'objects' of a temporal semantics are not the familiar things of our world, while the formal entities that do correspond to things are misleading called 'individual concepts'. ([32], pg. 281)

According to these considerations, at least in temporal logics we should interpret variables on individual concepts, that is, functions mapping each instant in time to the state of the individual at that instant. Notice that in presenting this idea, Garson takes for granted that *perdurantism* is the adequate ontological doctrine of physical objects, but this question is far from being settled. This theory, according to which physical objects stretch across time, as well as the three dimensions in space, is only one of the several proposals in the present debate on the ontology of physical objects, besides other accounts as *endurantism* and *sequentialism.* We shall thoroughly analyse these theories in the second part of the present work.

We conclude that there are sound arguments backing both a world-dependent interpretation of individual variables and an ontology of functions. From a logical point of view the two options are equivalent, in fact a world-dependent assignment $\sigma(x, w)$ to variable x tantamounts to assigning as value to x non-constant intension $I^\sigma(x)$. In the next section we investigate the features of such semantics, by referring

[12] [32], pg. 281.

in particular to results in [32] and [47]. We first analyse the *conceptual interpretation* of quantified modal logic. The extreme generality of the structures in this semantics prevents us from having an adequate calculus. To solve this problem, we consider the *substantial interpretation*, which offers the same advantages of the conceptual interpretation at a syntactic level, and for which there exist sound and complete systems.

3.4 Intensional Interpretations

In this section we deal with the conceptual and substantial interpretation of quantified modal logic. The main characteristic of these semantic accounts is that variables are interpreted in each world w, not on domain $D(w)$ of individuals in w, but on set $F(w)$ of functions on W s.t. $\mathbf{f}(w') \in D(w')$. This formal feature corresponds to the idea that the elements in our ontology are intensional objects, also called individual concepts, having different extensions in different worlds.

In par. 3.4.1 we consider the conceptual interpretation and the problems with it. In order to solve these problems we present the substantial interpretation in par. 3.4.2, then we analyse its proof-theory and completeness properties.

3.4.1 The conceptual interpretation

We give a meaning to formulas in language $\mathcal{L}^=$, according to the ideas developed in the previous section. We obtain a semantic account falsifying A22 and A23 - based on an ontology of intensional objects - by assigning to individual variables, not elements in outer domain $D(w)$ - that is no more intuitively considered as the domain of individuals, rather the set of (temporal, modal, etc...) individual states - but functions defined on set W of possible worlds s.t. for each $w' \in W$, $\mathbf{f}(w') \in D(w')$. We modify def. 1.3 of K-frame and introduce the notion of *conceptual frame*, or *cp*-frame in short.

Definition 3.7 (*cp*-Frame) *A conceptual frame $\mathcal{F}$ is an ordered 5-tuple $\langle W, R, D, d, F\rangle$ s.t.*

- *W, R, D, d are defined as for K-frames;*
- *F is a function assigning to every $w \in W$, function domain F of all the functions on W s.t. $\mathbf{f}(w') \in D(w')$.*

As it was the case for K-frames, we distinguish *cp*-frames with increasing (decreasing, constant) inner domains and increasing (constant) outer domains. As in Kripke semantics for languages with identity, we consider only normal interpretations in *cp*-frames, which actually interpret symbol '=' in $w \in W$ as equality on $D(w)$. Finally it is necessary to redefine also the notions of w-assignment and variant of a w-assignment, so that they fit in this new framework.

Definition 3.8 (Assignment) *A w-assignment σ for language $\mathcal{L}^=$, relevant to interpretation I, is a function from $Var(\mathcal{L}^=)$ to function domain F.*

A variant $\sigma\binom{x}{\mathbf{f}}$ of w-assignment σ is the assignment s.t. (i) it does not coincide with σ at most on x and (ii) it assigns element $\mathbf{f} \in F(w)$ to x.

We define models and valuations of terms w.r.t. *cp*-frames by adjusting the analogous definitions in section 3.2 to the present context. Finally we state truth conditions for formulas with identity in *cp*-models.

Definition 3.9 (Satisfaction) *The relation of satisfaction in w for formula $\phi \in \mathcal{L}^=$ w.r.t. valuation I^σ is inductively defined as follows:*

$$
\begin{aligned}
(I^\sigma, w) \models P^n(x_1, \ldots, x_n) \quad & iff \quad \langle \sigma(x_1)(w), \ldots, \sigma(x_n)(w) \rangle \in I(P^n, w) \\
(I^\sigma, w) \models x = y \quad & iff \quad \sigma(x)(w) = \sigma(y)(w) \\
(I^\sigma, w) \models \neg\psi \quad & iff \quad not\ (I^\sigma, w) \models \psi \\
(I^\sigma, w) \models \phi \to \psi \quad & iff \quad not\ (I^\sigma, w) \models \phi\ or\ (I^\sigma, w) \models \psi \\
(I^\sigma, w) \models \Box\phi \quad & iff \quad for\ every\ w' \in W,\ wRw'\ implies\ (I^\sigma, w') \models \phi \\
(I^\sigma, w) \models \forall x\phi \quad & iff \quad for\ every\ \mathbf{f} \in F, \mathbf{f}(w) \in d(w)\ implies\ (I^{\sigma\binom{x}{\mathbf{f}}}, w) \models \phi
\end{aligned}
$$

The definition of F guarantees that the evaluation clause for $\Box$-formulas is well-defined, that is, if wRw' then w-assignment σ is a w'-assignment from $Var(\mathcal{L}^=)$ to F as well. Moreover, in the case that $d(w) = D(w)$ for every $w \in W$, the evaluation clause for the universal quantifier reduces to

$$(I^\sigma, w) \models \forall x\phi \quad iff \quad for\ every\ \mathbf{f} \in F, (I^{\sigma\binom{x}{\mathbf{f}}}, w) \models \phi$$

as the antecedent of the implication becomes $\mathbf{f}(w) \in d(w) = D(w)$, which is trivially satisfied.

The conceptual interpretation faithfully formalizes intuitions in par. 3.3.2. First of all it is no more possible to prove soundness for Leibniz's Law, as we cannot substitute $\sigma(y)$ for $\sigma(x)$ in $(I^{\sigma\binom{x}{\sigma(x)}}, w) \models \phi$ to obtain $(I^{\sigma\binom{x}{\sigma(y)}}, w) \models \phi$. In fact from 3.2's premise $x = y$ we only deduce that $\sigma(x)(w) = \sigma(y)(w)$, not that $\sigma(x)$ and $\sigma(y)$ are the same function. Since σ is a non-rigid assignment, by identity of $\sigma(x)$ and $\sigma(y)$ in w we cannot conclude that in all $w' \in W$, $\sigma(x)$ is equal to $\sigma(y)$.

Leibniz's Law is not unrestrictedly valid, in particular it fails whenever ϕ is a modalized formula; hence necessity of identity A22 and of difference A23 admit counterexamples in the conceptual interpretation of quantified modal logic. As to the former, just consider w, w' s.t. wRw', and w-assignment σ s.t. $\sigma(x)(w) = \sigma(y)(w)$ but $\sigma(x)(w') \neq \sigma(y)(w')$. As to A23 just set $w'Rw$ instead of wRw'.

The conceptual interpretation satisfies Hughes and Cresswell's requirements for contingent identity systems, and is based on an ontology of individual concepts, thus it fulfils our *desiderata* in par. 3.3.2. Nonetheless this semantics reveals unsatisfactory features to a thorough analysis.

Problems with the conceptual interpretation

The conceptual interpretation validates some formulas, which highlight counterintuitive characteristics of the present semantics. The following formula

$$\Box\exists x\phi \to \exists x\Box\phi \tag{3.4}$$

is valid in the class of *cp*-frame with inner domains identical to outer ones. In fact valuation I^σ satisfies antecedent $\Box\exists x\phi$ in world w iff for every w', wRw' implies that there exists function $\mathbf{f}_{w'} \in F$ s.t. $(I^{\sigma\binom{x}{\mathbf{f}_{w'}}}, w') \models \phi$. In order to satisfy the consequent we have to show that there exists function $\mathbf{f} \in F$ s.t. for every w', wRw' implies $(I^{\sigma\binom{x}{\mathbf{f}}}, w') \models \phi$. For obtaining such an $\mathbf{f}$ we just set $\mathbf{f}(w') = \mathbf{f}_{w'}(w')$ for each w'. Function $\mathbf{f}$ belongs to F, as it is the set of all the functions on W.

But 3.4 seems to be particularly counterintuitive: as Quine remarks in [76], in certain games it is necessary that some player wins, but there is no player who is bound to win. Formula 3.4 recovers plausibility when we recall that our ontology is made of intensional objects, thus it admits the existence of individuals as 'the winner of the game', who is the one actually bound to win, even if this entity presents different concrete expressions according to the envisaged situation.

If our language contains individual constants as well, then 3.4 implies validity for further undesirable principles. In the class of classical *cp*-frames, formula $\Box\exists x(x = t)$ holds and by 3.4 also

$$\exists x \Box (x = t) \tag{3.5}$$

is valid[13]. Neither 3.5 is acceptable, whenever t is a non-rigid term. In fact if constant t stands for 'the author of counterpart theory', then 3.5 states that there exists an individual who is bound to be counterpart theory's author; contrarily to the intuition that being the author of counterpart theory is only a contingent property for individuals. Also in the present case 3.5 recovers likelihood, if we remember that our ontology contains intensional objects as the author of counterpart theory, who is necessarily counterpart theory's author, even if this name singles out different people in distinct counterfactual situations.

The last consequence of 3.4 we consider is the most counterintuitive of all. By A6 from $\exists x(x = t)$ we prove $\exists x \exists y(x = y)$, thus $\Box\exists x\exists y(x = y)$ by R2. Finally by 3.4 we obtain

$$\exists x \Box E(x) \tag{3.6}$$

which affirms that there is something - Garson hypothesizes that it is God - which necessarily exists.

We wonder whether we can get rid of these principles, by considering the class of all *cp*-frames. In this case formula 3.4 is no more valid, but Garson maintains that the interpretation of quantifiers is still different from their ordinary meaning. In fact 3.5 is true in every *cp*-model s.t. term t designates an existing object; whereas 3.6 is sound w.r.t. *cp*-models s.t. each inner domain $d(w)$ contains at least one individual.

These results depend on the extreme liberality of the ontology, on which the conceptual interpretation is based. In defining *cp*-frames we do not put any constraint on set F of functions defined on W. In addition in the evaluation clause for $\forall$-formulas, we just require that individual $\mathbf{f}(w)$ belongs to inner domain $d(w)$ of world w, without inquiring the relationship among the various exemplifications of individual concept $\mathbf{f}$. This means that in the conceptual interpretation any collection of elements from different worlds - David Lewis in w, a rock in w', a blade of grass in w'' - can be considered an individual concept and thus as an object in our ontology.

[13]Formula 3.5 is sound by the same reasons validating 3.4: in order to have function $\mathbf{f}$ s.t. for every w', wRw' implies $I^{\sigma\binom{x}{\mathbf{f}}}(x, w') = I^{\sigma\binom{x}{\mathbf{f}}}(t, w')$, just define $\mathbf{f}(w')$ as $I^{\sigma}(t)(w')$.

The lack of criteria for distinguishing in F well-formed individuals from mere collections of stages, gives reasons to some philosophers - Quine among them - to uphold a radical distinction between alethic and temporal logics[14]. Only in the latter, on the base of some plausible *trans-world identity* criterion - continuity in space, persistence in change, continuity in chemical composition - it is possible to isolate the genuine individual concepts and make use of them as constituents of our ontology.

Unaxiomatisability

The principal problem related to the choice of set F as quantification domain, consists in the impossibility of defining QML calculi complete w.r.t. the conceptual interpretation. In fact our language $\mathcal{L}^=$, when interpreted on *cp*-frames, has the expressive power of second-order arithmetic; that is, there exists formula SMA in language $\mathcal{L}^=$ s.t. SMA is true in a world w of a *cp*-model $\mathcal{M}$ iff $\mathcal{M}$ contains a standard model of arithmetic. If validity in the conceptual interpretation were axiomatisable, then the same would hold also for truth in structure $\mathbb{N}$ of natural numbers, against Gödel's incompleteness theorem.

We do not prove this result, but refer to the literature. Fine in [23] proves that second-order propositional modal logic - $SOPML$ in short - is incomplete if the modal base is $S4.2$ or weaker. Garson in [32] makes use of this result to demonstrate that second-order modal arithmetic ($SOMA$) is incomplete if the modal base is $S4.3$ or weaker. We wonder whether unaxiomatisability can be extended to modalities stronger than $S4.3$, as $S5$. But Kripke proved that the set of validities in the class of reflexive, transitive and symmetric *cp*-frames is axiomatisable[15], and in [23] Fine showed that $SOPML$ calculi are even decidable, when modality is $S5$.

By these last remarks we may think that the conceptual interpretation is not so bad, as $S5$ is usually considered the modality which faithfully represents the common notions of possibility and necessity. Thus axiomatisability, even decidability, for $S5$-type calculi gives evidence in favour of this semantic account. But this argument has two flaws. First of all it is not so plain that modal base $S5$ is actually the best choice to express all the meanings (logical, physical, temporal, epistemic, etc...) of 'possible' and 'necessary'. In addition 3.4, 3.5 and 3.6 are all sound principles, but we stressed that there is no plausible reading for these formulas. In the next paragraph we present a solution to the problems of the conceptual interpretation.

[14] "The devasting difference is that the series of momentary cross sections of our real world is uniquely imposed on us, for better or for worse, whereas all manner of paths of continuous gradation from one possible world to another are free for thinking up." [77], pg. 861.

[15] See [49].

3.4.2 The substantial interpretation

In the previous paragraph we listed some questions concerning the conceptual interpretation of quantified modal logic, as soundness of 3.4, 3.5 and 3.6 and unaxiomatisability of the set of validities. We noticed that these problems are due to the extreme liberality in formalizing the notion of intensional object: our domain of individuals is set F containing all the functions on W s.t. $\mathbf{f}(w') \in D(w')$. This remark provides useful hints on the way to modify the conceptual interpretation of language $\mathcal{L}^=$, in order to obtain a semantics more corresponding to our intuitions on the nature of objects and quantification. We adjust the notion of *cp*-frame, by defining each $F(w)$ as only a subset of the set of all the functions on W, that is, we consider a subset - the set of substances - of the class of all the possible intensional objects. This constraint corresponds to the idea that every temporal stage of Al belongs to one and the same individual, whereas Al at time t_1, Ben at time t_2 and Charlie at time t_3 are not stages of a unique substance and thus they do not constitute an object.

In this paragraph we analyse the remarkable advantages of the substantial interpretation in comparison to the conceptual one. We show that there exist sound and complete calculi w.r.t. this semantics, but first we introduce the technical details of individual substance semantics.

We modify definition 3.7 of *cp*-frames, according to the ideas above, and present the *substantial frames* or *s*-frames.

Definition 3.10 (*s*-Frame) *A substantial $\mathcal{F}$ is an ordered 5-tuple $\langle W, R, D, d, F\rangle$ s.t.*

- *W, R, D are defined as for K-frames, but there is no condition on D;*
- *F is a function assigning to every $w \in W$, a subset $F(w)$ of the class of functions on W, s.t. if wRw' then $F(w) \subseteq F(w')$;*
- *d is a function assigning to every $w \in W$, a subset $d(w)$ of $F(w)$.*

As it was the case for K- and *cp*-frames, we say that a *s*-frames has increasing (decreasing, constant) inner domains iff wRw' implies $d(w) \subseteq d(w')$ (resp. $d(w) \subseteq d(w')$, $d(w) = d(w')$); whereas in a *classical* *s*-frame $d(w) = F(w)$. Since variables are interpreted on intensional objects in $F(w)$, quantifiers ranges over functions in $d(w)$, which represents the set of substances that are instantiated in w. Finally notice that the class of K-frames is nothing but the class of *s*-frames, in which $F(w)$ is the set of constant functions on W. On the other hand, a *cp*-frame is a *s*-frame for which $F(w)$ is the not-proper subset of the class of functions on W.

A normal interpretation of language $\mathcal{L}^=$ ($\mathcal{L}^{E=}$) into *s*-frame $\mathcal{F}$ assigns a n-ary relation $I(P^n, w)$ to predicative constant P^n in w and define $I(E, w) = d(w)$. This interpretation of existence predicate E has some consequence: first of all it becomes an intensional predicate, that is, axiom A25 is no more valid if $\phi = E(x)$. Moreover formula $E(x)$ is no more definable as $\exists y(x = y)$, this is why we have to consider both language $\mathcal{L}^=$ and $\mathcal{L}^{E=}$.

A w-assignment σ assigns functions in $F(w)$ to variables in $\mathcal{L}^{=}$ ($\mathcal{L}^{E=}$). Truth conditions in world w for formula $\phi \in \mathcal{L}_0$ w.r.t. valuation I^σ are defined as in par. 3.4.1, but for the case of the universal quantifier that goes as follows:

$$(I^\sigma, w) \models \forall x\phi \quad iff \quad for\ every\ \mathbf{f} \in d(w), (I^{\sigma\binom{x}{\mathbf{f}}}, w) \models \phi$$

Finally we remark that the increasing function domain condition guarantees that the evaluation clause for quantified formulas is well-defined.

In the substantial interpretation axioms A22 and A23 fail by the same counterexamples in the conceptual interpretation, but none of 3.4, 3.5 and 3.6 holds even if we exclusively consider classical s-frames, because of the way function domain $F(w)$ is defined. As to 3.4, if antecedent $\Box\exists x\phi$ is true in world w, then for every $w' \in W$, wRw' implies that there exists function $\mathbf{f}_{w'} \in F(w')$ s.t. $(I^{\sigma\binom{x}{\mathbf{f}_{w'}}}, w') \models \phi$. In order to satisfy the consequent $\exists x\Box\phi$ as well, in the conceptual interpretation we considered function $\mathbf{f}$ s.t. for each w', $\mathbf{f}(w') = \mathbf{f}_{w'}(w')$; but now we have no hint that such a $\mathbf{f}$ belongs to $F(w)$. By the same reasoning we find counterexamples also to 3.5 and 3.6.

Finally it is easy to check that axioms A24, A25 are sound w.r.t. the substantial interpretation, but for $\phi = E(x)$. In this case we may have $\sigma(x)(w) = \sigma(y)(w)$, $\sigma(x) \in d(w)$ but $\sigma(x) \notin d(w)$; thus A25 has to be limited accordingly. In next paragraphs we define contingent identity QML^{ci} calculi and prove their adequacy w.r.t. the present semantics.

Contingent identity QML^{ci} calculi

We obtain QML^{ci} calculi either by adding axioms A24, A25 to QML calculi in chapter 1, or by deleting axioms A22, A23 from $QML^{=}$ calculi with identity in section 3.2.

Definition 3.11 (QML^{ci} calculi) *The following contingent identity QML^{ci} calculi consists in schemes of axioms A1, A2, A24, A25, and inference rules R1, R2, with in addition the respective postulates:*

calculi	*schemes of axioms*	*inference rules*
$Q.K^{ci}$	*A6,*	*R3*
$Q.K + BF^{ci}$	*A6, A12*	*R3*
$Q^\circ.K^{ci}$	*A8, A9, A10, A11*	*R5*
$Q^\circ.K + BF^{ci}$	*A8, A9, A10, A11, A12*	*R5*
$Q^\circ.K + CBF^{ci}$	*A8, A9, A10, A11, A13*	*R5*
$Q^\circ.K + CBF + BF^{ci}$	*A8, A9, A10, A11, A12, A13*	*R5*
$Q^E.K^{ci}$	*A7*	*R4*
$Q^E.K + BF^{ci}$	*A7, A12*	*R4*
$Q^E.K + CBF^{ci}$	*A7, A13*	*R4*
$Q^E.K + CBF + BF^{ci}$	*A7, A12, A13*	*R4*

We recall that A25 has to be restricted and formula $E(x)$ can no more be defined by means of identity; therefore we cannot make use of proofs in par. 3.2.3 to show that postulates A7 and R4 are provable in QML^{ci} calculi based on Kripke's theory of quantification. Thus each QML^{ci} calculus based on Kripke's theory of quantification is not equivalent to its free logic companion, differently from what happens for $QML^{=}$ calculi.

We can easily check soundness for QML^{ci} calculi w.r.t. the substantial interpretation. In the next paragraph we sketch a completeness proof and discuss completeness properties of QML^{ci} calculi. As usual we start with considering systems on modal base K.

Completeness for QML^{ci} calculi

The present completeness proof for QML^{ci} calculi is due to Garson [32], we extend his results to systems based on Kripke's theory of quantification. Here are the adequacy results to be proved.

Theorem 3.12 (Adequacy) *The following contingent identity QML^{ci} calculi are adequate w.r.t. the respective classes of s-frames:*

calculi	*inner domain*	*function domain*
$Q.K^{ci}$	*increasing*	$=$ *inner*
$Q.K+BF^{ci}$	*constant*	$=$ *inner*
$Q^{\circ}.K^{ci}$	*varying*	*constant*
$Q^{\circ}.K+BF^{=}$	*decreasing*	*constant*
$Q^{\circ}.K+CBF^{ci}$	*increasing*	*constant*
$Q^{\circ}.K+CBF+BF^{=}$	*constant*	*constant*
$Q^{E}.K^{ci}$	*varying*	*constant*
$Q^{E}.K+CBF^{ci}$	*increasing*	*constant*

In comparison to $QML^{=}$ calculi with identity, we have only the same result available for QML calculi without identity. In particular systems $Q^{E}.K+BF^{ci}$ and $Q^{E}.K+CBF+BF^{ci}$ with the Barcan formula are incomplete w.r.t. the substantial interpretation. In fact the necessity of non-existence A14 holds in any s-frame for a QML^{ci} calculus L with BF, as it satisfies the decreasing inner domain condition, but it is not provable in L. We cannot make use of the proof in par. 3.2.4, as the starting point is axiom A23; moreover in appendix A we prove that there are surjective c-models for L, which are not fictionally faithful, so that A14 fails.

In order to prove substantial-completeness for QML^{ci} calculi we make use of the canonical model method, as modified in par. 3.2.4, with in addition some minor changes to fit our new framework.

The canonical model Even in the present case we have to consider only normal s-models, thus in the canonical model $\mathcal{M}^{L}$ w.r.t. QML^{ci} calculus L, symbol '=' has to be interpreted as identity. Once again the elements in each world w of canonical frame $\mathcal{F}^L$ are equivalence classes of variables in language $\mathcal{L}^{=+}$, according to equivalence relation $\sim_w$ defined in par. 3.2.4. We define the canonical frame $\mathcal{F}^L$ for QML^{ci} calculus L on language $\mathcal{L}_0$ with an expansion $\mathcal{L}_0^+$ as follows:

- W^L is the class of L_w-saturated sets w of formulas in $\mathcal{L}_w$, for $\mathcal{L}_0 \subset_\infty \mathcal{L}_w \subset_\infty \mathcal{L}_0^+$;
- R^L is the relation on W^L s.t. wR^Lw' iff $\{\phi|\Box\phi \in w\} \subseteq w'$;
- for every $w \in W^L$, $D^L(w)$ is the set of equivalence classes of $Var(\mathcal{L}_w)$ in w;
- for every $w \in W^L$, $F^L(w)$ is the set of functions $y_\sim$, for $y \in \mathcal{L}_w$, s.t. $y_\sim(w') = y_{\sim_{w'}}$.
- for every $w \in W^L$, $d^L(w)$ is the set of functions $y_\sim$, for $y \in \mathcal{L}_w$, s.t. for every $\phi \in \mathcal{L}_w$, $\forall x\phi \to \phi[x/y] \in w$;

As usual we have to prove that the canonical frame $\mathcal{F}^L$ as defined above, is actually an s-frame. First of all by lemma 1.14 we prove that set W^L is non-empty whenever there exists an L-consistent set of formulas in $\mathcal{L}_0$, and each set $D^L(w)$ is non-empty either. Moreover for every $w \in W^L$, $F^L(w)$ is a subset of the class of functions on W, and $F(w) \subseteq F(w')$ as $\mathcal{L}_w \subseteq \mathcal{L}_{w'}$ for all $w, w' \in W$. Set $d(w)$ is trivially a subset of $F(w)$. We conclude that $\mathcal{F}^L$ is an s-frame, but it is left to prove that it is an s-frame for L.

Canonical interpretation I^L is normal and for $x_{1\sim_w}, \ldots, x_{n\sim_w} \in D^L(w)$, $\langle x_{1\sim_w}, \ldots, x_{n\sim_w}\rangle \in I^L(P^n, w)$ iff $P^n(x_1, \ldots, x_n) \in w$. Once more we prove that interpretation I^L is independent from the choice of representative x_i by postulate A25. Finally $I(E, w) = d(w)$. Canonical assignment σ^L assigns function $x_\sim$ to every variable $x \in Var(\mathcal{L}_w)$. In the next paragraph we write I, $\mathcal{M}$ and σ instead of I^L, $\mathcal{M}^L$, σ^L.

Truth lemma For proving completeness w.r.t. the substantial interpretation by the canonical model method, we have to show that the truth lemma holds, that is, a formula $\phi \in \mathcal{L}_w$ is satisfied in w by canonical valuation I^σ iff ϕ belongs to w. The proof is by induction on the length of $\phi \in \mathcal{L}_w$, we consider only the base of induction and the cases for the universal quantifier and the Box operator.

Consider atomic formula $P^n(x_1, \ldots, x_n)$. By definition of satisfaction $(I^\sigma, w) \models P^n(x_1, \ldots, x_n)$ iff $\langle\sigma(x_1)(w), \ldots, \sigma(x_n)(w)\rangle \in I(P^n, w)$ iff $\langle x_{1\sim_w}, \ldots, x_{n\sim_w}\rangle \in I(P^n, w)$. According to the definition of canonical interpretation $\langle x_{1\sim_w}, \ldots, x_{n\sim_w}\rangle \in I(P^n, w)$ iff $P^n(x_1, \ldots, x_n) \in w$.

As to identity formulas, assume that $(I^\sigma, w) \models x = y$. Since I is a normal interpretation, this is equivalent to $\sigma(x)(w) = \sigma(y)(w)$, that is $x_{\sim_w} = y_{\sim_w}$. This is the case iff $x = y \in w$.

Consider the case for $\phi = E(y)$: $I^\sigma \models E(y)$ iff $\sigma(y) \in d(w)$, that is, $y_\sim \in d(w)$. This is the case iff for every $\phi \in \mathcal{L}_w$, $\forall x\phi \to \phi[x/y] \in w$. By remark 1.31 it tantamounts to $E(y) \in w$.

Suppose that ϕ has form $\forall x\psi$. $\Leftarrow$ Assume that $\forall x\psi \in w$ and $y_\sim \in d(w)$. As w is $d(w)$-universal we have that $\psi[x/y] \in w$, in particular $\psi[x/y] \in \mathcal{L}_w$, and by induction hypothesis $(I^\sigma, w) \models \psi[x/y]$. By the conversion lemma $(I^{\sigma\binom{x}{y_\sim}}, w) \models \psi$, and given the arbitrariness of variant $\sigma\binom{x}{y_\sim}$ we have that $(I^\sigma, w) \models \forall x\psi$.

$\Rightarrow$ Assume that $\forall x\psi \notin w$. By the L_w-maximality of w, $\exists x\neg\psi \in w$ and since w is $d(w)$-rich, there exists $y_\sim \in d(w)$ s.t. $\neg\psi[x/y] \in w$ and $\neg\psi[x/y] \in \mathcal{L}_w$. By induction hypothesis not $(I^\sigma, w) \models \psi[x/y]$, and by the conversion lemma there exists $y_\sim \in d(w)$ s.t not $(I^{\sigma\binom{z}{y_\sim}}, w) \models \psi$, namely not $(I^\sigma, w) \models \forall x\psi$.

Suppose that ϕ has form $\Box\psi$. $\Leftarrow$ By definition of accessibility relation R.

$\Rightarrow$ We have to prove that if w is a world in the canonical model s.t. $\neg\Box\phi \in \mathcal{L}_w$ belongs to w, then there exists an $L_{w'}$-saturated set w', for $\mathcal{L}_{w'} \supseteq \mathcal{L}_w$, s.t. wRw' and $\neg\phi \in w'$. This result is obtained as in lemma 1.18.

Once we have proved the truth lemma, the canonical model theorem holds and if $\phi \in \mathcal{L}_0$ is not a theorem in QML^{ci} calculus L, then the canonical model $\mathcal{M}$ w.r.t L does not verify ϕ. Finally we check that $\mathcal{M}$ is based on an s-frame for L.

As regards systems $Q^\circ.K^{ci}$, $Q^E.K^{ci}$ there is nothing to prove. As to calculus $Q^E.K + CBF^{ci}$ we refer to remark 1.33; whereas calculi $Q^E.K + BF^{ci}$ and $Q^E.K + CBF + BF^{ci}$ are substantial-incomplete by the same reasons, which determined Kripke-incompleteness for $Q^E.K + BF$ and $Q^E.K + CBF + BF$: postulate A14 is a theorem in both of them, but is provable in none. Completeness for calculi based on Kripke's theory of quantification is proved by means of techniques and lemmas available for the analogous QML calculi. Finally systems $Q.K^{ci}$ and $Q.K + BF^{ci}$ have A6 as an axiom and by remark 1.28 for all $w \in W$, $d(w) = F(w)$.

We conclude that QML^{ci} calculi on modal base K have weaker completeness properties in comparison to $QML^=$ calculi.

Modal bases stronger than K

By lemma 1.41, contingent identity QML^{ci} calculi on modal bases T and $S4$ have the same completeness properties w.r.t. the substantial interpretation as quantified extensions of K. As to quantified extensions of B and $S5$, first of all we remark that calculi $Q.B^{ci}$ and $Q.B + BF^{ci}$ are equivalent and complete w.r.t. the class of s-frames with constant inner domains and outer domains identical to inner ones. System $Q^E.B^{ci}$ is complete w.r.t. the class of s-frames with varying inner domains and constant outer domains. Calculi $Q^\circ.B + CBF^{ci}$, $Q^\circ.B + CBF + BF^{ci}$ and $Q^E.B + CBF^{ci}$, $Q^E.B + CBF + BF^{ci}$ are pairwise equivalent, as they all prove A12; moreover A14 is a theorem in the latter ones, hence they are complete w.r.t. the class of s-frames with constant inner domains and constant outer domains. Finally notice that both $Q^\circ.K + BF^=$ and $Q^E.K + BF^=$ are substantial-incomplete: just consider a surjective, reflexive and symmetric counterpart model, which is not fictionally faithful. The completeness problem for calculus $Q^\circ.B^{ci}$ is still open. We summarize all these results in the following table.

Theorem 3.13 (Adequacy) *The following contingent identity QML^{ci} calculi are adequate w.r.t. the respective classes of reflexive, symmetric s-frames:*

calculi			inner domain	function domain
$Q.BK^{ci}$	$\equiv$	$Q.K + BF^{ci}$	*constant*	*= inner*
$Q^{\circ}.B + CBF^{ci}$	$\equiv$	$Q^{\circ}.B + CBF + BF^{=}$	*constant*	*constant*
$Q^{E}.B^{ci}$			*varying*	*constant*
$Q^{E}.B + CBF^{ci}$	$\equiv$	$Q^{E}.K + CBF + BF^{ci}$	*constant*	*constant*

For quantified extensions of $S5$ we have similar results, therefore QML^{ci} calculi on modal bases B and $S5$ have weaker completeness properties in comparison to modalities K, T and $S4$. Compare all these theorems with the ones in sections 1.2 and 1.3, and notice that these results are the same available for QML calculi: the introduction of identity and A24, A25 add nothing new to the deductive power of QML calculi.

