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Multi-agent Modal Logics for Knowledge Representation

Theoretical Aspects and Applications to Reasoning about Knowledge and Change

par

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"Success is going from failure to failure without losing enthusiasm."

Sir Winston Churchill

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Abstract

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In this work we review and analyse some recent advances on the application of multi-agent modal logics in the representation of individual and group knowledge, as well as temporal and spatial reasoning. The main theoretical contributions can be summarized as follows. In Chapter 3 we introduce the language of (multi-modal) second-order propositional modal logic (SOPML), an extension of standand modal logic with propositional quantification. We show how SOPML can be usefully adopted as a specification language to express local properties of Kripke frames in modal logic, including higher-order knowledge of agents in epistemic contexts, that is, knowledge agents have about other agents' knowledge. Further, we prove novel axiomatisation results with respect to several classes of Kripke frames. In particular, we highlight the key role played by common knowledge in obtaining a complete axiomatisation for second-order propositional epistemic logic (SOPEL), the epistemic version of SOPML. In Chapter 4 we introduce original notions of (bi)simulation and prove that they preserve the interpretation of formulas in SOPML. We also define (bi)simulation games and show them as powerful as (bi)simulation relations. Then, we apply this formal machinery to assess the expressive power of SOPML in representing spatial and temporal properties. In Chapter 5 we move to a dynamic setting and introduce second-order public announcement logic (SOPAL) by extending SOPEL with announcement operators. We make use of SOPAL to analyse the notions of knowability, successfulness of announcements, and preservation under arbitrary announcements. Interestingly, SOPAL is proved to be as expressive as SOPEL, but exponentially more succint. Finally, in Chapter 6, we return to a purely propositional setting and extend public announcement logic (PAL) with operators for both global and local announcements. We illustrate the formal machinery by means of scenarios in multi-agent systems and prove that this logic is stricly more expressive than PAL. The final outcome of these investigations is a family of expressive modal languages, suitable to reason about relevant concepts in knowledge representation, while still enjoying nice computational properties.

Acknowledgements

[...] And if you find her poor, Ithaka won't have fooled you.
Wise as you will have become, so full of experience,
you will have understood by then what these Ithakas mean.

Constantinos Kavafis

Thanks to all those who are sharing my journey to Ithaka.

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Dedicated to the women of my life
Domenica, Erica & Flora

Chapter 1

Introduction

In this chapter we review our main scientific contributions since obtaining the PhD in 2006. These are situated at the intersection of three relevant research areas:

1. Multi-agent systems (MAS) and logics for strategic abilities;
2. Data-aware systems, including auctioning mechanisms;
3. Formal methods and verification by model checking.

In order to position our contributions with respect to the state-of-the-art and to illustrate the comparative advancement, here we discuss significant, recent works in these three areas. Given the vastity of the subject matter, the discussion will necessarily be partial and oriented towards our research interests in the verification of multi-agent systems by means of agent-based techniques.

1.1 Logics for Multi-agent Systems

Multi-agent systems are open, distributed systems where the processes involved, or *agents*, show highly flexible and autonomous behaviour [127]. Agents in MAS are assumed to be proactive, endowed with beliefs about the surrounding environment, as well as their own private goals and plans to achieve them [101]. Researchers in artificial intelligence have adopted multi-agent systems to model and solve problems in several areas – including economics, game theory, planning, and robotics – that are difficult, viz. impossible, for an individual agent or a monolithic system to tackle [105]. Most importantly, the agent paradigm allows for a modular approach to system modelling, in which the interactions between agents are not hard-coded in the systems description, but emerge at run-time according to the agents' specification. Moreover, the description of agents in terms of intentional attitudes (e.g., beliefs, intentions, goals) allows us to abstract from actual implementation details and accounts for a high level of flexibility in the system description. The problems and techniques in the literature on MAS have much in common with distributed systems and software engineering, but contrary to these disciplines, here the emphasis is on the local, private information state of agents, as well as concepts such as individual and collective knowledge and belief. These features of MAS have been deemed extremely valuable in designing complex distributed applications, at least in the modelling stage.

Closely related to the verification of multi-agent stystems, logic-based formalisms for representing and reasoning about strategic abilities, both individual and coalitional, have been a thriving area of research in artificial intelligence and multi-agent system [51, 52, 74]. A diverse family of multi-modal logics has been introduced to provide a formal account of complex strategic reasoning and behaviours for individual agents and groups, including alternating-time temporal logic (ATL) [4], strategy logic [53], coalition logic [98], just to name a few. In parallel with these developments, a well-established tradition in knowledge representation focuses on extending formalisms for reactive systems with epistemic operators, so as to reason about the systems' evolution, as well as the knowledge agents have thereof [64]. Seminal contributions on extensions of linear- and branching-time temporal logics with agent-indexed epistemic modalities date back to the '80s [79, 80]. Since then, these investigations have matured into a solid body of works, which is nowadays rightly regarded as a key contribution of formal methods to computer science [95], particularly when combined with verification techniques [70, 89, 91].

In this broad research area our contributions have been aimed at

(i) analysing individual and group knowledge in multi-agent systems, with a specific focus on contexts of imperfect information;

(ii) specifying properties of data-aware and data-intensive MAS, where processes and data are seen as two equally relevant tenets of the system specification.

To develop these points I benefited from an IEF Marie Curie fellowship related to the FoMMAS project: First-order Modal Logics for the Specification and Verification of Multi-Agent Systems. Hereafter we review the key findings of the FoMMAS project, including later developments.

1.1.1 Temporal and Epistemic Logics for Multi-agent Systems

In this line we investigated expressive epistemic and temporal-epistemic multi-agent logics for MAS specification, proving both completeness [26–28, 31] and verification results [23–25]. Also related to general knowledge representation, a crucial distinction operated in game-theoretic contexts, including MAS, is whether players have perfect/imperfect information about the environment they are interacting in and with. Originally, most of the logics for strategies mentioned above have been introduced in contexts of perfect information, partly because this setting exhibits better computational properties. However, for many applications of interest, including autonomous agents, distributed computing, and economic theory, perfect information is either unrealistic as a working hypothesis or unattainable [63], and has to be dropped. Unfortunately, imperfect information is known to make the verification task computationally more costly. As an example, while we can model check ATL with perfect information in polynomial time, the corresponding problem for imperfect information is Δ_2^P-complete [88]. When perfect recall is included in the picture, the problem goes from being PTIME-complete to undecidable [62]. Thus, for logics of strategies to be adopted as specification languages in contexts of imperfect information it is key to develop efficient verification tools and techniques, capable of tackling model checking for practical cases of interest, possibly by means of appropriate approximations whenever the original problem is untractable or undecidable.

Our contribution on this subject consists of rich combinations of strategy and epistemic logic for MAS specifications [15, 16, 18]. Specifically, in [15] we introduced Epistemic Strategy Logic, an extension of strategy logic [96] with modal operators for individual and group knowledge, and we showed that the complexity of the model checking problem is no worse than in the purely strategic case. So, the extra expressivity for knowledge representation and reasoning comes at no computational cost, at least as long as the verification task is concerned. In [16] we provided a tableau-based decision procedure for ATEL, an epistemic extension of ATL, by building on previous work in [75, 76]. Finally, in [18] we proposed a context-sensitive semantics for ATL, in which epistemic alternatives are restricted according to the strategy currently used. This modelling choice allows us to recover some important validities that do not normally hold under imperfect information.

Directly related to the present work, we have recently started to investigate propositional quantification in epistemic logic. In [43] we introduced second-order propositional epistemic logic (SOPEL), where propositional quantification is used to express comparaisons between agents' knowledge such as "agent a knows at least as much as agent b".

The issues pertaining to the formalisation of such properties have already been considered in the literature (e.g., comparative epistemic logic [119]), but within a different formal account [120, 121]. The completeness results for SOPEL appearing in [43] are extended in Chapter 3 to general second-order propositional modal logic (SOPML). Further, in [44] we introduced (bi)simulation relations for SOPML, as well as (bi)simulation games. Then, we showed that the two notions, while preserving both the truth of formulas in SOPML, are not equivalent in general. In Chapter 4 we put forward a novel notion of (bi)simulation that is provably equivalent to (bi)simulation games. Finally, in [45] we defined an extension of public announcement logic (PAL) with propositional quantification, in order to express relevant epistemic concepts, including knowability, preservation under arbitrary announcements, and successfulness. These results appear as Chapter 5 hereafter.

1.1.2 First-order Extensions of MAS Logics

If agent-based logics are to be applied to the specification of multi-agent systems where the role played by data is key, these need to be extended with relational and first-order features to account for the data part. As an example, in auctions the behaviour of agents has to be checked against all admissible values for bids, asking prices, and true values, thus calling for (universal) quantification in the specification language for such properties. However, it is well-known that assuming naively unrestricted first-order quantification quickly leads to the undecidability of a number of problems, including satisfiability and model checking. Hence, more sophisticated methodologies have to be adopted to lift logics for agents to the first order. In fact, first-order logic includes some interesting fragments with nice computational properties (e.g., the monadic, guarded, and two-variable fragments [49]), which can be used to express specific behaviours of data-aware systems (DaS). For instance, quantification in DaS can be guarded by assuming that values range on appropriate subsets, suitably specified by predicates in the language. Also, whenever we want to compare two values that appear at different times of the system's execution, two variables are sufficient. Within the FoMMAS project we contributed to these investigations, by proving that some sound and complete axiomatisations for multi-agent temporal-epistemic logics can be lifted to the monodic fragment of first-order logic, i.e., a controlled form of quantification [29, 30, 32–34]. These investigations were further pursued with the EU STREP Project ACSI [1].

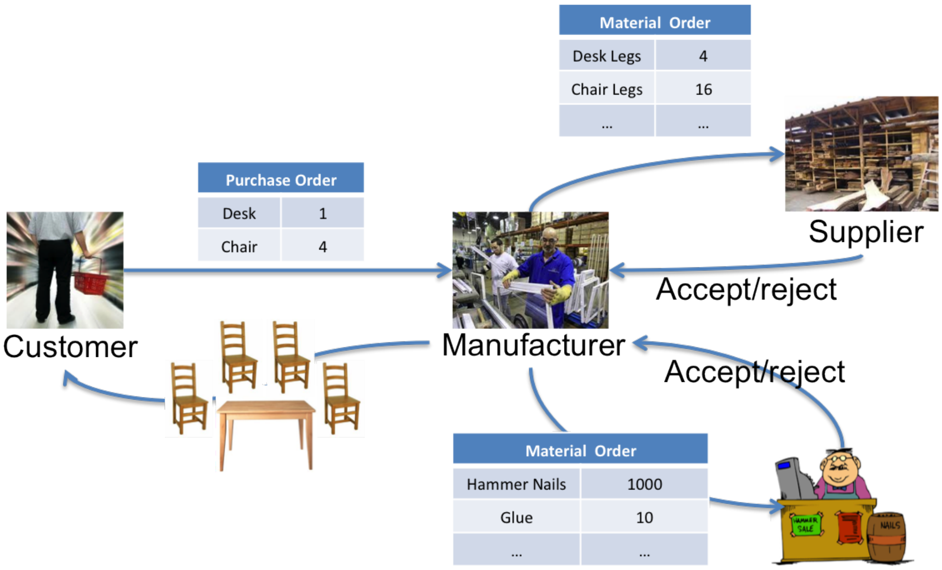


FIGURE 1.1: The order-to-cash scenario.

1.2 Data-aware Systems

Data-aware systems (DaS) are a novel paradigm for the design, implementation and integration of business processes in service-oriented computing [106]. The originality of this approach consists in "combin[ing] data and processes in a holistic manner as the basic building block[s]" of the system's description [56]. Typically data-aware systems include a *data model*, to account for the relational structure of data, as well as the *business processes* manipulating data. Both the data model and business processes are seen as equally important tenets of the system description. This setting is in marked contrast with most of the tradition on service architectures and composition, which usually abstracts data away to reduce the complexity of the system description and thus making the verification task amenable to standard model checking techniques [106]. As an example of DaS, here we briefly describe a business process inspired by a concrete IBM use case [87], illustrated in Fig. 1.1.

The *order-to-cash* scenario details the interactions of manufacturers, customers, and suppliers in an e-commerce situation involving the purchase and delivery of goods and services. At the start of the business process, a customer prepares and submits to some manufacturer a *purchase order* (PO), i.e., a list of products the customer requires, together with information about these products such as quantity, price, expected-by date, etc. Upon receiving a PO, the manufacturer prepares a *material order* (MO), i.e., a list of components needed to assemble the requested products, based on the information provided by the customer herself. The manufacturer then selects some suppliers and forwards them the appropriate material orders. Upon receiving an MO, a supplier evaluates the information provided therein and either accepts or rejects the order. In the former case she then proceeds to deliver the requested components to the manufacturer, according to the relevant specs. In the latter she notifies the manufacturer of her rejection. Finally, when the manufacturer receives the components, she assembles the product and, provided that the order has been paid for, she delivers it to the customer.

Observe that, even in such a plain scenario, all key components of data-aware systems are clearly represented. The data model includes the purchase and material orders, which can be encoded as some kind of data structure, typically a relation database; while the business processes detailing the evolution of orders from creation, through validation/rejection, to fulfilment, can be described by an appropriate set of operations on relational structures. Most importantly, the system's execution depends crucially on the data content of purchase and material orders: the supplier might chose to accept or reject a material order depending on whether she has enough resources for the requested quantity, whether the price is within a certain range of profitability, or whether she can meet the deadline for delivery. Thus, the agents' available actions and behaviour essentially depend on the information registered in the data model.

Our contributions in this area showed that the specification and verification of data-aware systems can benefit hugely from the adoption of a multi-agent perspective. Indeed, in the other-to-cash scenario above, the clients, manufacturers, and suppliers all have their private information, that they might want to share only partially or in a controlled way. They also have different goals (e.g., profit maximisation, timeliness), and they might have various plans available to achieve them. Similarly, in auction-based mechanisms bidders normally keep their true values private, as well as their bids in sealed auctions, for instance. This agent-based perspective on DaS has been explored in our works [40, 42], while most of current approaches still regard DaS as monolithic systems [59–61, 82]. Moreover, the agent approach allows for the application of modular abstraction techniques to tackle the model checking problem [77].

A significant advancement in the verification of data-aware systems has come from the EU STREP project ACSI [1], to which I contributed in 2011-12. The ACSI project focused on artifact-centric systems, a specific data-driven approach to the modelling and deployment of business processes, for which we produced a stream of fundamental contributions [38, 39, 41, 42, 59, 73, 82]. Among these results, a key finding is represented by the notion of *uniformity*, which has been used in [40, 42] to obtain a decidable model checking problem. Intuitively, a data-aware system is uniform whenever its evolution is determined only by data that are named explicitly in the system's description. Conversely, all data that are not exhibited can be deemed equivalent, as far as the system's execution is concerned. This allows to apply abstraction-based techniques to reduce the model checking problem to the finite case, provided that some additional constraints are met. Interestingly, the uniformity condition, which is related to the notion of *genericity* in database theory [2], is satisfied by a vast class of interesting systems, including some types of auctions. After the completion of the ACSI project, we pursued further these investigations into the verification of DaS. In particular, we applied these results and methodology to open multi-agent systems, where agents are assumed to join and leave the system at run-time. In [21, 22] we were able to prove that, under specific conditions, the model checking problem for these open MAS is decidable.

Albeit their success, we identify several criticalities regarding the methods made available by the ACSI project, as well as the current literature on DaS in general.

1. Although uniformity defines an important class of DaS, many systems of interest are not uniform. Indeed, most manipulations of data bring us outside the realm of uniformity. Even simple operations, such as increments on natural numbers, are sufficient to break uniformity. Hence, a first challenge is to find conditions more robust than uniformity, which still imply a decidable model checking problem. We tackled these issues in [17, 19], where we showed that some weaker results are available also for non-uniform systems.

2. A further assumption normally required to obtain decidability, besides uniformity, is *boundedness*, that is, the existence of an upper bound on the number of active elements in a data-aware system at any time in the execution [59]. However, in several scenarios assuming the existence of such a bound may appear arbitrary and artificial: databases can grow beyond any given size, by simply keeping on adding new entries (without removing any of the old ones). Thus, a further challenge with respect to the state-of-the-art is to identify classes of models, still general enough for representing most DaS of significance, but which can also be bounded in a natural way.

3. Related to the previous point, a third challenge is represented by unbounded systems (such as the expanding databases above). Again, in this case the decidability results of the ACSI project do not apply unconditionally, so novel techniques need to be explored. In [17] we obtained preliminary results on the verification of unbounded systems.

To provide principled answers to the questions raised above, we recently obtained a *Jeunes Chercheuses/Jeunes Chercheurs* fellowship from the *Agence Nationale de la Recherche* for our project SVeDaS: Specification and Verification of Data-aware Systems. The SVeDaS project is solidly set within the most recent advances on the verification of data-aware systems. Yet, it is meant to question the constraints imposed on DaS, namely uniformity and boundedness, in order to develop novel verification methods suitable for a wider class of DaS. Moreover, the SVeDaS project focuses on the particular data-aware systems represented by auction-based mechanisms. Indeed, auctions can be seen as a specific class of DaS: their outcome depends essentially on the values of bids, base prices, and true values. As a proof of concept, in [14, 35] we formalised a basic version of parallel and iterated English (ascending bid) auctions as DaS, then we successfully verified them against safety and liveness properties. These results validate the approach proposed in the SVeDaS project. However, more elaborate use cases, including real-time bidding, in which agents can modify their behaviour according to the outcome of previous auctions, are not covered by ACSI, since they suffer from limitation (1) detailed above. To overcome such issues, proposals have been put forward [17] that also support arithmetic operations [61]. Yet, these contributions neglect the imperfect knowledge that agents typically have of the system's global state, which limits the applicability of similar results to auctions.

1.3 Formal Verification by Model Checking

Formal methods are widely used to represent and analyse distributed and reactive systems. In combination with verification techniques by model checking, they have become one of the success stories in computer science [8, 55]. In the model checking approach, to verify whether a system S satisfies a property P (such as a safety, liveness, or secrecy requirement), first S is modelled as (some kind of) transition system $\mathcal{M}_S$, while property P is recast as a formula ϕ_P in some logical language of choice. Finally, verification is reduced to check whether the formula ϕ_P is true in the model $\mathcal{M}_S$, or $\mathcal{M}_S \vDash \phi_P$ formally. This verification procedure in outlined in Fig. 1.2. Nowadays, model checking is being successfully applied to the automated verification of real-life scenarios in safety critical systems, avionics, AUVs, robotics, and security protocols [85, 91, 99].

Similarly, the actual deployment of data-aware systems calls for the development of verification techniques. As an example, in designing auction-based mechanisms we might require that *bidders for a particular resource bid consistently with the true value they assign to the resource (i.e., they do not exceed it), without revealing this true value publicly*. Such requirements specify the behaviour of agents with respect to a possible infinite number of values for their bids and true values. However, verification techniques such as model checking are "mainly appropriate to control-intensive applications and less suited for data-intensive applications" [8]. Irrespectively of these difficulties, the model checking problem for auctions has received considerable attention recently [7, 78, 126,

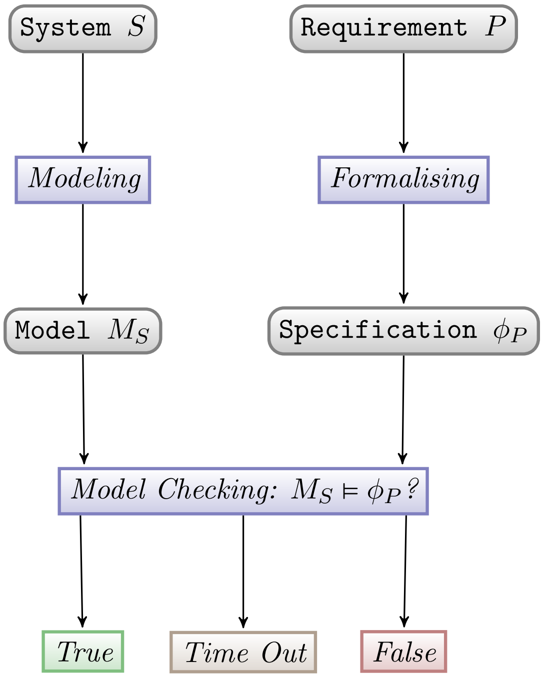


FIGURE 1.2: The model checking procedure in a nutshell.

128, 129]. Indeed, it is hard to overestimate the relevance of auctions and auction-based mechanisms in a wide range of distributed systems (e.g., task scheduling, power grid management, and resource allocation [57, 102]). However, with some notable exceptions, most of the research on this topic has focus on the design of auctioning mechanisms and the analysis of their formal properties, while the automated verification of these designs has only partially been addressed.

Our contributions to formal verification by model checking cover a wide range of methodologies and application domains. In [24, 25] we put forward an automata-theoretic technique for the verification of CTLK, an epistemic extension of the temporal logic CTL. This technique was implemented in an explicit-state model checker called ETAV – Epistemic Tree Automata Verifier. Finally, in [20] we developed an automated procedure for model checking quantum protocols, on top of the MCMAS model checker [92]. Further, in [36, 37] we introduced abtraction techniques for MAS verification, based on three-valued logics. While results along these lines are certainly of interest for model checking general multi-agent systems, we advocate a principled approch to the verification of data-aware systems that it is also capable of dealing with auction-based mechanisms. We reckon that, given the relevance that data representation and reasoning have gained in recent years, it is key for the deployment of business processes to provide data-aware systems with sound verification methodologies. In turn, this endeavour raises a number of challenges ranging from (i) the logic-based languages for specifying DaS behaviours, to (ii) the data structures to represent DaS symbolically, as well as (*iii*) efficient model checking algorithms to deal with relational and first-order features. We presented preliminaries results on model checking auctions in [35]. However, efficient verification algorithms are yet to be explored.

Chapter 2

Overview

The rest of the present work is devoted to one particular line of research among those reviewed in Chapter 1. Specifically, we develop and extend some of our contributions in the area of knowledge representation and reasoning [43–45]. This is meant to provide a coherent picture of some recent and interesting results, which still fit in the more general framework of modal logics for multi-agent systems.

Modal logics are nowadays a well-established area in mathematical logic, which has also become one of the most popular formal frameworks in artificial intelligence for knowledge representation and reasoning [47, 123]. This success is due to several reasons, including an expressive and flexible formal language, which enjoys nice computational properties [48, 109]. In particular, at the core of the semantics of modal logics lies the notion of *world*, or *state*. Indeed, this concept is very natural when analysing computational notions (a system evolving over time from a previous to a successive state), accounts of agency (states that are preferred, desired, or epistemically possible), and rational interactions (states that can be winning, losing, terminal, initial, etc.) Indeed, distributed computing [81], reactive systems [93], multi-agent systems [113], and game theory [112] have all benefited from the application of tools and techniques from modal logics, and this list is by no means exhaustive. Most importantly, the states in the models for modal logics are connected by means of indexed relations R_a, for some index a, which can model (program) transitions, epistemic or desired alternatives, or the effect of possible moves, where index a can assume a number of readings: a specific program, a dimension of time (say, future or past), an agent, a move, etc. Each accessibility relation R_a in the semantics is then paired with a necessity operator $\Box_a$ in the modal language, where a formula $\Box_a \varphi$ may be read as: after every execution of a, in each future time along dimension a, in every state considered possible or desired by agent a, or in every state that is the result of performing move a, formula φ holds.

The language of modal logics provides a crisp, variable-free way of expressing a variety of properties of interest. It is also important to realise that there is a multiplicity of modal logics: although the well-known normal axiomatisation **K** characterises the class $\mathcal{K}$ of validities on all models for modal logic, this does not mean that all logics for, say, agency, are the same and correspond to **K**. It only means that they are typically extensions of **K**. As a simple example, the scheme of formulas *(i)* $\Box_a\varphi \to \varphi$ appears reasonable when $\Box_a$ denotes 'agent a knows that ...', but perhaps it is less desirable when it is read as 'agent a believes that ...', as philosophically knowledge is analysed as truthful belief [83]. One of the reasons for the success of modal logics is that in many relevant cases a syntactic scheme corresponds to an additional constraint on the accessibility relation R_a: in the case of scheme *(i)*, reflexivity of R_a is, in a precise sense, a sufficient and necessary condition for its validity.

To appreciate this point, we use a little bit more detail (we assume some familiarity with modal logics, precise definitions are given in Section 3.1.) As already mentioned, central in the semantics of modal logic is the notion of (Kripke) frame $\mathcal{F}$, which comprises of a set W of states and some accessibility relations R_a, for indexes $a \in I$. We can then define a notion of validity $\vDash$ on frames and formulate the result mentioned above as follows:

$$R_a \text{ is reflexive} \quad \text{iff} \quad \mathcal{F} \vDash \Box_a\varphi \to \varphi, \text{ for all formulas } \varphi \tag{2.1}$$

Characterisations such as (2.1) are referred to as *correspondence results* [109], because they establish a mapping between a first-order property on frames (i.e., reflexivity) and a modal validity (i.e., scheme *(i)*). Another example of correspondence is between the first-order formula $\forall x \forall y(R_a(x,y) \to R_b(x,y))$ and modal scheme $\Box_b\varphi \to \Box_a\varphi$, which intuitively says that, e.g., whatever is achieved by program b, is also achieved by program a, or that agent a knows at least as much as agent b.

Mathematically elegant and powerful as correspondence theory may be, it also has shortcomings. Firstly, note that in the case of (2.1), correspondence is defined *globally*, i.e., scheme *(i)* has to be valid throughout the frame. This means that for instance (using a doxastic reading of *(i)*), we cannot model situations in which a's beliefs are true, but b does not know that. Indeed, if the truthfulness of agent a's beliefs is tantamount to the validity of *(i)*, then *(ii)* $K_b(\Box_a\varphi \to \varphi)$ is also a validity, enforcing agent b's knowledge. In particular, we cannot express that for all φ, $\Box_a\varphi \to \varphi$ is true, while for some some ψ, $\neg K_b(\Box_a\psi \to \psi)$ holds as well. Secondly, in (2.1) quantification appears at the meta-, and therefore the outermost, level. It is therefore impossible to distinguish (and to express in the language of modal logic) the following two situations: in the first, agent b knows that a has perfect information and is a perfect reasoner, and therefore, b knows a

priori that whatever a believes must be correct, that is, b knows that, for every formula φ, *(i)* holds. Informally, this would be represented as $K_b(\text{for all } \phi, \Box_a \phi \to \phi)$, which is not a well-formed formula in modal logic however. In the second situation agent b has systematically some empirical way to verify *a posteriori*, for every property φ, that whenever a believes it, then φ is true, that is, for every formula φ, b knows that *(i)* holds. This other situation can be represented as scheme *(ii)* above, which we remarked to be valid whenever *(i)* is.

As observed in [43], by allowing for quantification over propositions – and thus obtaining the language of second-order propositional modal logic (SOPML) – both issues mentioned above can be addressed. To wit, as regards the first example, the SOPML formula $\forall p(\Box_a p \to p) \wedge \neg K_b \forall p(\Box_a p \to p)$ intuitively expresses that all beliefs of agent a are correct, but b does not know this fact. Moreover, the two different readings in the second example can be represented by formulas $K_b \forall p(\Box_a p \to p)$ and $\forall p K_b(\Box_a p \to p)$, respectively. The reader may recognise here the distinction between *de dicto* and *de re* quantification. More generally, the truth of formula $\forall p(\Box_a p \to p)$ at state s enforces the truthfulness of agent a's beliefs in s only, therefore this is a *local* property of the frame, as opposed to the *global* validity of *(i)*. This fact allows agent b to consider (epistemically) possible a different state s' in which *(i)* does not hold.

The aim of the present work is to advance the application of propositional quantification and second-order propositional modal logic in knowledge representation and reasoning, through exploring and securing their theoretical fundations. We uphold the use of propositional quantification to express higher-order properties of knowledge, i.e., knowledge about other agents' knowledge, including truthfulness of knowledge, inclusions between the knowledge of agents, as well as dynamic properties such as knowability, successfulness, and preservation after arbitrary announcements. Specifically, the present work is structured as follows. In Part I we analyse second-order propositional modal logic from a static viewpoint. In Chapter 3 we introduce SOPML and interpret it on different classes of Kripke frames according to the features of the accessibility relations as well as the algebraic structure of the quantification domain of propositions. Most importantly, we provide completeness results with respect to various classes of Kripke frames. Results along this line are key to assess the computational properties of formal languages, as well as to develop automated reasoning methods. Here the main result shows that for SOPEL – the epistemic version of SOPML – the common knowledge operator acts as a universal modality on the class of *full* frames, thus allowing us to obtain a complete axiomatisation. In Chapter 4 we introduce (bi)simulation relations and prove that they preserve the interpretation of formulas in SOPML. Bisimulations are then applied to assess the expressive power of SOPML. Furthermore, part II is devoted to the dynamics of knowledge. In Chapter 5 we introduce second-order propositional announcement logic (SOPAL), an extension of SOPEL with public announcements. Notably, the interplay of announcement operators and propositional quantification allows us to express arbitrary announcements as well as notions such as knowability, preservation, and successfulness, which have been analysed previously within the framework of arbitrary public announcement logic (APAL) [9]. We thoroughly compare these two formalisms. Finally, in Chapter 6 we revert to a purely propositional setting to discuss the logic of global and local announcements (GLAL), an extension of public announcement logic (PAL) with operators for both global and local announcements. The final outcome of this work is a family of expressive modal languages, suitable to reason about relevant concepts in knowledge representation, while still enjoying nice computational properties.

A Note on Related Work. The present contribution builds on previously published works by the applicant and co-authors. Specifically, Chapter 3 extends [43] by considering general SOPML instead of just SOPEL. Indeed, here we tackle all normal modalities, rather than considering only multi-modal **S5**, suitable for the epistemic interpretation of modal logic, as it is done in [43]. Further, Chapter 4 draws from [44], but the (bi)simulation relations here considered are more general and provably equivalent to the (bi)simulation games. Finally, Chapter 5 is based on [45]. We refer to each chapter for an in-depth discussion of the differences with respect to the published papers.

Part I

Reasoning about Knowledge

Chapter 3

Completeness Results for Second-order Propositional Modal Logic

In this chapter we introduce the formal machinery on propositional quantification that will be used throughout the thesis. The main theoretical contribution consists in the completeness proof for second-order propositional modal logic with respect to several classes of Kripke frames. More specifically, in Section 3.1 we present the language of SOPML, including its epistemic version SOPEL, and provide it with a semantics in terms of Kripke frames extended with a domain D of sets of states for the interpretation of quantification. We consider various semantical constraint on our models, both on domain D and on the accessibility relations, and create a logical landscape that extends the state-of-the-art in a multi-agent direction, which we set to explore in the rest of the thesis. In Section 3.2 we illustrate the richness of the formal framework, particularly to express local properties in modal logic (LPML) [119, 120]. We compare and contrast our approach with [121], and show that the latter can be subsumed in the account here put forward. This validates our endeavour from the viewpoint of applications to knowledge representation. However, we maintain that for SOPML to be adopted as a specification language in artificial intelligence and knowledge representation, appropriate theoretical results and formal tools need to be developed. To this end, in Section 3.3 we present axiomatisations for a number of classes of validities and provide novel soundness and completeness results. The key finding here is that, while the *full* semantics for SOPML is incomplete for most of modal calculi, the same semantics admits a complete axiomatisation for SOPEL, where the accessibility relations are interpreted on equivalence relations. To obtain such a result, we make essential use of the fact that the common knowledge operator C acts as a universal modality. As a consequence, for reasoning about knowledge, SOPML admits sound and complete axiomatisations. To conclude, in Section 3.4 we discuss the results obtained and compare them with the state-of-the-art in SOPML. Our long-term aim for this thesis is to provide formal tools so as to facilitate the use of SOPML as a language for knowledge representation, as well as temporal and spatial reasoning in artificial intelligence.

3.1 Second-order Propositional Modal Logic

In this section we present the language of SOPML, some of its syntactic fragments, and their interpretation on Kripke frames and models. Then, we prove some preliminary results to be used hereafter.

3.1.1 The Formal Languages

To introduce second-order propositional modal logic, we fix a set AP of atomic propositions and a finite set I of indexes for modalities.

Definition 3.1 (SOPML)**.** The language $\mathcal{L}_{sopml}$ contains formulas ψ as defined by the following BNF:

$$\psi \quad ::= \quad p \mid \neg\psi \mid \psi \to \psi \mid \Box_a \psi \mid \Box^*_A \psi \mid \forall p \psi$$

where $p \in AP$, $a \in I$, and $A \subseteq I$.

The language $\mathcal{L}_{sopml}$ contains modal formulas $\Box_a \psi$, for every index $a \in I$. A general reading of this would be 'according to the aspect or dimension a, formula ψ holds'. The box can have more concrete interpretations, for instance dynamic (after execution of program or action a, ψ holds), temporal or spatial (along dimentions a, ψ), or deontic (in all situations that abide to norm a, ψ is true). Indices may also denote agents, in which case $\Box_a \psi$ can represent attitudes that relate to goals ('agent a desires ψ', or 'has ψ as a goal'), actions (agent a intends to achieve ψ), or information ('agent a believes ψ' or 'a knows that ψ'). The latter, epistemic interpretation of $\Box_a$ will obtain some special attention in this thesis, and we will write $K_a \psi$ rather than $\Box_a \psi$ for 'agent a knows that ψ'. Instead of $\Box^*_A \psi$, in the epistemic interpretation we will write $C_A \psi$ (in the group A of agents it is common knowledge that ψ). To give a hint of what this operator means in epistemic logic, define $E_A \psi$ (everybody in group A knows that ψ) as $\bigwedge_{a \in A} K_a \psi$. Then, formula $C_A \psi$ intuitively captures the infinite conjunction $\psi \wedge E_A \psi \wedge E_A E_A \psi \wedge E_A E_A E_A \psi \wedge \ldots$ (the standard definitions for $\top, \bot, \vee, \wedge$, and $\leftrightarrow$ apply). To sum up, whenever we consider the epistemic interpretation of modal operators, we write K_a and C_A, and define formulas ψ in the language $\mathcal{L}_{sopel}$ of second-order propositional epistemic logic (SOPEL) according to the following BNF:

$$\psi \quad ::= \quad p \mid \neg\psi \mid \psi \to \psi \mid K_a\psi \mid C_A\psi \mid \forall p\psi$$

for $p \in AP$, $a \in I$, and $A \subseteq I$. Then, the operator $\Box^*_A$ is interpreted intuitively as reachability along all dimensions in A, similarly to the interpretation of C_A. Also, we omit index A whenever it is equal to I and write $\Box^*\psi$ and $C\psi$ for $\Box^*_I\psi$ and $C_I\psi$. Standard references for modal logic are [47, 48], while for epistemic logic we refer to [64, 95].

In this paper we extend modal (epistemic) languages with propositional quantification. The quantified formula $\forall p\psi$ informally says that 'for all propositions, ψ is true', or, 'for all interpretations of p, ψ obtains'. As standard, the quantifier $\exists$ is dual to $\forall$: $\exists p\psi = \neg\forall p\neg\psi$. Analogously, in $\mathcal{L}_{sopml}$, $\Diamond_a\phi$ and $\Diamond^*_A\phi$ ares shorthands for $\neg\Box_a\neg\phi$ and $\neg\Box^*_A\neg\phi$, and in $\mathcal{L}_{sopel}$, M_a and $\overline{C}_A$ are dual to K_a and C_A. In what follows we use $\sharp$ as a placeholder for any unary operator $\neg$, $\Box_a$, $\Box^*$, and Q for any quantifier $\forall$, $\exists$. The name 'second-order propositional modal (epistemic) logic' is related to second-order quantification, as will become apparent in Section 3.2. In particular, this formalism has been studied in relation to monadic second-order logic – MSO [90, 107].

Example 3.1. *To give a flavour of the expressivity of $\mathcal{L}_{sopml}$, we present some specifications written in the language. We use variants of $\Box_a$ in our notation: their meaning will be clear from the context. Using $\mathcal{L}_{sopml}$ one can for instance express that agents a believes that agent b will always have some desire p that will remain unfulfilled: $B_a\Box^*\exists p(D_bp\wedge\neg p)$, where operators B_a and D_b are used to represent the doxastic and desire dimensions for agent a and b, respectively, whereas $\Box^*$ corresponds to the reachability relation with respect to all agents' moves. As a further example, formula* (i) $\forall p(\Box_ap \to \Box_bp)$ *expresses, in a dynamic context, that everything brought about by program a is also brought about by program b, or, provided a doxastic interpretation of the box operator, agent b believes everything that agent a believes. Deontically, the formula $\exists p(Op\wedge\neg p)$ expresses that the current world is not ideal: there are facts that ought to hold, but they don't. Finally, the doxastic-epistemic formula* (ii) $K_b\exists p(B_ap\wedge\neg p)$ *intuitively expresses that agent b knows that agent a's beliefs are incorrect, while* (iii) $\forall p(B_ap \to p)\wedge\Box_\alpha\exists q(B_aq\wedge\neg q))$ *denotes that currently, agent a's beliefs are correct, but after executing program α, this ceases to be the case. We remark that by using propositional quantification we can reason about general properties of knowledge, e.g., truthfulness, inclusion, equivalence of agents' knowledge and beliefs, as in specifications* (i), (ii), *and* (iii) *above.*

In this thesis we consider various fragments of $\mathcal{L}_{sopml}$ and $\mathcal{L}_{sopel}$. To begin with, the languages $\mathcal{L}_{ml}$ of (propositional) modal logic and $\mathcal{L}_{el}$ of (propositional) epistemic logic are obtained by removing clause $\forall p\psi$ from the definitions of $\mathcal{L}_{sopml}$ and $\mathcal{L}_{sopel}$, respectively. Likewise, the language $\mathcal{L}_{qbf}$ of quantified boolean formulas omits clauses $\Box_a\psi$ and $\Box^*_A\psi$ from $\mathcal{L}_{sopml}$; while propositional logic $\mathcal{L}_{pl}$ is defined as standard by considering only propositional connectives. Moreover, the *universal* fragment of $\mathcal{L}_{sopml}$ (or $\mathcal{L}_{a-sopml}$) is defined by the following BNF:

$$\psi \quad ::= \quad p \mid \neg p \mid \psi \wedge \psi \mid \psi \vee \psi \mid \Box_a\psi \mid \Box^*_A\psi \mid \forall p\psi$$

Notice that in $\mathcal{L}_{a-sopml}$ negation applies to atoms only. Hence, $\mathcal{L}_{a-sopml}$ contains no formula of the form $\exists p\psi$, $\Diamond_a\psi$, or $\Diamond^*_A\psi$. For convenience, we will also denote the set of atoms AP by $\mathcal{L}_{ap}$. A special role in this paper will be played by the languages $\mathcal{L}_x$ of sort x, with $x \in X = \{ap, pl, ml, sopml\}$. We summarise the main (syntactic) inclusions between languages in the following schema.

$$\begin{array}{ccccccccc}
 & & & \subsetneq & \mathcal{L}_{ml} & \subseteq & \mathcal{L}_{a-sopml} & \subsetneq & \\
\mathcal{L}_{ap} & \subseteq & \mathcal{L}_{pl} & & & & & & \mathcal{L}_{sopml} \\
 & & & \subsetneq & \mathcal{L}_{qbf} & & \subsetneq & &
\end{array}$$

We now introduce some syntactic notions that will be used throughout the chapter, starting with subformulas and free atoms.

Definition 3.2 (Subformula and free atoms)**.** The sets $Sub(\phi)$ and $fr(\phi)$ of subformulas and free atoms of formula $\phi \in \mathcal{L}_{sopml}$ are recursively defined as follows:

$$\begin{array}{lcl@{\qquad\qquad}lcl}
Sub(p) & = & \{p\} & fr(p) & = & \{p\} \\
Sub(\sharp\,\phi) & = & \{\sharp\,\phi\} \cup Sub(\phi) & fr(\sharp\,\phi) & = & fr(\phi) \\
Sub(\phi \to \phi') & = & \{\phi \to \phi'\} \cup Sub(\phi) \cup Sub(\phi') & fr(\phi \to \phi') & = & fr(\phi) \cup fr(\phi') \\
Sub(\forall p\phi) & = & \{\forall p\phi\} \cup Sub(\phi) & fr(\forall p\phi) & = & fr(\phi) \setminus \{p\}
\end{array}$$

A *sentence* is a formula ϕ with an empty set of free atoms, i.e., $fr(\phi) = \varnothing$. The set $bnd(\phi)$ of bound atoms in ϕ is defined as standard as the set of all atoms q appearing in the scope of any quantifier Qq. We assume that for each formula $\phi \in \mathcal{L}_{sopml}$, $fr(\phi)$ and $bnd(\phi)$ are disjoint. Actually, we impose that each quantifier binds a different variable. Both constraints can be enforced without loss of generality by renaming bound variables. Also, we introduce the set of atomic propositions in a formula ϕ as $AP(\phi) = fr(\phi) \cup bnd(\phi)$.