3.4.3 Remarks

In par. 3.4.1 we introduced the conceptual interpretation of quantified modal logic and underlined that this semantic account presents some interesting features:

- neither necessity of identity A22, nor necessity of difference A23 hold;
- axiom A6 is unrestrictedly valid , even if we consider non-rigid interpretations of individual constants;
- such semantics provides a suitable formal account to the perdurantist theory of physical objects.

Besides these attractive characteristics, the conceptual interpretation reveals less convincing aspects, in particular:

- Formulas $\Box\exists x\phi \rightarrow \exists x\Box\phi$, $\exists x\Box(x = t)$ and $\exists x\Box E(x)$ are all valid. But they respectively state that if in each world there exists an individual that satisfies ϕ, then there is a single individual satisfying ϕ in every possible world; there exists an individual who is necessarily t; there exists a necessary individual.
- The set of validities in the conceptual interpretation is not axiomatisable.

To solve these problems we presented the substantial interpretation in par. 3.4.2. The principal difference between the two accounts consists in limiting the notion of individual, by taking as domain of intensional objects in w not the whole set F of functions defined on W, but only a subset of F. The intuitive reasons for this choice are well-motivated and easy to understand: not all the aggregates of temporal or modal stages count as an individual, but only the ones having certain features of *continuity*. The conceptual interpretation can be thought of as the logical correspondent to an unrestricted perdurantist ontology, whereas the substantial interpretation constitutes a formal counterpart to the search for trans-world identity criteria.

From a technical point of view the substantial interpretation is not so satisfying, as we just have the same completeness results available for QML calculi. Moreover completeness properties for QML^{ci} calculi are weaker in comparison to their $QML^{=}$ companions, which are all complete w.r.t. Kripke semantics.

In the next section we attempt to give a meaning to formulas in language $\mathcal{L}^{=}$ by using counterpart semantics. We check which principles hold in counterpart frames, and the new sense controversial postulates A22, A23 acquire. Finally we verify completeness properties for typed calculi with identity.

3.5 Identity and Counterparts

In section 3.4 we provided semantics for systems with contingent identity, namely we introduced models for quantified modal logic validating neither A22 nor A23. These interpretations had some serious drawbacks. As to the conceptual interpretation, its assumptions are rather strong and we cannot have adequate theories describing these structures; whereas the substantial interpretation has weak completeness properties. In the present section we develop a semantic account for modal languages with identity by means of counterpart frames. We shall see that formulas A22, A23 are not valid for free in counterpart semantics, as it was the case in Kripke semantics, but correspond to precise constraints on c-frames. Moreover the accuracy of counterpart semantics in preserving distinctions, which are obliterated in the substantial account, yields to stronger completeness properties. Once more counterparts reveal nice semantic features.

In par. 3.5.1, 3.5.2 and 3.5.3 we respectively introduce typed language, counterpart semantics and proof-theory for identity; we consider both classical and contingent identity. Finally in par. 3.5.4 we prove completeness results by an adaptation of Ghilardi's method.

3.5.1 Typed language $\mathcal{L}_t^=$ with identity

Our starting point is once again alphabet $\mathcal{A}^=$ for quantified modal logic introduced in par. 3.2.1, but this time we deal with typed formulas, thus we define set $tFor_{\mathcal{A}^=}$ of typed first-order modal formulas on $\mathcal{A}^=$ as in par. 2.3.1, by modifying as follows the base of induction in def. 2.4:

- if t_1, t_2 are n-terms, then $t_1 = t_2$ is a formula of type n.

Formula $t_1 \neq t_2$ is a shorthand for $\neg(t_1 = t_2)$. Typed language $\mathcal{L}_t^=$ for quantified modal logic consists in alphabet $\mathcal{A}^=$ and set $tFor_{\mathcal{A}^=}$ of typed formulas. Even in language $\mathcal{L}_t^=$ we can get rid of predicative constant E: since our language contains symbol '=' for identity, for any n-term t in $\mathcal{L}_t^=$ we define formula $E(t) : n$ as an abbreviation of $\exists x_{n+1}(t = x_{n+1})$.

3.5.2 Counterpart semantics for identity

In the present paragraph we rely on the notions of c-frame, interpretation and finitary assignment in chapter 2; moreover interpretations for language $\mathcal{L}_t^=$ are normal, i.e. symbol '=' is interpreted as equality in the outer domain of c-models. It is not difficult to find an evaluation clause for identity statements in c-frames, which reflects the idea that n-valuation $\vec{a}$ satisfies formula $t_1 = t_2 : n$ in world w if and only if it ascribes the same individual to t_1, t_2. In fact we set:

$$(\vec{a}, w) \models t_1 = t_2 \quad iff \quad \vec{a}(t_1) = \vec{a}(t_2)$$

Now we examine which principles on identity are sound w.r.t. this interpretation.

Counterpart semantics makes true the following formulas on typed language $\mathcal{L}_t^=$, where ϕ has type m:

A24.	$t_1 = t_1$	self-identity,
A25.	$(t_1 = t_2) \to (\phi[x_1, \ldots, x_{m-1}, t_1] \to \phi[x_1, \ldots, x_{m-1}, t_2])$	Leibniz's Law.

which correspond to the homonymous postulates in language $\mathcal{L}^=$, this is why we keep the same names and enumeration. Principle A24 is trivially true. As to A25, if n-valuation $\vec{a}$ satisfies $(t_1 = t_2)$ in w, then $\vec{a}(t_1) = \vec{a}(t_2)$. Since

$$(\vec{a}, w) \models \phi[x_1, \ldots, x_{m-1}, t_1]$$

by the conversion lemma

$$(\langle \vec{a}(x_1), \ldots, \vec{a}(x_{m-1}), \vec{a}(t_1) \rangle, w) \models \phi$$

We substitute identicals and obtain $(\langle \vec{a}(x_1), \ldots, \vec{a}(x_{m-1}), \vec{a}(t_2) \rangle, w) \models \phi$, that is $(\vec{a}, w) \models \phi[x_1, \ldots, x_{m-1}, t_2]$.

It is important to remark that even if we apply SI and deduce the unrestricted version of Leibniz's Law, the following formulas

A22.	$x_1 = x_2 \to \Box(x_1 = x_2)$	necessity of identity,
A23.	$x_1 \neq x_2 \to \Box(x_1 \neq x_2)$	necessity of difference,

do not always hold, but tantamount to specific conditions on the counterpart relation. First of all we define a c-frame $\mathcal{F}$ *functional* iff for every $w, w' \in W$, if wRw', $C_{w,w'}(a, b)$ and $C_{w,w'}(a, b')$, then $b = b'$. Moreover $\mathcal{F}$ is *injective* iff for every $w, w' \in W$, if wRw', $C_{w,w'}(a, b)$ and $C_{w,w'}(a', b)$, then $a = a'$. Now we prove that a c-frame $\mathcal{F}$ is functional (resp. injective) iff $\mathcal{F}$ validates A22 (resp. A23).

Lemma 3.14 *A c-frame $\mathcal{F}$ is functional iff $x_1 = x_2 \to \Box(x_1 = x_2)$ is valid in $\mathcal{F}$.*

Proof. $\Leftarrow$ Suppose that $x_1 = x_2 \to \Box(x_1 = x_2)$ is valid in $\mathcal{F}$ and that wRw', $C_{w,w'}(a, b)$ and $C_{w,w'}(a, b')$. Valuation $\langle a, a \rangle$ satisfies antecedent $x_1 = x_2$ in w, and by A22 also $\Box(x_1 = x_2)$ is satisfied by $\langle a, a \rangle$ in w, thus counterparts b, b' of a in w' satisfies $x_1 = x_2$, i.e. they are identical.

$\Rightarrow$ Suppose that $x_1 = x_2 \to \Box(x_1 = x_2)$ is not valid in $\mathcal{F}$. Therefore there are an individual $a \in D(w)$, $w' \in W$ s.t. wRw', and two counterparts b, b' of a in w' s.t. $b \neq b'$.

Lemma 3.15 *A frame $\mathcal{F}$ is injective iff $x_1 \neq x_2 \to \Box(x_1 \neq x_2)$ is valid in $\mathcal{F}$.*

Proof. $\Leftarrow$ Suppose that A23 is valid in $\mathcal{F}$ and that wRw', $C_{w,w'}(a, b)$ and $C_{w,w'}(a', b)$. Valuation $\langle a, a' \rangle$ does not satisfy $\Box(x_1 \neq x_2)$ in w; by A23, $(x_1 \neq x_2)$ is neither sati sfied by $\langle a, a' \rangle$ in w, and this means that a is equal to a'.

$\Rightarrow$ Suppose that $x_1 \neq x_2 \to \Box(x_1 \neq x_2)$ is not valid in $\mathcal{F}$. Then there exist distinct individuals $a, a' \in D(w)$, $w' \in W$ s.t. wRw', and counterpart b in $D(w')$ of both a and a'.

Counterpart semantics validates A25 for every typed formula ϕ, but this fact does not imply soundness for A22, A23. Once more counterpart semantics is able to draw original distinctions on modal properties of individuals. The necessity of identity and the necessity of difference do not hold for free, as it was the case in Kripke semantics, but correspond to the functionality and injectivity condition on c-frames. But we do not have to limit principle 3.2 to non-modal formulas, as the notion of counterpart is extensional: two identical objects in w have the same counterparts. On the contrary in intensional interpretations being the concrete expression of an individual concept is not an extensional notion.

3.5.3 Typed QML_t^{ci} and $QML_t^{=}$ calculi with identity

We obtain typed calculi with identity adequate w.r.t. counterpart semantics, by adding schemes of axioms A22/A25 to typed QML_t calculi in par. 2.3.5. We have two available choices: either we add only postulates A24, A25 and obtain typed contingent identity QML_t^{ci} calculi, or we consider axioms A22, A23 as well, thus having typed $QML_t^{=}$ calculi with identity. We summarize in the following table the axiomatisation of QML_t^{ci} calculi, the one for $QML_t^{=}$ systems is immediate.

Definition 3.16 (**QML_t^{ci} calculi**) *The following typed contingent identity QML_t^{ci} calculi consist in axioms A1, A2, A16, A24, A25 and inference rules R1, R2, with in addition the respective postulates:*

calculi	*schemes of axioms*	*inference rules*
$Q.K_t^{ci}$	*A6,*	*R3*
$Q.K + BF_t^{ci}$	*A6, A12*	*R3*
$Q^E.K_t^{ci}$	*A7*	*R4*
$Q^E.K + BF_t^{ci}$	*A7, A12*	*R4*
$Q^E.K + CBF_t^{ci}$	*A7, A13*	*R4*
$Q^E.K + CBF + BF_t^{ci}$	*A7, A12, A13*	*R4*

Notice that postulates A22, A23 are not provable given version A25 of Leibniz's Law. In fact, by A25 we at most deduce

$$(x_1 = x_2) \to (\Box(x_1 = x_2)[x_1, x_1] \to \Box(x_1 = x_2)[x_1, x_2])$$

and by continuity principle A16 we have

$$(x_1 = x_2) \to (\Box(x_1 = x_2)[x_1, x_1] \to \Box(x_1 = x_2))$$

but in order to obtain A22, we should be able to infer

$$\Box(x_1 = x_1) \to \Box(x_1 = x_2)[x_1, x_1] \tag{3.7}$$

and then apply A24 and R1. But formula 3.7, which is an instance of the converse of A16, is not unrestrictedly valid, in fact it tantamounts to functionality.

In the next paragraph we prove that all these typed calculi are adequate w.r.t. counterpart semantics by using the techniques in section 2.

3.5.4 Adequacy results in counterpart semantics

Our point of reference in the completeness proof for typed QML_t^{ci} and $QML_t^{=}$ calculi is once more [10], we just extend Ghilardi's method to languages with identity. In particular we show that typed QML_t^{ci} calculi have stronger completeness properties than their untyped companions. We begin by listing the adequacy results to be proved.

Theorem 3.17 (Adequacy) *The following typed contingent identity QML_t^{ci} calculi are adequate w.r.t. the respective classes of c-frames for them:*

calculi	*c-frame*
$Q.K_t^{ci}$	*classical*
$Q.K + BF_t^{ci}$	*classical, surjective*
$Q^E.K_t^{ci}$	*all*
$Q^E.K + BF_t^{ci}$	*surjective*
$Q^E.K + CBF_t^{ci}$	*existentially faithful*
$Q^E.K + CBF + BF_t^{ci}$	*existentially faithful, surjective*

Notice that typed calculi on language $\mathcal{L}_t^{=}$ with identity are adequate w.r.t. the same classes of c-frames as their QML_t companions on languages $\mathcal{L}_t$. We recall that $\mathcal{F}$ is a c-frame *for* calculus L, appearing in the first column, iff every c-model based on $\mathcal{F}$ is c-model for L.

Ghilardi's method applied to identity

We apply Ghilardi's method to typed calculi with identity, in order to prove their completeness w.r.t. counterpart semantics. This proof basically consists in lemmas 2.27, 2.28 and 2.30, which are demonstrated by extending lemmas 2.22 and 2.25 to language $\mathcal{L}_t^{=}$. Even in the present case we start with introducing the first-order typed theory, the models of which constitute the worlds in our counterexample model.

Let L be a typed QML_t^{ci} calculus on $\mathcal{L}_t^{=}$, first-order (non-modal) language $\mathcal{L}_c$ is obtained as in par. 2.4.1, by adding to a first-order language a new n-ary predicative constant $P_{\Box\phi}$ for every formula $\Box\phi : n$ in $\mathcal{L}_t^{=}$. Translation $\psi_c : n \in \mathcal{L}_c$ of formula $\psi : n$ in $\mathcal{L}_t^{=}$ is defined as in par. 2.4.1, with in addition the following clause for identity:

- if ψ is atomic formula $t_1 = t_2$, then $\psi_c = \psi$.

For each typed QML_t^{ci} calculus L we consider first-order theory T_c with identity, containing as proper axiom each formula $\phi_c : n$ s.t. $\phi : n$ is provable in L. We extend lemma 2.22 to languages with identity; the proof is straightforward, hence we state the following result.

Lemma 3.18 *A formula $\phi : n \in \mathcal{L}_t^=$ is provable in L iff $\phi_c : n \in \mathcal{L}_c$ is provable in T_c.*

We only remark that postulates 3.1 and 3.2 are theorems in T_c, as A24 and A25 are axioms of L.

First-order models This time the worlds in the subordination frame w.r.t. calculus L with identity are normal models for first-order typed theory T_c, where a first-order normal model is defined as in def. 2.23 and in addition interpretation I is normal. Truth conditions in a normal $\mathcal{L}_c$-model $\mathcal{M}$ for identity statement $t_1 = t_2$ w.r.t. n-assignment $\vec{a}$ are defined as follows:

$$\mathcal{M} \models_{\vec{a}} t_1 = t_2 \quad iff \quad \vec{a}(t_1) = \vec{a}(t_2)$$

As it was the case for first-order typed theories, we state completeness for typed theories with identity in the following theorem.

Theorem 3.19 (First-order completeness) *If Δ is a T_c-consistent n-type in first-order typed language $\mathcal{L}_c$ with identity, then there exists a normal T_c-model $\mathcal{M}$ realizing Δ.*

This theorem is demonstrated by adapting the proof for theorem 2.25. In particular the normal T_c-model realizing Δ is classical, whenever T_c is obtained from a calculus $L \supseteq Q.K_t^{ci}$.

The subordination frame method As usual we prove completeness by contraposition. We assume that typed QML_t^{ci} calculus L does not prove formula $\phi : n$, hence n-type $\{\neg\phi_c\}$ is T_c-consistent and by theorem 3.19, there exists a normal T_c-model w realizing $\{\neg\phi_c\}$. This T_c-model w is the root of our subordination frame $\mathcal{F}^L$.

For every couple $\langle \vec{a}, \Box\psi : n \rangle$ s.t. not $w \models_{\vec{a}} P_{\Box\psi}(x_1, \ldots, x_n)$, we can find a normal T_c-model w', an admissible relation $C_{w,w'} \subseteq D_w \times D_{w'}$, and an n-tuple $\vec{b} \in D_{w'}^n$ s.t. $C_{w,w'}(\vec{a}, \vec{b})$ and not $w' \models_{\vec{b}} \psi_c$, by applying lemmas 2.27 and 2.28.

We define interpretation I^L of language $\mathcal{L}_t^=$ on subordination frame $\mathcal{F}^L$ by gluing together the various interpretations I_w. Notice that I^L is normal as each T_c-model $w \in W$ is normal. The ordered couple $\langle \mathcal{F}^L, I^L \rangle$ constitutes our counterexample model $\mathcal{M}^L$ w.r.t typed QML_t^{ci} calculus L. To simplify our notation we eliminate superscript L hereafter.

Truth lemma For proving the truth lemma in the present setting, we have to show that for every formula $\phi : n$ in $\mathcal{L}_t^=$, for every $w \in W$ and $\vec{a} \in D(w)^n$,

$$(\vec{a}, w) \models \phi \quad iff \quad w \models_{\vec{a}} \phi_L$$

The proof is by induction on the length of ϕ and completely identical to the one for lemma 2.30, but for the base of induction, as we have to consider also identity statements. Since w is a normal T_c-model $(\vec{a}, w) \models t_1 = t_2$ iff $\vec{a}(t_1) = \vec{a}(t_2)$, and this is the case iff $w \models_{\vec{a}} t_1 = t_2$.

By the truth lemma we show that $\mathcal{M}^L$ is a c-model for L and that unprovable formula ϕ does not hold in $\mathcal{M}^L$. In order to prove these facts with c-model $\mathcal{M}^L$ based on a c-frame for L, we need to show that for $L = Q^E.K + BF_t^{ci}$ (resp. $Q^E.K + CBF_t^{ci}$), $\mathcal{M}^L$ is surjective (resp. existentially faithful). This proof is the same as for typed QML_t calculi in chapter 2, hence theorem 3.17 holds.

For typed QML_t^{ci} calculi on modal bases stronger that K we have the same completeness results listed in theorems 2.39/2.42.

As regards typed $QML_t^{=}$ calculi we prove that the subordination frame, as defined above, is also functional and injective. As to functionality assume that $C_{w,w'}(a, b)$ and $C_{w,w'}(a, b')$; since w is a normal T_c-model, $w \models_{\langle a,a \rangle} x_1 = x_2 \rightarrow P_{\Box(x_1=x_2)}(x_1, x_2)$. Counterpart relation $C_{w,w'}$ is admissible, therefore $w' \models_{\langle b,b' \rangle} x_1 = x_2$ and $b = b'$.

As to injectivity suppose that $C_{w,w'}(a, b)$ and $C_{w,w'}(a', b)$; since w is a normal T_c-model, $w \models_{\langle a,a' \rangle} \neg P_{\Box(x_1 \neq x_2)}(x_1, x_2) \rightarrow x_1 = x_2$. Notice that $w' \models_{\langle b,b \rangle} x_1 = x_2$, thus by admissibility of the counterpart relation $w \models_{\langle a,a' \rangle} \neg P_{\Box(x_1 \neq x_2)}(x_1, x_2)$, and by hypothesis $w \models_{\langle a,a' \rangle} x_1 = x_2$. Therefore $a = a'$.

By these results we prove that the following typed $QML_t^{=}$ calculi are adequate w.r.t. the respective classes of functional and injective c-frames for them:

calculi	c-frame
$Q.K_t^{=}$	classical
$Q.K + BF_t^{=}$	classical, surjective
$Q^E.K_t^{=}$	all
$Q^E.K + BF_t^{=}$	surjective
$Q^E.K + CBF_t^{=}$	existentially faithful
$Q^E.K + CBF + BF_t^{=}$	existentially faithful, surjective

We remark that A14 is a theorem in every typed $QML_t^{=}$ calculus with BF, the proof is the same appearing in par. 3.2.4; but in $QML^{=}$ calculi with identity, there was no semantic correspondent to theoremhood of A14. For instance system $Q^E.K+BF$ is sound w.r.t. the class of K-frames with decreasing inner domains, and also $Q^E.K + BF^{=}$ is characterized by the same class of K-frames. On the contrary system $Q^E.K + BF_t$ is sound and complete w.r.t. the class of classical, surjective c-frames, thus invalidating A14; whereas $Q^E.K + BF_t^{=}$ is characterized by the class of functional, injective, classical and surjective c-frames which are provably fictionally faithful, thus validating A14. This is a further example of the distinctions available in counterpart semantics, which are completely obliterated by Kripke semantics.

Finally we list the completeness results for $QML_t^{=}$ calculi on modal bases stronger than K. For quantified extensions of T and $S4$ we have the same completeness results as above w.r.t the class of reflexive (reflexive and transitive) c-frames.

On the other hand we have just three non-equivalent typed $QML_t^=$ calculi on modal base B, which are adequate w.r.t. the following classes of functional, injective, reflexive and symmetric c-frames:

calculi					c-frame
$Q.B_t^=$	$\equiv$	$Q.B + BF_t^=$			classical, surjective
$Q^E.B_t^=$					all
$Q^E.B + BF_t^=$	$\equiv$	$Q^E.B + CBF_t^=$	$\equiv$	$Q^E.B + CBF + BF_t^=$	existentially faithful, surjective

For quantified extensions of $S5$ we have similar results. In conclusion, we proved counterpart-completeness for all our typed calculi with either classical or contingent identity.

3.6 Conclusions

In this chapter we tackled identity in modal settings. The interest of this subject lies in problems like trans-world identity, persistence and change of individuals in time. In particular we studied the formal assumptions, which support the interpretation of identity as either a necessary or contingent relation. This point will be useful in discussing the various ontological theories concerning persistence of physical objects through time. In section 3.1 we introduced identity and postulates 3.1 and 3.2 in a first-order framework, then considered the metatheoretic principles validating these formulas, as substitution of identicals *salva veritate*. In section 3.2 we extended self-identity and Leibniz's Law to modal language $\mathcal{L}^=$, and checked that these postulates are sound w.r.t. Kripke semantics. Furthermore $QML^=$ calculi with identity have strong completeness properties in this interpretation.

In section 3.3 we remarked that extending 3.1 and 3.2 to language $\mathcal{L}^=$ cannot be the last word on identity in modal settings. There are several reasons for rejecting unrestricted validity of Leibniz's Law, when talking about modal properties, the most common of which are contingent identities. But are these counterexamples also to the substitution of identicals? or we can save both 3.2 and SI? We saw that in the literature there are two strategies to solve these problems, while keeping SI universally valid: either we eliminate the implication from SI to 3.2 - as Frege does - by denying that identity of extensions is enough for substitution in intensional contexts; or we reform our language - as Russell does - so that counterexamples to 3.2 are not really counterexamples, as the premise is not a real identity.

There are further motivations to limiting Leibniz's Law, that we listed in par. 3.3.2. Validity of 3.2 is due to rigid assignments to variables, but in a number of contexts non-rigid - or world-dependent - assignments offer a more faithful modeling. We considered some of them as contingent identity systems of [47] and the perdurantist ontology of physical objects. Hence in section 3.4 we introduced semantics invalidating A22, A23: the conceptual and the substantial interpretations of [32]. But the former has unattractive features as validities $\Box\exists x\phi \rightarrow \exists x\Box\phi$, $\exists x\Box(x = t)$ and $\exists x\Box E(x)$, and unaxiomatisability; whereas the latter has weak completeness properties. Thus we were still looking for semantics adequate to contingent identity.

In section 3.5 we analysed the behaviour of identity in counterpart semantics. The semantics of counterparts can model both classical and contingent identity, by means of specific constraints - functionality and injectivity - to be imposed on c-frames. Moreover Leibniz's Law is unrestrictedly valid without implying A22, A23, and we have completeness results for all the typed $QML^=_t$ and QML^{ci}_t calculi. Even for identity, counterpart semantics is a handy and comprehensive tool for modeling modal properties of individuals. We conclude by recalling the completeness results we proved in the present chapter.

We showed that all the typed QML^{ci}_t calculi are counterpart-complete, whereas systems $Q^E.K+BF^{ci}$ and $Q^E.K+CBF+BF^{ci}$ containing BF are incomplete w.r.t. the substantial interpretation. The culprit is once more principle A14, which holds in each class of s-frames for these calculi, but is provable in none. By considering modality at least as strong as B, we recover substantial-completeness for $Q^E.B+$

$CBF + BF^{ci}$, whereas systems $Q^{\circ}.B + BF^{ci}$ and $Q^{E}.B + BF^{ci}$ are still incomplete.

We proved counterpart-completeness even for all the typed $QML_t^=$ calculi with identity. In this case Kripke semantics fares as well as counterparts, as we have Kripke-completeness for all the $QML^=$ calculi with identity. But we underlined some unsatisfactory features of $QML^=$ calculi, as the proof of A13 by a detour through identities, thus they violate the 'purity of methods' mentioned by Casari.

Part II

Logics and Ontology

Chapter 4

Ontologies for Physical Objects

4.1 The Interplay between Logics and Ontology

In this chapter we compare quantified modal logic and ontologies for physical objects. Specifically we aim at determining a precise correspondence between semantics for quantified modal logic we presented in the first part of the present work, and some theories on persistence conditions for material objects through change. The alleged relationship is not self-evident, but demands for a thorough analysis of the ontological assumptions basing our three semantic accounts. The whole discussion is fundamental to the present work, as it accounts for the applications of logical results to ontological issues in chapter 5.

First of all notice that the temporal interpretation of quantified modal logic, according to which the $\Box$ operator is interpreted as 'in every moment', and the ontologies for physical objects refer to the same domain of individuals - physical objects indeed - and they share the problems related to persistence and reidentification in time. Despite this close relationship and the relevant results obtained by connecting logics and ontology[1], researches on these topics developed independently, without any significant contribution joining together the two perspectives. This second part is intended to be a first step in this way, by making explicit the role logics can play in analysing ontological theories.

4.1.1 From logics to ontology

The development of quantified modal logic began with a series of papers by R. Barcan Marcus ([6], [7], [8]), and went on through contributions by J. Hintikka ([42], [43]), S. Kripke ([54]), A. Prior ([71], [72], [73]). In [55] and [56] Kripke made a major step forward, as he introduced possible worlds semantics for propositional and first-order modal logic respectively.

Kripke's semantics - based on Leibniz's intuition of defining necessity as truth in all the possible worlds - determined a deeper understanding of modality, while making natural both using model-theoretical techniques to prove proof-theoretical

[1] Just think of the actualism/possibilism issue and the discussion on the nature of possible worlds.

results[2], and applying modalities to various contexts, representable as Kripke's structures (instants in time, steps in a computation, deontologically perfect situations, epistemic state,...).

If we focus on the temporal interpretation of QML and assume that modal operators $\Box$ and $\diamond$ mean 'in every instant' and 'in some instant', then the domains of individuals in Kripke models contain objects appearing in time. Of course we find physical objects among them[3], as well as further things probably (events?). We conclude that in order to formulate a satisfactory semantics for quantified temporal logic, a clear ontology of physical objects is needed: we have to determine whether our individuals are wholly present in each moment, or we have just temporal parts; whether the same object may appear in different instants, or it is a plain non-sense to talk about individual identity across time. Every formal semantics presupposes an ontology, which most of the times is not explicitly stated. Thus we showed the first aspect of the interplay between logics and ontology, in which the ontological moment gives a shape the logical one.

4.1.2 From ontology to logics

In the last years, in analytical metaphysics, there has been a renewal of the debate on the nature of physical objects, with specific concern to their persistence conditions in time[4]. Several proposals are available, according to various authors, which can be roughly divided into three main accounts.

- The **perdurantist theory** of individuals has been recently upheld by authors as Hawley [39], Heller [41] and Sider [81], but it can be traced back to Quine [74], Russell [80] and Whitehead [97]. According to this account physical objects are extended in time as well as in space, with temporal parts formally analogous to spatial parts. This is why this theory is also named *four-dimensionalism*, after the three spatial dimensions and the temporal one. As a consequence individuals become *hunks of matters* or *spatio-temporal worms*, of which every-day objects are only temporary parts. Perdurantism virtually eliminates any distinction between objects and events, and some supporters maintain that it is more suitable to the image of space-time provided by contemporary physics[5].

- Four-dimensionalism confronts with the **endurantist theory** of material objects, also known as *three-dimensionalism*, one of the most influential representatives of which is D. Wiggins ([98]). Endurantism upholds the alleged traditional account of persistence - which in [65] Lowe ascribes even to Aristotle[6] - according to which physical objects extend only across space, and persist in time by remaining *wholly present* in every moment in which they

[2]Consider Reidhaar-Olson's proof of the fixed-point theorem for calculus GL.

[3]"The realm of the concrete precisely is the realm of time-bound existence." [65], p. 84.

[4]Consider for instance [63], [65].

[5]See [4], [5].

[6]*Physics*, Book IV, 10-14.

exist. Endurantists claim that their theoretical proposal is consistent with contemporary physics ([78]), and criticize the perdurantist notion of temporal part as meaningless ([65], [78]).

- Finally we have a third account - named **sequentialism** - the first explicit statement of which dates back to [17] by R. Chisholm; Lewis gave relevant contributions to its formal development in [59], [60]. Sequentialists agree with endurantists in thinking of objects as not extended in time, but differently from endurantists, in their opinion it is not legitimate to speak of strict identity of individuals in time as endurantists do; rather they deal with a *counterpart relation*, connecting different temporal stages into a unique *ens successivum.* In [94] Varzi discriminates between an endurantist version ([17], [65]) and a radical version ([59]) of sequentialism: according to the former *entia successiva* are made of three-dimensional basic constituents, called *continuants*; the latter denies the existence of such constituents.

It is quite natural to question the relationship among these different theories. We may wonder whether an account is reducible or irreducible to another one, which one has the greatest generality, which notions differ only verbally and which are genuinely different. Some work has been recently done on this topic, consider for instance [69] by K. Miller. In [45], [66] the authors uphold the equivalence between three- and four-dimensionalism. But in analysing these comparisons we have to make some preliminary remarks:

- As we said at the beginning of the paragraph, there exists no unique formulation for three-, four-dimensionalism and sequentialism, rather several proposals according to different authors. In some cases we confront with 'mixed views'[7].
- These ontological theses are often associated with orthogonal issues. For instance it is common opinion that while presentism implies endurantism by its actualist account of existence in time, perdurantism demands an eternalist framework, in which all the instants in time and temporal parts of objects are equally existing. The idea, according to which a particular theory of existence brings forth a specific ontology of physical objects, is present both in Loux [63] and in Lowe[8]. Furthermore a realist account of time is considered as a point of eternalism, thus supporting perdurantism. On the contrary a doctrine of time as Aristotle's, according to which time is engendered by the motion of enduring objects, is allegedly associated to three-dimensionalism. All these background assumptions make difficult to evaluate the arguments supporting or criticizing each persistence theory.

[7] Chisholm's sequentialism has three-dimensional basic constituents. On the other hand, Sider supports perdurantism in almost the whole [81], then turns to sequentialism in the last chapter.