Next we define when a formula is free for substitution.

Definition 3.3 (Free for ...)**.** Given an atom $p \in fr(\phi)$, a formula ψ is *free* for p in ϕ iff p does not appear in ϕ within the scope of any quantifier Qq for $q \in fr(\psi)$. Alternatively, we can define whether ψ *is free for* p *in* ϕ by induction on the structure of ϕ as follows:

for ϕ atomic,	ψ is free for p in ϕ
for $\phi = \sharp\,\phi'$,	ψ is free for p in ϕ iff it is in ϕ'
for $\phi = \phi' \to \phi''$,	ψ is free for p in ϕ iff it is in ϕ' and ϕ''
for $\phi = \forall q \phi'$,	ψ is free for p in ϕ iff $q \notin fr(\psi)$ and ψ is free for p in ϕ'

We finally introduce a notion of substitution for free formulas.

Definition 3.4 (Substitution)**.** Let ψ be free for $p \in fr(\phi)$, the substitution $\phi[p/\psi]$ is inductively defined as follows:

$$\begin{array}{lcl}
q[p/\psi] & = & \begin{cases} q & \text{for } q \text{ different from } p \\ \psi & \text{otherwise} \end{cases} \\
(\sharp\,\phi')[p/\psi] & = & \sharp(\phi'[p/\psi]) \\
(\phi' \to \phi'')[p/\psi] & = & (\phi'[p/\psi]) \to (\phi''[p/\psi]) \\
(\forall r \phi')[p/\psi] & = & \forall r(\phi'[p/\psi]), \text{ where } r \text{ is assumed different from } p \text{ as } p \in fr(\phi)
\end{array}$$

Intuitively, ψ being free for p in ϕ means that a substitution of p by ψ in ϕ does not create any new binding. As an example, $\neg q$ is free for p in $\exists r(r \to p)$ but not in $\phi = \exists q(p \leftrightarrow q)$. In the following we will see that, while $\exists q(p \leftrightarrow q)$ is actually a validity, if we were to blindly substitute p with $\neg q$ in ϕ, we would obtain $\exists q(\neg q \leftrightarrow q)$, which is tantamount to a contradiction. But note that, since $\neg q$ is not free for p in ϕ, by Definition 3.4, $\phi[p/\neg q]$ is not well-defined.

Example 3.2. *As a further example of the expressive power of SOPEL, consider the following specification: agent b knows everything that a knows, and agent c knows this fact, but d does not. This epistemic situation can be recast in $\mathcal{L}_{sopel}$ as the following formula:*

$$\forall p(K_a p \to K_b p) \land K_c \forall p(K_a p \to K_b p) \land \neg K_d \forall p(K_a p \to K_b p)$$

In particular, we can reason further about agent d's knowledge. Indeed, agent d might know that a knows something ignored by b, without being able to explicitly point out the content of a's extra knowledge. This can be recast in $\mathcal{L}_{sopel}$ by the following de dicto *formula:*

$$K_d \exists p(K_a p \land \neg K_b p) \tag{3.1}$$

However, d could actually know about a specific fact that a knows, but b ignores, as expressed in the de re *formula:*

$$\exists p K_d(K_a p \wedge \neg K_b p) \tag{3.2}$$

We remark that intuitively (3.2) is strictly stronger than (and entails) (3.1), in the sense that any model satisfying (3.2) also satisfies (3.1). Thus, among other things, SOPEL allows us to distinguish the two readings – de re *and* de dicto *– of individual knowledge.*

3.1.2 Kripke Frames and Models

To provide a meaning to formulas in $\mathcal{L}_{sopml}$ and fragments, we consider multi-modal Kripke frames and models, extended with a domain for the interpretation of quantifiers.

Definition 3.5 (Kripke frame)**.** A Kripke frame is a tuple $\mathcal{F} = \langle W, D, R \rangle$ where

- W is a set of *possible worlds*;
- D is the *domain of propositions*, i.e., a subset of 2^W;
- $R : I \to 2^{W \times W}$ assigns a *binary relation* on W to each index in I.

As standard in propositional modal logic (PML), for every index $a \in I$, R_a is an accessibility relation between worlds in W [47]. Differently from standard Kripke frames, Definition 3.5 includes a set $D \subseteq 2^W$ of "admissible" propositions for the interpretation of atoms and quantifiers. Clearly, the Kripke frames in Definition 3.5 are related to *general frames* [47, 94]. However, there are some notable differences. Firstly, in general frames the domain D of propositions is a boolean algebra with operators, whereas no such assumption holds in the present case. Secondly, the language interpreted on general frames is usually a plain modal logic, while here we address quantification as well. Indeed, propositional quantification makes our language strictly more expressive than propositional modal logic interpreted on general frames, as will become apparent later on.

The accessibility relations can satisfy various properties, e.g., seriality, symmetry, transitivity, reflexivity, etc. When interpreting the language $\mathcal{L}_{sopel}$ we assume that each R_a is an equivalence relation (i.e., symmetric, transitive and reflexive), in line with the epistemic reading of modal operators [95]. In the following, for a coalition $A \subseteq I$, we consider also the reflexive and transitive closure $R^*_A = (\bigcup_{a \in A} R_a)^*$ of the union of accessibility relations, to be used for the interpretation of operator $\Box^*_A$. Then, for agent a, coalition A, and $w \in W$, we set $R_a(w) = \{w' \mid R_a(w, w')\}$ and $R^*_A(w) = \{w' \mid R^*_A(w, w')\}$. If R_a is an equivalence relation for every $a \in A$, then $R^*_A(w)$ is the equivalence class of $w \in W$, it can be represented as the set $\mathcal{E}_A = \{R^*_A(w) \mid w \in W\}$ of its equivalence classes, and whenever A is a singleton $\{a\}$, we have that $R^*_A(w) = \{w' \mid R_{\{a\}}(w, w')\} = R_a(w)$.

To interpret formulas in $\mathcal{L}_{sopml}$ on Kripke frames, we introduce *assignments* as functions $V : AP \to D$. Also, for $U \in D$, the assignment V^p_U assigns U to p and coincides with V on all other atoms. Hence, atoms can only be assigned propositions in $D \subseteq 2^W$. A *Kripke model* over $\mathcal{F}$ is then defined as a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$. In the rest of the paper we consider specific classes of Kripke frames and models, which feature pre-eminently in the literature on SOPML [66, 94]. To introduce them, we first define operators $[a] : 2^W \to 2^W$, for every $a \in I$, such that $[a](U) = \{w \in W \mid R_a(w) \subseteq U\}$; while operator $[A]^* : 2^W \to 2^W$ is introduced so that $[A]^*(U) = \{w \in W \mid$ for every $n \in \mathbb{N}$, for every sequence $w_0, \dots, w_n$, if $w_0 = w$ and for every $i < n, w_i = w_{i+1}$ or $R_a(w_i, w_{i+1})$ for some $a \in A$, then $w_n \in U\}$.

Definition 3.6. A Kripke frame $\mathcal{F}$ is

boolean	iff	D is a boolean algebra, i.e., it is closed under intersection, union, and complement
modal	iff	D is a boolean algebra closed under operators $[a]$ and $[A]^*$, for every $a \in I$ and $A \subseteq I$
full	iff	$D = 2^W$

A Kripke model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ is *boolean* (*modal, full*, respectively) whenever the underlying frame $\mathcal{F}$ is. We distinguish the class $\mathcal{K}_{all}$ of all Kripke frames, the class $\mathcal{K}_{bool}$ of all boolean frames, the class $\mathcal{K}_{modal}$ of all modal frames, and the class $\mathcal{K}_{full}$ of all full frames. Observe that, by using an analogy with monadic second-order logic, the class of full frames corresponds to the basic interpretation of SOPML, where any frame is uniquely identified by fixing the set W of worlds and accessibility relations, as the domain D is equal to 2^W. On the other hand, the other classes of frames are related to the Henkin interpretation of MSO, where D can be a possibly strict subset of 2^W (cf. [108]).

Furthermore, within each of the classes in Definition 3.6, we will consider further conditions on the accessibility relations R_a: reflexivity r, transitivity t, and symmetry s. Hereafter, given type $y \in Y = \{all, bool, modal, full\}$ and subset $\tau \subseteq \{r, t, s\}$, $\mathcal{K}^\tau_y$ denotes the corresponding class of frames satisfying the properties in τ. For simplicity, $\mathcal{K}^e_y$ denotes class $\mathcal{K}^{\{r,t,s\}}_y$ (which we also write as $\mathcal{K}^{rts}_y$) of frames in which all accessibility relations are equivalences, that is, the class of epistemic frames for the interpretation of

SOPEL. We define a function $\widehat{\cdot}: X \to Y$ from language sort symbols to type symbols as follows: $\widehat{ap} = all$; $\widehat{pl} = bool$; $\widehat{ml} = modal$; and $\widehat{sopml} = full$. In total, we obtain 32 classes $\mathcal{K}_y^\tau$ of frames. However, we only consider 20 of them: the subsets $\tau \subseteq \{r, t, s\}$ corresponding to the 5 normal modalities **K**, **T**, **S4**, **B**, and **S5**, combined with the 4 types *all*, *bool*, *modal*, and *full*. Further classes of frames could be introduced, for instance the class where every formula in $\mathcal{L}_{sopml}$ defines a proposition in D. However, such a class is not directly relevant for the results below and its introduction requires a non-trivial generalisation of Kripke frames [94]. Thus, such extensions are beyond the scope of the present work.

We finally define the notion of satisfaction for formulas in $\mathcal{L}_{sopml}$.

Definition 3.7 (Semantics)**.** We define whether Kripke model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ *satisfies* formula $\varphi \in \mathcal{L}_{sopml}$ at world w, or $(\mathcal{M}, w) \vDash \varphi$, as follows:

$$
\begin{array}{lll}
(\mathcal{M}, w) \vDash p & \text{iff} & w \in V(p) \\
(\mathcal{M}, w) \vDash \neg\psi & \text{iff} & (\mathcal{M}, w) \nvDash \psi \\
(\mathcal{M}, w) \vDash \psi \to \psi' & \text{iff} & (\mathcal{M}, w) \nvDash \psi \text{ or } (\mathcal{M}, w) \vDash \psi' \\
(\mathcal{M}, w) \vDash \Box_a\psi & \text{iff} & \text{for all } w' \in R_a(w),\ (\mathcal{M}, w') \vDash \psi \\
(\mathcal{M}, w) \vDash \Box_A^*\psi & \text{iff} & \text{for all } w' \in R_A^*(w),\ (\mathcal{M}, w') \vDash \psi \\
(\mathcal{M}, w) \vDash \forall p\psi & \text{iff} & \text{for all } U \in D,\ (\mathcal{M}_U^p, w) \vDash \psi
\end{array}
$$

where $\mathcal{M}_U^p = \langle \mathcal{F}, V_U^p \rangle$.

By Definition 3.7, a quantified formula $\forall p\psi$ (respectively, $\exists p\psi$) is true at world w iff for every (respectively, some) assignment of propositions in D to atom p, ψ is true. Further, as it is the case for the common knowledge operator C_A, $(\mathcal{M}, w) \vDash \Box_A^*\psi$ iff $(\mathcal{M}, w') \vDash \psi$ for every world w' *reachable* from w, i.e., for every w' such that for some sequence $w_0, \ldots, w_k$ of worlds, *(i)* $w_0 = w$, *(ii)* $w_k = w'$, and *(iii)* for every $i < k$, $w_i = w_{i+1}$ or $R_a(w_i, w_{i+1})$ for some $a \in A$. Hence, in non-epistemic contexts, $\Box_A^*$ can be interpreted as a reachability operator, analogous to the common knowledge operator C_A. Indeed, in epistemic contexts, formulas $K_a\phi$ can be defined as $C_{\{a\}}\phi$.

The satisfaction set $[\![\varphi]\!]_{\mathcal{M}}$ of formula φ in model $\mathcal{M}$ is defined as $\{w \in W \mid (\mathcal{M}, w) \vDash \varphi\}$. We omit the subscript $\mathcal{M}$ whenever clear by the context. We now introduce various notions of truth and validity. First, we write $(\mathcal{F}, V, w) \vDash \phi$ as a shorthand for $(\langle \mathcal{F}, V \rangle, w) \vDash \phi$. Then, we say that ϕ is *true* at w, or $(\mathcal{F}, w) \vDash \phi$, iff $(\mathcal{F}, V, w) \vDash \phi$ for every assignment V; ϕ is *valid* in a frame $\mathcal{F}$, or $\mathcal{F} \vDash \phi$, iff $(\mathcal{F}, w) \vDash \phi$ for every world w in $\mathcal{F}$; ϕ is *valid* in a class $\mathcal{K}$ of frames, or $\mathcal{K} \vDash \phi$, iff $\mathcal{F} \vDash \phi$ for every $\mathcal{F} \in \mathcal{K}$. Also, ϕ is *true* in a model $\mathcal{M}$, or $\mathcal{M} \vDash \phi$, iff $(\mathcal{M}, w) \vDash \phi$ for every world w. Finally, ϕ is *satisfiable* iff for some model $\mathcal{M}$ and world w, $(\mathcal{M}, w) \vDash \phi$.

Observe that if we define $\mathsf{Th}(\mathcal{K}) = \{\phi \in \mathcal{L}_{sopml} \mid \mathcal{K} \models \phi\}$, then clearly

$$\mathsf{Th}(\mathcal{K}_{all}) \subseteq \mathsf{Th}(\mathcal{K}_{bool}) \subseteq \mathsf{Th}(\mathcal{K}_{modal}) \subseteq \mathsf{Th}(\mathcal{K}_{full})$$

We will show that these inclusions are strict, but first we illustrate some applications of SOPEL in reasoning about knowledge.

Example 3.3. *Consider sets $I = \{a, b, d\}$ of agents and $AP = \{p\}$ of atoms. The epistemice frame $\mathcal{F} = \langle W, D, R\rangle \in \mathcal{K}^e_{all}$ is given with components W and R as depicted in Fig. 3.1.*

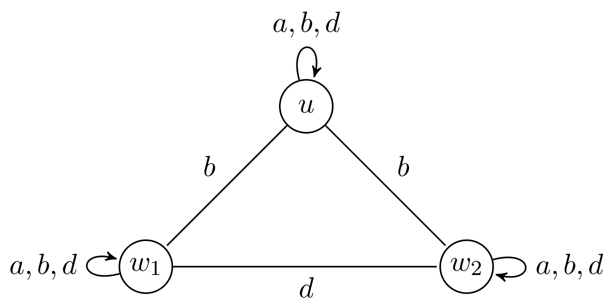


FIGURE 3.1: Frame $\mathcal{F}$ in Example 3.3.

Moreover, if we suppose that $D = \{\{w_1\}, \{w_2\}\}$, then for every assignment V, $(\mathcal{F}, V, w_1) \models \exists p(K_a p \wedge \neg K_b p)$, as $(\mathcal{F}, V^p_{\{w_1\}}, w_1) \models K_a p \wedge \neg K_b p$. Similarly, $(\mathcal{F}, V, w_2) \models \exists p(K_a p \wedge \neg K_b p)$ by considering assignment $V^p_{\{w_2\}}$. As a consequence, $(\mathcal{F}, w_1) \models K_d \exists p(K_a p \wedge \neg K_b p)$, that is, the de dicto *formula in Example 3.2 holds at w_1. However, this is not the case for its* de re *counterpart, as $(\mathcal{F}, w_1) \not\models \exists p K_d(K_a p \wedge \neg K_b p)$. To see the latter, note that $K_d(K_a p \wedge \neg K_b p)$ is not true at w_1 for any valuation of p with $\{w_1\}$ or $\{w_2\}$.*

On the other hand, if we suppose that $\mathcal{F}$ is a full frame in $\mathcal{K}^e_{full}$, that is, $D = 2^W$, then we obtain that $(\mathcal{F}, V, w_1) \models \exists p K_d(K_a p \wedge \neg K_b p)$, as both $(\mathcal{F}, V^p_{\{w_1,w_2\}}, w_1) \models K_a p \wedge \neg K_b p$ and $(\mathcal{F}, V^p_{\{w_1,w_2\}}, w_2) \models K_a p \wedge \neg K_b p$.

Example 3.4. *To assess the expressivity of SOPEL in knowledge representation, we contrast it with comparative epistemic logic (CEL) [121]. CEL extends the language $\mathcal{L}_{ml}$ of propositional modal logic with formulas $a \succcurlyeq b$, the intuitive interpretation of which is:* agent b knows at least as much as agent a. *Semantically, the clause for satisfaction of such formulas at world w in model $\mathcal{M}$ is given as*

$$(\mathcal{M}, w) \models a \succcurlyeq b \quad \textit{iff} \quad R_a(w) \supseteq R_b(w) \tag{3.3}$$

In this sense $a \succcurlyeq b$ also expresses a local property of frame $\mathcal{F}$, namely the inclusion $R_b(w) \subseteq R_a(w)$.

We show that the comparison between agent a's and agent b's knowledge can be recast in $\mathcal{L}_{sopel}$ as

$$\forall p(K_a p \to K_b p) \tag{3.4}$$

In particular, the RHS of (3.3) is tantamount to the satisfaction of (3.4) at w, whenever model $\mathcal{M}$ is full. More precisely, for an arbitrary model $\mathcal{M}$ we have

$$(\mathcal{M}, w) \vDash a \succcurlyeq b \quad \Rightarrow \quad (\mathcal{M}, w) \vDash \forall p(K_a p \to K_b p)$$

while the converse holds for full $\mathcal{M}$. As a result, formulas $a \succcurlyeq b$ and (3.4) have the same meaning in the class of full models, and therefore CEL can indeed be mimicked in SOPEL. We discuss this fact in more detail in Section 3.2.

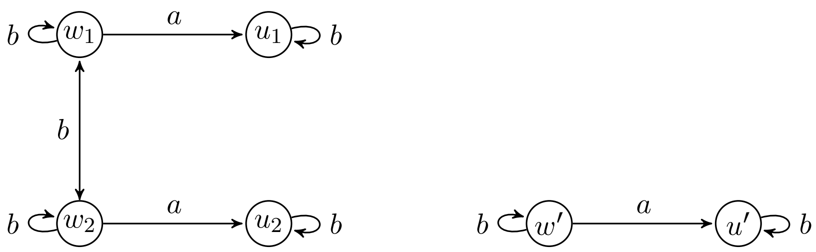
Moreover, in SOPEL we can make distinctions that are not expressible in epistemic logic. Related to Example 3.1, in $\mathcal{L}_{sopel}$ we can state that b knows that a's beliefs are not truthful *by using formula*

$$K_b \exists p(B_a p \wedge \neg p) \tag{3.5}$$

Notice that (3.5) expresses a de dicto *reading of quantification with respect to agent b's knowledge, that is, b knows that there exists some fact believed by a, which is false, possibly without being able to explicitly point out the actual content of a's false belief. On the other hand, b could actually be aware of some fact which is believed by a but false, as expressed in the following* de re *formula:*

$$\exists p K_b(B_a p \wedge \neg p) \tag{3.6}$$

We remark that (3.5) and (3.6) are not equivalent in general, (3.6) being strictly stronger than (3.5). Specifically, to account for the difference between (3.5) and (3.6), consider frame $\mathcal{G}$ in Fig. 3.2(a), where the W- and R-components are as depicted, and $D = \{\{w\} \mid w \in W\}$. Clearly, $(\mathcal{G}, V, w_1) \vDash B_a p \wedge \neg p$ for $V(p) = \{u_1\}$, and similarly $(\mathcal{G}, V', w_2) \vDash B_a p \wedge \neg p$ for $V'(p) = \{u_2\}$. Hence, $(\mathcal{G}, w) \vDash$ (3.5) for $w \in \{w_1, w_2\}$. On the other hand, for no $U \in D$, $(\mathcal{G}, V^p_U, w) \vDash B_a p \wedge \neg p$. Therefore, $(\mathcal{G}, w) \nvDash$ (3.6) for $w \in \{w_1, w_2\}$. Finally, we observe that $\exists p K_a \phi \to K_a \exists p \phi$ is a validity in every class of frames. As a result, in SOPEL formula (3.6) is strictly stronger than (3.5), and we can distinguish the de dicto *and* de re *readings of agent b's higher-level knowledge.*



(a) frame $\mathcal{G}$ (b) frame $\mathcal{G}'$

FIGURE 3.2: Frames $\mathcal{G}$ and $\mathcal{G}'$ in Example 3.4.