[8] "I argued that an 'endurance' theory of persistence goes naturally with a tensed view of time whereas a 'perdurance' theory is suited to a tenseless view." [65], p 106. On the contrary Sider ([81], p. 68) maintains that all the four combinations between presentism/eternalism and endurantism/perdurantism can be formulated as consistent theories, even if he remarks that the mix of presentism and perdurantism has not been considered by anyone.

- We lack a clear conceptual framework, most of times language is ambiguous. An example of this kind of problem is the discussion on the existence and nature of temporal parts, for which there is no unique definition[9].

We clearly see that there are strong reasons for applying logic to our ontological theories. In fact we can likely solve the difficulties listed above by providing a formal presentation of three-, four-dimensionalism and sequentialism. Such a presentation would have a threefold value, as (i) it would clarify the content of the different theoretical proposals, by stating them in the same (formal) language; (ii) it would make ontology comparison feasible, thus grounding any reduction or independence proof; (iii) it would create a common conceptual space, in which we would be able to talk about both temporal and modal properties of material objects.

Formalizing theories

At this point the question is: which is the 'right' formalization for our theories on persistence? There is always a gap between the intuitive content of a theory and its formal exposition, just think about the problems we listed in section 2.1 on the relationship between actualism and varying domain K-models. We thus proceed as follows:

1. In the first part of the present work we presented three different semantic accounts for quantified modal logic: (i) Kripke semantics in chapter 1, (ii) counterpart semantics in chapter 2 and (iii) the conceptual interpretation in chapter 3.

2. In this chapter we highlight that each formal proposal is based on a particular ontology for physical object: endurantism, sequentialism and perdurantism respectively, by referring to three main features of semantics: (i) the nature of individuals appearing in the domains of these structures, (ii) the principles sound with respect to each account, (iii) the representation and solution of ontological problems within these logical frameworks.

We aim at proving that our semantics for quantified modal logic actually constitute sound formalizations for the three ontological theses. Nonetheless there are some unsatisfactory aspect in our approach. In fact the notion of temporal part, which plays a major role in the literature, can not be expressed in formal language $\mathcal{L}^=$ introduced in chapter 3. We shall see that this relation is expressible in the metalanguage.

Finally we remark that, since we do not tackle the actualism/possibilism issue in the present discussion, we shall generally deem inner domains in our structures identical to outer ones.

Comparing theories

There is a further aspect of the interplay between logics and ontology. Once we have showed that Kripke semantics, the conceptual interpretation and counterpart

[9] Consider Sider's remarks on pp. 53-55 in [81].

semantics are sound formalizations for three-, four-dimensionalism and sequentialism respectively, then syntactic results provable for the formers can be used to deduce true statements concerning the latter. In particular, we can prove interesting facts related to the comparison of ontologies.

In chapter 5 we shall consider [27] by Fitting and a couple of papers by Kracht and Kutz ([53], [52]), in which the authors present translation functions from validities in counterpart semantics to formulas sound w.r.t. intensional structures. We develop their accounts and maintain that if a sequentialist accepts certain constraints on the counterpart relation, then she can agree with perdurantists by interpreting her modal discourse according to Fitting's and Kracht and Kutz's translation functions. We shall examine whether these results are strong enough to logically reduce perdurantism to sequentialism.

Our formal approach makes possible even to compare endurantism and sequentialism. According to the intuition by which the former is a limit case of the latter one, where counterpart relation is identity, we shall check whether we can reduce everything to sequentialism. In fact we prove a more interesting result: we need not to take identity as counterpart relation in order to falsify a formula not valid in Kripke frames, we can be content with much weaker conditions.

From an ontological point of view this result - formally proved in section 5.3 - implies that an endurantist need not to assume that something really persists through change, a sequence of suitably related objects is sufficient. Such an assertion may be surprising, as we said that one of the fundamental theses of endurantism postulates the persistence of objects identical to themselves. How could endurantists be content with 'a sequence of suitably related objects'?

Before comparing Kripke semantics, the conceptual interpretation and counterpart semantics to three-, four-dimensionalism and sequentialism in sections 4.2, 4.3 and 4.4 respectively, we remark that our analysis does not fall within the domain of ontological investigations, rather it is a task of metaontology. This means that we do not aim at solving ontological issues, by providing an answer to well-known puzzles; rather our purpose consists in comparing the various solutions proposed, and in assessing different degrees of generality. If our analysis in the present chapter is sound, then by the results in chapter 5 we maintain that three- and four-dimensionalism can be reduced to sequentialism. But this fact does not imply that that this theory solves all our problems, we only prove that whatever can be expressed by means of either three- or four-dimensionalism is expressible within sequentialism too.

4.2 Kripke Semantics and Endurantism

The present section is devoted to the analysis of the relationship between Kripke semantics and endurantism. We compare these accounts by considering two main aspects: in par. 4.2.1 we analyse the ontological assumptions underlying K-models, we claim that Kripke semantics is based on an endurantist account of individuals; whereas in par. 4.2.2 we consider some popular puzzles of individual change in time, and show that the endurantist solutions can be formalized within the kripkean framework. These two points are the content of the correspondence between endurantism and Kripke semantics. As regards the technical details for the latter, hereafter we refer to chapter 1.

4.2.1 An endurantist ontology

In order to motivate our claim that Kripke semantics is based on an endurantist ontology, we first analyse the assumptions on individuals in K-models, then we consider formulas sound w.r.t. Kripke semantics and check whether they are valid principle even for endurantists.

Wholly present objects

In the introduction we anticipated that according to endurantism an object persists in time by remaining *wholly present* in every instant in which it exists[10]. Some author tries to define what it means to be 'wholly present' in terms of temporal parts (see for instance [81], p. 64), but it is doubtful whether the notion of temporal part, which is a target for endurantists' criticisms, is clearer than the concept of a 'wholly present object'.

From chapter 1 we recall that Kripke semantics for quantified temporal logic is characterized by two ideas:

1. K-frames contain a set $D(t)$ of individuals for each instant t. We intuitively think of these individuals as the objects which it makes sense to talk about in instant t, as we interpret the variables for individuals in our language on each $D(t)$.

[10] "Something [...] *endures* iff it persists by being wholly present at more than one time." [60], p. 202.

"The endurantist claims that for a concrete particular to persist through time is to exist wholly and completely at different times." [63], p. 202.

"[...] according to the endurance account, an object persists through time by being 'wholly present' at each time at which it exists." [65], p. 98.

"I say that an object 'endures' if it persists by being wholly present at every time at which it exists." [78], p. 205.

"Three-dimensionalists say that things *endure*, that they have no temporal parts, that they are *wholly present* at every moment of their career." [81], p. 53.

"[...] things [three-dimensional continuants] with spatial parts and no temporal parts, which are conceptualized in our experience as occupying space but not time, and as persisting whole through time." [98], p. 31.

2. For evaluating in instant t a formula of type $\Box\phi[x_1, \ldots, x_n]$, where variables $x_1, \ldots, x_n$ are free, in every instant t' temporally related to t we refer to the same individuals $\sigma(x_1), \ldots, \sigma(x_n)$ in $D(t)$.

By the first point we conclude that individuals appearing in outer domain $D(t)$ are wholly present in instant t, as terms for individuals are interpreted on objects confined to t. As to the second point, we clearly see that the evaluation clause for modalized formulas is so structured that an individual is reidentified with itself in passing from one instant to another, that is, statement $\Box\phi[x_1, \ldots, x_n]$ is true of objects $a_1, \ldots, a_n$ in instant t iff in every instant accessible from t statement $\phi[x_1, \ldots, x_n]$ is true of the same individuals $a_1, \ldots, a_n$. In turn automatic reidentification is due to the increasing outer domain condition and rigidity of assignments, by which if $\sigma(x) = a \in D(t)$ and tRt' then $\sigma(x) = a \in D(t')$. Therefore reidentification through time becomes a primitive notion, as it is the case for endurantism; our modal language is so interpreted that it is originally guaranteed trans-world identity of objects.

These two features - individuals 'bound' to each instant t, which remain identical to themselves in passing from one instant to another - represent a first justification of the alleged endurantist character of Kripke semantics.

Endurantist principles

Kripke semantics validates some controversial principles, the intuitive interpretation of which is nonetheless accepted by endurantists. In chapter 3 we checked soundness w.r.t. K-frames for a well-known postulate in classical logic with identity: Leibniz's Law,

$$x = y \to (\phi \to \phi[x/y]) \qquad 3.2$$

In chapter 3 we remarked the relevance of 3.2 in establishing identity conditions for objects, as it is supposed to formalize the metalinguistic principle of substitution of identicals *salva veritate*. Then we showed that in Kripke semantics Leibniz's Law holds for every formula ϕ, even if modal operators appear therein. In fact if in instant t, $\sigma(x)$ is equal to $\sigma(y)$, then the objects denoted by variables x and y are one and the same individual and we can apply SI.

By the very same line of reasoning, endurantists deem Leibniz's Law unrestrictedly valid as well. By the antecedent in instant t the object denoted by variable x and the object denoted by y are equal, but endurantists consider only wholly present objects, thus if the denotations of x and y are equal in t then they are plain identical. Finally by SI every statement true of the denotation of x is true also of the denotation of y, and 3.2 holds.

We find this argument in [98], where Wiggins defends Leibniz's Law to uphold the *absoluteness of identity* against the supporters of relative identity, according to which "the notion of identity is concept- or sortal-relative, i.e. relative to the different possible answers to the question 'a is the same what as b?'[11]". Wiggins deduces the absoluteness of identity from 3.2 and makes use of it to develop an endurantist

[11] [98], p. 23.

theory of reidentification, thus we clearly see why most of the first chapter in [98] is devoted to motivate 3.2. In favour of Leibniz's Law, Wiggins maintains that:

> How, if a is b, could there be something true of *the object* a which was untrue of *the object* b? *They are the same object.* People sometimes speak of counterexamples to Leibniz's Law. But these are scarcely more impressive than the counter-examples to the Law of Non-Contradiction. ([98], p. 27)

But his defence of 3.2 is precisely based on an endurantist ontology, rejected by most of the supporters of relative identity. In fact from a is b - *in the present moment* - Wiggins infers that they are the same object, as objects are wholly present in every moment in which they exist. But this perspective, shared by Kripke semantics and endurantism, relies on hidden ontological assumptions 1 and 2 above.

In the class of K-frames two other relevant formulas with identity are valid: the necessity of identity and the necessity of difference, which we deal with in par. 3.2.2:

A22.	$x = y \rightarrow \Box(x = y)$	necessity of identity,
A23.	$x \neq y \rightarrow \Box(x \neq y)$	necessity of difference.

The soundness of A22, A23 is a direct consequence of the endurantist characteristics of Kripke semantics: (i) independence of assignments from instants and (ii) automatic reidentification of objects. Whenever $\sigma(x)$ is equal or different from $\sigma(y)$ in instant t, then the objects denoted by x and y are either the same or distinct individuals. By (i) and (ii) this will be the case in every instant t' accessible from t[12].

It is not by chance that both these principles are deemed sound by endurantists, on the base of considerations implying a Kripke-theoretical account of terms for individuals. If objects are identified with the material content appearing in a certain moment in time, then it is clear that all the identity and difference statements are necessarily true whenever true. The very same line of thoughts is followed by Kripke in [57], where he upholds an endurantist account of modality:

> It was clear from $(x)\Box(x = x)$ and Leibnitz's law that identity is an 'internal' relation: $(x)(y)(x = y \supset \Box x = y)$. (What pairs (x, y)) could be counterexamples? Not pairs of distinct objects, for then the antecedent is false; nor any pair of an object and itself, for then the consequent is true. ([57], p.3)

The same argument is at endurantist's disposal. We conclude that Kripke semantics and endurantism agree on the fact that individuals cannot separate in two distinct entities, nor merge and give life to a new individual. In the next paragraph we show which problems arise from this perspective and which are the strategies adopted by endurantists to solve them.

[12] We recall that A22 and A23 are provable by 3.2 and Brouwer's axiom.

4.2.2 Reidentifying in time

To further motivate our endurantist reading of Kripke semantics, we consider some conceptual problems related to change in individuals, for what concerns both their properties and their parts. Finally we present the puzzle of coincident but distinct objects. We show that the endurantist solutions to these questions can be formalized within Kripke-frames.

Puzzles of change: qualitative

For presenting the first type of puzzles, in [94] Varzi considers a table, which is clean in the morning and dirty at the evening. Substitution of identicals *salva veritate* - a revered principle in logic and ontology to which we referred in par. 4.2.1 - states that:

(SI) if individuals a and b are one and the same object, then every statement ϕ true of a is true also of b.

While the table is clean in the morning, it is dirty at the evening, thus by contraposition the table in the morning is supposed to be a different object from the table at the evening. This conclusion seems to deny the persistence of objects through qualitative change, and it turns out to be particularly awkward for an endurantist who is willing to affirm straight identity of individuals in passing time.

Before analysing the endurantist solutions to the puzzle of qualitative change, we consider how this situation is tackled in Kripke semantics. In K-frames we can ascribe different properties to the same object in distinct instants without any trouble: to say that in the morning the table is clean, we just affirm that at moment *morn* object **tab** belongs to the extension of predicate *Clean*, formally $\mathbf{tab} \in I(Clean, morn)$. On the other hand, the table is dirty at the evening iff $\mathbf{tab} \notin I(Clean, even)$. We clearly see that membership to $I(Clean, morn)$ and to $I(Clean, even)$ are not the same property, hence we do not confront with a violation of substitution of identicals. The identity between the table in the morning and the table at the evening can be soundly upheld.

This formal account does not explain what it means for individual x to have property P in moment t, but it reflects the endurantist solution obtained by introducing a temporal parameter[13]. Now the question is to philosophically motivate this introduction. Varzi lists four different analysis available to endurantists of the relationship among an individual, a property and an instant of time:

> We can interpret P either as a relational predicate relating object x to time t, or we can think that 'in t' acts as an adverbial modifier on the whole proposition 'x is P', or on predicate P, or on the copula 'is'. ([94], p. 11)

None of these solutions is free from criticisms, as Varzi remarks. The first one goes into Lewis' argument of temporal intrinsics[14], according to which if property

[13] For instance consider [98], p. 29; as well as [64], [88].
[14] [60], pp. 202-204.

P is a relational predicate relating object x to instant t, then there is no truly intrinsic property. Every statement of the form *subject-predicate* would have the logical structure *predicate(subject,time)*. The second solution shifts the problem from object-language to metalanguage: we define 'individual x has property P in moment t' true if and only if 'individual x has property P' is true in moment t; we are left with the problem of the meaning of definition's rightmost member. The third solution does not clarify the relationship between property P and the modification of P obtained by adjoining adverbial expression 'in t'. As to the fourth one, the temporal parameter should modify the exemplification relation represented by the copula, but thinking of exemplification as a relation causes a regress to infinity.

Notice that the logical analysis of statement 'individual x has property P in moment t' in Kripke semantics as $\sigma(x) \in I(P,t)$ is consistent with all these solutions, and thus subjected to each one of the criticisms listed above. In fact in Kripke semantics the interpretation $I(P)$ of property P, defined as $I(P)(t) = I(P,t)$, is a relation between individuals and time; moreover truth conditions for formula $P(x)$ at time t are given by metalinguistic statement $(I^\sigma, t) \models P(x)$, in which instant t appears; finally there is no constraint on the various $I(P,t)$, it can be any subset of $D(t)^n$.[15]

We conclude that for the puzzles of qualitative change there exists a logical account, that is, there are K-models that explain within the endurantist perspective how an individual can have inconsistent properties in distinct moments. The real question concerns the ontological level, as we do not have a clear endurantist account on what these models mean.

Puzzles of change: mereological

In the literature on persistence it is common to find puzzles, analogous to the ones for qualitative change, in which an individual is modified in some of its parts[16]. Consider again the table in the previous paragraph, this time it does not get dirty, but is deprived of a little piece; hence the table at the evening tab_e is a proper part of the table in the morning tab_m. We would like to maintain the following identities: (i) between the table in the morning tab_m and the table at the evening tab_e, (ii) between the table at the evening tab_e and the proper part $tab-$ of the table in the morning. But from (i), (ii) it is possible to derive a contradiction by using only endurantistically valid principles. This time we do not face an alleged violation of

[15] In order to clarify the relationship between predicate P and the extension of P in a certain instant t, Zalta modifies the definitions of K-frame and interpretation. In the former he introduces a denumerable set of functions $R^n_1, R^n_2, \ldots$, for every $n \in \mathbb{N}$, mapping each world w to a subset of $D(w)^n$; whereas interpretation I assigns a function R^n to every n-ary predicative constant P^n. In this way he tries to formally express the fact that the various extensions $I(P^n)(w)$ of predicate P^n are all instances of the same relation $I(P^n) = R^n$. Although Zalta's idea is interesting, from a logical point of view it adds nothing new to the present discussion, as we can prove that formula $\phi \in \mathcal{L}$ holds in the class of K-frames iff it is valid in the class of structures defined by Zalta.

[16] Next argument first appeared in [87] against arbitrary undetached parts. Since then it has been used with the most different aims: by Heller in [41] and Sider in [81] against endurantism; by van Inwagen in [88] to prove that perdurantism implies a counterpart-theoretic analysis of *de re* modality.

substitution of identicals *salva veritate*, as it was the case in qualitative change; rather we are in a situation in which either (a) the principle according to which the whole is strictly greater than each proper part fails, or (b) two distinct objects fill the same space. Even if language $\mathcal{L}^{=}$ does not include a symbol for proper-part relation, by simply postulating that the relation denoted by '$<$' is irreflexive, we can reconstruct this puzzle with our formal machinery and check which principles are considered and which are responsible for the inconsistency. First of all we fix our four hypotheses:

1. at the evening, the table minus a part is equal to the table at the evening;
2. at the evening, the table at the evening is equal to the table in the morning;
3. in the morning, the table minus a part is a proper part of the table in the morning;
4. no object is a proper part of itself.

The first hypothesis corresponds to (ii) and to the negation of (b), the second one to (i); whereas the third and the fourth one deny (a). These four hypotheses make true, in the morning, the following modal formulas:

$$\diamond(tab- = tab_e) \tag{4.1}$$
$$\diamond(tab_e = tab_m) \tag{4.2}$$
$$tab- < tab_m \tag{4.3}$$
$$tab_m \not< tab_m \tag{4.4}$$

where diamond $\diamond$ means 'there is a moment t s.t. ...'. Now consider the following proof:

1)	$\diamond(tab- = tab_e)$	by 4.1
2)	$\diamond(tab- = tab_e) \rightarrow (tab- = tab_e)$	A23
3)	$(tab- = tab_e)$	from 1,2 by R1
4)	$\diamond(tab_e = tab_m)$	by 4.2
5)	$\diamond(tab_e = tab_m) \rightarrow (tab_e = tab_m)$	A23
6)	$(tab_e = tab_m)$	from 4,5 by R1
7)	$(tab- = tab_m)$	from 3,6 by transitivity of identity
8)	$(tab- = tab_m) \rightarrow ((tab- < tab_m) \rightarrow (tab_m < tab_m))$	Leibniz's Law
9)	$(tab- < tab_m) \rightarrow (tab_m < tab_m)$	from 7,8 by R1
10)	$(tab- < tab_m)$	by 4.3
11)	$(tab_m < tab_m)$	from 9,10 by R1
12)	$(tab_m \not< tab_m)$	by 4.4
13)	$\perp$	from 11,12

Notice that the present proof makes use of only endurantistically valid principles. To avoid the inconsistency we have to reject either one of hypotheses 4.1/4.4, or one of the envisaged axioms and inference rules. The soundness of hypotheses 4.3 and

4.4 seems to be out of question, as these formulas reflect principles at the base of every theory of parts. Even the application of Leibniz's Law appears unquestionable as substitution does not occur within an intensional context. If we deem hypothesis 4.2 false, then we have to deny that an object can lose anyone of its part, though remaining identical to itself. We run into mereological essentialism[17]. Other authors think that the identity relation is not transitive[18]. Further authors, as van Inwagen in [87], maintain endurantism by denying that there can exist objects as the proper part of the table in the morning.

The perdurantist solution, with which we deal in details in the next section, invalidates the necessity of identity: from identity at the evening between the table at the evening and the table in the morning is not legitimate to infer that in the morning the two objects are equal.

If we reject 4.1 then we must admit that two distinct objects can fill the same space. Most endurantist solutions to the puzzle of mereological change develop this intuition. In [98] Wiggins makes use of his theory of sortals to uphold the invalidity of 4.1 and avoid inconsistencies. The basic idea of this theory is that "every thing is *something*, that is an entity of some type, and it is precisely this membership which determines identity conditions in time and space"[19]. According to Wiggins the table at the evening and the table minus a part belong to two different sortals, thus we can speak of spatial coincidence between them two at the evening, but not of identity. It is easy to understand how Wiggins can think of the table at the evening as distinct from the table minus a part: they have different modal properties and by Leibniz's Law, which is unrestrictedly valid for endurantists, the table at the evening differs from the table minus a part. The conclusion is that 4.1 is false[20].

Before considering our last puzzle, we notice that in the case of qualitative change we confronted with an ontological issue, which presented no trouble from a logical perspective. On the contrary in this case there are problems both from a logical and an ontological point of view.

Coincident but distinct objects

In this paragraph we tackle another problem often discussed in the literature on the ontology of physical objects, known as the puzzle of coincident but distinct objects[21]. For introducing it we quote from Heller:

> Consider two lumps of clay, one of which is shaped like the bottom half of a statue and the other like the top half of a statue. At the moment that those lumps are stuck together two distinct objects are brought into existence. One is a larger lump of clay composed of the two lumps. The other is a statue. [...] The larger lump and the statue are spatially and temporally coincident, but

[17]This is the opinion of Chisholm in [16].
[18]A similar account is upheld by Heller and Geach [33].
[19][92], p. 15.
[20]A similar solution is proposed by another endurantist, J. J. Thomson, in [85].
[21]Notice that this problem is reducible to fission of individuals.

they are not identical. For instance, it is true of the lump that it could have had a completely different shape, but this is not true of the statue. ([41], p. 30)

The version provided by Heller is of modal type, but it is easy to understand how we can paraphrase his argument according to time, by considering a moment in which the statue is deformed. An endurantist ontology implies counterintuitive consequences, in fact if the lumps of clay and the statue actually coincides and Leibniz's Law is a sound principle, then we cannot discriminate the lumps of clay from the statue on the base of some property. But this conclusion is inconsistent with the intuition that persistence conditions for statues are different from the ones for lumps of clay.

We can formalize even the present argument in language $\mathcal{L}^{=}$. We denote the lumps of clay and the statue with l and s respectively, predicative constant Dif stands for 'having a different form'. Our hypotheses are the following:

(a) $s = l$, the statue and the lumps of clay are identical;

(b) $\neg \diamond Dif(s)$, the statue does not survive deformations;

(c) $\diamond Dif(l)$, the lumps of clay survive deformations.

From these three premises, by endurantistically sound principles, we infer a contradiction.

1)	$(s = l) \to (\neg Dif(s) \to \neg Dif(l))$	Leibniz's Law
2)	$\Box(s = l) \to (\Box\neg Dif(s) \to \Box\neg Dif(l))$	from 1 by T3
3)	$(s = l) \to \Box(s = l)$	A22
4)	$(s = l) \to (\Box\neg Dif(s) \to \Box\neg Dif(l))$	from 2, 3 by transitivity of implication
5)	$s = l$	by (a)
6)	$\Box\neg Dif(s) \to \Box\neg Dif(l)))$	from 4, 5 by R1
7)	$\Box\neg Dif(s)$	by (b)
8)	$\Box\neg Dif(l)$	from 6, 7 by R1
9)	$\neg\Box\neg Dif(l)$	by (c)
10)	$\perp$	from 8, 9

There must be something wrong with either our postulates or hypotheses. It is unlikely that modus ponens, necessitation or necessity distribution are not valid, thus ontologists exclusively focus on premises (a)/(c) and the necessity of identity. As to the formers, (b) and (c) do not seem disprovable as they merely correspond to common intuitions on persistence conditions for statues and lumps of clay. Eliminating the necessity of identity tantamounts to restricting Leibniz's Law, but we have seen that the this principle is considered by endurantists so relevant for fixing identity conditions for individuals, that it can be neither invalidated nor restricted. Therefore we focus on hypothesis (a).

Even in the present case endurantists can make use Wiggins' theory of sortals to falsify hypothesis (a), thus avoiding inconsistencies. The statue and the lumps of clay belong to two different sortals, thus they maintain that they are different objects filling the same space and deny (a)[22].

4.2.3 Remarks

We conclude the present section by summarizing the main points in the relationship between Kripke semantics and endurantism. In par. 4.2.1 we highlighted the enduratist features of Kripke semantics:

1. In K-frames, for every instant t, there is an outer domain $D(t)$ of individuals, wholly present in t; we interpret the variables in our language on these individuals.

2. For evaluating in instant t modalized formula $\Box\phi[x_1, \ldots, x_n]$, where variables $x_1, \ldots, x_n$ are free, in every instant temporally related to t we refer to the same individuals $\sigma(x_1), \ldots, \sigma(x_n)$ belonging to $D(t)$. Individuals persist identical to themselves in passing time.

3. Leibniz's Law, the necessity of identity and the necessity of difference are sound principles both in Kripke semantics and endurantism.

In par. 4.2.2 we analysed some puzzles concerning qualitative and mereological change, and the puzzle of coincident but distinct objects. We remarked that all these puzzles - due to the 'traditional' endurantist account of persistence - can be formalized within the kripkean framework. As to the first one, there exist Kripke models that, while solving the logical problem, left untouched the ontological question. In the other two cases we also have logical difficulties as well as ontological. In order to find a solution we briefly presented Wiggins' theory of sortals, that sticks to an endurantist ontology. But Wiggins' proposal is not universally taken for granted: in the next section we consider the perdurantist account, which modifies the underlying logical perspective.

Before addressing the substantial interpretation of quantified modal logic and perdurantism, we refer to a theorem by Ghilardi which makes clear some limits of Kripke semantics. This result constitutes a reason to maintain that K-frames are not completely suitable for formalizing modalities in contexts with individuals, thus we look for sounder accounts.

Consider a modal propositional logic L and define the *predicative companion* LQ of L as the system obtained by adding the postulates of classical predicate calculus to L. In [35] Ghilardi proved that:

Theorem 4.1 *If L is an extension of $S4$ and the predicative companion LQ of L is complete w.r.t. Kripke semantics, then either L extends $S5$ or L is included in $S4.3$.*

[22] Consider the discussion of cases α and β on p. 37 in [98] and conclusions on p. 40: "The 'is' must mean 'is constituted of', and collection of parts will not function standardly as a normal covering concept in the locution 'is the same collection as', as it figures in our examples (α) and (β)."

The present result means that for L included between $S4.3$ and $S5$, LQ is incomplete w.r.t. any class of K-frames. Ghilardi's theorem acquires particular relevance for our work, when we consider that among the extensions of $S4.3$ there are temporal logics as $S4.3.1$ (discrete time) and $S4.4$ (end-of-the-world logic).

The Kripke-incompleteness of these quantified temporal systems can be deemed a sign that Kripke semantics is not completely satisfactory as a conceptual apparatus for reasoning about individuals in time. If we maintain that Kripke semantics is a sound formalization for endurantism, then logical problems with the former may reveal ontological troubles with the latter.

4.3 The Substantial Interpretation and Perdurantism

In the present section we analyse the relationship between the substantial interpretation of quantified modal logic and the perdurantist theory of physical objects. As it was the case for Kripke semantics and endurantism, we compare these accounts by first considering the ontological assumptions underlying s-models in par. 4.3.1, then in par. 4.3.2 we analyse the perdurantist solutions to the puzzles of change. We show that the substantial interpretation is based on a perdurantist ontology, and that the solutions of perdurantism can be formalized within the substantial framework.

In considering the substantial interpretation we make use of language $\mathcal{L}^=$ with identity, in this way we satisfy one of the *desiderata* listed in the introduction: we present the ontological theories in the same language, so that it is possible to make a comparison.

4.3.1 A perdurantist ontology

In this paragraph we aim at underlining the perdurantist characteristics of the substantial interpretation, through the analysis of the ontological assumptions on s-models and of formulas sound w.r.t. the substantial interpretation.

Distinct temporal parts in subsequent moments

In the introduction we anticipated that in the opinion of perdurantists, physical objects persist in time by having different temporal parts in subsequent moments[23]. Change for individuals in time is due to the sequence of these parts, whereas their persistence is founded on the fact that all these parts belong to the same individual.

Substance semantics precisely shows these two features:

1. Substantial frames contain for each instant t a domain $D(t)$, but in this case variables in language $\mathcal{L}^=$ are interpreted not on the various $D(t)$, rather on functions $\mathbf{f}$ defined on W s.t. $\mathbf{f}(t') \in D(t')$.

2. For evaluating in instant t a formula of type $\Box\phi[x_1, \dots, x_n]$, where variables $x_1, \dots, x_n$ are free, in every instant t' temporally related to t we refer to values $\sigma(x_1)(t'), \dots, \sigma(x_n)(t')$ in t' of functions $\sigma(x_1), \dots, \sigma(x_n)$.

As to the first point, if we think of W as the set of instants in time, and deem elements in the various $D(t')$ as concrete instances of objects in different moments,

[23] "On this view, a concrete particular is an aggregate or whole made up of different temporal parts, each existing at its own time; and for a particular to persist from one time to another is for it to have different temporal parts existing at those different times." [63], p. 202.

"According to the perdurance account, an object persists through time by having different temporal parts at different times at which it exists." [65], p. 98.

"I say [...] that an object 'perdures' if it persists by being only partially present at every time at which it exists." [78], p. 205.

"Four-dimensionalists say that things have *temporal parts*, that they *perdure* and that they are spread out over time." [81], p. 53.

then we clearly see that function domain $F(t)$ - over which we interpret terms for individuals - is made of objects extending in time. On the other hand the reference to the same object in different times is guaranteed as well by interpreting variables on functions in $F(t)$: for evaluating in instant t a formula of type $\Box\phi[x_1, \dots, x_n]$, in every instant t' accessible from t we refer to temporal parts $\sigma(x_1)(t'), \dots, \sigma(x_n)(t')$ in t' of individuals $\sigma(x_1), \dots, \sigma(x_n)$ belonging to $F(t')$.

Because of these characteristics of the substantial interpretation, many authors naturally associate it to a perdurantist ontology, consider for instance the following quotation from [32]:

> [...] imagine that our possible worlds are now states of our universe at a given time. The extension of a term at a given time will turn out to be a temporal slice of some thing, 'frozen' as it is at that instant. Notice that things, since they change, cannot be identified with term extensions. Instead, things are world-lines, or functions from times into time slices, and so they correspond to term intensions or individual concepts. Since our ontology takes thing, not their slices as ontologically basic, it is natural to quantify over term intensions in temporal logic. ([32], p. 281)

The differences between the substantial interpretation and Kripke semantics have relevant consequences on the set of validities w.r.t. s-frames, that we analyse in the next paragraph.

Perdurantist principles

As we remarked in chapter 3, in the substantial interpretation Leibniz's Law is no more unrestrictedly valid, in particular we have counterexamples to 3.2 whenever ϕ has form $\Box\psi$, as in the cases of the necessity of identity and the necessity of difference. This is not a problem for supporters of the substantial interpretation, as one of their aims is to develop *contingent identity systems*[24], in which it is possible to talk about fission and fusion of individuals.

Perdurantists, differently from endurantists, agree on restricting Leibniz's Law in intensional contexts and motivate this choice on the base of their theory of objects. For instance Heller maintains that modal distinctions do not depend on the physical structure of objects, but on our conventions on their nature[25]. Modal properties determine persistence conditions of individuals, and their characteristics in counterfactual situations. Therefore for perdurantists it is possible that two objects coincides with respect to their physical structure, but are subjected to different conventions determining diverse counterfactual properties, thus falsifying Leibniz's Law. In par. 4.3.2 we shall notice that the same reasoning is used to solve the puzzle of coincident but distinct objects.

[24] Consider [47].

[25] "Those putative modal properties are founded on our conventions, not on any actual properties of the object." [41], p. 32.

Furthermore it is possible to show that different accounts on the nature of four-dimensional objects fit into the framework of the substantial interpretation. Consider the class of s-frames s.t. each function domain $F(t)$ contains all the functions $\mathbf{f}$ defined on W s.t. $\mathbf{f}(t') \in D(t')$, then we obtain a formal framework sound with respect to the *unrestricted mereological composition* thesis. According to this specific version of perdurantism - upheld by Heller too[26] - every sequence of temporal parts counts as an individual, independently from considerations of spatio-temporal relationship or similarity, just like the aggregate composed by David Lewis in instant t_1, a rock in t_2 and a blade of grass in t_3 in par. 3.4.1. In the same way, in the envisaged class of s-frames we admit as individual in domain $F(t)$ any function $\mathbf{f}$, thus eliminating any constraint on the values of $\mathbf{f}$ for the various instants t.

In par. 3.4.1 we also remarked that this class of s-frames validates some formulas, the informal reading of which is particularly counterintuitive. Consider principle 3.4 for instance:

$$\Box\exists x\phi \rightarrow \exists x\Box\phi$$

the meaning of which is rather cumbersome, in fact even if in every moment there exists the highest mountain, this fact does not imply that there exists a mountain which is eternally the highest one. But perdurantists supporting the unrestricted mereological composition have no problem in finding object x satisfying 3.4: just consider the individual obtained by composing the highest mountains in the different geological ages.

The soundness of 3.4 has another relevant consequence. In par. 3.4.1 we said that Fine proved in [23] that the set of validities w.r.t. the class of s-frames s.t. each function domain $F(t)$ contains all the functions defined on W is unaxiomatisable. If we accept the correspondence thesis between semantic accounts for quantified modal logic and ontological theories, then perdurantists upholding unrestricted mereological composition must admit some kind of incompleteness in their theory.

4.3.2 Perdurantist solutions to the puzzles of change

In this paragraph we adduce further reasons to think of the substantial interpretation as a formal approach sound with respect to perdurantism. In particular we analyse the perdurantist solutions to the puzzles of change in section 4.2, and show that these ontological arguments can be reconstructed within the logical framework of substantial models.

Qualitative change

In par. 4.2.2 we noticed that the problems with change in properties of individuals are not logical, as there exist Kripke models in which inconsistent properties are ascribed to the same object in different moments. The main difficulty lies in the ontological opacity of the endurantist solutions. We show that there exist substantial models describing qualitative change, and check that they are sound w.r.t. the perdurantist account of change. Consider again the example of the table which is

[26] "[F]or *every* filled [spatio-temporal] region there is one object that exactly fills it." [41], p. 7.

clean in the morning and dirty at the evening. Perdurantists accept the conclusion obtained by applying (SI): if the table is clean in the morning and it is dirty at the evening, then we are dealing with two distinct objects, and precisely with the temporal part of the table in the morning and the temporal part of the table at the evening. In order to formalize the perdurantist solution, we define a substantial model $\mathcal{M}$ as follows:

- $W = \{morn, even\}$;
- $R = \{\langle morn, even\rangle\}$;
- $D(morn)\{\mathbf{tab}_{morn}\}$, $D(even)\{\mathbf{tab}_{even}\}$;
- $F(morn) = F(even) = \{\mathbf{Tab}\}$ s.t. $\mathbf{Tab}(morn) = \mathbf{tab}_{morn}$, $\mathbf{Tab}(even) = \mathbf{tab}_{even}$.

Moreover we maintain that $\mathbf{tab}_{morn} \in I(Clean, morn)$ and $\mathbf{tab}_{even} \notin I(Clean, even)$, without being forced to distinguish between properties $I(Clean, morn)$ and $I(Clean, even)$, as perdurantism is consistent with difference of temporal parts $\mathbf{tab}_{morn}$, $\mathbf{tab}_{even}$ of four-dimensional object $\mathbf{Tab}$.

Therefore the substantial interpretation solves the puzzle of qualitative change in a different way from Kripke semantics. Anyway, even in the present case, we have to examine the soundness of the perdurantist account at an ontological level; that is, we confront with criticisms in par. 4.2.2 to the different explanations of what it means having a property in an instant of time. Lewis' temporary intrinsic argument is harmless, as it is not necessary to introduce a temporal parameter to discriminate between properties 'being dirty in the morning' and 'being dirty at the evening', it is enough to speak of plain 'being dirty'. By the same reason there is no need to think of expression 'at moment t' as an adverbial modifier of either proposition 'x is P', or predicate P, or copula 'is'. Perdurantists analyse statement 'individual x has property P in moment t' as 'the temporal part of individual x in moment t has property P'. Therefore perdurantists only need to assume the existence of temporal parts of objects.

Mereological change

In par. 4.2.2 we remarked that to avoid inconsistencies in the puzzle of mereological change, we have to reject one of hypotheses 4.1/4.4, or the necessity of difference, or Leibniz's Law, or transitivity of identity. Wiggins' endurantist solution amounts to rejecting 4.1 on the base of his theory of sortals, on the contrary perdurantists deny validity to the necessity of difference: from identity at the evening between the table at the evening tab_e and its proper part $tab-$ does not follow that they are equal even in the morning, only those temporal parts are. Similarly, from identity at the evening between the table at the evening tab_e and the table in the morning tab_m it is not legitimate to infer that they are identical even in the morning. By this reasoning perdurantists avoid the contradiction.

Moreover the perdurantist solution to the puzzle can be formalized by means of a substantial model $\mathcal{M}$ satisfying 4.1/4.4, defined as follows:

- $W = \{morn, even\}$;
- $R = \{\langle morn, even\rangle\}$;
- $D(morn)\{\mathbf{tab}_{morn}, \mathbf{tab}-\}$, $D(even)\{\mathbf{tab}_{even}\}$;
- $F(morn) = F(even) = \{\mathbf{Tab}, \mathbf{Tab}-\}$ s.t. $\mathbf{Tab}(morn) = \mathbf{tab}_{morn}$, $\mathbf{Tab}-(morn) = \mathbf{tab}-$, $\mathbf{Tab}(even) = \mathbf{Tab}-(even) = \mathbf{tab}_{even}$.

In instant *morn* object **tab**− is a proper part of object $\mathbf{tab}_{morn}$, thus normal interpretation I assigns to predicate $<$ extension $\{\langle \mathbf{tab}-, \mathbf{tab}_{morn}\rangle\}$ in *morn*. Finally assignment σ is s.t. $\sigma(tab_m) = \mathbf{Tab}$, $\sigma(tab_e) = \mathbf{Tab}-$ and $\sigma(tab-) = \mathbf{Tab}-$. We clearly see that in substantial model $\mathcal{M}$ interpretation σ satisfies hypotheses 4.1/4.4.

We conclude that perdurantists avoid inconsistencies in the puzzle of mereological change, by suitably restricting Leibniz's Law, in particular the necessity of difference is not valid. This choice reflects the four-dimensionalist account of individuals, according to which individuals can share temporal parts, thus separating and merging. Moreover we dealt with the perdurantist strategy for explaining change in parts of individuals within the formalism of the substantial interpretation.

Coincident but distinct objects

Finally we consider the perdurantist solution to the puzzle of coincident but distinct objects. In par. 4.2.2 we remarked that sortal-theoreticians solve the problem by rejecting hypothesis (a): the lumps of clay l and the statue s are objects filling the same space, thus they are coincident. But $s = l$ is false as they have different modal properties, that is, they are distinct. On the contrary perdurantists deny that the necessity of identity is a valid principle: from $s = l$ it is not legitimate to infer $\Box(s = l)$. The proof in par. 4.2.2 is thus blocked on line (2).

Most important, it is easy to construct a substantial model $\mathcal{M}$ describing the perdurantist solution to the puzzle. Define $\mathcal{M}$ as follows:

- $W = \{t_1, t_2\}$;
- $R = \{\langle t_1, t_2\rangle\}$;
- $D(t_1)\{\mathbf{statue/of/clay}\}$, $D(t_2)\{\mathbf{statue}, \mathbf{lumps/of/clay}\}$;
- $F(t_1) = F(t_2) = \{\mathbf{s}, \mathbf{l}\}$ s.t. $\mathbf{s}(t_1) = \mathbf{l}(t_1) = \mathbf{statue/of/clay}$, $\mathbf{s}(t_2) = \mathbf{statue}$, $\mathbf{l}(t_2) = \mathbf{lumps/of/clay}$.

Normal interpretation I assigns extension $\{\mathbf{lumps/of/clay}\}$ to predicate Dif in moment t_2, and assignment σ is s.t. $\sigma(s) = \mathbf{s}$ and $\sigma(l) = \mathbf{l}$. We clearly see that for assignment σ, s-model $\mathcal{M}$ satisfies hypothesis (a)/(c). Even in the present case the perdurantist solution to our puzzle has an elegant presentation within the framework of the substantial interpretation.

We conclude the analysis of the puzzles of mereological change and coincident but distinct objects, by comparing the perdurantist and Wiggins' endurantist solution.

At a first glance it may seem that both theories avoid inconsistencies by admitting that objects filling the same space can be distinct, that is, the physical structure of an individual is not the only foundation to identity statements about it. For instance, in the case of mereological change both accounts maintain that at the evening the table at the evening coincides with the proper part of the table in the morning: for perdurantists these objects share a common temporal part, whereas in the opinion of sortal-theoreticians they just fill the same space. Nonetheless both theories deem the table at the evening different from the proper part of the table, as they discriminate these objects on the base of their modal properties. Similarly for the case of the statue and the lumps of clay. The two accounts seem to agree on distinguishing individuals according to principle SI.

Through a thorough analysis of the proofs above, we realize in which sense the two accounts actually differ. The endurantist theory of sortals rejects identities $\diamond(t- = t_e)$ and $s = l$: the table at the evening and the proper part of the table are essentially different three-dimensional objects, and this is the case also for the lumps of clay and the statue. On the contrary perdurantists accept identities $\diamond(t- = t_e)$ and $s = l$, as the temporal parts of these objects coincide, but it is not legitimate to apply Leibniz's Law to infer identity in the morning between the table at the evening and the proper part of the table, or between the lumps of clay and the statue after the deformation.

4.3.3 Remarks

We summarize the features of the relationship between the substantial interpretation and perdurantism. In par. 4.3.1 we claimed that the semantics of substances assumes a perdurantist ontology, and motivated this claim by the following remarks:

1. Substantial frames contain for each moment t a domain $F(t)$ of substances - functions $\mathbf{f}$ defined on instants in W s.t. $\mathbf{f}(t') \in D(t')$ - over which we interpret terms for individuals in our language. We can think of $\mathbf{f}(t')$ as the temporal part of four-dimensional object $\mathbf{f}$ in instant t'.

2. For evaluating in moment t modalized formula $\Box\phi[x_1, \ldots, x_n]$, where variables $x_1, \ldots, x_n$ are free, in every moment t' related to t we refer to temporal parts $\sigma(x_1)(t'), \ldots, \sigma(x_n)(t')$ in t' of individuals $\sigma(x_1), \ldots, \sigma(x_n)$ belonging to $F(t')$.

3. Leibniz's Law is not unrestrictedly valid in s-frames, in particular the necessity of identity and the necessity of difference fail. This means that the substantial interpretation is a sound framework to talk about perdurantist fusion and fission of individuals.

4. Within the substantial interpretation we can formalize the version of perdurantism accepting the unrestricted mereological composition.

In paragraph 4.3.2 we presented the perdurantist solutions to the puzzles of qualitative and mereological change and of coincident but distinct objects. As regards the first one, we defined a perdurantist model solving the logical problem as it was the case for endurantism and K-models. Differently from endurantism, the perdurantist account is not subjected to criticisms in par. 4.2.2 on what it means for individual x to have property P at time t. Moreover in the cases of mereological change and coincident but distinct objects, we have at our disposal s-models formalizing the perdurantist solutions. We conclude that the substantial interpretation is sound with respect to perdurantism.

4.4 Counterpart Semantics and Sequentialism

We have come to the last comparison between semantics for quantified modal logic and ontological theories: counterpart frames and sequentialism. As it was the case in sections 4.2 and 4.3, we consider the ontological assumptions underlying counterpart semantics for quantified modal logic, and the sequentialist solutions to the puzzles of change. We aim at proving that the sequentialist ontology underlies counterpart semantics.

Even in the present case we deal with language $\mathcal{L}^=$ with identity, thus present all the ontological theories in one same language.

4.4.1 A sequentialist ontology

In this paragraph we justify the alleged correspondence between semantics for quantified modal logic and ontologies for physical objects, by analysing the ontological assumptions on counterpart models and the principles on identity valid w.r.t. this interpretation.

Counterparts

Counterpart semantics is strongly philosophically motivated. The first formal proposal was made by Lewis in [59], though in an extensional language and referring to alethic modalities. Lewis motivated his counterpart theory by a sequentialist analysis of reidentification:

> [...] we might say, speaking causally, that your counterparts are you in other worlds, that they and you are the same; but this sameness is no more literal identity than the sameness between you today and you tomorrow. It would be better to say that your counterparts are men you *would have been*, had the world been otherwise. ([59], p. 112)

As regards necessity and possibility, Lewis comes to the same conclusions as sequentialists: we cannot properly speak of strict identity in counterfactual situations[27]. On the contrary, he seems to think of identity in time as a model for his counterpart theory, while taking for granted that sequentialism is sound. Thus counterpart semantics has been impregnated by the sequentialist perspective from its very beginning.

We analyse in detail the sequentialist features of counterpart semantics, as presented in chapter 2. First of all we summarize its basic characteristics:

1. As in Kripke semantics, counterpart frames contain for every instant t a domain $D(t)$ of individuals, over which we interpret the variables in our language.

2. Differently from Kripke semantics, for evaluating in instant t a formula of type $\Box\phi[x_1, \dots, x_n]$, where variables $x_1, \dots, x_n$ are free, in every instant t'

[27] Consider also chapter III in [17]. Moreover it is not a case that Chisholm as a sequentialist was also a supporter of counterparts, see for instance [16], pp. 592-594.

temporally related to w we do not necessarily refer to the same individuals $\sigma(x_1), \ldots, \sigma(x_n)$ in $D(w)$, but to their *counterparts* $\tau(x_1), \ldots, \tau(x_n)$ belonging to $D(w')$.

As it was the case in Kripke semantics, terms for individuals are interpreted on objects 'confined' to each instant t, not extending across different moments. But differently from Kripke semantics, reidentification is not taken for granted, individuals are not reidentified with themselves in passing from one instant to another. Just consider the evaluation clause for modal formulas as stated above, where σ is n-assignment $a_1, \ldots, a_n$ and τ is $b_1, \ldots, b_n$. When we evaluate in instant t formula $\Box\phi[x_1, \ldots, x_n]$, with respect to individuals $a_1, \ldots, a_n$, in every instant t' accessible from w we do not necessarily refer to the same $a_1, \ldots, a_n$ - it is even possible that they do not exist in instant t' - but to their *counterparts* $b_1, \ldots, b_n$ in t'.

Notice the differences between the present account and Lewis' counterpart theory. Nothing prevents us from assuming identity as counterpart relation, then we are back to the evaluation clause in Kripke semantics and endurantism can be seen as a limit case of sequentialism, where the only counterpart to an object is the object itself. Such a case is explicitly rejected by Lewis, as he denies that the same object exists in more than one world.

Sequentialist principles

In chapter 2 we remarked that neither the necessity of identity, nor the necessity of difference are sound principles in counterpart semantics. Then we defined two constraints to impose on counterpart relation:

Definition 4.2 (Functionality) *A c-frame $\mathcal{F}$ is* functional *iff for every $w, w' \in W$, for $a \in D(w)$, $b, b' \in D(w')$, if wRw', $C_{w,w'}(a, b)$ and $C_{w,w'}(a, b')$ then $b = b'$.*

Definition 4.3 (Injectivity) *A c-frame $\mathcal{F}$ is* injective *iff for every $w, w' \in W$, for $a, a' \in D(w)$, $b \in D(w')$, if wRw', $C_{w,w'}(a, b)$ and $C_{w,w'}(a', b)$ then $a = a'$.*

We showed that functionality is a necessary and sufficient condition for axiom A22 to hold, on the other hand a *c*-frame is injective if and only if A23 is valid. These features of counterpart semantics are consistent with sequentialists' claim that individuals can split and merge, that is, in counterpart models we can describe situations in which an individual has more than one counterpart in a subsequent moment, or it is counterpart to more than one individual appearing in a previous moment. This fact is relevant in relation to the puzzle of mereological change.

Differently from what happens in the substantial interpretation, the failure of A22 and A23 does not imply that Leibniz's Law is not universally valid. As we noticed in chapter 3, principle 3.2 holds for any typed formula ϕ, but by making explicit substitutions we at most prove

$$(x_1 = x_2) \to (\Box(x_1 = x_2)[x_1, x_1] \to \Box(x_1 = x_2))$$

which is harmless. In order to obtain A22, we should be able to infer

$$\Box(x_1 = x_2)[x_1, x_1] \tag{4.5}$$

but formula 4.5 tantamounts to functionality.

In particular Leibniz's Law is sound in the case that ϕ is a formula like $\Box P(x_1)$. This means that if the object denoted by x_1 is the same as the object denoted by x_2, then all the modal properties, not relations, of the former are also modal properties of the latter. As we will see in the next paragraph, this point gives problems in relation to the puzzle of coincident but distinct objects.

4.4.2 Sequentialist solutions to the puzzles of change

Hereafter we review the sequentialist solutions to the puzzle of change in section 4.2, and check whether counterpart semantics faithfully represent the intuitions of the sequentialist ontology. In this way we aim at further affirming the correspondence thesis between counterpart semantics and sequentialism.

Qualitative change

In order to explain change in properties of individuals with respect to time, sequentialists have at disposal a clearer account in comparison to endurantists, similar in many aspects to perdurantists' one. As perdurantists, sequentialists accept the radical consequence of applying SI to qualitative change: the table in the morning and the table at the evening are actually different objects. They are not temporal parts of the same four-dimensional individual, rather they are instantaneous objects related by the counterpart relation. This fact guarantees the reference to the 'same' table in the morning and at the evening.

Within the framework of counterparts, we can reconstruct the sequentialist solution to the puzzle of qualitative change. Consider the following c-model $\mathcal{M}$ s.t.:

- W, R and D are the same as in par. 4.3.2;
- $C_{morn,even} = \{\langle \mathbf{tab}_{morn}, \mathbf{tab}_{even} \rangle\}$.

Moreover we assume that $\mathbf{tab}_{morn} \in I(Clean, morn)$ and $\mathbf{tab}_{even} \notin I(Clean, even)$. In c-model $\mathcal{M}$ the sequentialist account is clearly described: sequentialists accept (SI) and discriminate between $\mathbf{tab}_{morn}$ and its counterpart $\mathbf{tab}_{even}$. Statement 'individual *tab* has property *Clean* in moment *morn*' is not analysed in terms of temporal parts of the table, rather as 'the counterpart of individual *tab* in moment *morn* has property *Clean*'[28]. This means that we are not compelled to distinguish between properties $I(Clean, morn)$ and $I(Clean, even)$. Since temporal modifiers act on individuals, sequentialists reject criticisms in par. 4.2.2 by means of considerations analogous to perdurantists' ones.

[28]Consider the solution in [81], p. 98 and in [96], p. 25: "[T]he new scheme by which we represent the temporalization of a statement having form (1) [x is P] can be formulated as follows: (50) the t-counterpart of x is P".

Mereological change

Since the necessity of difference fails in counterpart frames, it is not possible to derive the contradiction in the proof in par. 4.2.2, as it is blocked at line 1. Moreover sequentialists have at disposal a counterpart model satisfying hypotheses 4.1/4.4. Define c-model $\mathcal{M}$ from the substantial model in par. 4.3.2, by constructing the counterpart relation on substances **Tab** and **Tab**$-$. We fix the structure of $\mathcal{M}$ as follows:

- W, R and D are the same as in par. 4.3.2;
- $C(morn, even) = \{\langle \mathbf{tab}_{morn}, \mathbf{tab}_{even}\rangle, \langle \mathbf{tab}-, \mathbf{tab}_{even}\rangle\}$.

In c-model $\mathcal{M}$ normal interpretation I assigns extension $\{\langle \mathbf{tab}-, \mathbf{tab}_{morn}\rangle\}$ to predicate $<$ in moment $morn$. Finally σ is 3-assignment $\langle \mathbf{tab}-, \mathbf{tab}-, \mathbf{tab}_{morn}\rangle$, mapping individuals $\mathbf{tab}-$ and $\mathbf{tab}_{morn}$ to terms $tab_e, tab-, tab_m$ respectively. We easily verify that in c-model $\mathcal{M}$ assignment σ satisfies hypotheses 4.1/4.4.

This counterpart model formalizes the sequentialist intuition, according to which a single object can be counterpart to many individuals. Similarly to what happens in perdurantism, the sequentialist solution to the puzzle of mereological change can be soundly formalized within the framework of counterpart semantics.

Coincident but distinct objects

Finally we consider the puzzle of coincident but distinct objects. We anticipated that Leibniz's Law is unrestrictedly valid in counterpart semantics, in particular it holds for formulas expressing modal properties of individuals, hence we deduce a contradiction by means of sequentialistically valid principles as follows (notice that we have to adapt hypotheses (a)/(c) to our typed language, and that $s = x_i$ and $l = x_j$):

1)	$(s = l) \rightarrow ((\Box\neg Dif(x_i))[\vec{x}, s] \rightarrow (\Box\neg Dif(x_i))[\vec{x}, l])$	Leibniz's Law
2)	$s = l$	by (a)
3)	$(\Box\neg Dif(x_i))[\vec{x}, s] \rightarrow (\Box\neg Dif(x_i))[\vec{x}, l]$	from 1, 2 by R1
4)	$\Box\neg Dif(s) \rightarrow (\Box\neg Dif(x_i))[\vec{x}, s]$	as $s = x_i$
5)	$\Box\neg Dif(s) \rightarrow (\Box\neg Dif(x_i))[\vec{x}, l]$	from 3, 4 by transitivity
6)	$\Box\neg Dif(s)$	by (b)
7)	$(\Box\neg Dif(x_i))[\vec{x}, l]$	from 5, 6 by R1
8)	$(\Box\neg Dif(x_i))[\vec{x}, l] \rightarrow \Box\neg Dif(l)$	A16
9)	$\Box\neg Dif(l)$	from 7, 8 by R1
10)	$\neg\Box\neg Dif(l)$	by (c)
11)	$\perp$	from 9, 10

This inconsistency proof in counterpart semantics is somewhat unsatisfactory, as we would like to discriminate - as sequentialists do - counterparts of the statue from counterparts of the lumps of clay. This intuition brought forth formal accounts as the one developed by G. Ray and discussed in [15], in which counterpart relation and modal operators are indexed according to the different objects in our ontology.

Unrestricted validity for Leibniz's Law reveals that counterpart semantics is primarily concerned with *de re* modality: if individual a is identical to b, then every counterpart of the former is counterpart to the latter as well, as the notion of counterpart is extensional. It is interesting to compare the different behaviours of our semantics with respect to principle 3.2. Kripke semantics validates both 3.2 and A22, A23, as the notion 'the same individual as...' is extensional and uniquely defined; on the contrary the substantial interpretation rejects both 3.2 and A22, A23 as the notion 'the same four-dimensional object as...' is intensional. Finally in counterpart semantics 3.2 holds as the notion 'the same counterpart as...' is extensional, but A22, A23 fail as it is not uniquely defined.

4.4.3 Remarks

We conclude the analysis of the relationship between counterpart semantics and sequentialism by summarizing the results attained thus far. In par. 4.4.1 we considered the following points in order to uphold the sequentialist reading of counterpart semantics:

1. Lewis' counterpart theory is based on the idea of applying the sequentialist ontology of physical objects to alethic modalities, by substituting reidentification with the notion of counterpart.

2. For evaluating in moment t modal formula $\Box\phi[x_1, \ldots, x_n]$ with respect to individuals $a_1, \ldots, a_n \in D(t)$, in every moment t' temporally related to t we refer to counterparts $b_1, \ldots, b_n$ of $a_1, \ldots, a_n$ in $D(t')$.

3. The necessity of identity and the necessity of difference hold in the class of c-frames only under the constraints of functionality and injectivity respectively. On the other hand Leibniz's Law is unrestrictedly valid.

In par. 4.4.2 we formalized the sequentialist solutions to the puzzles of qualitative and mereological change. As it was the case for endurantism and perdurantism, even for sequentialism there exist counterpart models accounting for the first problem. Moreover the sequentialist solution does not run into the ontological criticisms to endurantism. As regards the puzzle of mereological change, counterpart models are a sound representation of the sequentialist solution. In the case of coincident but distinct objects, counterpart semantics does not reflect the sequentialist distinction among the various counterparts of an object, Leibniz's Law unrestrictedly holds and we derive an inconsistency in par. 4.2.2.

4.5 Conclusions

In this chapter we explicitly stated the deep relationship between the semantics for quantified modal logic, presented in the first part of this work, and the three main ontological accounts on persistence conditions for material objects. We compared Kripke semantics, the substantial interpretation and counterpart semantics with three-, four-dimensionalism and sequentialism respectively, by considering the following aspects:

- the ontological assumptions underlying the logical accounts;
- the agreement on valid principles for identity;
- the representability of the ontological solutions to the puzzles of change within our formal frameworks.

We showed that in Kripke semantics we have 'wholly present objects' and reidentification is automatic; that domains in the substantial interpretation are made of individuals extending across time, having different temporal parts in subsequent moments; that the notion of counterpart is common to sequentialism and counterpart semantics. Moreover our semantic accounts validate only identity principles, that are sound w.r.t. the respective ontological theses.

In Kripke semantics we formally reconstructed the arguments on mereological change and coincident but distinct objects, by using endurantistically valid principles. Then we made use of substantial and counterpart models to provide an elegant formalization to the perdurantist and sequentialist solutions to the puzzles of change. We also remarked that, as regards coincident but distinct objects, counterpart semantics departs from sequentialism.

After this discussion, we uphold the correspondence thesis between semantics for quantified modal logic and ontologies for physical objects. In the next and last chapter we make use of this thesis to formally compare our ontologies and determine degrees of generality and reducibility.

Chapter 5

Translations of Theories

In the present chapter we investigate translation functions for the sets of validities in Kripke semantics, the substantial interpretation and counterpart semantics. The importance of such an analysis is twofold: on the one hand we precisely state the necessary and sufficient conditions by which a formal approach is reducible to another one, thus distinguish different levels of generality among these accounts[1]. On the other one we make use of these logical results to clarify the relationships among the various ontological theories on persistence, in virtue of the analysis carried out in chapter 4.

In section 5.1 we provide the exact formulation of claims in chapter 2 and 3, according to which Kripke frames can be thought of as either counterpart frames where counterpart relation is everywhere-defined and it is identity, or as substantial frames with constant functions. We present translation function τ_n from untyped to typed languages, that appears in [21] and [34], and show that a formula ϕ in language $\mathcal{L}^=$ holds in the class of K-frames iff it holds in the class of constant s-frames, iff translation $\tau_n(\phi)$ holds in the class of c-frames where counterpart relation is everywhere-defined and it is identity.

By this first, easy result we conclude that Kripke frames can be seen as particular classes of substantial and counterpart frames. In virtue of the correspondence thesis between quantified modal logic and ontologies of physical objects, this means that we can deal with the endurantist account within both the perdurantist and sequentialist framework, by assuming specific constraints on four-dimensional objects and counterparts.

These constraints are nonetheless quite strong, thus in sections 5.2 and 5.3 we present the results attained by Fitting in [27] and by Kracht and Kutz in [52], concerning translations of formulas sound w.r.t. counterpart semantics into validities in the substantial interpretation. We single out the necessary and sufficient conditions to impose on c-frames, so that Fitting's and Kracht and Kutz's translation functions relate the two notions of validity.

[1] For instance consider Fitting's claim in [27], p.14: "[A]ny logic characterized by a class of simply connected Lewis counterpart frames is also characterized (under translation) by a class of simply connected $FOIL$ frames. When simply connected frames suffice, $FOIL$ includes, under translation, Lewis counterpart semantics."

In particular we prove for translation function + in [52], that a formula ϕ in language $\mathcal{L}^=$ holds in the class of perfect, injective, functional and reverse-counterpart faithful c-frames iff translation ϕ^+ holds in the class of monotonic, classical s-frames iff ϕ^+ holds in the class of classical K-frames. Later on we shall give all the formal definitions and details. For the time being notice that by the present result, we need not to make such a strong assumption as identity as counterpart relation in order to compare counterpart and Kripke semantics, we can be content with weaker conditions as perfection, injectivity, functionality and reverse-counterpart faithfulness.

Finally in section 5.4 we analyse the ontological consequences of these equivalences, that are due to a precise fact: the domains of objects in our interpretations are made of *bare particulars*, that is, we have no characterization of the inner structure of individuals, thus they can be thought of as interchangeable and the features of models are fixed only by the formal properties of the accessibility and counterpart relation. We may consider this fact as an unsatisfactory aspect of the present analysis, nonetheless the present study reveals the logical consequences of our pre-formal assumptions on the nature of physical objects and their persistence conditions.

5.1 Ghilardi's translation

In chapter 2 we anticipated that Kripke frames can be seen as counterpart frames, in which counterpart relation is everywhere-defined and it is identity. Now we precisely define this idea in terms of equivalence of the sets of validities w.r.t. the one and the other interpretation. We can at most attain equivalence - *modulo* translation function τ_n - as we deal with two different languages: $\mathcal{L}^=$ and $\mathcal{L}_t^=$. First of all we introduce translation function τ_n from untyped to typed languages, slightly modified in comparison with [21]. Let ϕ be a formula in $\mathcal{L}^=$, define $g(\phi)$ as the maximum k s.t. x_k occurs free in ϕ. Then for $n \geq g(\phi)$ we consider translated formula $\tau_n(\phi) : n$ in language $\mathcal{L}_t^=$, inductively defined as follows:

$$
\begin{aligned}
\tau_n(P^m(t_1, \ldots, t_m)) &:= P^m(t_1, \ldots, t_m) \\
\tau_n(t_1 = t_2) &:= t_1 = t_2 \\
\tau_n(\neg\psi) &:= \neg\tau_n(\psi) \\
\tau_n(\Box\psi) &:= \Box\tau_n(\psi) \\
\tau_n(\psi \to \psi') &:= \tau_n(\psi) \to \tau_n(\psi') \\
\tau_n(\forall x_i \psi) &:= \forall x_{n+1}(\tau_n(\psi)[x_i/x_{n+1}])
\end{aligned}
$$

Translation function τ_n assigns to formula ϕ in untyped language $\mathcal{L}^=$, formula $\tau_n(\phi) : n$ in typed $\mathcal{L}_t^=$, which intuitively has the same meaning as ϕ. The base of induction is clear; as to the inductive steps, translation τ_n commutes with propositional connectives and the modal operator. The only remarkable clause concerns the universal quantifier: since in typed languages we quantify only on the variable corresponding to the type of the formula, we have to substitute x_i with x_{n+1} in the translation of ϕ. Notice that we can avoid clashes of variables by renaming bounded occurrences, so that in $\tau_n(\forall x_i \psi) = \forall x_{n+1}(\tau_n(\psi)[x_i/x_{n+1}])$ variable x_{n+1} does not appear bounded in ψ.