Finally, consider frame $\mathcal{G}'$ in Fig. 3.2(b) with $D' = \{\{w'\} \mid w' \in W'\}$. We can check that $(\mathcal{G}', w') \vDash$ (3.6) (and (3.5) as well). However, $\mathcal{G}$ and $\mathcal{G}'$, taken as frames for modal logic (that is, suppressing the domains D and D' of interpretation, and using only the language $\mathcal{L}_{ml}$), are bisimilar, with bisimulation relation H such that $H(w', w_i)$ and $H(u', u_i)$ for $i \in \{1, 2\}$ [47]. Hence, $\mathcal{G}$ and $\mathcal{G}'$ cannot be distinguished by any propositional modal formula, implying that the de re *formula* (3.6) *cannot be expressed in $\mathcal{L}_{ml}$. We return to this example and the notion of bisimulation in Chapter 4.*

3.1.3 Preliminary Results

In this section we prove some preliminary results on the model theory of second-order propositional modal logic, that will be frequently used hereafter. To start with, in Lemma 3.8 we extend some basic but useful results in the theory of quantification.

Lemma 3.8.

1. *Let ϕ be a formula in $\mathcal{L}_{sopml}$ and $\mathcal{F}$ a frame in $\mathcal{K}_{all}$. If assignments V and V' coincide on $fr(\phi)$, then*

$$(\mathcal{F}, V, w) \vDash \phi \quad \textit{iff} \quad (\mathcal{F}, V', w) \vDash \phi$$

2. *Recall that $X = \{ap, pl, modal, sopml\}$ and $\widehat{\ } = \{(ap, all), (pl, bool), (ml, modal), (sopml, full)\}$. Let $x \in X$. Then,*

   (a) for every $\psi \in \mathcal{L}_x$ and model $\mathcal{M}$ over $\mathcal{F} \in \mathcal{K}_{\widehat{x}}$, we have $[\![\psi]\!]_{\mathcal{M}} \in D$;

   (b) if $\mathcal{F} \in \mathcal{K}_{\widehat{x}}$ and $\psi \in \mathcal{L}_x$ is free for p in ϕ, then

$$(\mathcal{F}, V^p_{[\![\psi]\!]_{\langle \mathcal{F}, V\rangle}}, w) \vDash \phi \quad \textit{iff} \quad (\mathcal{F}, V, w) \vDash \phi[p/\psi]$$

By Lemma 3.8(1) models built over the same frame and agreeing on the interpretation of free atoms, satisfy the same formulas. It follows in particular that a sentence ϕ is either satisfied by any assignment or none, that is, $(\mathcal{F}, w) \models \phi$ iff for every model $\mathcal{M}$ over $\mathcal{F}$, $(\mathcal{M}, w) \models \phi$, iff for some model $\mathcal{M}$ over $\mathcal{F}$, $(\mathcal{M}, w) \models \phi$. As a consequence of Lemma 3.8(2a), the domain of quantification in a model includes the set of denotations of formulas in that model, according to the various fragments of $\mathcal{L}_{sopml}$. Moreover, by Lemma 3.8(2b) the syntactic operation of substitution $\phi[p/\psi]$ corresponds to the semantic notion of reinterpretation $\mathcal{M}^p_{[\![\psi]\!]}$. These results, which show that quantification in SOPML is "well-behaved", will be extensively used hereafter.

As a further preliminary result, we show that if a formula $\phi \in \mathcal{L}_{sopel}$ is satisfied, then it is satisfied in a *universal* model, that is, a model where each state is reachable from any other state. Notice that in universal models, the operator C is the *universal modality*, defined so that $(\mathcal{M}, w) \models C\psi$ iff for all $w' \in W$, $(\mathcal{M}, w') \models \psi$. Although this is a standard result in propositional modal logic (e.g., [47, Proposition 2.6]), which makes use of the notion of generated submodel, in our case we need to ensure that taking submodels keeps the underlying frame in the same class as the original one. First of all, we define the submodel generated by a world.

Definition 3.9 (Submodel)**.** Given model $\mathcal{M} = \langle W, D, R, V\rangle$ and world $w \in W$, the *submodel* generated by w is the model $\mathcal{M}_w = \langle W_w, D_w, R_w, V_w\rangle$ such that

- W_w is the set of worlds reachable from w, i.e., $W_w = R^*_I(w)$;
- $D_w = \{U_w \subseteq W_w \mid U_w = U \cap W_w \text{ for some } U \in D\}$;
- for every $a \in I$, $R_{w,a} = R_a \cap W_w^2$;
- for every $p \in AP$, $V_w(p) = V(p) \cap W_w$.

We remark that Definition 3.9 applies to any model $\mathcal{M}$ and it is tantamount to restricting the components of $\mathcal{M}$ to the set of states reachable from the selected world w.

We now state the following preservation result on the submodel construction.

Proposition 3.10. *For $y \in \{all, bool, modal, full\}$ and $\tau \subseteq \{r, t, s\}$, if a frame $\mathcal{F}$ belongs to $\mathcal{K}^\tau_y$ then also $\mathcal{F}_w \in \mathcal{K}^\tau_y$. In particular, $\mathcal{F} \in \mathcal{K}^e_y$ implies $\mathcal{F}_w \in \mathcal{K}^e_y$.*

By the following lemma formulas in $\mathcal{L}_{sopml}$ are preserved on generated submodels.

Lemma 3.11. *Let $\mathcal{M}$ be a model and consider submodel $\mathcal{M}_w$ for $w \in W$. For every $v \in W_w$ and $\phi \in \mathcal{L}_{sopml}$,*

$$(\mathcal{M}, v) \models \phi \quad \textit{iff} \quad (\mathcal{M}_w, v) \models \phi$$

As an immediate corollary of Lemma 3.11 we have that, for every $w \in W$, $(\mathcal{M}, w) \models \phi$ iff $(\mathcal{M}_w, w) \models \phi$.

Now let $\mathcal{K}_{univ}$ be the class of *universal* frames, that is, the frames where every world is accessible from any other world. In universal (epistemic) frames the common knowledge operator C acts clearly as the universal modality. Also, every generated submodel is based on a frame in $\mathcal{K}_{univ}$. As a consequence of Lemma 3.11, we have the following result.

Corollary 3.12. *For $y \in \{all, bool, modal, full\}$,*

$$\mathsf{Th}(\mathcal{K}^e_y) \quad = \quad \mathsf{Th}(\mathcal{K}^e_y \cap \mathcal{K}_{univ})$$

By Corollary 3.12 we can assume without loss of generality that, as long as we are interested in validity, the common knowledge operator C acts as the universal modality on the set W of worlds. This fact will be used in Section 3.3 for the completeness proof of SOPEL.

3.2 Local Properties in Modal Logic

In the introduction we discussed the difference between a global property as expressed by the modal schema *(i)* $\Box_a \varphi \to \varphi$, whose validity entails that the accessibility relation in a given frame is reflexive, and a local property such as the one represented by the SOPML formula $\forall p(\Box_a p \to p)$ that, as we shall see, implies that $R_a(w, w)$ holds in all and only worlds w where the formula is evaluated to true. Along this line, [119–121] put forward a sophisticated account to express local properties, by introducing dedicated modal operators to a basic propositional modal logic. In order to present the language of local properties in modal logic, or LPML, to compare the two approaches, and more generally to discuss the expressive power of SOPML, we consider a monadic second-order logic and a first-order fragment interpreted on Kripke frames.

Given a frame $\mathcal{F} = \langle W, D, R \rangle$ and a set AP of atoms, we define an MSO alphabet containing binary predicate constants R^*_A and R_a for every index $a \in I$ and set $A \subseteq I$, a unary predicate variable P for every atom $p \in AP$, and a set $\mathcal{X}$ of individual variables. Then, MSO formulas $\Theta \in \mathcal{L}_{mso}$ are defined in BNF as follows:

$$\Theta \quad ::= \quad P(x) \mid x = y \mid R_a(x, y) \mid R^*_A(x, y) \mid \neg\Theta \mid \Theta \to \Theta \mid \forall x \Theta \mid \forall P \Theta$$

where $a \in I$, $A \subseteq I$, and $x, y \in \mathcal{X}$.

We also consider the following first-order fragment $\mathcal{L}_{fo}$ of MSO:

$$\Theta \quad ::= \quad x = y \mid R_a(x,y) \mid \neg\Theta \mid \Theta \to \Theta \mid \forall x \Theta$$

The first-order language $\mathcal{L}_{fo}$ is obtained from $\mathcal{L}_{mso}$ by removing predicate quantification obviously, but also the clauses for unary predicates as well as binary relation R^*_A. This is indeed the first-order language considered in [121]. Moreover, we denote as $\mathcal{L}^1_{fo}$ the fragment of $\mathcal{L}_{fo}$ containing formulas with at most one free individual variable. This fragment is well-known to be rich enough to express properties of frames such as reflexivity, symmetry, and transitivity.

As regards the interpretation of MSO and FO formulas, an *assignment* ρ now is a function associating a world $w \in W$ to every individual variable x, and a set $U \in D$ to every predicate variable P. For $w \in W$ and $U \in D$, the variants ρ^x_w and ρ^P_U are defined similarly to SOPML.

Definition 3.13 (Semantics of MSO)**.** We define whether frame $\mathcal{F} = \langle W, D, R\rangle$ *satisfies* formula $\Theta \in \mathcal{L}_{mso}$ for an assignment ρ, or $(\mathcal{F}, \rho) \vDash \Theta$, as follows:

$(\mathcal{F},\rho) \vDash P(x)$	iff	$\rho(x) \in \rho(P)$
$(\mathcal{F},\rho) \vDash x = y$	iff	$\rho(x) = \rho(y)$
$(\mathcal{F},\rho) \vDash R_a(x,y)$	iff	$R_a(\rho(x), \rho(y))$
$(\mathcal{F},\rho) \vDash R^*_A(x,y)$	iff	$R^*_A(\rho(x), \rho(y))$
$(\mathcal{F},\rho) \vDash \neg\Theta$	iff	$(\mathcal{F},\rho) \nvDash \Theta$
$(\mathcal{F},\rho) \vDash \Theta \to \Theta'$	iff	$(\mathcal{F},\rho) \nvDash \Theta$ or $(\mathcal{F},\rho) \vDash \Theta'$
$(\mathcal{F},\rho) \vDash \forall x\Theta$	iff	for all $w \in W$, $(\mathcal{F},\rho^x_w) \vDash \Theta$
$(\mathcal{F},\rho) \vDash \forall P\Theta$	iff	for all $U \in D$, $(\mathcal{F},\rho^P_U) \vDash \Theta$

Obviously Definition 3.13 induces an interpretation of formulas in $\mathcal{L}_{fo}$ as well. In particular, for a formula $\Theta(x) \in \mathcal{L}^1_{fo}$, we write $(\mathcal{F}, w) \vDash \Theta$ to denote that $(\mathcal{F},\rho) \vDash \Theta$ for $\rho(x) = w$, and $\mathcal{F} \vDash \Theta$ if $(\mathcal{F}, w) \vDash \Theta$ for all $w \in W$. The different interpretations of the satisfaction relation $\vDash$ for SOPML and MSO respectively will be clear from the context.

We now briefly recall some basic modal theory on local definability: we refer the interested reader to [47, 48] for further details. We use θ (or $\theta(\vec{a},\vec{p})$ to emphasise sequences $\vec{a}$ of indices and $\vec{p}$ of atoms) for formulas in $\mathcal{L}_{ml}$. Likewise, we use $\Theta \in \mathcal{L}^1_{fo}$ for first-order formulas with at most one free variable interpreted over states (or $\Theta(\vec{a}, x)$ to denote that Θ mentions $\vec{a}$ as indices and has x as the free variable).

Definition 3.14. Let $\theta \in \mathcal{L}_{ml}$ and $\Theta \in \mathcal{L}^1_{fo}$,

1. θ *defines* frame property Θ iff for all frames $\mathcal{F}$, $\mathcal{F} \vDash \theta$ iff $\mathcal{F} \vDash \Theta$.
2. θ *locally defines* Θ iff for all $\mathcal{F}$ and all $w \in \mathcal{F}$, $(\mathcal{F}, w) \vDash \theta$ iff $(\mathcal{F}, w) \vDash \Theta$.

As examples of Definition 3.14, consider the well-known schemes **T** $\Box_a\varphi \to \varphi$, **4** $\Box_a\varphi \to \Box_a \Box_a \varphi$, and **B** $\varphi \to \Box_a \Diamond_a \varphi$, that (locally) define the properties of reflexivity, transitivity, and symmetry on frames.

In the theory of PML, when formula θ locally defines Θ and some other mild conditions hold, one obtains the following connection between axiomatisation and completeness: if an axiomatisation $\mathbf{Ax}$ is complete for a class $\mathcal{K}$ of frames, then $\mathbf{Ax} + \theta$ is complete for class $\{\mathcal{F} \in \mathcal{K} \mid \mathcal{F} \vDash \forall x \Theta\}$ of frames satisfying condition Θ. So for instance, taking the basic modal logic $\mathbf{K}$, which is sound and complete with respect to the class $\mathbb{K}$ of all frames, the logic $\mathbf{K} + \mathbf{T}$ is sound and complete with respect to class $\{\mathcal{F} \in \mathbb{K} \mid \mathcal{F} \vDash \forall x R_a(x,x)\}$, that is, the class of reflexive frames. As further examples, whereas $\mathbf{S5} = \mathbf{K} + \mathbf{T} + \mathbf{4} + \mathbf{B}$ is sound and complete with respect to class $\mathbb{S}5 = \{\mathcal{F} \in \mathbb{K} \mid R_a \text{ is an equivalence relation}\}$, the logic $\mathbf{S5} + (\Box_b\varphi \to \Box_c\varphi)$ is sound and complete with respect to $\{\mathcal{F} \in \mathbb{S}5 \mid \mathcal{F} \vDash \forall x(R_c(x) \subseteq R_b(x))\}$.

This is an appealing modular feature of modal logic. Yet, as also remarked in ([119–121]), this can only be applied if one adds formula θ as a *global* property: assuming θ as an axiom implies that it becomes a validity. For instance, adding formula $B_a\varphi \to \varphi$ to an axiom system, in order to model that agent a's beliefs are correct, implies that in the resulting logic, it is common knowledge that a's beliefs are correct, and this fact will remain true no matter what happens. Likewise, by adding $K_b\varphi \to K_c\varphi$ as an axiom for modelling that c is at least as knowledgeable as b, in the resulting frames and models it will be common knowledge that c knows whatever b knows, and this will again remain true no matter what happens. Thus, it is of interest to study formalisms that can express properties like: 'although a's beliefs are correct, b does not know this' and 'c knows everything that b knows, but after b opens the letter, this ceases to hold'. By using propositional quantification we can intuitively formalise such expressions as

$$\forall p(B_a p \to p) \wedge \neg K_b \forall p(B_a p \to p) \tag{3.7}$$

and

$$\forall p(K_b p \to K_c p) \wedge [\mathtt{read}_b] \exists q(K_b q \wedge \neg K_c q) \tag{3.8}$$

respectively.

To compare our approach based on SOPML to van Ditmarsch et al.'s LPML, we first provide a brief account of the latter.

$\theta(\vec{a},\vec{p})$	$\Theta(\vec{a},x)$	$\boxdot(\vec{a})$
$\Box_a p \to \Box_b p$	$\forall y(R_b(x,y) \to R_a(x,y))$	$\mathit{Sup}(a,b)$
$\Box_c p \to \Box_a \Box_b p$	$\forall y,z(R_a(x,y) \wedge R_b(y,z) \to R_c(x,z))$	$\mathit{Trans}(a,b,c)$
$\neg \Box_a \bot$	$\exists y R_a(x,y)$	$\mathit{Ser}(a)$
$\Box_a p \to p$	$R_a(x,x)$	$\mathit{Refl}(a)$
$\neg \Box_a p \to \Box_b \neg \Box_c p$	$\forall y,z(R_a(x,y) \wedge R_b(x,z) \to R_c(y,z))$	$\mathit{Eucl}(a,b,c)$
$\neg \Box_a p \to \neg \Box_b \Box_c p$	$\forall z(R_a(x,z) \to \exists y R_b(x,y) \wedge R_c(y,z))$	$\mathit{Dens}(a,b,c)$
$(\neg \Box_a p \wedge \neg \Box_b q) \to \neg \Box_c (p \vee q)$	$\forall y,z((R_a(x,y) \wedge R_b(x,z)) \to (y = z \wedge R_c(x,y))$	$\mathit{Func}(a,b,c)$

TABLE 3.1: as in [12, Table 1] $\Theta(\vec{a},x)$ is a property of state x, and $\boxdot(\vec{a})$ is a name in the object language such that $\boxdot(\vec{a})$ holds at w iff $\Theta(\vec{a},x)$ holds in $\mathcal{M}$ for $\rho(x) = w$.

3.2.1 Local properties and LPML

This section on LPML is based on [119–121]: we refer the reader to these references for a more extensive exposition. The term 'logic' is maybe not appropriate for LPML, rather, it is a specific approach to 'connect', in a modal object language, a modal formula $\theta \in \mathcal{L}_{ml}$ and a first-order property $\Theta \in \mathcal{L}^1_{fo}$ through the introduction of a relational atom $\boxdot$ (or $\boxdot(\vec{a})$), in such a way that on Kripke models $\boxdot$ is interpreted as Θ locally. More precisely, the language of LPML extends $\mathcal{L}_{ml}$ with formulas of type $\boxdot(\vec{a})$, whose interpretation is provided by an associated formula $\Theta_{\boxdot}(\vec{a},x) \in \mathcal{L}^1_{fo}$, according to the following satisfaction clause:

$$(\mathcal{F},V,w) \vDash \boxdot(\vec{a}) \quad \text{iff} \quad (\mathcal{F},w) \vDash \Theta_{\boxdot}(\vec{a},x) \tag{3.9}$$

By clause (3.9) we say that formula $\boxdot(\vec{a})$ *expresses locally* first-order property $\Theta_{\boxdot}$ (at w).

Then, LPML investigates how operator $\boxdot$ can help us, in the object language, to build a bridge between modal formulas $\theta_{\boxdot}$ on the one hand, and first-order properties $\Theta_{\boxdot}$ that $\theta_{\boxdot}$ locally defines on the other. So, for instance, we can have $\boxdot(a) = \mathit{Refl}(a)$ for $\Theta_{\boxdot}(a,x) = R_a(x,x)$, or $\boxdot(b,c) = \mathit{Sup}(b,c)$ for $\Theta_{\boxdot}(b,c,x) = \forall y(R_c(x,y) \to R_b(x,y))$ (for more examples, see Table 3.1). In LPML, property (3.7) is then represented as $\mathit{Refl}(a) \wedge \neg K_b \mathit{Refl}(a)$, while property (3.8) is given as $\mathit{Sup}(b,c) \wedge [\mathtt{read}_b]\neg \mathit{Sup}(b,c)$.

Recalling that operator $\boxdot$ is part of the object language of LPML, [121] then adds to the basic modal logic **K**, for specific formulas $\theta_{\boxdot} \in \mathcal{L}_{ml}$, an axiom $\mathbf{Ax}_{\boxdot}$ and an inference rule $\mathbf{R}_{\boxdot}$. Further, [121, Theorem 2] provides a sufficient condition on the relationship between $\theta_{\boxdot}, \boxdot$ and $\Theta_{\boxdot}$, called *local harmony*, under which $\mathbf{K} + \mathbf{Ax}_{\boxdot} + \mathbf{R}_{\boxdot}$ is a sound and complete axiomatisation for the class of models that satisfy $\Theta_{\boxdot}$.