5.1.1 From Kripke to counterpart validity...

We immediately prove the following theorem, that clearly states the relationship between validity in Kripke and counterpart semantics.

Theorem 5.1 *A formula $\phi \in \mathcal{L}^=$ is valid in the class of K-frames iff translation $\tau_n(\phi) \in \mathcal{L}_t^=$ holds in the class of everywhere-defined c-frames s.t. for every $w, w' \in W$, $C_{w,w'}$ is identity.*

Proof. $\Leftarrow$ If there exists a K-model $\mathcal{M}$ falsifying ϕ, then we construct a suitable c-model $\mathcal{M}'$ s.t. not $\mathcal{M}' \models \tau_n(\phi)$. Define $\mathcal{M}'$ as follows:

- W', R', D' and d' are the same as for $\mathcal{M}$;
- for $a \in D(w)$, $b \in D(w')$, $C_{w,w'}(a, b)$ iff $a = b$;
- interpretation I' is identical to I in $\mathcal{M}$, that is, if P^n is an n-ary predicative constant and $w \in W'$, then $I'(P^n, w) = I(P^n, w)$ is an n-ary relation on $D'(w)$.

It is trivial to check that $\mathcal{M}'$ is a c-model. By the increasing outer domain condition on $\mathcal{M}$, counterpart relation $C_{w,w'}$ is everywhere defined and it is identity by definition. Now we prove equivalence between satisfaction in $\mathcal{M}$ and in $\mathcal{M}'$.

Lemma 5.2 *Let $w \in W$, $\phi \in \mathcal{L}^=$ and $\sigma(\vec{x}) = \vec{a}$,*

$$(I^\sigma, w) \models \phi \quad iff \quad (\vec{a}, w) \models \tau_n(\phi)$$

Proof. The proof is by induction on the length of ϕ. For the base of induction suppose that ϕ has form $P^m(t_1, \ldots, t_m)$, then $(I^\sigma, w) \models \phi$ iff $\langle \sigma(t_1), \ldots, \sigma(t_m) \rangle \in I(P^m, w) = I'(P^m, w)$. Notice that $\vec{a}(t_i) = \sigma(t_i)$, thus $\langle \sigma(t_1), \ldots, \sigma(t_m) \rangle \in I(P^m, w)$ iff $\langle \vec{a}(t_1), \ldots, \vec{a}(t_m) \rangle \in I'(P^m, w)$, and this is the case iff $(\vec{a}, w) \models P^m(t_1, \ldots, t_m)$.

The case for identity statements is similar, and the inductive cases for propositional connectives are straightforward.

If ϕ has form $\forall x_i \psi$, then $(I^\sigma, w) \models \phi$ iff for every $a^* \in d(w)$, $(I^{\sigma\binom{x_i}{a^*}}, w) \models \psi$. By induction hypothesis and the conversion lemma $(\vec{a} \cdot a^*, w) \models \tau_n(\psi)[x_i/x_{n+1}]$ i.e. $(\vec{a}, w) \models \tau_n(\forall x_i \psi)$.

If ϕ has form $\Box\psi$, then $(I^\sigma, w) \models \phi$ iff for all $w' \in W$, wRw' implies $(I^\sigma, w') \models \psi$. By induction hypothesis this is the case iff for all $w' \in W'$, $wR'w'$ implies $(\vec{a}, w') \models \tau_n(\psi)$. By definition of $C_{w,w'}$ this is the case iff for all $w' \in W'$, for all $\vec{b} \in D'(w')^n$, $wR'w'$ and $C_{w,w'}(\vec{a}, \vec{b})$ imply $(\vec{b}, w') \models \tau_n(\psi)$. Therefore $(\vec{a}, w) \models \tau_n(\Box\psi)$.

We supposed that not $\mathcal{M} \models \phi$, this means that there is assignment σ and world $w \in W$, s.t. $(I^\sigma, w) \models \neg\phi$. By lemma 5.2 $(\sigma(\vec{x}), w) \models \neg\tau_n(\phi)$, i.e. c-model $\mathcal{M}'$ falsifies $\tau_n(\phi)$.

$\Rightarrow$ We show that if a suitable c-model $\mathcal{M}$ falsifies $\tau_n(\phi) \in \mathcal{L}_t^=$, then we can define K-model $\mathcal{M}'$ s.t. not $\mathcal{M}' \models \phi$ as follows:

- W', R', D', d' and I' are the same as for $\mathcal{M}$.

We check that $\mathcal{M}'$ satisfies the increasing outer domain condition. Let $a \in D'(w)$ and $wR'w'$; $C_{w,w'}$ is everywhere defined, thus there exists $b \in D'(w')$ s.t. $C_{w,w'}(a, b)$, and since $C_{w,w'}$ is identity $b = a \in D'(w')$. We conclude that $\mathcal{M}'$ is a K-model. Also in the present case we prove equivalence between satisfaction in $\mathcal{M}$ and in $\mathcal{M}'$.

Lemma 5.3 *Let $w \in W$, $\phi \in \mathcal{L}^=$ and $\sigma(\vec{x}) = \vec{a}$,*

$$(\vec{a}, w) \models \tau_n(\phi) \quad iff \quad (I^\sigma, w) \models \phi$$

Proof. The proof is by induction on the length of ϕ and completely identical to the one for lemma 5.2, we consider only the case for the modal operator.

If ϕ has form $\Box\psi$, then $(\vec{a}, w) \models \Box\tau_n(\psi)$ iff for all $w' \in W$, for all $\vec{b} \in D(w')^n$, wRw' and $C_{w,w'}(\vec{a}, \vec{b})$ imply $(\vec{b}, w') \models \tau_n(\psi)$. By assumptions on $C_{w,w'}$, this is the case iff for all $w' \in W$, wRw' implies $(\vec{a}, w') \models \tau_n(\psi)$. By induction hypothesis this is equivalent to: for all $w' \in W'$, $wR'w'$ implies $(I^\sigma, w') \models \psi$, that is $(I^\sigma, w) \models \phi$.

We supposed that not $\mathcal{M} \models \tau_n(\phi)$, this means that there is n-assignment $\vec{a}$ and world $w \in W$ s.t. $(\vec{a}, w) \models \tau_n(\neg\phi)$. By lemma 5.3 assignment σ s.t. $\sigma(\vec{x}) = \vec{a}$ satisfies $\neg\phi$ in w, that is, K-model $\mathcal{M}'$ falsifies ϕ. Therefore theorem 5.1 holds.

Theorem 5.1 exactly states in which sense counterpart semantics is more general than Kripke's account: we mimic the notion of validity in K-frames by considering everywhere-defined c-frames, in which counterpart relation is identity. We can even deal with specific classes of structures, as by the proof of the theorem we clearly see that formula $\phi \in \mathcal{L}^=$ is valid in the class of K-frames with increasing (resp. decreasing, constant) inner domains and increasing (resp. constant) outer domains iff translation $\tau_n(\phi) : n$ in $\mathcal{L}_t^=$ holds in the class of everywhere-defined c-frames, in which counterpart relation is identity, with in addition the corresponding inclusion constraints on inner and outer domains.

5.1.2 ...from Kripke to substantial validity...

For what concerns the relationship between Kripke semantics and the substantial interpretation, in chapter 3 we remarked that we can think of K-frames as s-frames, where function domain $F(w)$ consists in constant functions from W to $\bigcup_{w' \in W} D(w')$, that we call *constant* s-frame. Even in the present case we specify this claim, but this time we actually prove identity between sets of validities in Kripke and in constant s-frames. In what follows we assume that in an s-frame $\mathcal{F}$, for every $a \in D(w)$, there exists $\mathbf{f} \in F(w)$ s.t. $\mathbf{f}(w) = a$ without any loss of generality. In fact if $\mathcal{F}$ is an s-frame falsifying $\phi \in \mathcal{L}^=$, then we can construct an s-frame $\mathcal{F}'$ satisfying the condition above, s.t. ϕ does not hold in $\mathcal{F}'$.

Theorem 5.4 *A formula $\phi \in \mathcal{L}^=$ is valid in the class of K-frames iff ϕ holds in the class of constant s-frames.*

Proof. $\Leftarrow$ Suppose that $\mathcal{M}$ is a K-model falsifying ϕ, then consider constant s-model $\mathcal{M}'$ defined as follows:

- W', R' and D' are the same as for $\mathcal{M}$;
- for every $w \in W'$, $F'(w)$ is the set of constant functions $\mathbf{f}$ s.t. $\mathbf{f}(w') \in D(w')$;
- for every $w \in W'$, $d'(w) \subseteq F'(w)$ s.t. $\mathbf{f} \in d'(w)$ iff $\mathbf{f}(w) \in d(w)$;
- interpretation I' is defined as I, i.e. for an n-ary predicative constant P^n and $w \in W$, $I'(P^n, w) = I(P^n, w)$.

We have to prove that for every $w, w' \in W'$, $F'(w) \subseteq F'(w')$: this is guaranteed by the definition of function domains and the increasing outer domain condition on $\mathcal{M}$. Hence $\mathcal{M}'$ is a constant s-model.

Let σ be an assignment in $\mathcal{M}$, define assignment σ': $Var(\mathcal{L}^=) \to F'(w)$ as $\sigma'(x) = \mathbf{f}$ iff $\mathbf{f}(w) = \sigma(x)$. Since $F'(w)$ is the set of constant functions, assignment σ' is well-defined, that is, if $\mathbf{f}(w) = \sigma(x) = \mathbf{g}(w)$ then $\mathbf{f} = \sigma'(x) = \mathbf{g}$. We can easily prove next equivalence lemma.

Lemma 5.5 *Let $w \in W$ and $\phi \in \mathcal{L}^=$,*

$$(I^\sigma, w) \models \phi \quad iff \quad (I'^{\sigma'}, w) \models \phi$$

Proof. The proof is by induction on the length of ϕ. As to the base case, if ϕ has form $P^m(t_1, \ldots, t_m)$, then $(I^\sigma, w) \models \phi$ iff $\langle \sigma(t_1), \ldots, \sigma(t_m) \rangle \in I(P^m, w) = I'(P^m, w)$. By definition of σ', $\langle \sigma(t_1), \ldots, \sigma(t_m) \rangle \in I(P^m, w)$ iff $\langle \sigma'(t_1)(w), \ldots, \sigma'(t_m)(w) \rangle \in I'(P^m, w)$ iff $(I'^{\sigma'}, w) \models P^m(t_1, \ldots, t_m)$.

The proof for identity statements is similar, and inductive cases for propositional connectives and the modal operator are straightforward.

If ϕ has form $\forall x \psi$, then $(I^\sigma, w) \models \phi$ iff for all $a^* \in d(w)$, $(I^{\sigma\binom{x}{a^*}}, w) \models \psi$. By induction hypothesis this is the case iff $(I'^{\sigma\binom{x}{a^*}'}, w) \models \psi$, but notice that $\sigma\binom{x}{a^*}'$ is equal to $\sigma'\binom{x}{\mathbf{f}}$, where $\mathbf{f}(w) = a^* \in d(w)$. Therefore $\mathbf{f} \in d'(w)$ and for all $\mathbf{f} \in d'(w)$, $(I'^{\sigma'\binom{x}{\mathbf{f}}}, w) \models \psi$, i.e. $(I'^{\sigma'}, w) \models \forall x \psi$.

By assumption not $\mathcal{M} \models \phi$, this means that there is assignment σ and world $w \in W$ s.t. $(I^\sigma, w) \models \neg\phi$. By lemma 5.5 $(I'^{\sigma'}, w) \models \neg\phi$, that is, s-model $\mathcal{M}'$ falsifies ϕ.

$\Rightarrow$ Suppose that $\mathcal{M}$ is a constant s-model falsifying ϕ, then consider K-model $\mathcal{M}'$ defined as follows:

- W', R' and d' are the same as for $\mathcal{M}$;
- for every $w \in W'$, $D'(w)$ is function domain $F(w)$;
- interpretation I' is so defined that for n-ary predicative constant P^n and $w \in W$, $I'(P^n, w) = \{\mathbf{f} \in F(w) | \mathbf{f}(w) \in I(P^n, w)\}$ is an n-ary relation on $D'(w)$.

Model $\mathcal{M}'$ has increasing outer domains, by assumption on F and definition of D', hence it is a K-model. Moreover assignment σ in $\mathcal{M}$ from $Var(\mathcal{L}^=)$ to $F(w)$ is also an assignment in $\mathcal{M}'$ from $Var(\mathcal{L}^=)$ to $D'(w)$. Thus we prove the equivalence lemma.

Lemma 5.6 *Let $w \in W$ and $\phi \in \mathcal{L}^=$,*

$$(I^\sigma, w) \models \phi \quad iff \quad (I'^{\sigma}, w) \models \phi$$

Proof. Once more the proof is by induction on the length of ϕ. As to the base of induction, if ϕ has form $P^m(t_1, \ldots, t_m)$ then $(I^\sigma, w) \models \phi$ iff $\langle \sigma(t_1)(w), \ldots, \sigma(t_m)(w) \rangle \in I(P^m, w)$. This is the case iff $\langle \sigma(t_1), \ldots, \sigma(t_m) \rangle \in I'(P^m, w)$, i.e. $(I'^\sigma, w) \models P^m(t_1, \ldots, t_m)$.

The proof for identity statements is similar, and inductive cases for propositional connectives the modal operator and the universal quantifier are straightforward.

We supposed that not $\mathcal{M} \models \phi$, this means that there exist assignment σ and world $w \in W$, s.t. $(I^\sigma, w) \models \neg\phi$. By lemma 5.6 $(I'^\sigma, w) \models \neg\phi$, that is, K-model $\mathcal{M}'$

falsifies ϕ. This fact concludes the proof of theorem 5.4.

Theorem 5.4 precisely defines the relationship between Kripke semantics and the substantial interpretation: we mimic the notion of validity in K-frames by considering the class of s-frames s.t. function domains contain only constant functions. As it was the case for theorem 5.1, we can even consider specific classes of structures: formula $\phi \in \mathcal{L}^=$ holds in the class of K-frames with increasing (resp. decreasing, constant) inner domains and increasing (resp. constant) outer domains iff ϕ is valid in the class of constant s-frames, with in addition the corresponding inclusion constraints on inner and function domains.

We summarize theorems 5.1, 5.4 in the following result on the relationship among Kripke semantics, the substantial interpretation and counterpart semantics.

Theorem 5.7 *A formula $\phi \in \mathcal{L}^=$ is valid in the class of K-frames iff it is valid in the class of constant s-frames iff translation $\tau_n(\phi)$ in $\mathcal{L}_t^=$ is valid in the class of everywhere-defined c-frames s.t. for $w, w' \in W$, $C_{w,w'}$ is identity.*

By theorem 5.7 validity in K-frames is equivalent to validity in constant s-frames and to validity in everywhere-defined c-frames s.t. for $w, w' \in W$, $C_{w,w'}$ is identity, *modulo* translation function τ_n.

Finally notice that it is possible to prove a strengthened version of theorem 5.4, as it easy to check that the assumption on constant s-frames can be weakened to monotonicity: an s-frame is said to be *monotonic* iff for every function $\mathbf{f}$, $\mathbf{g}$ in $F(w)$, $\mathbf{f}(w') = \mathbf{g}(w')$ implies $\mathbf{f} = \mathbf{g}$. We devote the last part of the present paragraph to the proof of the following result.

Theorem 5.8 *A formula $\phi \in \mathcal{L}^=$ is valid in the class of K-frames iff ϕ holds in the class of monotonic s-frames.*

Proof. The second part of the proof goes as for theorem 5.4. As regards the first one, suppose that $\mathcal{M}$ is an K-model falsifying ϕ, then consider s-model $\mathcal{M}'$ defined as in the first part of theorem 5.4, but:

- for every $w \in W'$, $F'(w)$ is the set of monotonic functions on W'.

Trivially $\mathcal{M}'$ is a monotonic s-model. Moreover let σ be an assignment in $\mathcal{M}$ and define assignment $\sigma' : Var(\mathcal{L}^=) \to F'(w)$ as $\sigma'(x) = \mathbf{f}$ iff $\mathbf{f}(w) = \sigma(x)$. This time assignment σ' is well-defined by monotonicity condition on $\mathcal{M}'$. Finally we can prove lemma 5.5, thus $\mathcal{M}'$ falsifies ϕ.

By theorem 5.8 we state the following strengthening of theorem 5.7.

Theorem 5.9 *A formula $\phi \in \mathcal{L}^=$ is valid in the class of K-frames iff it is valid in the class of monotonic s-frames iff translation $\tau_n(\phi)$ in $\mathcal{L}_t^=$ is valid in the class of everywhere-defined c-frames s.t. for $w, w' \in W$, $C_{w,w'}$ is identity.*

Our conclusion is that we can deal with Kripke's approach within both counterpart semantics and the substantial interpretation, by respectively assuming everywhere defined c-frames where counterpart relation is identity, and constant (monotonic) intensional objects. As a consequence even the counterpart-theoretic and substantial account are equivalent, *modulo* translation function τ_n.

5.1.3 ...and back.

In the previous paragraphs we showed that counterpart semantics can mimic the notion of validity in Kripke semantics and in the substantial interpretation, as for each sound formula $\phi \in \mathcal{L}^=$ in K-frames and in constant (monotonic) s-frames, translation $\tau_n(\phi)$ holds in suitable c-frames. We may interpret this result the other way round: if we restrict our attention to everywhere-defined c-frames s.t. for $w, w' \in W$, $C_{w,w'}$ is identity, then we can faithfully describe these structure within both Kripke's and the substantial account. But the results proved thus far are not sufficient to justify such a claim, as it can be the case that some typed formula ϕ - valid in our class of c-frames - is the translation of no validity ψ in either Kripke semantics or constant (monotonic) s-frames.

In the present paragraph we prove that this can not happen. We start with introducing translation function τ^- from typed to untyped languages, modelled on function c in chapter 2 and defined as follows:

$$\begin{aligned}
\tau^-(P^m(t_1, \dots, t_m)) &:= P^m(t_1, \dots, t_m) \\
\tau^-(t_1 = t_2) &:= t_1 = t_2 \\
\tau^-(\neg\psi) &:= \neg\tau^-(\psi) \\
\tau^-(\Box\psi(t_1, \dots, t_m)) &:= \Box\tau^-(\psi)[t_1, \dots, t_m] \\
\tau^-(\psi \to \psi') &:= \tau^-(\psi) \to \tau^-(\psi') \\
\tau^-(\forall x_{n+1}\psi) &:= \forall x_{n+1}\tau^-(\psi)
\end{aligned}$$

We show that τ^- is dual to τ_n, that is, if $\phi : n$ is a formula in $\mathcal{L}_t^=$ then $\tau_n(\tau^-(\phi))$ is equal to ϕ.

First of all notice that in an everywhere-defined c-frame $\mathcal{F}$ s.t. for $w, w' \in W$, $C_{w,w'}$ is identity, substitution commutes even with the modal operator, i.e. principle A16' $\Box(\phi[t_1, \dots, t_m]) \to \Box\phi[t_1, \dots, t_m]$ holds in $\mathcal{F}$. By means of this fact we prove the following lemma.

Lemma 5.10 *If ϕ is a formula in $\mathcal{L}^=$ s.t. $g(\phi) \leq m$ and $g(\phi[t_1, \dots, t_m]) \leq n$, and $\mathcal{F}$ is an everywhere-defined c-frame s.t. for $w, w' \in W$, $C_{w,w'}$ is identity, then*

$$\mathcal{F} \models \tau_n(\phi[t_1, \dots, t_m]) \leftrightarrow \tau_m(\phi)[t_1, \dots, t_m]$$

Proof. The proof is by induction on the length of ϕ. As to the base case, $\tau_n(P^k(u_1, \dots, u_k)[t_1, \dots, t_m]) = \tau_n(P^k(s_1, \dots, s_k))$, where $s_j = u_j[t_1, \dots, t_m]$. By definition of τ_n, $\tau_n(P^k(s_1, \dots, s_k)) = P^k(s_1, \dots, s_k)$, that is, $P^k(u_1, \dots, u_k)[t_1, \dots, t_m] = \tau_m(P^k(u_1, \dots, u_k))[t_1, \dots, t_m]$.

As to negation,

$$\tau_n(\neg\psi[t_1,\ldots,t_m]) = \neg\tau_n(\psi[t_1,\ldots,t_m])$$

By induction hypothesis $\neg\tau_n(\psi[t_1,\ldots,t_m]) = \neg\tau_m(\psi)[t_1,\ldots,t_m]$, and then

$$\neg\tau_m(\psi)[t_1,\ldots,t_m] = \tau_m(\neg\psi)[t_1,\ldots,t_m]$$

The proof for implication is similar.

As to the modal operator,

$$\tau_n(\Box\psi[t_1,\ldots,t_m]) = \Box\tau_n(\psi[t_1,\ldots,t_m])$$

By induction hypothesis $\Box\tau_n(\psi[t_1,\ldots,t_m]) = \Box(\tau_m(\psi)[t_1,\ldots,t_m])$, and since $\mathcal{F}$ is everywhere-defined and for $w, w' \in W$, $C_{w,w'}$ is identity,

$$\Box(\tau_m(\psi)[t_1,\ldots,t_m]) = \Box\tau_m(\psi)[t_1,\ldots,t_m] = \tau_m(\Box\psi)[t_1,\ldots,t_m]$$

As to the universal quantifier,

$$\tau_n(\forall x_i\psi[t_1,\ldots,t_m]) = \forall x_{n+1}(\tau_n(\psi[t_1,\ldots,t_m])[x_i/x_{n+1}])$$

By induction hypothesis $\forall x_{n+1}(\tau_n(\psi[t_1,\ldots,t_m])[x_i/x_{n+1}]) = \forall x_{n+1}(\tau_m(\psi)[t_1,\ldots,t_m][x_i/x_{n+1}])$, moreover

$$\begin{aligned}
\forall x_{n+1}(\tau_m(\psi)[t_1,\ldots,t_m][x_i/x_{n+1}]) &= \\
\forall x_{n+1}(\tau_m(\psi)[x_i/x_{m+1}][t_1,\ldots,t_m,x_{n+1}]) &= \\
\forall x_{m+1}(\tau_m(\psi)[x_i/x_{m+1}])[t_1,\ldots,t_m] &= \tau_m(\forall x_i\psi)[t_1,\ldots,t_m]
\end{aligned}$$

This concludes the proof of lemma 5.10. In the next theorem we state the duality between functions τ_n and τ^-.

Theorem 5.11 *Let $\phi : n$ be a formula in $\mathcal{L}_t^{=}$, and $\mathcal{F}$ an everywhere-defined c-frame s.t. for $w, w' \in W$, $C_{w,w'}$ is identity. It is the case that*

$$\mathcal{F} \models \tau_n(\tau^-(\phi)) \leftrightarrow \phi$$

Proof. Once again the proof is by induction on the length of ϕ. As to the base case, $\tau_n(\tau^-(P^m(t_1,\ldots,t_m)) = \tau_n(P^m(t_1,\ldots,t_m)) = P^m(t_1,\ldots,t_m) : n$.

The cases for propositional connectives are straightforward, thus we consider the modal operator:

$$\tau_n(\tau^-(\Box\psi(t_1,\ldots,t_m))) = \tau_n(\Box\tau^-(\psi)[t_1,\ldots,t_m])$$

We assumed that $\mathcal{F}$ is an everywhere-defined c-frame where $C_{w,w'}$ is identity, thus by lemma 5.10

$$\tau_n(\Box\tau^-(\psi)[t_1,\ldots,t_m]) = \tau_m(\Box\tau^-(\psi))[t_1,\ldots,t_m] = \Box\tau_m(\tau^-(\psi))[t_1,\ldots,t_m]$$

By induction hypothesis $\Box\tau_m(\tau^-(\psi))[t_1, \ldots, t_m] = \Box\psi[t_1, \ldots, t_m]$, that is $\Box\psi(t_1, \ldots, t_m)$.

As regards the universal quantifier:

$$\tau_n(\tau^-(\forall x_{n+1}\psi)) = \tau_n(\forall x_{n+1}\tau^-(\psi)) = \forall x_{n+1}(\tau_n(\tau^-(\psi))[x_{n+1}/x_{n+1}])$$

By induction hypothesis $\forall x_{n+1}\tau_n(\tau^-(\psi)) = \forall x_n\psi$.

Therefore theorem 5.11 holds - τ_n and τ^- are dual functions - and by theorems 5.7, 5.9 we prove the following result:

Theorem 5.12 *A formula $\tau_n(\tau^-(\phi)) = \phi : n$ in language $\mathcal{L}_t^=$ is valid in the class of everywhere-defined c-frames s.t. for $w, w' \in W$, $C_{w,w'}$ is identity iff translation $\tau^-(\phi)$ in $\mathcal{L}^=$ is valid in the class of constant (monotonic) s-frames, iff it is valid in the class of K-frames.*

We conclude that if a counterpart-theoretician restricts her attention to everywhere-defined c-frames s.t. for $w, w' \in W$, $C_{w,w'}$ is identity, then either a follower of Kripke or a supporter of individual substances can agree with her, since in case that statement ϕ holds in this particular class of c-frames, then translation $\tau^-(\phi)$ is sound w.r.t both K-frames and constant (monotonic) s-frames. We have a complete correspondence among these three semantic accounts.

5.1.4 Ontological consequences

In this paragraph we investigate the ontological consequences of the results obtained thus far. If our argument in chapter 4 is accepted, and we deem the various semantic accounts for quantified modal logic as faithful representations of the different ontological thesis on persistence, then the theorems proved so far specify the formal relationships among sequentialism, perdurantism and endurantism. In particular we maintain that a sequentialist, who thinks of the counterpart relation as identity and everywhere-defined, a perdurantist, who restricts intensional objects to constant (monotonic) individual concept, and an endurantist can agree on the set of truths, by interpreting their modal discourse according to translation functions τ_n and τ^-.

This equivalence does not imply any reduction of one of these approaches to another, as though formula $\phi \in \mathcal{L}^=$ is valid in the class of K-frames iff it is valid in the class of monotonic s-frames, iff translation $\tau_n(\phi) : n$ in $\mathcal{L}_t^=$ is valid in the class of everywhere-defined c-frames s.t. for $w, w' \in W$, $C_{w,w'}$ is identity, it also holds that formula $\phi \in \mathcal{L}_t^=$ is valid in that class of c-frames iff translation $\tau^-(\phi)$ in $\mathcal{L}^=$ is valid in the class of K-frames, iff it is valid in the class of monotonic s-frames. On the one hand sequentialists can affirm that every truth of endurantism and restricted perdurantism can be expressed in reformed sequentialism; on the other one endurantists or perdurantists may say that reformed sequentialists do not have to introduce counterparts, as their theories contain all that is needed to express each sound statement $\phi \in \mathcal{L}_t^=$ as $\tau^-(\phi)$. From a formal point of view these three theories are completely equivalent.

We may wonder whether the hypotheses underlying the translation functions are acceptable. It seems rather unlikely that sequentialists can restrict counterpart relation to everywhere-defined identity, as counterpart theory was first proposed by Lewis just because according to him the different concrete instances of an individual in time may not be related by mere identity. Furthermore many sequentialists think of the counterpart relation as not everywhere-defined, nor functional, nor injective. As regards the substantial interpretation, even limiting quantification to monotonic functions can be considered an artificial condition, but it is not always the case. In [2] Aloni singles out the class of s-frames, where function domains $F(w)$ consist in *conceptual covers*, i.e. they are sets of functions on W s.t.

$$for\ w \in W,\ for\ a \in D(w),\ there\ exactly\ exists\ one\ \mathbf{f} \in F(w)\ s.t.\ \mathbf{f}(w) = a$$

This condition tantamounts to monotonicity, but notice that Aloni is primarily concerned with epistemic logic.

Finally we remark that even if the constraints on c and s-frames do not seem convincing, all the results stated above provide the necessary and sufficient conditions by which our three accounts can be faithfully embedded one into another. These conditions are nonetheless quite strong, as we noticed, we wonder whether there exist translation functions among validities in the various semantics, assuming weaker constraints. In the next section we analyse Fitting's and Kracht and Kutz's translations and the assumptions needed in order to make them work.

5.2 Fitting's *First Order Intensional Logic*

In [27] Fitting introduces *First Order Intensional Logic* - $FOIL$ in short - defined on language $\mathcal{L}_{FOIL}$ containing all the logical and descriptive symbols in $\mathcal{L}^{=}$, as well as variables $f_1, f_2, \ldots,$ for intensional objects and the λ-abstractor. The definition of formulas in language $\mathcal{L}_{FOIL}$ goes as usual, moreover if ϕ is a formula, x an individual variable and f an intensional variable, then $\lambda x.\phi(f)$ is a formula.

Fitting presents a relational semantics for this language by means of $FOIL$-frames, a particular version of substantial frames. The main result proved in [27] concerns translation function $*$: Fitting shows that a formula $\phi \in \mathcal{L}_t^{=}$ holds in a *simply connected* c-frame $\mathcal{F}$ iff translation $\phi^* \in \mathcal{L}_{FOIL}$ holds in the $FOIL$-frame companion to $\mathcal{F}$; thus the set of validities in the class of $FOIL$-frames is a subset of validities in the class of simply connected c-frame, *modulo* translation function $*$. Then he proves that simply connectedness condition is eliminable, by suitably generalizing the notion of $FOIL$-frame, and concludes that counterpart semantics can be plainly embedded into $FOIL$-frames[2].

In this section we are interested in the reverse implication between validity in $FOIL$- and in counterpart frames. In par. 5.2.2 we show that translation ϕ^* of formula ϕ holds in a $FOIL$-frame $\mathcal{F}$ iff ϕ holds in a suitable c-frame $\mathcal{F}'$. This way we prove that ϕ is sound w.r.t. a particular class of c-frames iff translation ϕ^* holds in class of $FOIL$-frames. In order to obtain such a result the simply connectedness condition on c-frames is too strong and too weak at the same time, in fact the c-frame defined on a $FOIL$-frame $\mathcal{F}$ cannot be proved simply connected. In par. 5.2.1 we provide a new constraint for counterpart frames, and show that it is necessary and sufficient for our aims. In particular we prove lemma 5.20, analogous to Proposition 6.6 in [27].

As to the definition of $FOIL$-frames and of all the other notions, we refer to [27] with some minor change. A $FOIL$-frame $\mathcal{F}$ is an ordered 6-tuple $\langle W, R, D, d_o, F, d_i \rangle$ s.t.:

- W, R, D, d_i, F are defined as for s-frames, in addition the increasing outer domain condition holds;
- d_o is a function assigning to every $w \in W$, a subset $d_o(w)$ of $D(w)$.