Definition 3.15 (Local Harmony). Formulas $\theta(\vec{a},\vec{p}) \in \mathcal{L}_{ml}$, $\Theta(\vec{a},x) \in \mathcal{L}^1_{fo}$, and $\boxdot(\vec{a})$ in LPML are in *local harmony* iff *(i)* θ (locally) defines Θ, and *(ii)* $\boxdot$ expresses Θ locally.

Now, one could say that LPML as a language is at least as expressive as first-order logic over binary relations, as there are no restrictions, in the object language, on the relational atoms $\boxdot$ that can be added to standard PML. However, the aim of LPML is not to express arbitrary first-order properties Θ, but to reason locally about properties like veridicality of agent a's beliefs, or an agent c knowing more than b. In particular, there has to exist a modal formulas θ that (locally) defines Θ. LPML expresses such first-order properties by adding atoms like $\mathit{Refl}(a)$ and $\mathit{Sup}(b,c)$, respectively. We reckon that SOPML, allowing for quantification over propositions, is an alternative way to study local properties which is at least as natural as LPML, provably more expressive.

3.2.2 Local properties, LPML and SOPML

We now show that if formulas $\Theta_\boxdot(\vec{a},x)$ and $\theta_\boxdot(\vec{a},\vec{p})$ are in local harmony with some atom $\boxdot(\vec{a})$, then $\boxdot(\vec{a})$ is equivalent to $\forall\vec{p}\theta(\vec{a},\vec{p}) \in \mathcal{L}_{sopml}$, within the class of full frames. Hence, SOPML is at least as expressive as LPML in expressing local properties.

Theorem 3.16. *Suppose that $\theta(\vec{a},\vec{p})$, $\boxdot(\vec{a})$, and $\Theta(\vec{a})$ are in local harmony. Then, for every full model $\mathcal{M}$ and world w,*

$$(\mathcal{M},w) \vDash \forall\vec{p}\theta(\vec{a},\vec{p}) \quad \textit{iff} \quad (\mathcal{M},w) \vDash \boxdot(\vec{a})$$

Theorem 3.16 implies in a sense that what can be done in LPML, can also be done in SOPML: if $\theta(\vec{a},\vec{p})$, $\boxdot(\vec{a})$ and $\Theta(\vec{a})$ are in local harmony, then, to reason locally about a scheme θ, one can either use the universal closure $\forall\vec{p}\theta$ in SOPML, or atom $\boxdot(\vec{a})$ in LPML. The result also suggests ways in which SOPML may be more appropriate to reason about local properties, namely cases where Θ is not locally defined by any formula $\theta \in \mathcal{L}_{ml}$, or, conversely, when θ does not define a first-order property Θ locally. Hereafter we consider such cases.

Example 3.5. *Consider the following first-order formulas:*

- $\Theta_1 = \neg R_a(x,x)$ *(irreflexivity)*
- $\Theta_2 = \exists x_1,\dots,x_n \bigwedge_{i\leq n}(R_a(x,x_i) \wedge \bigwedge_{i\neq j\leq n} x_i \neq x_j)$ *(having at least n R_a-successors)*
- $\Theta_3 = \forall y(R_a(x,y) \wedge R_a(y,x) \to x = y)$ *(anti-symmetry)*
- $\Theta_4 = \forall y(R_b(x,y) \to \neg R_a(x,y))$ *(R_a and R_b have empty intersection)*
- $\Theta_5 = \forall y(R_a(x,y) \wedge R_b(x,y) \to R_c(x,y))$ *(R_c contains the intersection of R_a and R_b).*

It is well-known that these first-order properties are not definable in modal logic [47, 48]. However, consider the following formulas in SOPML

- $\varphi_1 = \exists p(\Box_a p \wedge \neg p)$
- $\varphi_2 = \exists p_1, \dots, p_n(\bigwedge_{i \leq n} \Diamond_a(p_i \wedge \bigwedge_{j \leq n, j \neq i} \neg p_j))$
- $\varphi_3 = \exists p(p \wedge \forall q(\Diamond_a(q \wedge \Diamond_a p) \to q))$
- $\varphi_4 = \exists p(\Box_a p \wedge \Box_b \neg p)$
- $\varphi_5 = \forall p(\Box_c p \to \exists q(\Box_a q \wedge \Box_b(q \to p)))$

which locally define Θ_1 to Θ_5 on full frames. We show this result in the following lemma.

Lemma 3.17. *Consider formulas $\varphi_i \in \mathcal{L}_{sopml}$ and $\Theta_i \in \mathcal{L}^1_{fo}$ in Example 3.5, for $i = 1, \dots, 5$. Let $\mathcal{F}$ be a full frame, then*

$$\mathcal{F} \models \varphi_i \quad \textit{iff} \quad \mathcal{F} \models \Theta_i$$

As a consequence of Lemma 3.17, SOPML is strictly richer than LPML, as in the former we can express properties Θ_1-Θ_5 that cannot be expressed in the latter.

Example 3.6 (Distributed Knowledge)**.** *To come back to an example from epistemic logic, an interesting notion in collective knowledge is that of* distributed knowledge *$D\varphi$. The intuition here is that distributed knowledge is the knowledge of a 'wise man' (cf. [64]) with whom all agents have shared their knowledge. The typical example is a situation where, for instance, one agent knows φ, another knows that $\varphi \to \psi$, implying distributed knowledge of ψ. A more concrete example goes as follows: it is distributed knowledge in every group of agents (provided everybody knows its own birthday) whether two agents share their birthday. The notion of* distributed knowledge *$D\varphi$ for n agents has an axiomatisation that is sound and complete with respect to models where the corresponding relation R_D is the intersection of all the individual agents' accessibility relations. However, intersection is not locally definable in modal logic (for more on a discussion on modal properties of distributed knowledge, or implicit knowledge as it is sometimes called, see for instance [84, 103].) However, in SOPML, by using formula φ_5 in Example 3.5 we can express that agent c knows exactly what the distributed knowledge of agents a and b is:*

$$\forall p(K_a p \to K_c p) \wedge \forall p(K_b p \to K_c p) \wedge \forall p(K_c p \to \exists q(K_a q \wedge K_b(q \to p))) \tag{3.10}$$

Notice that (3.10) *uses exactly the idea of the typical example of distributed knowledge between two agents discussed above: if agent c knows some fact p, i.e., p is distributed* *knowledge between a and b, then there exists some fact q such that a knows q and b knows $q \to p$. So, they are able to derive p by pooling together their knowledge.*

Can we generalise this to n agents? Indeed we can, as follows. Define

$$\varphi = \forall p(Dp \to \exists q_1 \dots \exists q_{n-1}(K_1 q_1 \wedge \dots \wedge K_{n-1} q_{n-1} \wedge K_n(\bigwedge_{i<n} q_i \to p))$$

and let

$$\Theta = \forall y((R_1(x,y) \wedge \dots \wedge R_n(x,y) \to R_D(x,y))$$

Then, we can prove the following result.

Proposition 3.18. *For every full frame $\mathcal{F}$, $(\mathcal{F}, w) \models \varphi$ iff $(\mathcal{F}, w) \models \Theta(x)$.*

The proof is a generalisation of the proof of Lemma 3.17(5). It follows that operator D locally expresses the distributed knowledge of ψ among agents $1, \dots, n$:

$$\bigwedge_{i \le n} \forall p(K_i p \to Dp) \wedge \varphi \wedge D\psi$$

From Examples 3.5 and 3.6 it follows that SOPML is more expressive than propositional modal logic, and it can also express local properties that cannot be dealt with in LPML. Example 3.5 also indicates when SOPML can axiomatise frames that cannot be characterised in PML: for instance, formula $\exists p(\Box p \wedge \neg p)$ characterises *irreflexive* frames, in the same way as $\exists p(\Box p \wedge \Diamond \Diamond \neg p)$ characterises *intransitive* frames. Venema [124] calls such characterisations *negatively definable*. The idea here is the following: suppose that formula $\theta \in \mathcal{L}_{ml}$ locally defines some property Θ; is there a modal formula that locally defines $\neg\Theta$? As an example, whereas $R_a(x,x)$ is (locally) defined by $\Box_a p \to p$, the negation $\neg R(x,x)$ is not (locally) defined by $\neg(\Box p \to p)$, or equivalently, $\Box_a p \wedge \neg p$, since this would require that on frames for this formula, atom p were false. Gabbay [69] came up with a derivation rule, rather than an axiom, to characterise irreflexivity; while [124] analyses more generally when a negative characterisation of some class of frames also leads to an axiomatisation of such class. For our discussion, it is important to realise that reflexivity is actually characterised by a modal *scheme* $\Box_a \varphi \to \varphi$, and, in contrast, by formula $\forall p(\Box_a p \to p)$ in SOPML. But then, irreflexivity is characterised by the negation of that SOPML formula: $\exists p(\Box p \wedge \neg p)$. Moreover, notice that SOPML allows us to interpret such formulas locally, so that we can reason about models that have both reflexive and irreflexive points.

From Example 3.5 we also learn that there are first-order properties Θ that cannot be characterised by any modal formula $\theta \in \mathcal{L}_{ml}$, while we do have a formula in SOPML characterising it. It is also possible to come up with formulas in SOPML that do not correspond to any first-order formula (hence, in SOPML one could reason locally about them, but not in LPML.) A first example of such formulas is $\exists p \neg\delta$ for $\delta = (\Diamond p \wedge \Diamond\Box\neg p) \to \Diamond(\Box^{-1}\Diamond p \wedge \Box\neg p)$ (here $\Box^{-1}$ is interpreted as the converse of relation R for $\Box$). As argued in [124], although δ as a scheme characterises Dedekind-complete frames among the linear orderings, the frames for $\neg\delta$ are not elementary, and hence not first-order definable. A further example is Löb formula $\forall p(\Box(\Box \to p) \to \Box p)$: this formula characterises frames with R being transitive and its converse well-founded [124, p. 8].

To conclude our comparison between SOPML and LPML, we observe that the $\boxdot$ operators act in fact as a sort of linguistic black boxes, bringing the metalanguage of the theory of first-order logic into the object language of modal logic. In constrast, SOPML is more transparent, as everything is done in the object language. In addition, for the first-order conditions in [121] there must always be a suitable modal counterpart. Indeed, the axioms $\mathbf{Ax}_{\boxdot} : \boxdot(\vec{a}) \to \theta_{\boxdot}(\vec{a}, \vec{p})$ in [121] make sense only as long as there is a propositional modal formula $\theta_{\boxdot}$ related to $\boxdot$, and this is not always the case as discussed above. We will see later that none of the above has to be assumed to axiomatise SOPML.

3.2.3 Monadic Second-order Logic

We conclude this section by analysing the expressiveness of second-order propositional modal logic through a correspondence between SOPML and monadic second-order logic (MSO), that extends the standard translation between modal and first-order logic [47].

More specifically, ST is the translation between SOPML and MSO defined as follows:

$$
\begin{array}{lcl}
ST_x(p) & = & P(x) \\
ST_x(\neg\phi) & = & \neg ST_x(\phi) \\
ST_x(\phi \to \phi') & = & ST_x(\phi) \to ST_x(\phi') \\
ST_x(\Box_a \phi) & = & \forall y(R_a(x,y) \to ST_y(\phi)) \\
ST_x(\Box^*_A \phi) & = & \forall y(R^*_A(x,y) \to ST_y(\phi)) \\
ST_x(\forall p \phi) & = & \forall P(ST_x(\phi))
\end{array}
$$

Clearly, for every formula $\phi \in \mathcal{L}_{sopml}$, $ST_x(\phi) \in \mathcal{L}_{mso}$ is a formula where x is the only free individual variable. If $\psi \in \mathcal{L}_{ml}$ is a purely propositional modal formula, then $ST_x(\psi) \in \mathcal{L}_{fo}$ is a first-order formula, as obtained via the standard translation. In particular, $ST_x(\psi)$ belongs to $\mathcal{L}^1_{fo}$.

We can now prove the following preservation result for the standard translation, that will be used later in the completeness proof.

Lemma 3.19. *For every model $\mathcal{M} = \langle \mathcal{F}, V \rangle$, world $w \in W$, and formula $\psi \in \mathcal{L}_{sopml}$,*

$$(\mathcal{M}, w) \vDash \psi \quad \textit{iff} \quad (\mathcal{F}, \rho) \vDash ST_x(\psi)$$

whenever $\rho(x) = w$ and $\rho(P_i) = V(p_i)$.

As a consequence of Lemma 3.19, there is a one-to-one correspondence between formulas in SOPML and their standard translations in MSO in the following sense: a frame $\mathcal{F}$ validates the universal closure $\forall \vec{p}\psi$ of a formula $\psi \in \mathcal{L}_{sopml}$ iff property $\forall x \forall \vec{P} ST_x(\psi) \in \mathcal{L}_{mso}$ holds in $\mathcal{F}$, where $\vec{P}$ are all the unary predicates appearing in $ST_x(\psi)$. We observe that this is not the case for propositional modal logic in general. For instance, for the McKinsey formula $\Box \Diamond p \to \Diamond \Box p$ there is no first-order principle Θ such that Θ holds in all and only frames validating the McKinsey formula.

3.3 Completeness and Model Checking

In this section we present two theoretical results regarding second-order propositional modal logic. First, by building on [43, 66] we provide sound and complete axiomatisations for various classes of validities, defined on a number of classes of models for SOPML. In particular, we will show that the operator $\Box^*$ (or, rather, common knowledge) is key to obtain a complete axiomatisation of epistemic models with respect to SOPEL. Further, we will analyse the model checking problem and prove that its complexity is no worse than for quantified boolean logic. These results give us useful insights into the computational complexity of SOPML, as well as its amenability for knowledge representation and reasoning. Specifically, we are able to demonstrate that in many cases of interest the computational properties of SOPML are no worse than that of the purely propositional case.

3.3.1 Semantic Completeness

This section is devoted to axiomatise several classes of validities on Kripke frames built on sets I of indexes and AP of atomic propositions. We first present a class $\mathbf{K}_x$ of logics, one for each $x \in \{ap, pl, ml, sopml\}$. In the following, $\Box$ is a placeholder for any of modal operators $\Box_a$ and $\Box^*_A$, for $a \in I$ and $A \subseteq I$.

Definition 3.20 (Logic $\mathbf{K}_x$)**.** For each $x \in \{ap, pl, ml, sopml\}$, the axioms and inference rules of $\mathbf{K}_x$ are as follows:

Prop	all instances of propositional tautologies
K	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
T	$\Box^*_A\phi \to \phi$
4	$\Box^*_A\phi \to \Box^*_A \Box^*_A \phi$
C1	$\Box^*_A\phi \to \bigwedge_{a\in A} \Box_a(\phi \wedge \Box^*_A\phi)$
C2	from $\phi \to \bigwedge_{a\in A} \Box_a(\psi \wedge \phi)$ infer $\phi \to \Box^*_A\psi$
$\mathbf{Ex}_x$	$\forall p\phi \to \phi[p/\psi]$, where $\psi \in \mathcal{L}_x$
BF	$\forall p \Box \phi \to \Box\forall p\phi$
MP	from $\phi \to \psi$ and ϕ infer ψ
Nec	from ϕ infer $\Box\phi$
Gen	from $\phi \to \psi$ infer $\phi \to \forall p\psi$, for p not free in ϕ

On top of that, $\mathbf{K}_{sopml}$ also includes axiom

$$\textbf{At} \quad \exists p(p \wedge \forall q(q \to \Box^*(p \to q)) \wedge \textstyle\bigwedge_{a\in I} \forall r(\Diamond_a r \to \Box^*(p \to \Diamond_a r)))$$

The axioms **Prop** and **K** are standard of any modal logic, as are rules Modus Ponens (**MP**) and Necessitation (**Nec**). Axioms **T**, **4**, **C1** and inference rule **C2** guarantee that $\Box^*_A$ is interpreted as the reflexive and transitive closure of the various $\Box_a$, for $a \in A$: **T** and **4** ensure reflexivity and transitivity respectively, **C1** is the 'elimination' rule for $\Box^*_A$, as it enables to derive arbitrary iterations $\Box_{a_1} \dots \Box_{a_n} \phi$ from $\Box^*_A\phi$, and **C2** is the 'introduction' or 'induction' rule for $\Box^*_A$. Axiom **BF** is known as the Barcan formula and it intuitively says that the domain D of interpretation is the same for all worlds in W: if for all assignments something holds in all reachable worlds, then in all reachable worlds, it holds for all assignments (this would not be the case if in a a-successor world, the domain D' of intepretation were bigger than D, for instance.) In Example 3.7 we will demonstrate that the converse of **BF** is derivable in all $\mathbf{K}_x$. The scheme of axioms $\mathbf{Ex}_x$ and the generalisation rule **Gen** are typical of first-order settings. Scheme $\mathbf{Ex}_x$ is the elimination axiom for $\forall$: if something holds for all valuations, then it also holds for each instance from the domain (which can be the set of all atoms, boolean formulas, modal formulas, or SOPML formulas.) The rule **Gen** of generalisation is the introduction rule for $\forall$: if ψ follows from ϕ for an arbitrary p, we infer that $\forall p\psi$ follows from ϕ.

Notice that axiom **At** is only part of logic $\mathbf{K}_{sopml}$. It modifies the homonymous axiom appearing in [66] to take into account the multi-modal setting. Semantically, this formula is true at a world w if $\{w\} \in D$. To see this, take assignment V for which $V(p) = \{w\}$. Then whichever property ψ holds in w (q and $\Diamond r$ being specific instances), in all worlds reachable from w, property $(p \to \psi)$ will hold: for w itself, this is so because both p and ψ hold; for any world reachable from w, different from w, this is so because p is false at that successor. More generally, **At** holds on frames that are *atomic*:

Definition 3.21 (Atomicity)**.** A frame $\mathcal{F} = \langle W, D, R\rangle$ is *atomic* if for every $w \in W$ there is $U \in D$ such that *(i)* $w \in U$; *(ii)* for all $U' \in D$, $w \in U'$ implies $U \subseteq U'$; and *(iii)* for all $u \in U$, $U' \in D$, and $a \in I$, if $R_a(w, w')$ for some $w' \in U'$, then $R_a(u, u')$ holds for some $u' \in U'$. We call such $U \in D$ an *atom* (of $\mathcal{F}$).

Loosely speaking, **At** holds in w if there is an atom U containing w and such that all worlds in U have access to the same sets U' in D as w. Clearly, for every $w \in W$, the atom containing w is unique.

As customary in PML, by considering a suitable combination of axioms

T	$\Box\phi \to \phi$
B	$\phi \to \Box \diamond \phi$
4	$\Box\phi \to \Box\Box\phi$

we can introduce the following normal extensions of $\mathbf{K}_x$, for $x \in \{ap, pl, ml, sopml\}$:

$$\begin{array}{lcl} \mathbf{T}_x & := & \mathbf{K}_x + \mathbf{T} \\ \mathbf{S4}_x & := & \mathbf{K}_x + \mathbf{T} + \mathbf{4} \\ \mathbf{B}_x & := & \mathbf{K}_x + \mathbf{T} + \mathbf{B} \\ \mathbf{S5}_x & := & \mathbf{K}_x + \mathbf{T} + \mathbf{B} + \mathbf{4} \end{array}$$

This gives us our 20 logics, 5 for each type $x \in X$.

The notions of *proof* and *theoremhood* are defined as standard. A formula ϕ is *derivable* in logic $\mathbf{L}$ from a set Δ of formulas, or $\Delta \vdash_{\mathbf{L}} \phi$, iff for some $\phi_0, \dots, \phi_m \in \Delta$, formula $\bigwedge_{i \leq m} \phi_i \to \phi$ is a theorem in $\mathbf{L}$, or $\vdash_{\mathbf{L}} \bigwedge_{i \leq m} \phi_i \to \phi$.

Example 3.7. *As an example, we provide a proof of the converse of the Barcan formula* **CBF** $\Box\forall p\phi \to \forall p \Box \phi$ *in* $\mathbf{K}_{ap}$*:*

1. $\forall p\phi \to \phi$	*by axiom* $\mathbf{Ex}_{ap}$
2. $\Box(\forall p\phi \to \phi) \to (\Box\forall p\phi \to \Box\phi)$	*by axiom* **K**
3. $\Box(\forall p\phi \to \phi)$	*from (1) by rule* **Nec**
4. $\Box\forall p\phi \to \Box\phi$	*from (2), (3) by rule* **MP**
5. $\Box\forall p\phi \to \forall p \Box \phi$	*from (4) by rule* **Gen***, as* p *is not free in* $\Box\forall p\phi$

Since **CBF** *is derivable in* $\mathbf{K}_{ap}$*, it is derivable in all the other 19 logics mentioned so far.*

We now prove the soundness and completeness results for some of the logics $\mathbf{L}_x$ with respect to the corresponding class $\mathcal{K}^{\tau}_{\widehat{x}}$ of Kripke frames, starting with soundness. Hereafter, given a logic $\mathbf{L}_x$, let $\tau(\mathbf{L}_x)$ be a subset of $\{r, t, s\}$, such that $\mathbf{L}_x$ includes axiom $\mathbf{T}$ iff τ contains r (for reflexivity), $\mathbf{L}_x$ includes axiom $\mathbf{4}$ iff τ contains t (for transitivity), and $\mathbf{L}_x$ includes axiom $\mathbf{B}$ iff τ contains s (for symmetry).

Theorem 3.22 (Soundness)**.** *Recall that* $X = \{ap, pl, modal, sopml\}$ *and* $\widehat{\ } = \{(ap, all), (pl, bool), (ml, modal), (sopml, full)\}$. *For* $x \in X$ *and every formula* $\phi \in \mathcal{L}_{sopml}$,

$$\vdash_{\mathbf{L}_x} \phi \textit{ implies } \mathcal{K}^{\tau(\mathbf{L}_x)}_{\widehat{x}} \vDash \phi$$

As a consequence of Theorem 3.22, all our 20 logics are sound with respect to the corresponding classes of frames.

Before moving to the completeness proof, we state some negative results already stated in [66].

Theorem 3.23. *For* $|I| = 1$, *the theories* $\mathsf{Th}(\mathcal{K}^{\tau}_{full})$ *are not axiomatisable for* $\tau \subseteq \{r, t\}$ *and* $\tau = \{rs\}$.

No result is available in the literature for classes $\mathcal{K}^{s}_{full}$ and $\mathcal{K}^{ts}_{full}$ of symmetric and symmetric, transitive full frames. Here we provide a complete axiomatisation for the class $\mathcal{K}^{e}_{full}$ of epistemic full frames, among others. Hence, our result is in contrast with Theorem 3.23. We immediately state the completeness result. Here we use the notation of Theorem 3.22.

Theorem 3.24 (Completeness)**.** *For* $x \in \{ap, pl\}$, *and every* $\phi \in \mathcal{L}_{sopml}$,

$$\mathcal{K}^{\tau(\mathbf{L}_x)}_{\widehat{x}} \vDash \phi \quad \textit{implies} \quad \vdash_{\mathbf{L}_x} \phi$$

Moreover,

$$\mathcal{K}^{e}_{full} \vDash \phi \quad \textit{implies} \quad \vdash_{\mathbf{S5}_{sopml}} \phi$$

Theorem 3.24 guarantees completeness of logics of type x, with respect to models of sort $\widehat{x}$, for the types of atomic propositions and propositional formulas. Completeness also holds if we add properties such as reflexivity, transitivity, and symmetry to the frames, as long as we add the corresponding axioms from $\{\mathbf{T}, \mathbf{4}, \mathbf{B}\}$ to the logic. The theorem also states completeness for logic $\mathbf{S5}_{sopml}$ with respect to full frames with equivalence relations. To our knowledge this is the first completeness result for SOPML in a multi-agent setting. Most importantly, by using common knowledge and axiom

At we are able to prove that logic $\mathbf{S5}_{sopml}$ is complete, differently to what stated in Theorem 3.23 for modalities weaker than **S5**. On the other hand, our method does not apply to modal frames. Hence, the completeness result for the class of modal frames remains an open problem.

To prove Theorem 3.24 for $x \in \{ap, pl\}$, we show that if a formula ϕ is $\mathbf{L}_x$-consistent, then we can construct an appropriate model $\mathcal{M}_{\mathbf{L}_x}$ that satisfies ϕ. For logic $\mathbf{L}_{ap}$ (respectively, $\mathbf{L}_{pl}$) this amounts to finding a model whose underlying frame is any frame (respectively, a boolean algebra). As regards $\mathbf{S5}_{sopml}$, more work is needed.

3.3.1.1 Completeness of $\mathbf{L}_{ap}$ and $\mathbf{L}_{pl}$

In this section we show that if a formula ϕ is $\mathbf{L}_x$-consistent for $x \in \{ap, pl\}$, that is, $\nvdash_{\mathbf{L}_x} \neg\phi$, then we can construct a (canonical) model $\mathcal{M}_{\mathbf{L}_x} = \langle \mathcal{F}, V \rangle$ that satisfies ϕ. Moreover, $\mathcal{F}$ is shown to belong to the appropriate class $\mathcal{K}_{\widehat{x}}^{\tau(\mathbf{L}_x)}$ of frames. This implies that $\mathcal{K}_{\widehat{x}}^{\tau(\mathbf{L}_x)} \nvDash \neg\phi$. Hereafter we prove this result first for $x = ap$, and then introduce the changes to be made for $x = pl$. For readability, we omit the subscript ap whenever clear by the context. First of all, we present some useful notions. We define $Sub^*(\phi)$ as the set of all subformulas of ϕ, together with formulas $\theta \wedge \Box_A^*\theta$ and $\Box_a(\theta \wedge \Box_A^*\theta)$ for every subformula $\Box_A^*\theta$ of ϕ and $a \in A$. Further, let $Sub^{\neg}(\phi)$ be the set of all formulas in $Sub^*(\phi)$ as well as their negations. More formally,

$$
\begin{aligned}
Sub^*(\phi) &:= Sub(\phi) \cup \{\theta \wedge \Box_A^*\theta, \Box_a(\theta \wedge \Box_A^*\theta) \mid \Box_A^*\theta \in Sub(\phi) \text{ and } a \in A\} \\
Sub^{\neg}(\phi) &:= Sub^*(\phi) \cup \{\neg\theta \mid \theta \in Sub^*(\phi)\}
\end{aligned}
$$

Notice that $Sub^*(\phi)$ and $Sub^{\neg}(\phi)$ are finite by construction.

Next we introduce some terminology on sets of formulas.

Definition 3.25. Let $\Lambda, \Sigma \subseteq \mathcal{L}_{sopml}$ be sets of formulas over set AP of atoms, and Y a denumerable set of atoms. Let $\phi \in \mathcal{L}_{sopml}$ and define

$$
\sim \phi = \begin{cases} \phi' & \text{if } \phi = \neg\phi' \\ \neg\phi & \text{otherwise} \end{cases}
$$

We say that Λ is closed under single negation if for each $\phi \in \Lambda$, either ϕ is of the form $\neg\phi'$ for some $\phi' \in \Lambda$, or else $\neg\phi \in \Lambda$. Moreover, Λ is

$\mathbf{L}_x$-consistent $\quad$ iff $\quad$ $\Lambda \nvdash_{\mathbf{L}_x} \bot$

Σ-complete	iff	for every formula $\phi \in \Sigma$, $\phi \in \Lambda$ or $\sim \phi \in \Lambda$
Σ-maximal	iff	Λ is consistent and Σ-complete
Y-rich	iff	for every ϕ over AP, if $\exists p \phi \in \Lambda$ then $\phi[p/q] \in \Lambda$ for some $q \in Y$
Y-universal	iff	for every ϕ over AP, if $\forall p \phi \in \Lambda$ then $\phi[p/q] \in \Lambda$ for every $q \in Y$
Σ–saturated	iff	Λ is Σ-maximal, Y-rich and Y-universal for some $Y \subseteq AP$

Hereafter we omit the subscript $\mathbf{L}_x$ whenever clear by the context. We may also omit reference to Σ if $\Sigma = \mathcal{L}_{sopml}$.

By the next lemma every consistent set of formulas can be extended to a maximal set.

Lemma 3.26 (Maximality). *Suppose that Σ is closed under single negation. If a set $\Delta \subseteq \Sigma$ is consistent, then there exists a Σ-maximal set $\Phi \supseteq \Delta$.*

Now we show that a Σ-maximal set Φ can be extended to obtain a Ω-saturated set Γ, for some $\Omega \subseteq \mathcal{L}_{sopml}$. First, we present the procedure informally, referring to the construction in the saturation lemma below. The issue with Φ is that it might contain existential formulas $\exists p \psi$, for which there is no witness q such that $\psi[p/q] \in \Phi$. Hence, we consider such a new atom q and add $\psi[p/q]$ to Φ: this can be done consistently as q, being new, does not appear in ψ, and therefore it can substituted for p in ψ. We repeat the procedure for all existential formulas in Φ. The result is a consistent set Γ_1^{pre}, which also contains a witness for every existential formula in Φ. However, this procedure has two consequences: on one hand, we add formulas $\psi[p/q]$ to Φ, whose subformulas are not necessarily decided by Φ. On the other, for every universal formula $\forall p \theta$, the substitution $\theta[p/q]$ is admissible, but subformulas of $\theta[p/q]$ are not decided either. Hence, we consider all subformulas of the various $\psi[p/q]$ and $\theta[p/q]$, and proceed to maximise Γ_1^{pre} against these subformulas, similarly as shown in Lemma 3.26. The outcome is a maximal and rich set Γ_1, with witnesses for all existential formulas in Φ. However, new existential formulas might have been introduced in the step from Γ_1^{pre} to Γ_1, which have to be taken care of. Therefore, we repeat these steps and build an infinite sequence $\Phi = \Gamma_0, \Gamma_1^{\mathsf{pre}}, \Gamma_1, \Gamma_2^{\mathsf{pre}}, \Gamma_2 \dots$ of sets of formulas. Finally, we show that the union of all the sets in the sequence is a saturated set as desired.

Lemma 3.27 (Saturation). *Let Σ be closed under single negation, and let Φ be Σ-maximal. Then there exists a set $\Omega \supseteq \Sigma$ of formulas in $\mathcal{L}_{sopml}$, closed under negation, and a set Γ of formulas over $AP \cup Y$, where Y is an infinite set of new variables, such that $\Gamma \supseteq \Phi$, and Γ is Ω-saturated.*

Observe that, in the construction of Γ, only witnesses for existential formulas and exemplifications of universal formulas are introduced (as well as subformulas thereof).

Hence, Γ really contains only alphabetical variants of the formulas in $Sub^{\neg}(\phi)$. On the other hand, all modal formulas in Γ have bounded modal depth. These features are key for the completeness proof, specifically for the truth lemma, and it also prevents the application of the method developed here to the class of modal frames. Indeed, saturated sets are infinite in general because of quantification; whereas there is only a finite number of types of modal formulas.

We now describe the construction of the canonical model for a formula ϕ such that $\nvdash \neg\phi$. First, define W as the set of all saturated sets w of formulas over $AP \cup Y$ as obtained in Lemmas 3.26 and 3.27 starting from $Sub^{\neg}(\phi)$. Notice that W is non-empty as the set $\{\phi\} \subseteq Sub^{\neg}(\phi)$ is consistent by hypothesis, and by Lemma 3.27 there exists a saturated set $\Gamma \supseteq \{\phi\}$ in W. Further, for $w, w' \in W$ and $a \in I$, define $R_a(w, w')$ iff $\{\phi \mid \Box_a \phi \in w\} \subseteq w'$. Finally, for every new atom p in Lemma 3.27, we consider a set $U_p = \{w \in W \mid p \in w\} \subseteq W$ and define the domain D of propositions as $\{U_p \mid p \in Q\}$.

We can then introduce the canonical model for a consistent formula ϕ.

Definition 3.28 (Canonical Model)**.** The canonical model for an $\mathbf{L}$-consistent formula ϕ is a tuple $\mathcal{M}_{\mathbf{L}} = \langle W, D, R, V \rangle$ where *(i)* W, D and R are defined as above; and *(ii)* V is the assignment such that $w \in V(p)$ iff $p \in w$.

Next we prove that the canonical model with respect to any logic $\mathbf{L}_{ap}$ is indeed a model based on a frame in $\mathcal{K}_{all}^{\tau(\mathbf{L}_{ap})}$ (recall that $\widehat{ap} = all$ and that $\mathbf{L}_{ap}$ represents 5 different logics: $\mathbf{K}_{ap}, \mathbf{T}_{ap}, \mathbf{S4}_{ap}, \mathbf{B}_{ap}$, and $\mathbf{S5}_{ap}$).

By the remarks above, W is a non-empty set of saturated sets and D is a subset of 2^W. Moreover, axiom $\mathbf{T}$ (respectively, $\mathbf{4}$, $\mathbf{B}$) enforces relation R_a on W to be reflexive (respectively, transitive, symmetric), as it is the case for propositional modal logic. Hence we obtain the following result.

Lemma 3.29. *The canonical model $\mathcal{M}_{\mathbf{L}_{ap}}$ in Definition 3.28 is a Kripke model based on a frame in $\mathcal{K}_{all}^{\tau(\mathbf{L}_{ap})}$.*

We can finally prove the truth lemma for logics $\mathbf{L}_{ap}$. Here we adapt the proof in [64] for propositional epistemic languages with common knowledge. Indeed the key case concerns operators $\Box_A^*$.

Lemma 3.30 (Truth lemma)**.** *For every logic $\mathbf{L}_{ap}$, in the canonical model $\mathcal{M}_{\mathbf{L}_{ap}}$, for every $w \in W$ and every formula $\psi \in \Omega$,*

$$(\mathcal{M}_{\mathbf{L}_{ap}}, w) \models \psi \quad \textit{iff} \quad \psi \in w$$

By Lemma 3.30, if $\nvdash_{\mathbf{L}_{ap}} \neg\phi$ then there exists a saturated set $w \supseteq \{\phi\}$ such that in the canonical model $\mathcal{M}_{\mathbf{L}_{ap}}$, we have $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \vDash \phi$. Moreover, $\mathcal{M}_{\mathbf{L}_{ap}}$ is based on a frame $\mathcal{F} \in \mathcal{K}_{ap}^{\tau(\mathbf{L}_{ap})}$. Thus, $\mathcal{K}_{ap}^{\tau(\mathbf{L}_{ap})} \nvDash \neg\phi$. This concludes the completeness proof for $\mathbf{L}_{ap}$.

As regards logic $\mathbf{L}_{pl}$, we need to modify the definition of saturation, the proof of Lemma 3.27, the definition of the canonical model, and the proof of the truth lemma.

First, a set Λ of formulas in $\mathcal{L}_{sopml}$ is

Y-universal	iff	for every formula ϕ over AP, if $\forall p\phi \in \Lambda$ then $\phi[p/\psi] \in \Lambda$ for every propositional formulas $\psi \in \mathcal{L}_{pl}$, possibly by propositional equivalence, built over Y
Σ–saturated	iff	Λ is Σ-maximal, Y-rich and Y-universal for some $Y \subseteq AP$

In particular, for $\mathbf{L}_{pl}$ witnesses of existential formulas and instantiations of universal formulas are propositional formulas $\psi \in \mathcal{L}_{pl}$, built over set Y of atoms, which without loss of generality can be assumed to be in some canonical representation, for instance, in conjunctive normal form. By doing so, we have that there is only a finite number of equivalent propositional formulas.

We now state the appropriate version of the saturation lemma for $\mathbf{L}_{pl}$.

Lemma 3.31 (Saturation)**.** *Let Σ be closed under single negation, and let Φ be Σ-maximal. Then there exists a set $\Omega \supseteq \Sigma$ of formulas in $\mathcal{L}_{sopml}$, closed under negation, and a set Γ of formulas over $AP \cup Y$, where Y is an infinite set of new variables, such that $\Gamma \supseteq \Phi$, and Γ is Ω-saturated.*

Notice that the statement of Lemma 3.31 is the same as for Lemma 3.27, but here we are using a different notion of saturation. In particular, the proof of the lemma is different in the following: Γ_{i+1} is obtained by saturating $\Gamma_{i+1}^{\mathsf{pre}}$ with propositional formulas, rather than simply atoms.

We now introduce the canonical model for $\mathbf{L}_{pl}$.

Definition 3.32 (Canonical Model)**.** The canonical model for an $\mathbf{L}_{pl}$-consistent formula ϕ is a tuple $\mathcal{M}_{\mathbf{L}_{pl}} = \langle W, D, R, V \rangle$ where

- W is the set of all saturated sets w of formulas over $AP \cup Y$ as obtained in Lemma 3.31, starting from $Sub^{\neg}(\phi)$;
- D is the domain of sets $U_\psi = \{w \in W \mid \psi \in w\} \subseteq W$, for every propositional formula $\psi \in \mathcal{L}_{pl}$ over Q;

- R and V are given as in Def. 3.28.

Given Def. 3.32 of canonical model, we can show that D is closed under boolean operations, and therefore $\mathcal{M}_{\mathbf{L}_{pl}}$ is indeed a boolean model.

Lemma 3.33. *The canonical model $\mathcal{M}_{\mathbf{L}_{pl}}$ is boolean.*

As a consequence of Lemma 3.33, $\mathcal{M}_{\mathbf{L}_{pl}}$ is based on a frame in $\mathcal{K}_{bool}^{\tau(\mathbf{L}_{pl})}$. Moreover, we are able to prove the following version of the truth lemma.

Lemma 3.34 (Truth lemma)**.** *For every logic $\mathbf{L}_{pl}$, in the canonical model $\mathcal{M}_{\mathbf{L}_{pl}}$, for every $w \in W$ and every formula $\psi \in \Omega$,*

$$(\mathcal{M}_{\mathbf{L}_{pl}}, w) \vDash \psi \quad \textit{iff} \quad \psi \in w$$

As a consequence of Lemma 3.34, the truth lemma also holds for boolean frames and we obtain a completeness proof for $\mathbf{L}_{pl}$.

We now discuss briefly why the method developed above fails for modal frames. In modal models each modal formula ψ defines an element in D, that is, the set U_ψ of states satisfying ψ, and by axiom $\mathbf{Ex}_{ml}$, ψ can be substitued in any formula. As a result, in the construction of the canonical model we obtain formulas of arbitrary modal depth, while a key feature of the proof for arbitrary and boolean frames is the fact that the construction of the canonical model make use only of formulas of bounded modal depth. As a result, the completeness proof for boolean frames is left as an open problem.

3.3.1.2 Completeness of $\mathbf{S5}_{sopml}$

We now establish the completeness of $\mathbf{S5}_{sopml}$ with respect to class $\mathcal{K}^e_{full}$. To do this, apart from atomicity of frames, we introduce an additional property on them: *completeness.*

Definition 3.35 (Completeness of frames)**.** A frame $\mathcal{F}$ is *complete* iff the domain D is closed under infinite unions and intersections.

Notice that we use 'complete' with two different meanings: *(i)* semantical completeness of a logic, and *(ii)* algebraic completeness of a frame. The context will disambiguate. We use *at* and *com* as subscripts to denote the respective classes of frames. Clearly, every full frame is boolean, atomic and complete. Hence, $\mathsf{Th}(\mathcal{K}^e_{bool,at,com}) \subseteq \mathsf{Th}(\mathcal{K}^e_{full})$. The converse inclusion follows from the next well-known algebraic result.

Theorem 3.36 ([72])**.** *Every complete atomic boolean algebra is isomorphic to the powerset of some set.*

By Theorem 3.36 we can prove the following lemma.

Lemma 3.37. $\mathsf{Th}(\mathcal{K}^e_{full}) = \mathsf{Th}(\mathcal{K}^e_{bool,at,com})$.

In [66] Fine remarks that the language of second-order propositional mono-modal logic is not rich enough to express algebraic completeness. We prove that this is the case for SOPML as well. To do so, we introduce the *Dedekind-MacNeille completion* $\mathcal{F}^+$ of an epistemic frame $\mathcal{F}$. First of all, given a set $Z \subseteq D$, let $Z^u = \{U \in D \mid U' \subseteq U$ for every $U' \in Z\}$ denote the set of upper bounds of Z, and let $Z^l = \{U \in D \mid U \subseteq U'$ for every $U' \in Z\}$ be the set of lower bounds. The Dedekind-MacNeille completion $\mathcal{F}^+$ of $\mathcal{F}$ has components

- $W^+ = D$;
- $D^+ = \{Z \subseteq D \mid (Z^u)^l = Z\}$.
- for every $a \in I$ and atoms U, U', $R^+_a(U, U')$ iff $R_a(u, u')$ for some $u \in U$ and $u' \in U'$.