An interpretation I for language $\mathcal{L}_{FOIL}$ assigns to each n-ary predicative constant P^n a set $I(P^n, w) \subseteq D(w)^n$. As it was the case in Kripke semantics for languages with identity, we consider only normal interpretations in $FOIL$-frames, which actually interpret symbol '$=$' as equality on $D(w)$.

A w-assignment σ for language $\mathcal{L}_{FOIL}$ is a function from individual variables to outer domain $D(w)$, and from intensional variables to function domain $F(w)$. The definition of variants $\sigma\binom{x}{\mathbf{f}}$ and $\sigma\binom{x}{a}$ of w-assignment σ is straightforward, whereas valuation $I^\sigma(s, w)$ is $\sigma(s)$ whenever s is an individual variable, it is $\sigma(s)(w)$ whenever s is an intensional variable.

[2] "When simply connectedness frames suffice, $FOIL$ includes, under translation, Lewis counterpart semantics." [27], p. 14.

Finally the relation of satisfaction in w for formula $\phi \in \mathcal{L}_{FOIL}$ w.r.t. valuation I^σ is inductively defined as follows:

$$\begin{array}{rll}
(I^\sigma, w) \models P^n(s_1, \dots, s_n) & iff & \langle I^\sigma(s_1, w), \dots, I^\sigma(s_n, w)\rangle \in I(P^n, w) \\
(I^\sigma, w) \models s_1 = s_2 & iff & I^\sigma(s_1, w) = I^\sigma(s_2, w) \\
(I^\sigma, w) \models \neg\psi & iff & not\ (I^\sigma, w) \models \psi \\
(I^\sigma, w) \models \phi \to \psi & iff & not\ (I^\sigma, w) \models \phi\ or\ (I^\sigma, w) \models \psi \\
(I^\sigma, w) \models \Box\phi & iff & for\ every\ w' \in W,\ wRw'\ implies\ (I^\sigma, w') \models \phi \\
(I^\sigma, w) \models \forall x\phi & iff & for\ every\ a \in d_o(w), (I^{\sigma\binom{x}{a}}, w) \models \phi \\
(I^\sigma, w) \models \forall f\phi & iff & for\ every\ \mathbf{f} \in d_i(w), (I^{\sigma\binom{x}{\mathbf{f}}}, w) \models \phi \\
(I^\sigma, w) \models \lambda x.\phi(f) & iff & (I^{\sigma\binom{x}{\mathbf{f}(w)}}, w) \models \phi
\end{array}$$

The constraints on D and F guarantee well-definiteness of the evaluation clause for $\Box$-formulas. Now we have all the formal details concerning $FOIL$-frames, then we go on considering some notions specifically involved in comparing $FOIL$- and c-frames.

Let $\mathcal{F}$ be a counterpart frame and $w, w' \in W$, a *path* $P_{w,w'}$ from w to w' is a sequence $w_1, \dots, w_n$ of worlds in W s.t.:

(i) $w_1 = w$ and $w_n = w'$;

(ii) for all $1 \leq i \leq n-1$, w_iRw_{i+1} or $w_{i+1}Rw_i$.

We remark that a path from w to w' is also a path from w' to w. A function $\mathbf{f}$ defined on a subset of W is called *$\mathcal{F}$-compatible* iff for every $w, w' \in Dom(\mathbf{f})$, if wRw' then $\mathbf{f}(w')$ is a counterpart of $\mathbf{f}(w)$, i.e. $C_{w,w'}(\mathbf{f}(w), \mathbf{f}(w'))$. Notice that $\mathcal{F}$-compatible functions are the *objects* of counterpart frames in [52]. Since the solution to the problems in this chapter is trivial whenever we consider c-frames for which $|W| = 1$, hereafter we assume $|W| \geq 2$.

5.2.1 From simply connectedness to goodness

In order to prove Proposition 6.6 in [27], that is, a formula ϕ holds in a counterpart frame $\mathcal{F}$ iff translation ϕ^* holds in the $FOIL$-frame companion to $\mathcal{F}$, Fitting assumes that $\mathcal{F}$ is *simply connected*: for every $w, w' \in W$, there exists one and only one path from w to w'. As Fitting remarks, it is not fundamental that there is at least a path, as if in a c-frame there are worlds w, w' connected by no path, then $\mathcal{F}$ can be split into two c-frames $\mathcal{F}_1$ and $\mathcal{F}_2$: a formula ϕ holds in $\mathcal{F}$ iff it is valid both in $\mathcal{F}_1$ and $\mathcal{F}_2$. Hereafter we consider only c-frames s.t. for every $w, w' \in W$, there exists at least one path from w to w' without any loss of generality. The strong assumption consists in the uniqueness of such a path. But we show that this condition does not adequately reflect Fitting's idea, as the example of a simply connected c-frame on p. 14 in [27] is not strictly as such, since it is always possible to arbitrarily extend a path by going to and fro two worlds. The following definition actually formalizes the property of simply connectedness used by Fitting.

Definition 5.13 (Simply connectedness) *A c-frame $\mathcal{F}$ is* simply connected *iff for every distinct $w, w' \in W$, the set of paths from w to w' ordered by inclusion relation $\subseteq$ has a minimum.*

We stress the fact that w and w' have to be different. This condition can not be generalized to every $w, w' \in W$, or else a counterpart frame would be simply connected iff $|W| \leq 2$.

By simply connectedness of c-frame $\mathcal{F}$ we should be able to prove that every $\mathcal{F}$-compatible function $\mathbf{f}$, defined on $W_0 \subseteq W$, can be extended to an $\mathcal{F}$-compatible function $\mathbf{f}'$ defined on the whole W. This result is needed to prove that if formula $\Box\phi$ does not hold in a simply connected c-frame $\mathcal{F}$, then translation $(\Box\phi)^*$ does not hold in the $FOIL$-frame companion to $\mathcal{F}$, that is, if counterparts $\vec{b}$ to $\vec{a}$ falsify ϕ in w', then $\mathcal{F}$-compatible functions $\mathbf{f}_i$ s.t. $\mathbf{f}_i = \{(w, a_i), (w', b_i)\}$, can be extended to intensional objects defined on the whole W, thus falsifying translation ϕ^*.

First of all we show that simply connectedness is too strong and too weak as a condition at the same time, as there exist c-frames that are not simply connected, in which it is nonetheless possible to extend every $\mathcal{F}$-compatible function, so that $\mathcal{F}$-compatibility is preserved. Moreover there exist simply connected c-frames, for which this is not true. An example of the first kind is c-frame $\mathcal{F}$ s.t.

- $W = \{w_1, w_2, w_3, w_4\}$;
- $R = \{(w_1, w_2), (w_1, w_3), (w_2, w_4), (w_3, w_4)\}$;
- $D(w_1) = D(w_2) = D(w_3) = D(w_4) = \{a\}$;
- $d = D$;
- $C_{w_1,w_2} = C_{w_1,w_3} = C_{w_2,w_4} = C_{w_3,w_4}$ is identity.

The present c-frame is not simply connected, as from w_1 to w_4 there exist path w_1, w_2, w_4 and path w_1, w_3, w_4; but is it possible to extend any $\mathcal{F}$-compatible function to an $\mathcal{F}$-compatible function defined on the whole W. On the other hand the following c-frame $\mathcal{F}'$ is simply connected:

- $W' = \{w_1, w_2, w_3\}$;
- $R' = \{(w_1, w_2), (w_2, w_3)\}$;
- $D(w_1) = D(w_2) = D(w_3) = \{a, b\}$;
- $d = D$;
- $C'_{w_1,w_2} = \{(a, a)\}, C'_{w_2,w_3} = \{(b, b)\}$.

But $\mathcal{F}$-compatible functions $\mathbf{f} = \{(w_1, a)\}$, $\mathbf{f}' = \{(w_2, b)\}$ and $\mathbf{f}'' = \{(w_3, a)\}$ can not be extended to the whole W', so that $\mathcal{F}$-compatibility is preserved.

In order to solve these problems we redefine the constraint on counterpart frames. Similarly to paths, we introduce the notion of *chain of individuals*, chain in short.

Definition 5.14 (Chain of individuals) *Let $a \in w$, $a' \in w'$, a chain of individuals $P_{(a,w),(a',w')}$ from a to a' is a sequence of ordered couples $(a_1, w_1), \ldots, (a_n, w_n)$ s.t.*

(a) $w_1, \ldots, w_n$ is a path from w to w',

(b) $a_1 = a$ and $a_n = a'$,

(c) for every $1 \leq i \leq n-1$, if $w_i R w_{i+1}$ then $C_{w_i,w_{i+1}}(a_i, a_{i+1})$, and if $w_{i+1} R w_i$ then $C_{w_{i+1},w_i}(a_{i+1}, a_i)$.

As it was the case for paths, a chain from $a \in w$ to $a' \in w'$ is also a chain from $a' \in w'$ to $a \in w$. Now we introduce the constraint to be imposed on counterpart frames.

Definition 5.15 (Goodness) *A c-frame $\mathcal{F}$ is* good *iff for every $w, w' \in W$, for every $a \in D(w)$, $b \in D(w')$, if wRw' and $C_{w,w'}(a, b)$, then*

1. $w = w'$ implies $a = b$;

2. for paths $P_{w,w'}$, $P'_{w,w'}, \ldots,$ from w to w', there exist corresponding chains $P_{(a,w),(b,w')}$, $P'_{(a,w),(b,w')}, \ldots,$ from a to b s.t. $w_j = w_i$ implies $a_{w_j} = a_{w_i}$.

This property eliminates frames like $\mathcal{F}'$ and the cases considered by Fitting. In order to prove that if $\mathcal{F}$ is good, then every function can be extended to an $\mathcal{F}$-compatible function defined on the whole W, the hypothesis of $\mathcal{F}$-compatibility is not enough. Consider the following good c-frame $\mathcal{F}$ s.t.

- $W = \{w_1, w_2, w_3\}$;
- $R = \{(w_1, w_2), (w_2, w_3)\}$;
- $D(w_1) = D(w_2) = D(w_3) = \{a, b, c\}$;
- $d = D$;
- $C_{w_1,w_2} = \{(a, a), (b, b)\}, C_{w_2,w_3} = \{(a, a), (b, b)\}$.

Functions $\mathbf{f} = \{(w_1, a), (w_3, b)\}$, $\mathbf{f}' = \{(w_1, b), (w_3, a)\}$ and $\mathbf{f}'' = \{(w_2, c)\}$ are $\mathcal{F}$-compatible, but they can not be extended to the whole W so that $\mathcal{F}$-compatibility is preserved. Thus we have to restrict our definition as follows.

Definition 5.16 ($\mathcal{F}$-compatibility*) *A function* $\mathbf{f}$ *on counterpart frame $\mathcal{F}$ is $\mathcal{F}$-*compatible* *iff*

- $\mathbf{f}$ *is $\mathcal{F}$-compatible;*
- *either $Dom(\mathbf{f}) = \{w\}$ and wRw, or $Dom(\mathbf{f}) = \{w, w'\}$ and wRw'.*

By definition of $\mathcal{F}$-compatibility, we deduce that if $\mathbf{f}$ is an $\mathcal{F}$-compatible* function, then either $Dom(\mathbf{f}) = \{w\}$ and $C_{w,w}(\mathbf{f}(w), \mathbf{f}(w))$, or $Dom(\mathbf{f}) = \{w, w'\}$ and $C_{w,w'}(\mathbf{f}(w), \mathbf{f}(w'))$. The present condition is rather cumbersome, nonetheless it is all that we need, as it will be clear by the proof of lemma 5.20.

Now we prove that goodness is a necessary and sufficient condition for extending every $\mathcal{F}$-compatible* functions $\mathbf{f}$ to an $\mathcal{F}$-compatible function $\mathbf{f}'$, defined on the whole W.

Theorem 5.17 *A c-frame $\mathcal{F}$ is good iff every $\mathcal{F}$-compatible* function $\mathbf{f}$ can be extended to an $\mathcal{F}$-compatible function $\mathbf{f}'$, defined on the whole W.*

Proof. $\Rightarrow$ Suppose that c-frame $\mathcal{F}$ is good and $\mathbf{f}$ is an $\mathcal{F}$-compatible* function. By definition of $\mathcal{F}$-compatibility* there exist $w, w' \in Dom(\mathbf{f})$ s.t. wRw' and $C_{w,w'}(\mathbf{f}(w), \mathbf{f}(w'))$; then consider the set composed by all the paths $P_{w,w'}, P'_{w,w'}, \ldots$, from w to w'. Each $w'' \in W$ appears in some of them, as by hypothesis there is at least one path from w to w'' and one from w'' to w'. By goodness there exist corresponding chains $P_{(a,w),(b,w')}, P'_{(a,w),(b,w')}, \ldots$, s.t. $w_j = w_i$ implies $a_{w_j} = a_{w_i}$; we define $\mathbf{f}'(w'') = a_{w''}$. The condition on chains guarantees the well-definiteness of $\mathbf{f}'$, that is, $w_j = w_i$ implies $\mathbf{f}'(w_j) = \mathbf{f}'(w_i)$. Thus $\mathbf{f}'$ is defined on the whole W and $\mathbf{f}'(w) = a = \mathbf{f}(w)$, $\mathbf{f}'(w') = b = \mathbf{f}(w')$. We prove that for every $w_1, w_2 \in W$, if $w_1 R w_2$ then $C_{w_1,w_2}(\mathbf{f}(w_1), \mathbf{f}(w_2))$. By our hypotheses there is path $w, \ldots, w_1, w_2, \ldots, w'$ in the set of all the paths from w to w', and chain $P_{(a,w),(b,w')}$ s.t. in particular $C_{w_1,w_2}(\mathbf{f}(w_1), \mathbf{f}(w_2))$.

$\Leftarrow$ Suppose that $\mathcal{F}$ is not good, we show that there exists an $\mathcal{F}$-compatible* function $\mathbf{f}$, that can not be extended to an $\mathcal{F}$-compatible function $\mathbf{f}'$ defined on the whole W. Assume that there exist $w, w' \in W$, $a \in w$, $b \in w'$ s.t. wRw' and $C_{w,w'}(a, b)$, and $w = w'$ implies $a = b$; moreover there exists a set S of paths from w to w', a world w'' in some $P_{w,w'} \in S$, but there is no $a_{w''} \in w''$ appearing in all the chains of individuals $P_{(a,w),(b,w')}$ from a to b. We define function $\mathbf{f}$ on $\{w, w'\}$ by setting $\mathbf{f}(w) = a$ and $\mathbf{f}(w') = b$. This function is $\mathcal{F}$-compatible*, but $\mathbf{f}$ cannot be extended to an $\mathcal{F}$-compatible function defined on the whole W. If that were the case, there would be $a_{w''} = \mathbf{f}(w'')$, appearing in all the chains of individuals from a to b, for every path from w to w', against hypothesis.

We conclude the present paragraph with some remarks on theorem 5.17.

1. The construction of $\mathbf{f}'$ starting from $\mathbf{f}$ does not guarantee its uniqueness: we showed that for every $w \in W$, it is possible to find a value $a_w \in D(w)$ for $\mathbf{f}'$, so that $\mathcal{F}$-compatibility constraint is satisfied, but we did not prove that a_w is unique. Of course this fact does not imply any problem: as long as $\mathcal{F}$-compatibility is preserved, the same function can be extended in different ways.

2. The substitution of $\mathcal{F}$-compatibility with $\mathcal{F}$-compatibility* does not cause any trouble to the proof of Proposition 6.6 in [27]. We shall give a formal proof in the following paragraph, for the time being notice that $\mathcal{F}$-compatibility is

used to show that if formula $\Box\phi$ does not hold in a good c-frame $\mathcal{F}$, then translation $(\Box\phi)^*$ does not hold in the $FOIL$-frame companion to $\mathcal{F}$. In particular, if there exist $w' \in W$ and $\vec{b} \in D(w')^n$ s.t. wRw', $C_{w,w'}(\vec{a}, \vec{b})$ and not $(\vec{b}, w') \models \phi$, then we can define on set $\{w, w'\}$ $\mathcal{F}$-compatible* functions $\mathbf{f}_i$, by setting $\mathbf{f}_i = \{(w, a_i), (w', b_i)\}$. By theorem 5.17 the various $\mathbf{f}_i$ can be extended to $\mathcal{F}$-compatible functions $\mathbf{f}'_i$, defined on the whole W. These functions are individuals belonging to function domain $F'(w)$ in the $FOIL$-frame companion to $\mathcal{F}$, which falsify translation ϕ^*.

3. Finally we remark that def. 5.14 is very similar to the definition of *thread* in [52], even if the former is more general than the latter. Kracht and Kutz maintain that if accessibility relation R on possible worlds has no cycle, then for every *thread* t there exists $\mathcal{F}$-compatible function o s.t. for every couple (a, w) in t, $o(w) = a$. But this claim is false and to see that, it is enough to consider Fitting's example in [27], p. 15. It follows that the proofs of theorems 23 and 29 in [52] do not hold. In particular the authors do not single out the necessary and sufficient conditions, by which their translation function between formulas sound w.r.t. c-frames and validities in *coherence*-frames works. We attempt to solve this problem in section 5.3.

5.2.2 Fitting's translation

In the present paragraph we investigate the translation of validities in the class of good c-frames into formulas sound w.r.t. $FOIL$-frames. We introduce translation function $*$ from language $\mathcal{L}_t^=$ to $\mathcal{L}_{FOIL}$ as it appears in [27], with some minor change due to the fact that Fitting does not make use of typed languages.

Definition 5.18 (Translation function $*$) *Let $\phi : n$ be a formula in $\mathcal{L}_t^=$, translation $\phi^* \in \mathcal{L}_{FOIL}$ is inductively defined as follows:*

- *if ϕ is an atomic formula, then $\phi^* = \phi$;*
- $(\neg\psi)^* = \neg\psi^*$;
- $(\psi \to \psi')^* = \psi^* \to \psi'^*$;
- $(\forall x_{n+1}\psi)^* = \forall x_{n+1}\psi^*$;
- $(\Box\psi(t_1, \ldots, t_m))^* = \forall f_1, \ldots, f_m((f_1 = t_1 \wedge \ldots \wedge f_m = t_m) \to \Box(\lambda x_1, \ldots, x_m.\psi^*(f_1, \ldots, f_m))$, *where $f_1, \ldots, f_m$ are intensional variables not appearing in ψ^*.*

All these clauses are clear, but the one for $\Box$-formulas. It states a correspondence between the notions of *counterpart* and *instance of an intensional object*. In the next theorem we prove that for every world w' accessible from w, counterparts $\vec{b}$ of individuals $\vec{a}$ satisfy ψ in w' iff ψ^* is satisfied in w' by all the intensional objects, having $\vec{a}$ as value in w and $\vec{b}$ as value in w'.

Theorem 5.19 *A formula $\phi \in \mathcal{L}_t^=$ holds in the class of good c-frames iff translation $\phi^* \in \mathcal{L}_{FOIL}$ holds in the class of FOIL-frames.*

Proof. $\Leftarrow$ Suppose that ϕ does not hold in a c-model $\mathcal{M}$, based on a good c-frame $\mathcal{F}$; we define model $\mathcal{M}'$ on $FOIL$-frame $\mathcal{F}'$ s.t. translation ϕ^* does not hold in $\mathcal{M}'$:

- $W' = W$, $R' = R$, $d_o = d$;
- for $w \in W'$, $D'(w) = \bigcup_{w' \in W} D(w')$;
- for $w \in W'$, $F'(w)$ is the set of all $\mathcal{F}$-compatible functions on W;
- for $w \in W'$, $d_i(w)$ is the set of functions $\mathbf{f}$ in $F'(w)$ s.t. $\mathbf{f}(w) \in d_o(w)$;
- I' is the normal interpretation extending I.

The so-defined $\mathcal{M}'$ is a $FOIL$-model: the increasing outer domain and the increasing function domain conditions are both satisfied by definition. Moreover if ϕ is a formula in $\mathcal{L}_t^=$, then in ϕ^* there is no free intensional variable; thus if $\vec{a}$ is an n-assignment from individual variables in $\mathcal{L}_t^=$ to outer domain $D(w)$, then we have only two possible cases: either any assignment σ extending $\vec{a}$ to intensional variables s.t. $\sigma(\vec{x}) = \vec{a}$ satisfies ϕ^*, or ϕ^* is satisfied by no such extension of $\vec{a}$. The following lemma corresponds to Proposition 6.6 in [27].

Lemma 5.20 *Let $w \in W$, $\phi : n \in \mathcal{L}_t^=$ and $\sigma(\vec{x}) = \vec{a}$,*

$$(\vec{a}, w) \models \phi \quad iff \quad (I'^{\sigma}, w) \models \phi^*$$

Proof. The proof is by induction on the length of ϕ. As to the base case, consider atomic ϕ: n-assignment $\vec{a}$ satisfies $P^m(t_1, \ldots, t_m)$ in w iff $\langle \vec{a}(t_1), \ldots, \vec{a}(t_m)\rangle \in I(P^n, w) = I'(P^n, w)$ iff $\langle \sigma(t_1), \ldots, \sigma(t_n)\rangle \in I'(P^n, w)$, that is, $(I'^{\sigma}, w) \models P^n(t_1, \ldots, t_n)$. The case for identity statements is similar.

The inductive cases for propositional connectives are straightforward, hence we consider only the universal quantifier and the modal operator.

If ϕ has form $\forall x_{n+1}\psi$, then $(\vec{a}, w) \models \phi$ iff for all $a^* \in d(w)$, $(\langle \vec{a} \cdot a^* \rangle, w) \models \psi$. By induction hypothesis it tantamounts to: for all $a^* \in d_o(w)$, $(I'^{\sigma\binom{x_{n+1}}{a^*}}, w) \models \phi^*$, that is, $(I'^{\sigma}, w) \models \forall x_{n+1}\phi^*$.

Formula ϕ as form $\Box\psi(t_1, \ldots, t_m)$. $\Rightarrow$ Suppose that there exist $w' \in W'$, $\mathbf{f}_1, \ldots, \mathbf{f}_m \in d_i(w)$ s.t. wRw', $\mathbf{f}_i(w) = \sigma(t_i)$ but not $(I'^{\sigma\binom{x_1,\ldots,x_m}{\mathbf{f}_1(w'),\ldots,\mathbf{f}_m(w')}}, w') \models \psi^*$. By induction hypothesis neither $(\langle \mathbf{f}_1(w'), \ldots, \mathbf{f}_m(w')\rangle, w') \models \psi$, and notice that $\mathbf{f}_i(w')$ is a counterpart of $\vec{a}(t_i) = \mathbf{f}_i(w)$ by $\mathcal{F}$-compatibility hypothesis on functions in $F'(w)$. Thus not $(\vec{a}, w') \models \psi$.

$\Leftarrow$ Suppose that there exist $w' \in W'$ and individuals $\vec{b} \in D(w')^m$ s.t. wRw', $C_{w,w'}(\vec{a}(t_i), b_i)$ and not $(\vec{b}, w') \models \psi$. If b_i is a counterpart of $\vec{a}(t_i)$, then we define function $\mathbf{f}_i$ s.t. $\mathbf{f}_i(w) = \vec{a}(t_i)$ and $\mathbf{f}_i(w') = b_i$. Each $\mathbf{f}_i$ is an $\mathcal{F}$-compatible* function, that by theorem 5.17 can be extended to an $\mathcal{F}$-compatible function $\mathbf{f}'_i$, defined on the whole W. Furthermore $\mathbf{f}'_i \in d_i(w)$ as $\vec{a}(t_i) \in d(w)$, and n-assignment $\vec{b}$ coincides with $\sigma\binom{x_1,\ldots,x_m}{\mathbf{f}'_1(w'),\ldots,\mathbf{f}'_m(w')}(\vec{x})$, thus by induction hypothesis not $(I'^{\sigma\binom{x_1,\ldots,x_m}{\mathbf{f}'_{x_1}(w'),\ldots,\mathbf{f}'_{x_m}(w')}}, w') \models \psi^*$.

By lemma 5.20 we establish half of the result we are interested in: if ϕ does not hold in c-model $\mathcal{M}$, based on good c-frame $\mathcal{F}$, then translation ϕ^* is not valid in $\mathcal{M}'$, based on the $FOIL$-frame companion to $\mathcal{F}$.

$\Rightarrow$ Suppose that ϕ^* does not hold in some $FOIL$-model $\mathcal{M}$, we construct a c-model $\mathcal{M}'$ on a good c-frame $\mathcal{F}'$ s.t. ϕ does not hold in $\mathcal{M}'$ either:

- $W' = W$, $R' = R$, $D' = D$, $d' = d_o$;
- for $w, w' \in W$, for $a \in D'(w), b \in D'(w')$, $C_{w,w'}(a, b)$ iff there exists $\mathbf{f} \in d_i(w)$ s.t. $\mathbf{f}(w) = a$ and $\mathbf{f}(w') = b$.
- interpretation I' coincides with I.

The so-defined model $\mathcal{M}'$ is a c-model, in addition c-frame $\mathcal{F}'$ is good: if $a \in D'(w)$ and $b \in D'(w')$, wRw' and $C_{w,w'}(a, b)$, then by definition of counterpart relation $C_{w,w'}$, there exists $\mathbf{f} \in d_i(w)$ s.t. $\mathbf{f}(w) = a$ and $\mathbf{f}(w') = b$. For each set S of paths from w to w', we define the corresponding chains of individuals $P_{(a,w),(b,w')}, P'_{(a,w),(b,w')}, \ldots$, by setting $a_{w_i} = \mathbf{f}(w_i)$.

Finally let σ be an assignment in w for individual and intensional variables of $\mathcal{L}_{FOIL}$ into domains $D(w)$ and $F(w)$ respectively, then $\vec{a} = \sigma(\vec{x})$ s.t. $x_1, \ldots, x_n$ are individual variables. Now we prove the following equivalence lemma.

Lemma 5.21 *Let $w \in W$, $\phi : n \in \mathcal{L}_t^=$ and $\sigma(\vec{x}) = \vec{a}$,*

$$(I^\sigma, w) \models \phi^* \quad iff \quad (\vec{a}, w) \models \phi$$

Proof. The proof is by induction on the length of ϕ. The base case follows directly by construction of $\mathcal{M}'$, and is proved as in lemma 5.20; the inductive cases for propositional connectives are trivial.

If ϕ has form $\forall x_{n+1}\psi$, $(I^\sigma, w) \models \forall x_{n+1}\psi^*$ iff for every $a^* \in d_o(w)$, $(I^{\sigma\binom{x_{n+1}}{a^*}}, w) \models \psi^*$. By induction hypothesis it tantamounts to: for every $a^* \in d'(w)$, $(\langle \vec{a} \cdot a^* \rangle, w) \models \psi$, that is, $(\vec{a}, w) \models \forall x_{n+1}\psi$.

Formula ψ has form $\Box\psi$. $\Leftarrow$ Suppose that there exist $w' \in W$, $\mathbf{f}_1, \ldots, \mathbf{f}_m \in d_i(w)$ s.t. wRw', $\mathbf{f}_i(w) = \sigma(t_i)$, but not $(\langle \mathbf{f}_1(w'), \ldots, \mathbf{f}_m(w') \rangle, w') \models \psi$ by induction hypothesis. Each $\mathbf{f}_i(w')$ is a counterpart of $\vec{a}(t_i) = \mathbf{f}_i(w)$ by definition of $C_{w,w'}$, thus not $(\vec{a}, w') \models \phi$.

$\Rightarrow$ Suppose that there exist $w' \in W'$ and $\vec{b} \in D'(w')^m$ s.t. wRw', $C_{w,w'}(\vec{a}(t_i), b_i)$ and not $(\vec{b}, w') \models \psi$. If b_i is a counterpart of $\vec{a}(t_i)$, then there exists function $\mathbf{f}_i \in d_i(w)$ s.t. $\mathbf{f}_i(w) = \vec{a}(t_i)$ and $\mathbf{f}_i(w') = b_i$, thus neither $(\langle \mathbf{f}_1(w'), \ldots, \mathbf{f}_m(w') \rangle, w') \models \psi$. By induction hypothesis not $(I^{\sigma\binom{x_1, \ldots, x_m}{\mathbf{f}_1(w'), \ldots, \mathbf{f}_m(w')}}, w') \models \psi^*$.

This concludes the proof of lemma 5.21, by which if translation ϕ^* does not hold in a $FOIL$-model $\mathcal{M}$, then ϕ is not valid in c-model $\mathcal{M}'$, based on the good c-frame companion to $\mathcal{F}$. Therefore theorem 5.19 holds.

By the proofs of lemmas 5.20 and 5.21, we can show that theorem 5.19 holds for various subclasses of good c-frames and $FOIL$-frames, that is, a formula $\phi \in \mathcal{L}_t^=$ holds in the class of good c-frames with increasing (decreasing, constant) inner domains iff translation ϕ^* in $\mathcal{L}_{FOIL}$ holds in the class of $FOIL$-frames with increasing (decreasing, constant) d_o domains.

The definition of *goodness* is rather cumbersome, but if we assume that our modal base is $S5$, then it simplifies as follows:

- for every $w, w' \in W$, for every $a \in D(w)$, $b \in D(w')$, if wRw' and $C_{w,w'}(a, b)$, then for every $w'' \in W$ there exists $c \in D(w'')$ s.t. $C_{w,w''}(a, c)$, and if $w'' = w' = w$ then $a = b = c$.

We prove that with this hypothesis we recover the whole goodness condition. Consider paths $P_{w,w'}, P'_{w,w'}, \ldots$, from w to w', since our modal base is $S5$ we have $wRw_1R \ldots Rw_{n-1}Rw'$ iff $wRw_1RwR \ldots Rw_{n-1}RwRw'$. Now we define a chain of individuals for the latter path, then prove that it works also for the former. By our hypothesis and properties of counterpart relation in S5, we have that

$$aC_{w,w_1}c_1C_{w_1,w}aC_{w,w_2} \ldots C_{w,w_{n-1}}c_{n-1}C_{w_{n-1},w}aC_{w,w'}b$$

which tantamounts to chain $aC_{w,w_1}c_1C_{w_1,w_2} \ldots C_{w_{n-2},w_{n-1}}c_{n-1}C_{w_{n-1},w'}b$ from a to b. Moreover $w_j = w_i$ implies $a_{w_j} = a_{w_i}$.

The present remark shows that the goodness condition is not so strange as it may appear at first sight, as in S5 modality it tantamounts to a kind of conditional everywhere-definiteness.

5.2.3 Remarks

In the previous paragraph we proved that a formula $\phi \in \mathcal{L}_t^=$ is sound w.r.t good c-frames iff translation ϕ^* holds in the class of $FOIL$-frames. In comparison to Fitting, we showed also the implication from validity in counterpart frames to validity in $FOIL$-frames. In particular, in par. 5.2.1 we singled out the necessary and sufficient conditions by which c-frames and $FOIL$-frames validate the same set of formulas, *modulo* translation function $*$.