Notice that once again each R^+_a is well-defined by the definition of atomicity, that is, if $R^+_a(U, U')$ then $R_a(u, u')$ for some $u \in U$ and $u' \in U'$. Hence, for every $v \in U$, $R_a(v, v')$ for some $v' \in U'$. Moreover, R^+_a is an equivalence relation whenever R_a is, and $\mathcal{F}^+$ is atomic by construction, as for each atom $U \in D$, $(\{U\}^u)^l = \{U\}$ is an atom in D^+. It is also complete (it is well-known that the completion of a partially ordered set is the smallest complete lattice that contains the given partial order, see for instance [72].)

Given this background, we are now able to prove the following result.

Lemma 3.38. $\mathsf{Th}(\mathcal{K}^e_{bool,at,com}) = \mathsf{Th}(\mathcal{K}^e_{bool,at})$

By combining Lemma 3.37 and 3.38 we obtain that $\mathsf{Th}(\mathcal{K}^e_{full}) = \mathsf{Th}(\mathcal{K}^e_{bool,at})$. Thus, for our purposes it is sufficient to prove completeness of $\mathbf{S5}_{sopml}$ with respect to $\mathcal{K}^e_{bool,at}$.

To this aim, we introduce the canonical model $\mathcal{M}_{\mathbf{S5}_{sopml}}$.

Definition 3.39 (Canonical Model)**.** The canonical model for an $\mathbf{S5}_{sopml}$-consistent formula ϕ is a tuple $\mathcal{M}_{\mathbf{S5}_{sopml}} = \langle W, D, R, V \rangle$ where

- R and V are given as in Def. 3.28.
- W is the restriction of the set of all saturated sets w of formulas over $AP \cup Y$ as obtained in Lemma 3.31, starting from $Sub^{\neg}(\phi)$, to the states reachable from $\Pi \supseteq \{\phi\}$ through the reachability relation R^*_I;

- D is the domain of sets $U_\psi = \{w \in W \mid \psi \in w\} \subseteq W$, for every propositional formula $\psi \in \mathcal{L}_{pl}$ over Q;

By Corollary 3.12 and following discussion, the restriction on W can be operated without loss of generality. We now prove that axiom **At** ensures atomicity of $\mathcal{M}$ in particular.

Lemma 3.40. *The canonical model $\mathcal{M}_{\mathbf{S5}_{sopml}}$ is boolean and atomic.*

By Lemma 3.40, we obtain that the canonical model is built on a boolean and atomic frame. Finally, we state the truth lemma for $\mathcal{M}_{\mathbf{S5}_{sopml}}$.

Lemma 3.41 (Truth lemma)**.** *In the canonical model $\mathcal{M}_{\mathbf{S}5_{sopml}}$, for every $w \in W$ and every formula $\psi \in \Omega$,*

$$(\mathcal{M}_{\mathbf{S}5_{sopml}}, w) \vDash \psi \quad \textit{iff} \quad \psi \in w$$

We omit the proof as it goes as in Lemma 3.34. This completes the proof for $\mathbf{S5}_{sopml}$. Observe the essential use of common knowledge in the proof of Lemma 3.40. Specifically, operator C acts as the universal modality in the canonical frame for $\mathbf{S5}_{sopml}$, and this allows us to quantify over all states belonging to the same atom. This remark fails for modalities strictly weaker than **S5**. Thus, common knowledge is key to obtain a complete axiomatisation for epistemic frames.

We conclude this section by summarising the soundness and completeness results for our logics with respect to the relevant classes of frames.

Theorem 3.42 (Soundness and Completeness)**.** *For $x \in \{ap, pl\}$, each logic $\mathbf{L}_x$ is sound and complete with respect to the class $\mathcal{K}_{\widehat{x}}^{\tau(\mathbf{L}_x)}$ of frames that are reflexive (respectively, transitive, symmetric), whenever $\mathbf{L}_x$ includes axiom **T** (respectively, **4**, **B**).*

Moreover, the logic $\mathbf{S5}_{sopml}$ is a sound and complete axiomatisation of $\mathcal{K}^{e}_{full}$.

As a result, for types *ap* and *pl* we are able to prove soundness and completeness for all normal modalities in a multi-modal setting. More interestingly, for type *sopml* we obtain such result for the epistemic interpretation of modalities only. To our knowledge, these are the first results of this kind on SOPML in a multi-modal setting. Finally, the completess question for type $x = ml$ is still open.

3.3.2 Generalised Completeness

We now extend the completeness results in the previous section by considering extra axioms expressing properties of frames. Specifically, let $\mathbf{L}$ be any axiomatisation mentioned in Theorem 3.42. Then, if we extend $\mathbf{L}$ with the universal closure $\forall\vec{p}\psi$ of a formula $\psi \in \mathcal{L}_{sopml}$, the resulting calculus $\mathbf{L} + \forall\vec{p}\psi$ is sound and complete with respect to the class of frames satisfying the MSO condition $\forall x \forall \vec{P} ST_x(\psi)$, where $\vec{P}$ are all the unary predicates appearing in $ST_x(\psi)$.

Theorem 3.43. *Let ψ be a formula in SOPML, then the logic $\mathbf{L} + \forall\vec{p}\psi$ is sound and complete with respect to the corresponding class $\mathcal{K}$ of frames satisfying $\forall x \forall \vec{P} ST_x(\psi)$.*

By the result above we immediately obtain that for every formula $\theta(\vec{a}, \vec{p})$ appearing in Table 3.1, $\mathbf{L} + \forall p \theta(\vec{a}, \vec{p})$ is sound and complete with respect to the class of frames satisfying $\forall x \forall \vec{P} \Theta(\vec{a}, x)$. More generally, there is a one-to-one correspondence between a SOPML axiom $\forall\vec{p}\theta$ and the MSO condition $\forall x \forall \vec{P} ST_x(\theta)$ on the corresponding class of sound and complete frames. Notice that this is not the case in propositional modal logic.

3.3.3 The Model Checking Problem

To explore further the computational properties of SOPML, in this section we tackle the corresponding model checking problem, defined as follows.

Definition 3.44 (Model Checking)**.** Given a formula $\phi \in \mathcal{L}_{sopml}$ and a finite model $\mathcal{M}$, determine whether $\mathcal{M} \vDash \phi$.

Then, by using also Algorithm 1 we are able to prove the following complexity result.

Theorem 3.45 (Model Checking)**.** *The model checking problem for SOPML is PSPACE-complete.*

As a result, the model checking problem for SOPML is no more computationally complex than the corresponding problem for quantified boolean formulas. Thus, the enhanced expressiveness comes at no extra computational cost, when compared with QBF. With respect to propositional modal logic, the complexity increases from PTIME to PSPACE. However, this is something to be expected given the extra expressive power of propositional quantification.

Algorithm 1 Computation of the satisfaction set $[\![\phi]\!]_{\mathcal{M}}$

switch (ϕ):
case p**:**
  return $V(p)$;
case $\neg\psi$**:**
  return $W \setminus [\![\psi]\!]_{\mathcal{M}}$;
case $\psi \wedge \psi'$**:**
  return $[\![\psi]\!]_{\mathcal{M}} \cap [\![\psi']\!]_{\mathcal{M}}$;
case $\Box_a \psi$**:**
  return $\{w \in W \mid R_a(w) \subseteq [\![\psi]\!]_{\mathcal{M}}\}$;
case $\Box^*_A \psi$**:**
  return $\{w \in W \mid R^*_A(w) \subseteq [\![\psi]\!]_{\mathcal{M}}\}$;
case $\forall p \psi$**:**
  return $\bigcap_{U \in D}\{[\![\psi]\!]_{\mathcal{M}^p_U}\}$;

3.3.4 The Finite Model Property

We now briefly argue why the logic SOPML does not have the final model property. Consider the following set of formulas:

$$\Gamma = \{\Diamond_a \top, \Box_a \Diamond_a \top, \forall p(\Box_a p \to \Box_a \Box_a p), \Box_a \exists p(p \wedge \Box_a \neg p)\}$$

The first two formulas of Γ requires R_a to be serial, the third enforces transitivity of R_a, and by the fourth and Example 3.5, we know that R_a is irreflexive. But it is easy to verify that a transitive, serial, and irreflexive relation requires the underlying model to be infinite. In other words, we found a finite set Γ of SOPML formulas which only has infinite models.

Theorem 3.46. *The logic SOPML does not have the finite model property.*

3.4 Discussion and Related Literature

In this section we upheld and motivated the use of second-order propositional modal logic as a specification language for reasoning about knowledge as well as spatial and temporal properties in artificial intelligence. Specifically, we aimed at developing proof- and model-theoretic techniques, notably complete axiomatisations, to support the use of SOPML in applications. In Section 3.1 we introduced 20 different classes of Kripke frames, according to the structure of the domain D of quantification and the features of the accessibility relations. In Section 3.3 we provided complete axiomatisations for 11 of these classes. Specifically, whenever the frames are general or boolean, we obtain complete axiomatisations for all normal modalities. On the other hand, for full frames we know from previous results that normal modalities weaker that **S5** are unaxiomatisable [66, 90]. Nonetheless, for **S5** and the epistemic interpretation of modalities we are able to provide a complete system of axioms. An interesting feature of the proof is the essential use of common knowledge as universal operator: this fact, which is exploited in the completeness proof, only holds for **S5** modalities. On the other hand, we left the completeness of modal frames as an open problem.

The contributions in this chapter are inspired by a series of papers on LPML, an extension of propositional modal logic to express local properties [119–121]. Here, instead of introducing an *ad hoc* language (with an adjustment for each local property one has in mind), we make use of the general framework of second-order propositional multi-modal logic. In Section 3.2 we provided a detailed comparison of the two approaches. In particular, we showed that SOPML subsumes LPML. Moreover, in cases where first-order properties are not definable by modal formulas (e.g. irreflexivity), or modal formulas express properties not definable in first-order logic (e.g. Dedekind completeness), SOPML is strictly more expressive than LPML and allows to reason about such properties locally.

Mono-modal SOPML was first considered by Bull and Fine [50, 66], mainly in relation with axiomatisability and (un)decidability questions. However, the high computational complexity of SOPML and some undecidability and non-axiomatisability results might partially explain why SOPML has been studied far less than propositional modal logic, and it has been virtually unexplored as a specification language for knowledge representation and reasoning. Here we considered a multi-modal version of SOPML, and its epistemic counterpart – SOPEL, which had originally been introduced in [43, 44].

Amongst more recent contributions, in [90] the authors proved that the expressive power of SOPML (for modalities weaker than 4.2) is the same as second-order predicate logic, and thus undecidable; while [107] provided SOPML with analogues of the van Benthem-Rosen and Goldblatt-Thomason theorems. In [67] propositional quantification and bisimulations are analysed in the context of modal logic. However, the type of quantification there considered preserve standard bisimulations, and therefore the resulting logic is provably as expressive as epistemic logic, strictly weaker than SOPML.

More directly related to the present contribution are [43, 44] by the same authors. In [43] we introduced epistemic quantified boolean logic (EQBL), an epistemic variant of SOPML, and provided axiomatisability and model-checking results. Differently from the reference, here we tackle general SOPML, defined also on modalities strictly weaker than **S5**. Indeed, in this chapter we analysed all normal modalities. Moreover, in Appendix A.1 we provide full details on the construction of the canonical models to prove completeness, and discuss key issues on the interaction between quantification and common knowledge in the completeness proof.

Chapter 4

Simulations and Games

In this chapter we investigate the expressive power of second-order propositional modal logic by introducing truth-preserving (bi)simulation relations for SOPML. Bisimulations are an essential tool for the model theory of propositional modal logic, as they exactly describe the conditions under which two models satisfy the same formulas in PML. Moreover, propositional modal logic is characterized by the well-known van Benthem theorem as the bisimulation-invariant fragment of first-order logic [47]. Hereafter we introduce (bi)simulations for SOPML and prove that they are indeed truth-preserving. Further, in Section 4.1.1 we present abstractions for frames and show that these are similar. In Section 4.2 we define games for (bi)simulations and prove the equivalence of the two approaches: model- and game-theoretic. Finally, in Section 4.3 we provide examples on the application of (bi)simulations to the analysis of the expressive power of SOPML in spatial and temporal reasoning.

4.1 Simulations and Bisimulations

We define the notion of (bi)simulations on frames, although it is immediate to extend this definition to models. In the rest of the chapter we consider frames $\mathcal{F} = \langle W, D, R \rangle$, $\mathcal{F}' = \langle W', D', R' \rangle$, and models $\mathcal{M} = \langle \mathcal{F}, V \rangle$, $\mathcal{M}' = \langle \mathcal{F}, V \rangle$ defined on $\mathcal{F}$ and $\mathcal{F}'$ respectively. Hereafter we use Σ to denote a relation on domain D, differently from Section 3.3, where it stands for a set of formulas. The distinction will be clear by the context.

Definition 4.1 (Frame Simulation)**.** Given frames $\mathcal{F}$ and $\mathcal{F}'$, a *simulation* is a pair (σ, Σ) of relations $\sigma \subseteq W \times W'$, $\Sigma \subseteq D \times D'$ such that *(i)* for every $U \in D$, $\Sigma(U, U')$ for some $U' \in D'$; and *(ii)* $\sigma(w, w')$ implies

1. for every $v \in W$, $a \in I$, if $R_a(w, v)$ then $\sigma(v, v')$ for some $v' \in R'_a(w')$;

2. for every $U \in D$, $U' \in D'$, $\Sigma(U, U')$ implies $w \in U$ iff $w' \in U'$.

Notice that condition 1 in Definition 4.1 expresses the standard notion of simulation in PML. Hence, simulations for SOPML extend the corresponding definition for PML (we devote more discussion to this point later on.) Moreover, the definition of simulation above differs from a similar notion put forward in [44]. Specifically, in [44] only a relation on states is considered, thus obtaining a strictly weaker notion. This will become apparent when analysing simulation games in Section 4.2.

We say that state w' *simulates* w, or $w \preceq w'$, iff $\sigma(w, w')$ holds for some simulation pair (σ, Σ). Similarly, a set U' *simulates* U, or $U \preceq U'$, iff $\Sigma(U, U')$ holds for some simulation pair (σ, Σ). Differently from what happens in PML, the pair $(\preceq, \preceq)$ is not a simulation generally. To check this, consider isomorphic[1] frames $\mathcal{G}_1 = \langle \{w_1, w_2\}, \{\{w_1\}, \{w_2\}\}, \{(w_1, w_2), (w_2, w_1)\} \rangle$ and $\mathcal{G}_2 = \langle \{x_1, x_2\}, \{\{x_1\}, \{x_2\}\}, \{(x_1, x_2), (x_2, x_1)\} \rangle$. Clearly, $w_1 \preceq x_1$ and $\{w_1\} \preceq \{x_2\}$. However, it is not the case that $w_1 \in \{w_1\}$ iff $x_1 \in \{x_2\}$. Nonetheless, each $\preceq$ is a preorder, i.e., a reflexive and transitive relation. Finally, a frame $\mathcal{F}'$ *simulates* $\mathcal{F}$, or $\mathcal{F} \preceq \mathcal{F}'$, iff for every $w \in W$, $w \preceq w'$ for some $w' \in W'$.

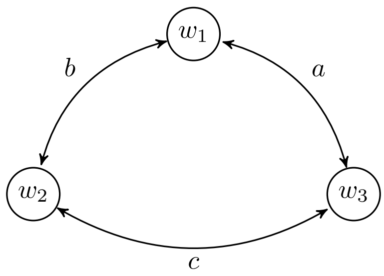
We illustrate the newly introduced notion by an example.

Example 4.1. *Consider frames $\mathcal{G} = \langle W, R, D \rangle$ and $\mathcal{G}' = \langle W', R', D' \rangle$ over set $I = \{a, b, c\}$ of indexes, depicted in Figure 4.1, with*

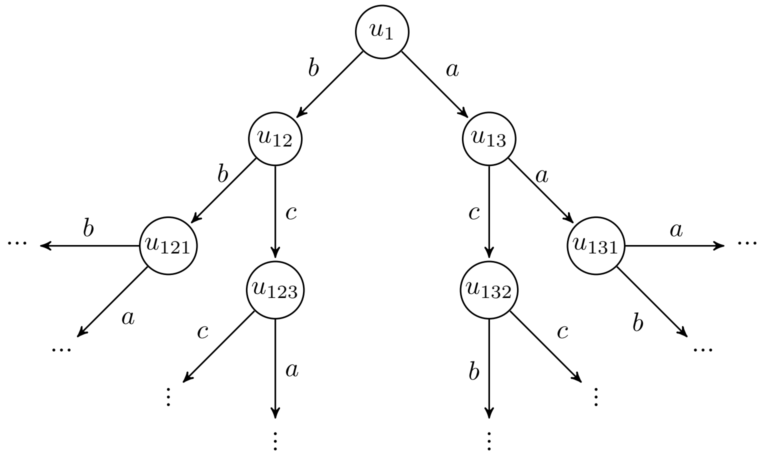
- $W = \{w_1, w_2, w_3\}$;
- $R_a = \{(w_1, w_3), (w_3, w_1)\}$, $R_b = \{(w_1, w_2), (w_2, w_1)\}$, $R_c = \{(w_2, w_3), (w_3, w_2)\}$;
- $D = \{\{w_1\}, \{w_2\}, \{w_3\}\}$;
- $W' = \{u_s \mid s$ *is a finite sequence on* $\{1, 2, 3\}$ *starting with 1, with no adjacent repetition*$\}$;
- *for every* $i \in I$, $R'_i = \{(u_s, u_{s'}) \mid s' = s \cdot m$ *and* $R_i(w_{last(s)}, w_m)\}$;
- *let* $U'_n = \{u_s \mid last(s) = n\}$, *then* $D' = \{U'_1, U'_2, U'_3\}$.

Intuitively, frame $\mathcal{G}$ can be thought of as a scenario where robots a, b, and c move around locations w_1, w_2, w_3 (robot a moves between w_1 and w_3, etc.) Frame $\mathcal{G}'$ would then be a structure that allows one to capture the same scenario but with the additional possibility to reason about some notion of history, or time. One might for instance add an atom p_i which is true exactly at nodes at level i. To do this, one needs to make

[1]We do not provide a formal definition of isomorphism. It suffices to say that it is a one-to-one correspondence that preserve accessibility relations and sets in D.



(a) frame $\mathcal{G}$



(b) frame $\mathcal{G}'$

FIGURE 4.1: Frames $\mathcal{G}$ and $\mathcal{G}'$ in Example 4.1 (D components are omitted for clarity).

appropriate assumptions about D' in $\mathcal{G}'$, like requiring that the frame is full. We do not consider these matters further.

Now consider the pair (σ, Σ) of relations $\sigma \subseteq W \times W'$ and $\Sigma \subseteq D \times D'$ such that $\sigma(w_n, u_s)$ holds iff $last(s) = n$ and $\Sigma(\{w_n\}, U'_m)$ holds iff $n = m$. We check that (σ, Σ) is indeed a simulation. Firstly, for every $\{w_n\} \in D$, we have $\Sigma(\{w_n\}, U'_n)$ for $U_n \in D'$. Secondly, if $\sigma(w_n, u_s)$ and $R_i(w_n, w_m)$, then $s' = s \cdot m$ is such that $R'_i(u_s, u_{s'})$ and $\sigma(w_m, u_{s'})$. Thirdly, if $\sigma(w_n, u_s)$ and $\Sigma(\{w_k\}, U'_m)$, then $last(s) = n$ and $k = m$. Therefore, $w_n \in \{w_k\}$ iff $n = k$, iff $last(s) = m$, iff $u_s \in U'_m$.

Finally, we observe that for every $w_n \in W$, $\sigma(w_n, u_s)$ for $last(s) = n$. Thus, frame $\mathcal{G}'$ simulates $\mathcal{G}$.

We now consider the following remark on the relation between simulations and properties of frames.

Remark 4.2. If a frame $\mathcal{F}'$ simulates a boolean (respectively modal, full) frame $\mathcal{F}$, then $\mathcal{F}'$ need not to be boolean (respectively modal, full). Nor does $\mathcal{F}'$ being boolean (modal, full) imply that $\mathcal{F}$ is also boolean (modal, full).