By our remark on the goodness condition in $S5$ modality, we maintain that, whenever a counterpart-theoretician accepts

(i) a Leibnizian universe, where accessibility and counterpart relation are equivalence relations;

(ii) whenever individual a in the present world w has a counterpart b in a world w' accessible from w, then it has a counterpart c in every world w'' accessible from w, and a is the only counterpart to itself in w;

then it follows that the counterpart-theoretician and the supporter of modal occurrents can come to terms, by interpreting their words through translation function $*$. There is no genuine difference. We can deal with sequentialism within a perdurantist framework, whenever we accept (i), (ii) and restrict our modal language to $\mathcal{L}_t^=$.

Nonetheless the present account demands further investigation. In Fitting's $FOIL$-frames there are two kinds of objects: intensions and individuals; this fact is mirrored in language $\mathcal{L}_{FOIL}$, where we have two different types of variables. But we may wonder which part of counterpart theory is still expressible by the substantial account, if we admit in our ontology only four-dimensional objects. We devote the next section to answering this question, by basically referring to [52] by Kracht and Kutz. We consider their translation function $+$ from language $\mathcal{L}_t^=$ of counterpart frames to language $\mathcal{L}^=$ of substantial frames.

5.3 From Goodness to Perfection

In section 5.2 we assumed the goodness condition on counterpart frames, so that validities in the class of good c-frames tantamount to validities in the class of $FOIL$-frames, *modulo* translation function $*$. In order to obtain an analogous result, having at our disposal language $\mathcal{L}^=$ containing only individual variables, no intensional variable, we need to introduce a strengthening of goodness.

Definition 5.22 (Perfection) *A c-frame $\mathcal{F}$ is* perfect *iff $\mathcal{F}$ is good, everywhere-defined and classical.*

Notice that everywhere-definiteness corresponds to Kracht and Kutz's assumption in [52], that they call *Counterpart-Existence Property.* By theorem 5.17, perfection of c-frame $\mathcal{F}$ implies that for every $a \in D(w)$, there exists a $\mathcal{F}$-compatible function $\mathbf{f}$ defined on the whole W s.t. $\mathbf{f}(w) = a$. In fact by everywhere-definiteness there exist $w' \in W$, $b \in D(w')$ s.t. $C_{w,w'}(a, b)$, thus we define function $\mathbf{f}$ as $\mathbf{f}(w) = a$ and $\mathbf{f}(w') = b$; $\mathbf{f}$ is well-defined by the goodness condition and is $\mathcal{F}$-compatible*. By theorem 5.17, $\mathbf{f}$ can be extended to an $\mathcal{F}$-compatible function defined on the whole W.

This strengthening of goodness is due to the fact that in our language there is only one kind of variables - individual variables - as in our s-models we only talk about individual concepts. Hereafter we consider s-frames in which for every $w \in W$, for every $a \in D(w)$, there exists function $\mathbf{f} \in F(w)$ s.t. $\mathbf{f}(w) = a$, without any loss of generality, as it was assumed in par. 5.1.2.

5.3.1 Kracht and Kutz's translation

We introduce translation function $+$ from formulas in language $\mathcal{L}_t^=$ to formulas in $\mathcal{L}^=$, as it appears in [52].

Definition 5.23 (Translation function $+$) *Let $\phi : n$ be a formula in $\mathcal{L}_t^=$, translation $\phi^+ \in \mathcal{L}^=$ is inductively defined as follows:*

- *if ϕ is an atomic formula, then $\phi^+ = \phi$;*
- $(\neg\psi)^+ = \neg\psi^+$;
- $(\psi \to \psi')^+ = \psi^+ \to \psi'^+$;
- $(\forall x_{n+1}\psi)^+ = \forall x_{n+1}\psi^+$;
- $(\Box\psi(t_1, \dots, t_m))^+ = \forall f_1, \dots, f_m((f_1 = t_1 \wedge \dots \wedge f_m = t_m) \to \Box(\psi^+[f_1, \dots, f_m]))$, *where $f_1, \dots, f_m$ are variables not appearing in ϕ^+.*

All these clauses are identical to the ones for translation function $*$, but the case for the modal operator. Translation function $+$ need not to make use of the λ-abstractor, as variables $x_1, \dots, x_n$ are to be interpreted on intensional objects and thus $\lambda x_1, \dots, x_m.\psi^+(f_1, \dots, f_m)$ semantically tantamounts to $\psi^+[f_1, \dots, f_m]$.

Now it is possible to prove the following theorem on equivalence between the notions of validity in perfect c-frames and in classical s-frames, *modulo* translation function $+$.

Theorem 5.24 *A formula $\phi \in \mathcal{L}_t^=$ is valid in class of perfect c-frames iff $\phi^+ \in \mathcal{L}^=$ is valid in the class of classical s-frames.*

Proof. $\Leftarrow$ Suppose that ϕ does not hold in some c-model $\mathcal{M}$, based on a perfect c-frame $\mathcal{F}$, we construct s-model $\mathcal{M}'$ s.t. translation ϕ^+ does not hold in $\mathcal{M}'$, as follows:

- $W' = W$, $R' = R$, $D' = D$;
- for $w \in W'$, $F'(w)$ is the set of $\mathcal{F}$-compatible functions defined on W;
- for $w \in W'$, $d'(w) = F'(w)$;
- interpretation I' coincides with I.

Notice that $\mathcal{M}'$ is a classical s-model, as it satisfies the increasing function domain condition and for all $w \in W$, $d'(w) = F'(w)$.

Let $\vec{a}$ be an n-assignment for variables in $\mathcal{L}_t^=$ into set $D(w)$, define assignment σ for variables in $\mathcal{L}^=$ into set $F'(w)$ as $\sigma(x_i)(w) = a_i$, for $1 \leq i \leq n$. By a previous remark on perfect c-frames, it is always possible to find functions $\sigma(\vec{x})$ and such an assignment σ, even if it is not necessarily unique.

Lemma 5.25 *Let $\phi : n \in \mathcal{L}_t^=$, $w \in W$ and $\sigma(\vec{x})(w) = \vec{a}$,*

$$(\vec{a}, w) \models \phi \quad iff \quad (I'^{\sigma}, w) \models \phi^+$$

Proof. The proof is by induction on the length of ϕ. As to the base case, n-assignment $\vec{a}$ satisfies $P^m(t_1, \ldots, t_m)$ in w iff $\langle \vec{a}(t_1), \ldots, \vec{a}(t_m) \rangle \in I(P^m, w) = I'(P^m, w)$ iff $\langle \sigma(t_1)(w), \ldots, \sigma(t_m)(w) \rangle \in I'(P^m, w)$, that is, $(I'^{\sigma}, w) \models P^m(t_1, \ldots, t_m)$. The case for identity statements is similar, and the inductive cases for propositional connectives are trivial.

If ϕ has form $\forall x_{n+1}\psi$, then $(\vec{a}, w) \models \forall x_{n+1}\psi$ iff for every $a^* \in D(w)$, $(\langle \vec{a} \cdot a^* \rangle, w) \models \psi$. By our assumptions for every $a^* \in D(w)$, there exists $\mathbf{f}_{a^*} \in F'(w)$ s.t. $\mathbf{f}_{a^*}(w) = a^*$. By induction hypothesis for every $\mathbf{f} \in F'(w)$, $(I'^{\sigma\binom{x_{n+1}}{\mathbf{f}}}, w) \models \psi^+$, i.e. $(I'^{\sigma}, w) \models \forall x_{n+1}\psi^+$.

Formula ϕ has form $\Box\psi$. $\Rightarrow$ Consider $\mathbf{f}_1, \ldots, \mathbf{f}_m \in F'(w)$ s.t. $\mathbf{f}_i(w) = \sigma(t_i)(w)$. By $\mathcal{F}$-compatibility of $\mathbf{f}_i$, $\mathbf{f}_i(w')$ is counterpart to $\vec{a}(t_i)$ and since $(\vec{a}, w) \models \phi$, then $(\langle \mathbf{f}_1(w'), \ldots, \mathbf{f}_m(w') \rangle, w') \models \psi$. By induction hypothesis $(I'^{\sigma\binom{x_1, \ldots, x_m}{\mathbf{f}_1, \ldots, \mathbf{f}_m}}, w') \models \psi^+$, thus $(I'^{\sigma}, w) \models \phi^+$.

$\Leftarrow$ Suppose that there are $w' \in W$, $\vec{b} \in D(w')^m$ s.t. wRw', $C_{w,w'}(\vec{a}(t_i), b_i)$ but not $(\vec{b}, w') \models \psi$. As b_i is a counterpart of $\vec{a}(t_i)$, we define functions $\mathbf{f}_i$ s.t. $\mathbf{f}_i(w) = \vec{a}(t_i)$ and $\mathbf{f}_i(w') = b_i$. Each $\mathbf{f}_i$ is an $\mathcal{F}$-compatible* function, that by theorem 5.17 can be extended to an $\mathcal{F}$-compatible function $\mathbf{f}_i' \in F'(w)$ defined on the whole W. Then define assignment σ s.t. $\sigma(t_i) = \mathbf{f}_i$, we have that $\mathbf{f}_i(w) = \sigma(t_i)(w)$ and $\sigma(\vec{t})(w') = \vec{b}$, thus by induction hypothesis neither $(I'^{\sigma\binom{x_1,\dots,x_m}{\mathbf{f}_1,\dots,\mathbf{f}_m}}, w') \models \psi^+$.

By lemma 5.25 we establish half of the result we are interested in, as whenever $\phi \in \mathcal{L}_t^=$ does not hold in a c-model $\mathcal{M}$ on a perfect c-frame $\mathcal{F}$, then $\phi^+ \in \mathcal{L}^=$ is not valid in s-model $\mathcal{M}'$, based on the s-frame companion to $\mathcal{F}$.

$\Rightarrow$ Suppose that ϕ^+ is not valid in some classical s-model $\mathcal{M}$, then construct a c-model $\mathcal{M}'$ on a perfect c-frame $\mathcal{F}'$ s.t. ϕ does not hold in $\mathcal{M}'$:

- $W' = W$, $R' = R$, $D' = d' = D$;
- for $w, w' \in W$, for $a \in D'(w), b \in D'(w')$, $C_{w,w'}(a, b)$ iff there exists $\mathbf{f} \in F(w)$ s.t. $\mathbf{f}(w) = a$ and $\mathbf{f}(w') = b$;
- interpretation I' coincides with I.

The so-defined model $\mathcal{M}'$ is a counterpart model; in addition c-frame $\mathcal{F}'$ is perfect: it is everywhere-defined as by assumption on s-frames for every $a \in D(w)$, there exists $\mathbf{f} \in F(w)$ s.t. $\mathbf{f}(w) = a$, and if wRw', for finding $b \in D'(w')$ s.t. $C_{w,w'}(a, b)$, we only have to set $b = \mathbf{f}(w')$. It is classical by definition and as regards the goodness condition, the proof that it holds in $\mathcal{F}'$ is identical to the one in par. 5.2.2.

Let σ be an assignment for variables in language $\mathcal{L}^=$ into function domain $F(w)$, we define n-assignment $\vec{a}$ for variables in $\mathcal{L}_t^=$ into outer domain $D(w)$ s.t. $\vec{a} = \sigma(\vec{x})(w)$.

Lemma 5.26 *Let $\phi : n \in \mathcal{L}_t^=$, $w \in W$ and $\vec{a} = \sigma(\vec{x})(w)$,*

$$(I^\sigma, w) \models \phi^+ \quad iff \quad (\vec{a}, w) \models \phi$$

Proof. The proof is once more by induction on the length of ϕ. We start with considering the base case for atomic ϕ. We have that $(I^\sigma, w) \models P^m(t_1, \dots, t_m)$ iff $\langle \sigma(t_1)(w), \dots, \sigma(t_m)(w) \rangle \in I(P^m, w) = I'(P^m, w)$, that is, $\langle \vec{a}(t_1), \dots, \vec{a}(t_m) \rangle \in I'(P^m, w)$. This is the case iff n-assignment $\vec{a}$ satisfies $P^m(t_1, \dots, t_m)$ in w.

The case for identity statements is completely similar, and the inductive cases for propositional connectives are trivial.

If ϕ has form $\forall x_{n+1}\psi$, then $(I^\sigma, w) \models \forall x_{n+1}\psi^+$ iff for every $\mathbf{f} \in F(w)$, $(I^{\sigma\binom{x_{n+1}}{\mathbf{f}}}, w) \models \psi^+$. By our assumption on s-frames every $a^* \in D(w)$ is equal to $\mathbf{f}(w)$ for some $\mathbf{f} \in F(w)$, thus by induction hypothesis we have that for every $a^* \in D(w)$, $(\langle \vec{a} \cdot a^* \rangle, w) \models \psi$, i.e. $(\vec{a}, w) \models \forall x_{n+1}\psi$.

Formula ϕ has form $\Box\psi$. $\Leftarrow$ Suppose that there exist $\mathbf{f}_1, \dots, \mathbf{f}_m \in F(w)$, $w' \in W$ s.t. $\mathbf{f}_i(w) = \sigma(t_i)(w)$, wRw' but not $(I^{\sigma\binom{x_1,\dots,x_m}{\mathbf{f}_1,\dots,\mathbf{f}_m}}, w') \models \psi^+$. By induction hypothesis neither $(\langle \mathbf{f}_1(w'), \dots, \mathbf{f}_m(w') \rangle, w') \models \psi$, moreover each $\mathbf{f}_i(w')$ is a counterpart of $\vec{a}(t_i) = \mathbf{f}_i(w)$ by definition of $C_{w,w'}$; thus not $(\vec{a}, w) \models \phi$.

$\Rightarrow$ Assume that there exist $w' \in W'$, $\vec{b} \in D'(w')^m$ s.t. wRw', $C_{w,w'}(\vec{a}(t_i), b_i)$ but not $(\vec{b}, w) \models \psi$. By definition of the counterpart relation there are functions

$\mathbf{f}_1, \ldots, \mathbf{f}_m \in F(w)$ s.t. $\mathbf{f}_i(w) = \vec{a}(t_i)$ and $\mathbf{f}_i(w') = b_i$, then we define assignment σ as $\sigma(t_i) = \mathbf{f}_i$. Therefore $\mathbf{f}_i(w) = \sigma(t_i)(w)$ and by induction hypothesis neither $(I^{\sigma\binom{x_1,\ldots,x_m}{\mathbf{f}_1,\ldots,\mathbf{f}_m}}, w') \models \psi^+$.

Once we have proved lemma 5.26, we have theorem 5.24, that is, formula $\phi \in \mathcal{L}_t^=$ holds in the class of perfect c-frames iff translation ϕ^+ is valid in the class of classical s-frames.

Also in the present case the definition of *perfection* can be simplified, if we consider $S5$ as modal base, in fact goodness and everywhere-definiteness are equivalent to:

- for every $w, w' \in W$, for every $a \in D(w)$, if wRw' then there exists $b \in D(w')$ s.t. $C_{w,w'}(a, b)$, and if $w' = w$ then $a = b$.

In $S5$ perfection tantamounts to a modified version of everywhere-definiteness, that we call *everywhere-definiteness*$^+$, with in addition $d(w) = D(w)$ for every $w \in W$.

5.3.2 Perfect c-frames and Kripke semantics

Now we consider a particular version of theorem 5.24, concerning specific classes of perfect c-frames and classical s-frames. The following result will be useful in comparing Kripke and counterpart semantics, as it provides translation conditions further more interesting than the ones listed in theorem 5.1. First of all we say that a c-frame $\mathcal{F}$ is *inverse-counterpart faithful* iff for $w, w' \in W$, wRw' and $w'Rw$ imply $\breve{C}_{w,w'} = C_{w',w}$; then we prove the following result.

Theorem 5.27 *A formula $\phi \in \mathcal{L}_t^=$ is valid in the class of perfect, injective, functional and inverse-counterpart faithful c-frames iff $\phi^+ \in \mathcal{L}^=$ is valid in the class of monotonic, classical s-frames.*

Proof. $\Leftarrow$ Suppose that ϕ does not hold in some c-model $\mathcal{M}$, based on a perfect, injective, functional and inverse-counterpart faithful c-frame $\mathcal{F}$, then we construct a monotonic, classical s-model $\mathcal{M}'$ s.t. ϕ^+ does not hold in $\mathcal{M}'$:

- $W' = W$, $R' = R$, $D' = D$;
- for $w \in W'$, $F'(w)$ is the set of $\mathcal{F}$-compatible, monotonic functions on W;
- for $w \in W'$, $d'(w) = F'(w)$;
- I' is the interpretation coinciding with I.

The so-defined model $\mathcal{M}'$ is a monotonic and classical s-model. In order to prove lemma 5.25 for case $\phi = \Box\psi$, we have to show that if there are $w' \in W$, $\vec{b} \in D(w')^m$ s.t. wRw' and $C_{w,w'}(\vec{a}(t_i), b_i)$, then $\mathcal{F}$-compatible* functions $\mathbf{f}_i$ s.t. $\mathbf{f}_i(w) = \vec{a}(t_i)$ and $\mathbf{f}_i(w') = b_i$, can be uniquely extended to $\mathcal{F}$-compatible functions $\mathbf{f}'_i$, defined on the whole W.

Assume for reduction that there are $\mathbf{f}'_i$, $\mathbf{g}'_i$ s.t. $\mathbf{f}'_i(w) = \vec{a}(t_i) = \mathbf{g}'_i(w)$ and $\mathbf{f}'_i(w') = b_i = \mathbf{g}'_i(w')$, but there is $w'' \in W$ s.t. $\mathbf{f}'_i(w'') \neq \mathbf{g}'_i(w'')$. This means that there exist chains of individuals $P_{(\vec{a}(t_i),w),(b_i,w')}$ and $P'_{(\vec{a}(t_i),w),(b_i,w')}$, in which couples $(\mathbf{f}'_i(w''), w'')$, $(\mathbf{g}'_i(w''), w'')$ appear. Hence in these chains there are worlds w_1, w_2 and individuals a_1, a_2, a'_2 s.t. one of the following cases holds:

- $w_1 R w_2$, $C_{w_1,w_2}(a_1, a_2)$ and $C_{w_1,w_2}(a_1, a'_2)$;
- $w_2 R w_1$, $C_{w_2,w_1}(a_2, a_1)$ and $C_{w_2,w_1}(a'_2, a_1)$;
- $w_1 R w_2$ and $w_2 R w_1$, $C_{w_1,w_2}(a_1, a_2)$ and $C_{w_2,w_1}(a'_2, a_1)$.

The first case violates functionality, the second one injectivity and the third one inverse-counterpart faithfulness, against our hypotheses. Therefore $\mathbf{f}'_i(w'') = \mathbf{g}'_i(w'')$ and lemma 5.25 holds.

$\Rightarrow$ Suppose that ϕ^+ is not valid in some s-model $\mathcal{M}$, based on a monotonic, classical s-frame $\mathcal{F}$; then we construct c-model $\mathcal{M}'$ on a perfect, injective, functional and inverse-counterpart faithful c-frame $\mathcal{F}'$ s.t. ϕ does not hold in $\mathcal{M}'$, as in the second part of theorem 5.24.

We proved that $\mathcal{M}'$ is a perfect c-model, it is left to show that it is also injective, functional and inverse-counterpart faithful. We give the proof only for injectivity, as the other cases are similar. Assume that wRw', $C_{w,w'}(a, b)$ and $C_{w,w'}(a, b')$; by definition of counterpart relation in $\mathcal{M}'$, this means that there are $\mathbf{f}, \mathbf{g} \in \mathcal{M}$ s.t. $\mathbf{f}(w) = a = \mathbf{g}(w)$, $\mathbf{f}(w') = b$ and $\mathbf{g}(w') = b'$. By the monotonicity hypothesis on $\mathcal{M}$, $\mathbf{f} = \mathbf{g}$ and thus $b = b'$. It is easy to see that lemma 5.26 holds, thus we prove theorem 5.27.

By considering also theorem 5.8, we have the following result concerning the relationships among the notions of validity in Kripke and in counterpart semantics, and in the substantial interpretation.

Theorem 5.28 *A formula $\phi \in \mathcal{L}^=_t$ is valid in the class of perfect, injective, functional and inverse-counterpart faithful c-frames iff translation $\phi^+ \in \mathcal{L}^=$ is valid in the class of classical, monotonic s-frames, iff it is valid in the class of classical K-frames.*

Finally notice that if our modal base is $S5$, then a c-frame is perfect, injective, functional and inverse-counterpart faithful iff it is everywhere defined$^+$, classical, injective and functional. At the end of par. 5.3.1 we remarked that a leibnizian c-frame - i.e. reflexive, transitive and symmetric - is perfect iff it is everywhere-defined$^+$ and classical; moreover it is easy to check that an injective and functional leibnizian c-frame is also inverse-counterpart faithful. Therefore we strengthen theorem 5.28 as follows:

Corollary 5.29 *A formula $\phi \in \mathcal{L}^=_t$ is valid in the class of everywhere-defined$^+$, classical, injective and functional leibnizian c-frames iff translation $\phi^+ \in \mathcal{L}^=$ is valid* *in the class of classical, monotonic leibnizian s-frames iff it is valid in the class of classical leibnizian K-frames.*

As it was the case for translation τ_n, theorem 5.28 does not imply the equivalence of the three accounts, as it can be the case that for some $\phi \in \mathcal{L}^=$ there is no $\psi \in \mathcal{L}_t^=$ s.t. $\phi = \psi^+$, thus there would be some truth in classical, monotonic s-frames and in classical K-frames that is not expressible as the translation of some formula sound w.r.t. perfect, injective, functional and inverse-counterpart faithful c-frames.

In par. 5.1.3 we defined dual translation τ^- and proved that for every $\phi \in \mathcal{L}_t^=$, if $\mathcal{F}$ is an everywhere-defined c-frame, then $\mathcal{F}$ validates $\tau_n(\tau^-(\phi)) \leftrightarrow \phi$; thus τ^- was the dual translation to τ_n. By theorem 5.12 we had complete equivalence among counterpart semantics, the substantial interpretation and Kripke semantics. In the next paragraph we prove a similar result for translation $+$.

5.3.3 Equivalence

In the present paragraph we prove that translation τ_n is dual not only to τ^-, but also to function $+$. From this result equivalence of Kripke semantics, the substantial interpretation and counterpart semantics immediately follows.

Lemma 5.30 *Let ϕ be a formula in $\mathcal{L}^=$ and $\mathcal{F}$ a classical K-frame or a monotonic, classical s-frame. It is the case that $\mathcal{F}$ validates $(\tau_n(\phi))^+$ iff ϕ holds in $\mathcal{F}$.*

Proof. The proof is by induction on the length of ϕ and similar to the one for lemma 5.11, as $+$ is identical to τ^- but for the case of the modal operator.

As to the base of induction $(\tau_n(P^m(t_1,\ldots,t_m)))^+ = (P^m(t_1,\ldots,t_m))^+ = P^m(t_1,\ldots,t_m)$.

The inductive cases for propositional connectives are straightforward, as both $+$ and τ^- commute with negation and implication, whereas for the universal quantifier we have

$$(\tau_n(\forall x_i\psi))^+ = (\forall x_{n+1}(\tau_n(\psi)[x_i/x_{n+1}]))^+ = \forall x_{n+1}(\tau_n(\psi)[x_i/x_{n+1}])^+$$

we can prove that substitution and translation $+$ commute, that is, $(\theta[x_i/x_j])^+$ is equal to $\theta^+[x_i/x_j]$ for every $\theta \in \mathcal{L}_t^=$; therefore

$$\forall x_{n+1}(\tau_n(\psi)[x_i/x_{n+1}])^+ = \forall x_{n+1}((\tau_n(\psi))^+[x_i/x_{n+1}])$$

and by induction hypothesis it tantamounts to $\forall x_i\psi$.

As regards the modal operator, we have that

$$(\tau_n(\Box\psi))^+ = \forall f_1,\ldots,f_n((f_1 = x_1 \wedge \ldots \wedge f_n = x_n) \to \Box((\tau_n(\psi))^+[f_1,\ldots,f_n]))$$

that by induction hypothesis tantamounts to

$$\forall f_1,\ldots,f_n((f_1 = x_1 \wedge \ldots \wedge f_n = x_n) \to \Box(\psi[f_1,\ldots,f_n])) \tag{5.1}$$

We prove that $\mathcal{F}$ validates 5.1 iff $\Box\psi$ holds in $\mathcal{F}$. From left to right the implication follows by A6, by substituting $f_1, \ldots, f_n$ with $x_1, \ldots, x_n$. As to the implication from right to left by A25 we have

$$(f_1 = x_1 \wedge \ldots \wedge f_n = x_n) \rightarrow \Box(\psi[f_1, \ldots, f_n])$$

and by R5 the desired result follows.

Thus lemma 5.30 holds, and we prove the following corollary to theorem 5.28.

Corollary 5.31 *A formula $\tau_n(\phi) \in \mathcal{L}_t^=$ is valid in the class of perfect, injective, functional and inverse-counterpart faithful c-frames iff translation $(\tau_n(\phi))^+ = \phi \in \mathcal{L}^=$ is valid in the class of classical, monotonic s-frames, iff it is valid in the class of classical K-frames.*

by which we deduce the complete equivalence of the three accounts.

5.3.4 Remarks

In par. 5.3.1 we proved that a formula $\phi \in \mathcal{L}_t^=$ holds in the class of perfect c-frames iff translation $\phi^+ \in \mathcal{L}^=$ is sound w.r.t. classical s-frames. By this result we deduce that if a counterpart-theoretician accepts points (i) in par. 5.2.3, and in addition:

(iii) a possibilist point of view, by which in every possible world w set $d(w)$ of existing objects is equal to set $D(w)$ of possible individuals;

(iv) every individual a in the present world w has a counterpart b in each world w', accessible from w, and a is the only counterpart to itself in w;

then she can meaningfully express her theses within the modal occurrent framework, by interpreting her modal discourse through translation function $+$.

Furthermore by theorem 5.28, we derive that if our counterpart-theoretician accepts (i), (iii), (iv) and that:

(v) each individual is at most counterpart to one individual;

(vi) each individual has at most one counterpart;

then she can express her truths not only within a framework of modal occurrents, but also by means of enduring objects.

These remarks imply that under the present conditions, we can deal with counterpart semantics in the substantial interpretation and Kripke semantics. But even if these constraints are much weaker than those in section 5.1, they seem to be unacceptable to a counterpart-theoretician.

On the other hand, by theorem 5.31 we maintain that Kripke's and the substantial account can be reduced to perfect, injective, functional and inverse-counterpart faithful c-frames, as by lemma 5.30 every formula $\phi \in \mathcal{L}^=$ sound w.r.t. either classical K-frames or classical, monotonic s-frames, can be considered as translation $(\tau_n(\phi))^+$ for formula $\tau_n(\phi) \in \mathcal{L}_t^=$, valid in the aforementioned class of c-frames. These three accounts are completely equivalent from a formal point of view, whenever we restrict our modal discourse to languages $\mathcal{L}_t^=$ and $\mathcal{L}^=$.

We conclude by remarking a disappointing feature of translation function $+$. For proving theorem 5.28 we have to assume classical K-, s- and c-frames. This condition is unsatisfactory, especially in comparison to theorems 5.8, 5.9 and 5.19, from which we deduce several corollaries concerning the various inclusion relationships of inner domains. It does not seem possible to generalize the present result in the same way, as we have to consider classical s-frames in order to prove that the c-frame companion to s-frame $\mathcal{F}$ is everywhere-defined. In particular, for every $a \in D(w)$, we find $\mathbf{f} \in d(w)$ s.t. $\mathbf{f}(w) = a$, only by assuming that $d(w) = F(w)$.

5.4 Conclusions

In this chapter we introduced four translation functions - though three are pairwise dual one to another - for formulas sound w.r.t. Kripke semantics, the substantial interpretation and counterpart semantics. Then we presented the necessary and sufficient conditions for these functions to work, thus faithfully embedding one account into another.

In section 5.1 we formally proved the intuitive equivalence among counterpart semantics, with identity as counterpart relation, the substantial interpretation, with constant intensional objects, and Kripke semantics. By theorem 5.1 a formula $\phi \in \mathcal{L}^=$ holds in the class of K-frames iff it is valid in constant s-frames, iff translation $\tau_n(\phi) : n$ in $\mathcal{L}_t^=$ is sound w.r.t. c-frames s.t. for every $w, w' \in W$, $C_{w,w'}$ is everywhere-defined and it is identity. Then we proved that the condition on s-frames can be weakened to monotonicity.

If our argument in chapter 4 is sound, and we think of the various semantics for quantified modal logic as faithful representations of sequentialism, perdurantism and endurantism, then this result highlights the formal relationship among these different theories, that is, an endurantist, or a perdurantist admitting only monotonic intensional objects in her ontology, can express their modal discourse within counterpart semantics. Moreover by translation function τ^- and theorem 5.11, we can go the other way round as well: a sequentialist, who consider identity as the everywhere-defined counterpart relation, can express the whole set of her truths within a perdurantist or endurantist framework. Therefore we can consider these three different approaches as completely equivalent

In par. 5.1.4 we pointed out that the hypothesis underlying translation functions τ_n and τ^- are hardly acceptable, thus we went on analysing Fitting's and Kracht and Kutz's translation functions, which require weaker assumptions on c-frames. In section 5.2 we introduced Fitting's translation function $*$ and the class of good c-frames, that is, counterpart frames s.t. if wRw' and $C_{w,w'}(a, b)$, then

1. $w = w'$ implies $a = b$;
2. for every paths $P_{w,w'}, P'_{w,w'}, \ldots$ from w to w', there exist corresponding chains $P_{(a,w),(b,w')}, P'_{(a,w),(b,w')}, \ldots$ from a to b s.t. $w_j = w_i$ implies $a_{w_j} = a_{w_i}$.

We proved that a formula $\phi \in \mathcal{L}_t^=$ is valid in good c-frames iff translation ϕ^* holds in the class of $FOIL$-frames. By our argument in chapter 4 we maintain that, whenever sequentialists accept

(i) a leibnizian universe, where the accessibility and counterpart relation are equivalence relations;

(ii) if individual a in the present world w has a counterpart b in a world w' accessible from w, then it has a counterpart c in every world w'' accessible from w, and a is the only counterpart to itself in w;

then it follows that sequentialists and perdurantists can agree on the set of true statements, by interpreting their words through translation function $*$.

In $FOIL$-frames there are two kinds of objects: intensions and individuals; but if a perdurantist admits only individual concepts, we may wonder whether it is still possible to translate counterpart theory into such an account. In section 5.3 we presented Kracht and Kutz's translation function $+$ and a refinement of goodness: *perfection.* We proved that a formula $\phi \in \mathcal{L}_t^=$ holds in perfect c-frames iff translation $\phi^+ \in \mathcal{L}^=$ is sound w.r.t. classical s-frames. By our correspondence thesis between semantics for quantified modal logic and ontological theories on persistence, we affirm that if a sequentialist accepts (i) and:

(iii) a possibilist point of view, by which in every possible world w set $d(w)$ of existing objects is equal to set $D(w)$ of possible individuals;

(iv) every individual a in the present world w has a counterpart b in each world w', accessible from w, and a is the only counterpart to itself in w;

then she can express her truths within a perdurantist framework, made of only intensional objects, by interpreting her modal discourse through translation $+$.

We even compared sequentialism and perdurantism to endurantism, as by theorem 5.28 we maintain that if a sequentialist accepts in addition that:

(v) each individual is at most counterpart to one individual;

(vi) each individual has at most one counterpart;

then sound statements according to her can be expressed not only within a perdurantist framework, but also in endurantist terms.

In par. 5.3.3 we showed that translation function τ_n is dual also to $+$, so that every formula $\phi \in \mathcal{L}^=$ sound w.r.t. either classical K-frames or classical, monotonic s-frames, can be considered as translation $(\tau_n(\phi))^+$ of formula $\tau_n(\phi) \in \mathcal{L}_t^=$, valid in perfect, injective, functional and inverse-counterpart faithful c-frames. We concluded that both the endurantist and perdurantist account can be formalized within the sequentialist framework: it is not necessary to assume that the counterpart relation is everywhere-defined and it is identity, as the constraints in theorem 5.28 are enough.

Finally we would like to stress the fact that these results do not provide reasons for preferring an ontological approach rather than another. Our aim in the present section was not to support sequentialism, rather than perdurantism or enduratism. These logical investigations are not intended to push anyone toward a specific ontological account, as they concern only formal features of the various theories, without saying anything about their correspondence to our everyday experiences. Nonetheless we deem such an analysis useful, as it clearly states the relationship among sequentialism, perdurantism and enduratism, by highlighting their degree of generality and the constraints by which they are reducible one another. Even if ours is a merely formal analysis, it is not unimportant to check whether these diverse ontological accounts genuinely differ; whether they exactly say the same thing, though by using different terms; whether one of them encompasses the other ones. We hope to have provided a first, partial insight into these questions.

Summing Up

The work is over! In the present conclusion we summarize the problems, which we dealt with, and the results attained.

At the end of chapter 1 we remarked that Kripke semantics reveals deep limitations when applied to quantified modal logic. In particular, we pointed out the gap between systems based on Kripke's theory of quantification and free logic:

- We proved Kripke-completeness for quantified extensions of K, T and $S4$; but we had to adapt the canonical model method to systems based on Kripke's theory of quantification, as $Q^{\circ}.K+BF$, $Q^{\circ}.K+CBF$ and $Q^{\circ}.K+CBF+BF$. Moreover calculi based on free logic, containing Barcan formula, turned out to be Kripke-incomplete.

- As regards quantified extensions of B, we had completeness proofs only for pairwise equivalent calculi $Q^{\circ}.B+CBF$, $Q^{\circ}.B+CBF+BF$ and $Q.B$, $Q.B+BF$. Systems $Q^{\circ}.B+BF$ and $Q^{E}.B+BF$ are Kripke-incomplete, whereas the completeness problem for $Q^{\circ}.B$ is still open. Analogous results hold for the corresponding extensions of $S5$.

- By means of canonical models with constant outer domains, in section 1.4 we proved Kripke-completeness for $Q^{E}.B$ and equivalent calculi $Q^{E}.B+CBF$, $Q^{E}.B+CBF+BF$, but this technique is at our disposal only for systems on language $\mathcal{L}^{E}$. The question for $Q^{\circ}.B$ and $Q^{\circ}.S5$ is once more left unanswered.

By these theorems and Ghilardi's incompleteness results we demand a more satisfactory semantic account of quantified modal logic. That is why we turned to counterpart semantics.

In chapter 2 we underlined the advantages of counterpart semantics when we deal with individuals in modal settings:

- By using finitary assignments and types, we evaluate a modal formula w.r.t. all and only the individuals actually appearing therein.

- In counterpart frames we discriminate formulas which are deemed equivalent in Kripke semantics, as BF and the necessity of non-existence.

- Counterpart semantics seems to be the best available formalization for actualism.

In addition typed QML_t calculi for quantified modal logic have strong completeness properties. We applied Ghilardi's method to prove counterpart-completeness for systems $Q.K_t$ and $Q.K+BF_t$. Moreover calculi $Q^{E}.K+BF_t$ and $Q^{E}.K+CBF+BF_t$ are counterpart-complete, differently from what happens to their QML companions in Kripke semantics. As regards modalities stronger than K the advantages of counterpart semantics are even more striking, as we proved counterpart-completeness for systems $Q^{E}.B+BF_t$ as well as $Q^{E}.S5+BF_t$, while the corresponding QML calculi are Kripke-incomplete. By these facts we maintain that counterpart semantics represent a major improvement - in comparison to Kripke's - in assigning a meaning to necessity and possibility, when we deal with individuals.

In chapter 3 we investigated identity in modal settings. In section 3.2 we extended self-identity and Leibniz's Law - the postulates for identity in first-order logic - to modal language $\mathcal{L}^=$, and checked soundness for these principles w.r.t. Kripke semantics. Moreover we proved that $QML^=$ calculi with identity have strong completeness properties in this semantics. Nonetheless in section 3.3 we remarked that extending Leibniz's Law to language $\mathcal{L}^=$ cannot be the last word on identity in modal settings. There are philosophically motivated reasons for limiting Leibniz's Law; in par. 3.3.2 we considered some of them as contingent identity systems, languages with individual constants and the perdurantist ontology of physical objects. Thus in section 3.4 we introduced the substantial interpretations of QML, in which necessity of identity A22 and necessity of difference A23 fail. The problem with this account are its weak completeness properties: systems $Q^E.K + BF^{ci}$, $Q^E.K + CBF + BF^{ci}$ containing BF, and systems $Q^\circ.B + BF^{ci}$, $Q^E.B + BF^{ci}$ turned out to be substantial-incomplete. Finally in section 3.5 we analysed identity in counterpart semantics. The semantics of counterparts can model both classical and contingent identity, by means of specific constraints - functionality and injectivity - on *c*-frames. Leibniz's Law is unrestrictedly valid without implying A22, A23, and we have completeness results for all the typed $QML_t^=$ and QML_t^{ci} calculi with classical and contingent identity.

In the second part of the present work we dealt with the *correspondence thesis* between semantics for quantified modal logic we presented in the first part, and three ontological accounts on persistence conditions for material objects.

In chapter 4 we compared Kripke semantics, the substantial interpretation and counterpart semantics to three-, four-dimensionalism and sequentialism respectively, by considering the following points:

- The domains of individuals in Kripke semantics contain 'wholly present' objects, whereas in the substantial interpretation we have individuals extending across time; counterparts are common both to sequentialism and counterpart semantics.

- Our semantic accounts validate only principles on identity, that are sound w.r.t. the corresponding ontological thesis.

- In Kripke semantics we formally reconstructed the arguments of mereological change and coincident but distinct objects, by using endurantistically valid principles. Then we made use of substantial and counterpart models to formalize the perdurantist and sequentialist solutions to the puzzles of change.

In the end we maintained that our semantics for quantified modal logic are sound formalizations of the three ontologies for physical objects.

In chapter 5 we made use of the argument in chapter 4 to compare three-, four-dimensionalism and sequentialism. By theorem 5.19 and the *correspondence thesis* we affirm that, whenever sequentialists accept:

(i) a leibnizian universe, where the accessibility and counterpart relation are equivalence relations;

(ii) if individual a in the present world w has a counterpart b in a world w' accessible from w, then it has a counterpart c in every world w'' accessible from w, and a is the only counterpart to itself in w;

then sequentialists and perdurantists can agree, by interpreting their words through translation function $*$ in [27]. In section 5.3 we presented Kracht and Kutz's translation function $+$, for which we do not need the λ-abstractor in [27]. By our *correspondence thesis* and theorem 5.24, this time we maintain that if a sequentialist accepts (i) and:

(iii) a possibilist point of view, by which in every possible world w set $d(w)$ of existing objects is equal to set $D(w)$ of possible individuals;

(iv) every individual a in the present world w has a counterpart b in each world w', accessible from w, and a is the only counterpart to itself in w;

then she can express her truths within a perdurantist framework. We compared also sequentialism and perdurantism to endurantism: by theorem 5.28 we deduce that if a sequentialist accepts in addition that:

(v) each individual is at most counterpart to one individual;

(vi) each individual has at most one counterpart;

then sound statements according to her can be expressed not only within a perdurantist framework, but also in endurantist terms.

In conclusion, we investigated three semantic approaches to quantified modal logic, proved many completeness results and reviewed the pros and cons of each account. In addition we applied these formal frameworks to the debate on persistence conditions of material objects, and provided some results of this application. Once more we stress that the present work is not intended to give reasons to prefer one ontological theory rather than another. We just aimed at clearly stating the relationship among sequentialism, perdurantism and enduratism, by highlighting their degree of generality and the constraints by which they are reducible one to another.

Appendix A

Kripke-incompleteness with the Barcan formula

In this appendix we prove the incompleteness results we mentioned in section 1.2, concerning QML calculi based on free logic and containing the Barcan formula. Specifically, in section A.1 we prove Kripke-incompleteness for calculus $Q^E.K+BF$. This proof is inspired to a similar result first appeared in [20], and makes use of techniques and lemmas from [21], to which we refer for the details. In section A.2 we shall see how to extend this incompleteness result to systems on modal bases stronger than K.

A.1 Kripke-incompleteness of $Q^E.K + BF$

We begin by showing that calculus $Q^E.K+BF$ is incomplete w.r.t. Kripke semantics, that is, there is no class of Kripke frames which validates all and only the theorems of $Q^E.K + BF$. In particular we prove that the necessity of non-existence:

A14. $\neg E(x_1) \rightarrow \Box\neg E(x_1)$

holds in any K-frame for $Q^E.K + BF$, but A14 is not a theorem in $Q^E.K + BF$. We immediately state the main result in this appendix.

Theorem A.1 *Calculus $Q^E.K + BF$ is Kripke-incomplete, that is, every K-frame for $Q^E.K + BF$ validates A14, but $Q^E.K + BF \nvdash \neg E(x_1) \rightarrow \Box\neg E(x_1)$.*

Proof. We first show that every K-frame for $Q^E.K + BF$ validates postulate A14. Let $\mathcal{F}$ be a K-frame for $Q^E.K+BF$, this means that $\mathcal{F}$ satisfies the decreasing inner domain condition: if wRw' then $d(w') \subseteq d(w)$. Suppose that $(I^\sigma, w) \models \neg E(x_1)$, that is $\sigma(x_1) \notin d(w)$. For every w', wRw' implies that $\sigma(x_1) \notin d(w')$ by the decreasing inner domain condition, thus $(I^\sigma, w') \models \neg E(x_1)$ and $(I^\sigma, w) \models \Box\neg E(x_1)$.

In order to show that calculus $Q^E.K + BF$ does not prove the necessity of non-existence, we need two lemmas. By the first one if formula ϕ in language $\mathcal{L}^E$ is a theorem in $Q^E.K + BF$, then translation $\tau_n(\phi)$ of ϕ - as defined in chapter 5 -

holds in a suitable c-frame $\mathcal{F}$. By the second lemma this suitable c-frame $\mathcal{F}$ does not validate formula $\neg E(x_1) \to \Box\neg E(x_1)$, i.e. the translation of A14 for $n = 1$. By contraposition we obtain that $Q^E.K + BF \nvdash \neg E(x_1) \to \Box\neg E(x_1)$.

We start with the proof of the second lemma as it is immediate.

Theorem A.2 *There exists an everywhere-defined, surjective c-frame $\mathcal{F}$ s.t. not $\mathcal{F} \models \neg E(x_1) \to \Box\neg E(x_1)$.*

Proof. Consider the following c-frame $\mathcal{F}$:

- $W = \{w, w'\}$;
- $R = \{\langle w, w'\rangle\}$;
- $D(w) = \{a, a'\}$, $D(w') = \{b\}$;
- $d(w) = \{a\}$, $d(w') = \{b\}$;
- $C_{w,w'} = \{\langle a, b\rangle, \langle a', b\rangle\}$.

We easily check that $\mathcal{F}$ is everywhere-defined, as individuals a, a' in $D(w)$ have counterpart b in $D(w')$, and surjective, as individual b in $d(w')$ is counterpart to a in $d(w)$. But postulate A14 fails in $\mathcal{F}$, as it is not fictionally faithful. Consider 1-assignment a' by which $(a', w) \models \neg E(x_1)$; it is the case that $C_{w,w'}(a', b)$ and $b \in d(w')$, thus $(a', w) \models \Diamond E(x_1)$ and not $\mathcal{F} \models \neg E(x_1) \to \Box\neg E(x_1)$.

Now we prove the implication from theoremhood in $Q^E.K + BF$ and validity in everywhere defined, surjective c-frames, *modulo* translation function τ_n. In order to obtain this result we need lemma 4.3 in [21], that we state without proving.

Lemma A.3 *If ϕ is a formula in $\mathcal{L}^E$, $\mathcal{F}$ is an everywhere-defined c-frame and $x_{i_1}, \ldots, x_{i_m}$ are free for $x_1, \ldots, x_m$ in ϕ, then*

$$\mathcal{F} \models \tau_m(\phi)[x_{i_1}, \ldots, x_{i_m}] \to \tau_n(\phi[x_{i_1}, \ldots, x_{i_m}]).$$

By means of lemma A.3 we prove our second partial result, the proof of which closely follows lemma 4.4 in [21].

Theorem A.4 *Let $\phi \in \mathcal{L}^E$, $n = max(1, g(\phi))$ and $\mathcal{F}$ an everywhere-defined, surjective c-frame, then*

$$Q^E.K + BF \vdash \phi \quad \textit{implies} \quad \mathcal{F} \models \tau_n(\phi)$$

Proof. The proof is by induction on the length of the proof of ϕ. We consider only postulates A7 and R4, for the other cases we refer to [21].

As regards A7, we have to prove that $\mathcal{F}$ validates $\tau_n(\forall x_i\psi \to (E(x_j) \to \psi[x_i/x_j]))$, that is

$$\mathcal{F} \models \forall x_{n+1}\tau_n(\psi)[x_i/x_{n+1}] \to (E(x_j) \to \tau_n(\psi[x_i/x_j]))$$

By lemma A.3, this is the case IF

$$\mathcal{F} \models \forall x_{n+1}\tau_n(\psi)[x_i/x_{n+1}] \to (E(x_j) \to \tau_n(\psi)[x_i/x_j])$$

which is equivalent to

$$\mathcal{F} \models \forall x_{n+1}\tau_n(\psi)[x_i/x_{n+1}] \to (E(x_j) \to \tau_n(\psi)[x_i/x_{n+1}][x_{n+1}/x_j])$$

Define $\theta = \tau_n(\psi)[x_i/x_{n+1}]$, then we have to prove that

$$\mathcal{F} \models \forall x_{n+1}\theta \to (E(x_j) \to \theta[x_{n+1}/x_j]) \tag{A.1}$$

We know that axiom A7 is valid in $\mathcal{F}$, thus

$$\mathcal{F} \models \forall x_{n+1}\theta[x_1, \ldots, x_n] \to (E(x_{n+1}) \to \theta)$$

by substitution we obtain

$$\mathcal{F} \models \forall x_{n+1}\theta[x_1, \ldots, x_n][x_{n+1}/x_j] \to (E(x_{n+1})[x_{n+1}/x_j] \to \theta[x_{n+1}/x_j])$$

that is

$$\forall x_{n+1}\theta \to (E(x_j) \to \theta[x_{n+1}/x_j])$$

which is formula A.1.

As to R4, suppose that $\mathcal{F}$ validates $\tau_n(\psi \to (E(x_j) \to \phi))$, where x_j does not appear in ψ, by induction hypothesis this means that

$$\mathcal{F} \models \tau_n(\psi) \to (E(x_j) \to \tau_n(\phi))$$

and by substitution

$$\mathcal{F} \models \tau_n(\psi)[x_j/x_{n+1}] \to (E(x_j)[x_j/x_{n+1}] \to \tau_n(\phi)[x_j/x_{n+1}])$$

Since x_j does not appear in ψ

$$\mathcal{F} \models \tau_n(\psi)[x_1, \ldots, x_n] \to (E(x_{n+1}) \to \tau_n(\phi)[x_j/x_{n+1}])$$

we apply R4 and obtain

$$\mathcal{F} \models \tau_n(\psi) \to \forall x_{n+1}(\tau_n(\phi)[x_j/x_{n+1}])$$

Therefore $\mathcal{F}$ validates $\tau_n(\psi \to \forall x_j\phi)$.

By lemma A.2, it is not the case that our everywhere-defined, surjective c-frame $\mathcal{F}$ validates $\tau_1(\neg E(x_1) \to \Box\neg E(x_1))$. By lemma A.4 calculus $Q^E.K + BF$ does not prove A14, which is nonetheless valid in K-frames for $Q^E.K + BF$. We conclude the proof of theorem A.1 by stating the Kripke-incompleteness of $Q^E.K + BF$.

A.2 Further incompleteness results

In this paragraph we extend the incompleteness proof for system $Q^E.K + BF$ to some other QML calculi based on free logic, containing BF. First of all notice that even calculus $Q^E.K + CBF + BF$ is Kripke-incomplete, as the necessity of non-existence trivially holds in K-frames for $Q^E.K + CBF + BF$, but postulate A14 is not a theorem in $Q^E.K + CBF + BF$ either. In order to prove the latter fact we remark that lemmas A.2 and A.4 hold even if $\mathcal{F}$ is an everywhere-defined, surjective and existentially faithful c-frame.

Moreover if we strengthen our modal base to T or $S4$, Kripke-incompleteness holds as well, as lemmas A.2 and A.4 are provable even for reflexive and transitive c-frames.

Finally, in par. 1.3.2 we showed that calculi $Q^E.B + CBF + BF$ and $Q^E.S5 + CBF + BF$ are complete w.r.t. Kripke semantics, in particular they both prove A14. On the contrary systems $Q^E.B + BF$ and $Q^E.S5 + BF$ are Kripke-incomplete. For instance consider the case for $Q^E.S5 + BF$. Define counterpart frame $\mathcal{F}$ as follows:

- $W = \{w, w'\}$;
- $R = W^2$;
- $D(w) = \{a, a'\}$, $D(w') = \{b\}$;
- $d(w) = \{a\}$, $d(w') = \{b\}$;
- $C_{w,w'} = D(w) \times D(w')$, $C_{w',w} = D(w') \times D(w)$, $C_{w,w} = D(w)^2$ and $C_{w',w'} = D(w')^2$.

By definition c-frame $\mathcal{F}$ is reflexive, transitive and symmetric, and it is easy to check that $\mathcal{F}$ is everywhere-defined and surjective, so it is a c-frame for $Q^E.S5 + BF$. But the translation of postulate A14 fails in $\mathcal{F}$, by the same reason it failed in lemma A.2. Therefore by lemma A.4 calculus $Q^E.S5 + BF$ is Kripke-incomplete.

We conclude the present appendix by remarking that Kripke semantics is particularly unsatisfactory with respect to QML calculi based on free logic and containing BF: systems $Q^E.K + BF$ and $Q^E.K + CBF + BF$ are Kripke-incomplete, and the same holds for modal bases T and $S4$. As to modalities B and $S5$, we have incompleteness results for $Q^E.B + BF$ and $Q^E.S5 + BF$.

Appendix B

Completeness of T_c-theories

In this appendix we prove theorem 2.25 in the case that first-order typed theory T_c is based on free logic, that is, we show that whenever Δ is a T_c-consistent n-type in first-order typed language $\mathcal{L}_c$, there exists a T_c-model $\mathcal{M}$ realizing Δ.

First of all we recall the definitions in chapter 2. A formula $\phi : n \in \mathcal{L}_c$ is *derivable* in T_c from set Δ of typed formulas in $\mathcal{L}_c$ - $\Delta \vdash_{T_c} \phi$ in short - iff there are $\phi_1, \ldots, \phi_n \in \Delta$ and substitutions $[\vec{x}_1], \ldots, [\vec{x}_n], [\vec{x}]$ s.t. $\vdash_{T_c} \phi_1[\vec{x}_1] \wedge \ldots \wedge \phi_n[\vec{x}_n] \rightarrow \phi[\vec{x}]$.

Now let Λ be a set of typed formulas in language $\mathcal{L}_c$,

Λ is T_c-*consistent*	iff	$\Lambda \nvdash_{T_c} \perp$;
Λ is T_c-*complete*	iff	for every formula $\phi \in \mathcal{L}_c$, $\phi \in \Lambda$ or $\neg\phi \in \Lambda$;
Λ is T_c-*maximal*	iff	Λ is T_c-consistent and T_c-complete.

Notice that if n-type Δ is T_c-consistent as a set, then it is also T_c-consistent as a type, that is, no finite conjunction of formulas in Δ is refutable in T_c.

Furthermore let Y be a set of variables in $\mathcal{L}_c$,

Λ is Y-*rich*	iff	for $\phi \in \mathcal{L}_c$, if $\exists x_{n+1}\phi \in \Lambda$ then there is $y \in Y$ s.t. $\phi[\vec{x}, y] \in \Lambda$;
Λ is T_c-*saturated*	iff	Λ is T_c-maximal and Y-rich for some $Y \subseteq Var(\mathcal{L}_c)$.

In what follows $\Delta[\vec{x}]$, for n-type $\Delta = \{\psi_1, \psi_2, \ldots\}$, is m-type $\{\psi_1[\vec{x}], \psi_2[\vec{x}], \ldots\}$.

Lemma B.1 *If Δ is a T_c-consistent n-type in $\mathcal{L}_c$, then there exists a T_c-consistent, Y-rich set $\Gamma \supseteq \Delta$ of formulas in $\mathcal{L}_c$, for $Y \subseteq Var(\mathcal{L}_c)$.*

Proof. Assume that there is an enumeration of existential formulas in $\mathcal{L}_c$, then define by recursion a chain of types in $\mathcal{L}_c$ s.t.

$$\Gamma_0 = \Delta$$

$$\Gamma_{k+1} = \begin{cases} \Gamma_k[x_1, \ldots, x_m] \cup \{\theta_k[x_1, \ldots, x_j, x_{m+1}] \wedge E(x_{m+1})\} & \text{if } \Gamma_k \cup \{\exists x_{j+1}\theta_k[x_1, \ldots, x_j]\} \\ & \text{is } T_c\text{-consistent;} \\ \Gamma_k & \text{otherwise.} \end{cases}$$

Without any loss of generality we assume that j is minor or equal to type m of formulas in Γ_k. In fact we can first consider existential formulas of type 0, then those of type 1, … . Moreover for each $n \in \mathbb{N}$, $\exists x_{n+1}\top : n$ is a formula in $\mathcal{L}_c$; thus the type is bound to increase.

Type $\Gamma_0 = \Delta$ is T_c-consistent. If $\Gamma_{k+1} = \Gamma_k[x_1, \ldots, x_m] \cup \{\theta_k[x_1, \ldots, x_j, x_{m+1}] \wedge E(x_{m+1})\}$ were not T_c-consistent, then there would be $\psi_1, \ldots \psi_p \in \Gamma_k$ s.t.

$$\vdash_{T_c} \bigwedge \psi_l[x_1, \ldots, x_m] \rightarrow (E(x_{m+1}) \rightarrow \neg\theta_k[x_1, \ldots, x_j, x_{m+1}])$$

and by R4

$$\vdash_{T_c} \bigwedge \psi_l \rightarrow \forall x_{m+1}(\neg\theta_k[x_1, \ldots, x_j, x_{m+1}])$$

that is

$$\vdash_{T_c} \bigwedge \psi_l \rightarrow \forall x_{j+1}\neg\theta_k[x_1, \ldots, x_j]$$

Therefore $\Gamma_k \cup \{\exists x_{j+1}\theta_k[x_1, \ldots, x_j]\}$ would not be T_c-consistent against hypothesis.

Now we show that set $\Gamma = \bigcup_{k\in\mathbb{N}} \Gamma_k$ is T_c-consistent. If it were not the case, there would be $\psi_1 \in \Gamma_{k_1}$, …, $\psi_h \in \Gamma_{k_h}$ s.t.

$$\vdash_{T_c} \psi_1[\vec{x}_1] \wedge \ldots \wedge \psi_h[\vec{x}_h] \rightarrow \bot \tag{B.1}$$

Suppose that B.1 has type m, then $\psi_1[\vec{x}_1], \ldots, \psi_h[\vec{x}_h] \in \Gamma_k$ for some $k \geq k_h + (m-q)$, where q is the type of ψ_h, which is assumed to be the greatest. Therefore Γ_k would not be T_c-consistent against hypothesis. Moreover Γ is Y-rich for $Y = \{y | E(y) \in \Gamma\}$.

Lemma B.2 *If Γ is a T_c-consistent set of formulas in $\mathcal{L}_c$, then there exists a T_c-maximal set $\Pi \supseteq \Gamma$ of formulas in $\mathcal{L}_c$.*

Proof. Assume that there is an enumeration of formulas in $\mathcal{L}_c$, then define by recursion a chain of sets in $\mathcal{L}_c$ s.t.

$$\begin{array}{lcl}
\Pi_0 & = & \Gamma \\
\Pi_{k+1} & = & \begin{cases} \Pi_k \cup \{\theta_k\} & \text{if } \Pi_k \cup \{\theta_k\} \text{ is } T_c\text{-consistent;} \\ \Pi_k \cup \{\neg\theta_k\} & \text{otherwise.} \end{cases}
\end{array}$$

Set $\Pi_0 = \Gamma$ is T_c-consistent by hypothesis. On the other hand, whenever $\Pi_k \cup \{\theta_k\}$ is not T_c-consistent, there exist $\psi_1, \ldots, \psi_h \in \Pi_k$ s.t.

$$\vdash_{T_c} \psi_1[\vec{x}_{\psi_1}] \wedge \ldots \wedge \psi_h[\vec{x}_{\psi_h}] \rightarrow \neg\theta_k[\vec{x}_1] \tag{B.2}$$

If $\Pi_k \cup \{\neg\theta_k\}$ were not T_c-consistent either, then there would be $\phi_1, \ldots, \phi_j \in \Pi_k$ s.t.

$$\vdash_{T_c} \phi_1[\vec{x}_{\phi_1}] \wedge \ldots \wedge \phi_j[\vec{x}_{\phi_j}] \rightarrow \theta_k[\vec{x}_2] \tag{B.3}$$

Suppose that B.2 has type m while B.3 : n, and $m \geq n$. This means that there are substitutions $[\vec{x}_1], \ldots, [\vec{x}_j], [\vec{x}_3]$ s.t.

$$\vdash_{T_c} (\psi_1[\vec{x}_{\psi_1}] \wedge \ldots \wedge \psi_h[\vec{x}_{\psi_h}] \wedge \phi_1[\vec{x}_{\phi_1}][\vec{x}_1] \wedge \ldots \wedge \phi_j[\vec{x}_{\phi_j}][\vec{x}_j]) \rightarrow (\neg\theta_k[\vec{x}_1] \wedge \theta_k[\vec{x}_2][\vec{x}_3]) \tag{B.4}$$

By B.4 set Π_k is not T_c-consistent against hypothesis.

Moreover $\Pi = \bigcup_{k \in \mathbb{N}} \Pi_k$ is T_c-consistent and T_c-complete by construction.

We supposed that n-type Δ is T_c-consistent, by lemma B.1 there exists a T_c-consistent set $\Gamma \supseteq \Delta$, which is Y-rich for $Y = \{y|E(y) \in \Gamma\}$. By applying lemma B.2 on Γ, we obtain a T_c-maximal set $\Pi \supseteq \Gamma$, which is Y-rich for $Y = \{y|E(y) \in \Pi\}$ and $\Pi \supseteq \Delta$.

Finally we define a model for language $\mathcal{L}_c$ on Π as follows:

- $D_\Pi = Var(\mathcal{L}_c)$;
- $d_\Pi = \{y|E(y) \in \Pi\}$;
- individuals $t_1, \ldots, t_m$ belongs to $I_\Pi(P^m)$ iff $P^m(t_1, \ldots, t_m) \in \Pi$.

At this point we prove the following lemma, from which completeness immediately follows.

Lemma B.3 *For every $\phi : n$ in $\mathcal{L}_c$,*

$$\Pi \models_{\langle x_1, \ldots, x_n \rangle} \phi \quad iff \quad \phi \in \Pi$$

Proof. The proof is by induction on the length of $\phi \in \mathcal{L}_c$.

As to the base of induction, consider atomic formula $P^m(t_1, \ldots, t_m)$. By definition of satisfaction $\Pi \models_{\langle x_1, \ldots, x_n \rangle} P^m(t_1, \ldots, t_m)$ iff $\langle \vec{x}(t_1), \ldots, \vec{x}(t_m) \rangle \in I_\Pi(P^m)$ iff $\langle t_1, \ldots, t_m \rangle \in I_\Pi(P^m)$. According to the definition of interpretation, $\langle t_1, \ldots, t_m \rangle \in I_\Pi(P^m)$ iff $P^m(t_1, \ldots, t_m) \in \Pi$.

As to the inductive step, we separately consider each connective and the universal quantifier.

If ϕ has form $\neg\psi$, then $\Pi \models_{\langle x_1, \ldots, x_n \rangle} \neg\psi$ iff not $\Pi \models_{\langle x_1, \ldots, x_n \rangle} \psi$ iff by induction hypothesis $\psi \notin \Pi$. Since Π is T_c-maximal, this is the case iff $\neg\psi \in \Pi$.

If ϕ has form $\psi \to \psi'$, then $\Pi \models_{\langle x_1, \ldots, x_n \rangle} \psi \to \psi'$ iff not $\Pi \models_{\langle x_1, \ldots, x_n \rangle} \psi$ or $\Pi \models_{\langle x_1, \ldots, x_n \rangle} \psi'$. By induction hypothesis it tantamounts to $\psi \notin \Pi$ or $\psi' \in \Pi$; in both cases we have $\psi \to \psi' \in \Pi$ because Π is T_c-maximal.

Suppose that ϕ has form $\forall x_{n+1}\psi$. $\Leftarrow$ Assume that $\forall x_{n+1}\psi \in \Pi$ and y is an individual in d_Π. Since Π is T_c-maximal, $\forall x_{n+1}\psi[\vec{x}, y] \to (E(y) \to \psi[\vec{x}, y]) : m \in \Pi$; but $\forall x_{n+1}\psi[\vec{x}, y]$, $E(y)$ belong to Π by hypothesis, hence $\psi[\vec{x}, y] : m \in \Pi$. By induction hypothesis $\Pi \models_{\langle x_1, \ldots, x_m \rangle} \psi[\vec{x}, y]$, that is, $\Pi \models_{\langle x_1, \ldots, x_n, y \rangle} \psi$ for each $y \in d_\Pi$. Therefore $\Pi \models_{\langle x_1, \ldots, x_n \rangle} \forall x_{n+1}\psi$.

$\Rightarrow$ Assume that $\forall x_{n+1}\psi \notin w$. Since Π is L-maximal $\exists x_{n+1}\neg\psi \in w$ and Π is d_Π-rich, then there exists $y \in d_\Pi$ s.t. $\neg\psi[\vec{x}, y] : m \in \Pi$. By induction hypothesis not $\Pi \models_{\langle x_1, \ldots, x_m \rangle} \psi[\vec{x}, y]$, and by the conversion lemma there exists $y \in d_\Pi$ s.t. not $\Pi \models_{\langle x_1, \ldots, x_n, y \rangle} \psi$, i.e. not $\Pi \models_{\langle x_1, \ldots, x_n \rangle} \forall x_{n+1}\psi$.

By lemma B.3 we have that Π is a T_c-model; moreover $\Pi \models_{\langle x_1, \ldots, x_n \rangle} \phi$ for every $\phi \in \Delta \subseteq \Pi$. Thus, whenever an n-type Δ is T_c-consistent, there exists a T_c-model $\mathcal{M}$ realizing Δ.

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