To see this, consider that a simulation $\mathcal{F}'$ may contain sets of states that do not simulate any state in $\mathcal{F}$, which are not closed under set-theoretic operations. The other implication can be proved by a similar line of reasoning. Hence, similar frames need not to belong to the same class. Below we compare these results with those available for bisimulations.

We now state that simulations preserve the satisfaction of the universal fragment of SOPML.

Theorem 4.3. *If $w \preceq w'$, then for every $\varphi \in \mathcal{L}_{a-sopml}$,*

$$(\mathcal{F}', w') \vDash \varphi \quad \textit{implies} \quad (\mathcal{F}, w) \vDash \varphi$$

As an immediate consequence of Theorem 4.3 we obtain the following corollary.

Corollary 4.4. *If $\mathcal{F} \preceq \mathcal{F}'$, then for every $\varphi \in \mathcal{L}_{a-sopml}$,*

$$\mathcal{F}' \vDash \varphi \quad \textit{implies} \quad \mathcal{F} \vDash \varphi$$

Thus, the notion of simulation introduced in Definition 4.1 preserves the universal fragment of SOPML, similarly to the case for standard simulations and PML.

Example 4.2. *Consider again frames $\mathcal{G}$ and $\mathcal{G}'$ in Example 4.1. We showed that $\mathcal{G}'$ simulates $\mathcal{G}$. Moreover, we can easily check that $\mathcal{G}'$ validates the following formula in $\mathcal{L}_{a-sopml}$:*

$$\forall p \left(p \to \bigvee_{i \in I} \Box_i \neg p \right) \tag{4.1}$$

which intuitively says that at each position some agent moves to a different position. By Corollary 4.4 we deduce that (4.1) is valid in $\mathcal{G}$ as well.

Example 4.3. *Consider the notion of submodel $\mathcal{M}_w$ generated by world w given in Definition 3.9. It is easy to check that the pair (σ, Σ), where σ is the identity relation and $\Sigma(U, U')$ holds iff $U = U' \cap W_w$ is the restriction of U' to the worlds accessible from w, is a simulation between $\mathcal{F}_w$ and $\mathcal{F}$. In particular, for every $U \in D_w$, $\Sigma(U, U')$ whenever $U = U' \cap W_w$ for $U' \in D$. Moreover, $\sigma(w, w)$ implies that, for $v \in W$, $a \in I$, if $R_{w,a}(w, v)$ then $R_a(w, v)$ and $\sigma(v, v)$. Finally, if $\sigma(w, w)$ and $\Sigma(U, U')$, then $U = U' \cap W_w$ and $w \in U$ iff $w' \in U'$. As an immediate consequence of Theorem 4.3, we obtain the implication from left to right of Lemma 3.11, restricted to universal SOPML.*

Simulations can naturally be extended to bisimulations. Also in this case, our focus is at the level of frames. In the following the *converse* of a relation R is the relation $R^{-1} = \{(u, v) \mid R(v, u)\}$.

Definition 4.5 (Frame Bisimulation)**.** Given frames $\mathcal{F}$ and $\mathcal{F}'$, a *bisimulation* is a pair (ω, Ω) of relations $\omega \subseteq W \times W'$, $\Omega \subseteq D \times D'$ such that both (ω, Ω) and $(\omega^{-1}, \Omega^{-1})$ are simulations. That is, *(i)* for every $U \in D$, $\Omega(U, U')$ for some $U' \in D'$, and for every $U' \in D'$, $\Omega(U, U')$ for some $U \in D'$; and *(ii)* $\omega(w, w')$ implies

1. for every $v \in W$, $a \in I$, if $R_a(w, v)$ then $\omega(v, v')$ for some $v' \in R'_a(w')$;
2. for every $v' \in W'$, $a \in I$, if $R'_a(w', v')$ then $\omega(v, v')$ for some $v \in R_a(w)$;
3. for every $U \in D$, $U' \in D'$, $\Omega(U, U')$ implies $w \in U$ iff $w' \in U'$.

States w and w' are *bisimilar*, or $w \approx w'$, iff $\omega(w, w')$ holds for some bisimulation pair (ω, Ω). Similarly, sets U' and U are *bisimilar*, or $U \approx U'$, iff $\Omega(U, U')$ holds for some bisimulation pair (ω, Ω). Again, the pair $(\approx, \approx)$ is not necessarily a bisimulation, similarly to what was shown above for simulations, but each $\approx$ is an equivalence relation. Finally, frames $\mathcal{F}$ and $\mathcal{F}'$ are *bisimilar*, or $\mathcal{F} \approx \mathcal{F}'$, iff *(i)* for every $w \in W$, $w \approx w'$ for some $w' \in W'$; and *(ii)* for every $w' \in W'$, $w \approx w'$ for some $w \in W$.

Example 4.4. *Notice that frames $\mathcal{G}$ and $\mathcal{G}'$ in Example 4.1 are actually bisimilar. To prove this fact, we show that the converse relations $\sigma^{-1} \subseteq W' \times W$ and $\Sigma^{-1} \subseteq D' \times D$ form a simulation pair. Firstly, for every $U'_n \in D'$, the set $U = \{w_n\} \in D$ is such that $\Sigma(U, U')$. Secondly, if $\sigma^{-1}(u_s, w_n)$ and $R'_i(u_s, u_{s'})$ then $last(s) = n$ and $s' = s \cdot m$ for $w_m \in W$ such that $R_i(w_n, w_m)$. Hence, $\sigma^{-1}(u_{s'}, w_m)$. As to (3), the proof is identical as for simulations.*

We now state the following remark on the relationship between properties of frames and bisimulations.

Remark 4.6. Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are bisimilar. Then, $\mathcal{F}$ is boolean (respectively modal) iff $\mathcal{F}'$ is. However, if $\mathcal{F}$ is full, then $\mathcal{F}'$ need not to be full. Nor does $\mathcal{F}'$ being full imply that $\mathcal{F}$ is also full. Moreover, if $\mathcal{F}$ and $\mathcal{F}'$ are both bisimilar and full, then they are isomorphic, that is, any bisimulation between full $\mathcal{F}$ and $\mathcal{F}'$ is an isomorphism.

Compare the situation for bisimulations with the weaker results available in Remark 4.2 for simulations. Specifically, bisimulations preserve the class of boolean and modal frames.

We now state the main preservation result of this section.

Theorem 4.7. *If $w \approx w'$, then for every formula $\varphi \in \mathcal{L}_{sopml}$,*

$$(\mathcal{F}, w) \models \varphi \quad \textit{iff} \quad (\mathcal{F}', w') \models \varphi.$$

As an immediate consequence of Theorem 4.7 we obtain the following.

Corollary 4.8. *If $\mathcal{F} \approx \mathcal{F}'$, then for every $\varphi \in \mathcal{L}_{sopml}$,*

$$\mathcal{F} \models \varphi \quad \textit{iff} \quad \mathcal{F}' \models \varphi$$

We can now infer that bisimulations in SOPML are 'stronger' than the corresponding notion for PML: whereas we noticed that the frames of Figure 3.2 are bisimilar in PML, as a consequence of Theorem 4.7, and Example 3.4, which says that the frames do not agree on formula (3.6), we conclude that they are not bisimilar in the SOPML sense.

Example 4.5. *We now consider two graph-theoretic properties. First, the notion of 3-colorability, as formalised by the following SOPML formula, where operator $\Box$ is interpreted on the edges $E \subseteq W^2$ of a graph $G = \langle W, E \rangle$, while $\Box^*$ is interpreted on the reflexive and transitive closure of E as standard:*

$$\exists p_1, p_2, p_3 \left(\Box^*(p_1 \vee p_2 \vee p_3) \wedge \Box^* \bigwedge_{i \neq j} \neg(p_i \wedge p_j) \wedge \bigwedge_{1,2,3} \Box^*(p_i \to \neg \Diamond p_i) \right) \tag{4.2}$$

The truth of this formula in a vertex $v \in G$ implies that (i) *all vertices in the subgraph generated by v are either p_1, p_2, or p_3;* (ii) *each vertex has at most one colour; and* (iii) *no two adjacent vertices have the same colour. Thus, the subgraph generated by v is 3-colorable. Observe that frame $\mathcal{G}$ in Figure 4.1(a) is indeed 3-colorable, and since states w_1 and u_1 are bisimilar, as an immediate consequence of Theorem 4.7, also frame $\mathcal{G}'$ is 3-colorable.*

To illustrate further the (in)expressivity of SOPML through simulations, we consider one more graph-theoretic property: the existence of a Hamiltonian path, i.e., a path that visits all vertices in a graph exactly once. Again, frame $\mathcal{G}$ in Figure 4.1(a) has a Hamiltonian path w_1, w_2, w_3. On the other hand, frame $\mathcal{G}'$ in Figure 4.1(b) has no such path. Since $\mathcal{G}$ and $\mathcal{G}'$ are bisimilar, the following result immediately follows.

Lemma 4.9. *The property of having a Hamiltonian path is not expressible in SOPML.*

Indeed, it is known that such property is expressible in the language MSO$_2$, an extension of MSO, which is strictly more expressive than SOPML [58, Proposition 5.13].

Discussion. We now compare our definition of (bi)simulation for SOPML, with the standard notion of (bi)simulation for PML [47]. Observe that if a frame $\mathcal{F}'$ simulates $\mathcal{F}$ in SOPML, with simulation pair (σ, Σ), then for every model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ based on $\mathcal{F}$, model $\mathcal{M}' = \langle \mathcal{F}', \Sigma(V) \rangle$ on $\mathcal{F}'$ PML-simulates $\mathcal{M}$. In particular, if $\sigma(w, w')$ then for every $v \in W$, $a \in I$, $R_a(w, v)$ implies that $\sigma(v, v')$ for some $v' \in R'_a(w')$ by condition *(ii)*.1 in Definition 4.1. Moreover, $w \in V(p) \in D$ iff $w' \in \Sigma(V)(p) \in D'$ by conditions *(i)* and *(ii)*.2. Therefore, if $\mathcal{M}'$ satisfies any universal formula ϕ in PML, then ϕ also holds in $\mathcal{M}$. Hence, Definition 4.1 of simulation for frames in SOPML is indeed a generalisation of the model-theoretic notion in PML. Furthermore, if frames $\mathcal{F}'$ and $\mathcal{F}$ are bisimilar in SOPML, with bisimulation pair (ω, Ω), then models $\mathcal{M} = \langle \mathcal{F}, V \rangle$ and $\mathcal{M}' = \langle \mathcal{F}', \Omega(V) \rangle$ are also bisimilar in PML. Likewise, models $\mathcal{M}' = \langle \mathcal{F}', V' \rangle$ and $\mathcal{M} = \langle \mathcal{F}, \Omega^{-1}(V) \rangle$ are PML-bisimilar as well. Also in this case, SOPML bisimulations on frames generalise PML bisimulations on models.

4.1.1 Abstraction

This section is devoted to the definition of a notion of abstraction for Kripke frames. Abstractions are deemed useful for system verification, as they allow to ignore some selected features of the system, thus focusing only on the properties relevant for the verification task [54]. Indeed, a key fact about abstractions is that they simulate the original system. Hereafter we prove such a result for SOPML, starting with a family of equivalence relations on states.

Definition 4.10 (Equivalence)**.** Given a frame $\mathcal{F}$, consider an equivalence relation $\sim$ on W such that for every state $w, w' \in W$, $w \sim w'$ implies that for every $U \in D$, $w \in U$ iff $w' \in U$. Further, we denote by $[w] = \{w' \in W \mid w' \sim w\}$ the equivalence class of w in $\mathcal{F}$, and for a set $U \subseteq W$, we let $[U]$ be $\{[w] \mid w \in U\}$.

Clearly, if we replace 'implies' in Definition 4.10 by 'iff', we obtain the coarsest equivalence relation satisfying the conditions therein.

Definition 4.11 (Abstraction)**.** Given a frame $\mathcal{F}$, the *abstraction* $\mathcal{F}^A = \langle W^A, D^A, R^A \rangle$ of $\mathcal{F}$ (according to equivalence relation $\sim$) is the frame such that

- $W^A = \{[w] \mid w \in W\}$;
- $D^A = \{[U] \mid U \in D\}$;
- for every $a \in I$, $R^A_a([w], [w'])$ iff $R_a(v, v')$ for some $v \in [w]$, $v' \in [w']$.

Notice that the coarsest abstraction $\mathcal{F}^A$ is finite whenever the interpretation domain D in $\mathcal{F}$ is, and of size $|W^A| = \mathcal{O}(D)$ at most.

Example 4.6. *To illustrate abstractions, we show that the frame $\mathcal{G}$ in Example 4.1 is (isomorphic to) the coarsest abstraction $\mathcal{G}'^A$ of $\mathcal{G}'$. First of all, two worlds u_s and*

$u_{s'}$ are equivalent according to the coarsest equivalence $\sim$ iff for all $U'_n \in D'$, $u_s \in U'_n$ iff $u_{s'} \in U'_n$, iff last(s) = last(s'). So, in abstraction $\mathcal{G}'^A$ we have three equivalence classes $[u_{t\cdot 1}]$, $[u_{t\cdot 2}]$, and $[u_{t\cdot 3}]$, for sequences $t \in \{1,2,3\}^*$ beginning with 1. As to the accessibility relations, $R'^A_i([u_{t\cdot n}][u_{t'\cdot m}])$ iff for $u_{t\cdot n}$, $u_{t'\cdot m}$ in W', $R'_i(u_{t\cdot n}, u_{t'\cdot m})$, that is, $t' = t\cdot n$ and $R_i(w_n, w_m)$. Hence, for instance, for agent a, we have $R'^A_a([u_{t\cdot 1}][u_{t'\cdot 3}])$ and $R'^A_a([u_{t\cdot 3}][u_{t'\cdot 1}])$, as required. Finally, $D'^A = \{[U'_n] \mid U'_n \in D'\} = \{\{[u_{t\cdot 1}]\}, \{[u_{t\cdot 2}]\}, \{[u_{t\cdot 3}]\}\}$. Clearly, the abstraction $\mathcal{G}'^A$ of $\mathcal{G}'$ is isomorphic to $\mathcal{G}$, with mapping $w_i \mapsto [u_{t\cdot i}]$ for $i = 1,2,3$.

We now extend a standard result in modal logic, namely that abstractions are indeed simulations, to SOPML.

Lemma 4.12. *Given a frame $\mathcal{F}$ with abstraction $\mathcal{F}^A$, the pair of mappings $w \mapsto [w]$ and $U \mapsto [U]$ is a simulation.*

We remark that the abstraction $\mathcal{F}^A$ of a full frame $\mathcal{F}$ is isomorphic to $\mathcal{F}$. In fact, for every $w \in W$, the set $\{w\}$ belongs to D, and since $w \sim w'$ iff for all $U \in D$, $w \in U$ iff $w' \in U$, $w \sim w'$ implies in particular that $w \in \{w'\}$, that is, $w = w'$. As a consequence, $w \mapsto \{w\}$ is the only simulation on states between $\mathcal{F}$ and $\mathcal{F}^A$, and it is also an isomorphism. Further, in Example 4.6 we observed that frame $\mathcal{G}$ is (isomorphic to) the coarsest abstraction of $\mathcal{G}'$. Hence, Lemma 4.12 provides an alternative proof of the fact that $\mathcal{G}$ simulates $\mathcal{G}'$, that was discussed in Example 4.4.

The following corollary follows immediately from Lemmas 4.3 and 4.12.

Corollary 4.13. *Let $\mathcal{F}$ be a frame with abstraction $\mathcal{F}^A$. For every universal formula $\varphi \in \mathcal{L}_{a-sopml}$,*

$$(\mathcal{F}^A, [w]) \vDash \varphi \quad \textit{implies} \quad (\mathcal{F}, w) \vDash \varphi$$

The results presented above have an impact that goes beyond their theoretical interest. As an example, we observed that relevant properties P of frames (such as reflexivity, transitivity, symmetry, etc.) are definable in propositional modal logic in the sense that for some formula ϕ in PML, a frame $\mathcal{F}$ validates ϕ iff $\mathcal{F}$ satisfies property P. In SOPML more properties become frame-definable within the class of full frames. For instance, in Section 3.2 we showed that a full frame $\mathcal{F}$ is irreflexive iff $\mathcal{F} \vDash \exists p(\Box p \land \neg p)$. On the other hand, whenever we consider the class of all frames, several properties remain non-definable. For instance, the frame $\mathcal{G}$ in Example 4.1 is symmetric, while $\mathcal{G}'$ is irreflexive. Since, $\mathcal{G}$ and $\mathcal{G}'$ are bisimilar, and therefore satisfy the same formulas in SOPML, we conclude that neither of these properties is definable in the class of all frames. Such results provide us with further knowledge on the expressive power of SOPML.

4.2 Simulation Games for SOPML

The relation between (bi)simulations and zero-sum games is well-known in propositional modal logic [47], where bisimulation games are routinely applied to derive (in)expressivity results. In this section we introduce (bi)simulation games for SOPML. Similarly to the case for PML, the existence of a winning strategy for Duplicator guarantees the preservation of (universal) formulas in SOPML. We start by considering simulation games played by Spoiler and Duplicator.

Definition 4.14 (Simulation Game)**.** A *simulation game* G starting from pointed frames $(\mathcal{F}, w)$ and $(\mathcal{F}', w')$ is defined as follows. Let $(\mathcal{F}, v, \vec{U})$, $(\mathcal{F}', v', \vec{U}')$ be the current state of the game, where $v \in W$ (respectively, $v' \in W'$) and $\vec{U}$ (respectively, $\vec{U}'$) is a possibly empty tuple of sets in D (respectively, D').

Then, the game proceeds according to the following rules:

1. Either Spoiler picks a set $U \in D$ and Duplicator has to reply with a set $U' \in D'$ such that $v \in U$ iff $v' \in U'$. The new state of the game is $(\mathcal{F}, v, \vec{U} \cdot U)$, $(\mathcal{F}', v', \vec{U}' \cdot U')$.
2. Or, for some $a \in I$, Spoiler picks a state $u \in R_a(v)$ and Duplicator has to reply with state $u' \in R'_a(v')$ such that for every i, $u \in U_i$ iff $u' \in U'_i$. The new state of the game is $(\mathcal{F}, u, \vec{U})$, $(\mathcal{F}', u', \vec{U}')$.

If Duplicator cannot match a Spoiler's move, then Spoiler wins the game. Otherwise, Duplicator wins the game. A *winning strategy* is a strategy whereby Duplicator can reply to all of Spoiler's moves, thus winning the game.

We now show that the existence of a winning strategy is tantamount to the existence of a simulation.

Theorem 4.15. *Duplicator has a winning strategy for the simulation game starting in* $(\mathcal{F}, w)$*,* $(\mathcal{F}', w')$ *iff* $(\mathcal{F}, w) \preceq (\mathcal{F}', w')$*.*

Notice that this result is in marked contrast with [44], where the notion of simulation there provided entails the existence of a winning strategy for Duplicator, but the existence of a winning strategy does not imply the existence of a simulation. On the contrary, here we have a perfect match between the two concepts. This is due to the novel notion of simulation put forward in Definition 4.1.

As a direct consequence of Theorem 4.3 and 4.15, we obtain that the existence of a winning strategy for Duplicator implies the preservation of formulas in $\mathcal{L}_{a-sopml}$.

Corollary 4.16. *If Duplicator has a winning strategy for the game starting in state $(\mathcal{F}, w)$, $(\mathcal{F}', w')$, then for every universal formula $\varphi \in \mathcal{L}_{a-sopml}$,*

$$(\mathcal{F}', w') \vDash \varphi \quad \textit{implies} \quad (\mathcal{F}, w) \vDash \varphi$$

Next, simulation games can be easily generalized to bisimulation games.

Definition 4.17 (Bisimulation Game)**.** A *bisimulation game G* starting from pointed frames $(\mathcal{F}, w)$ and $(\mathcal{F}', w')$ is defined as follows. Let $(\mathcal{F}, v, \vec{U})$, $(\mathcal{F}', v', \vec{U}')$ be the state of the game, where $v \in W$ (respectively, $v' \in W'$) and $\vec{U}$ (respectively, $\vec{U}'$) is a possibly empty tuple of sets in D (respectively, D').

Then, the game proceeds according to the following rules:

1. Either Spoiler picks a set $U \in D$ (respectively, $U' \in D'$) and Duplicator has to reply with a set $U' \in D'$ (respectively, $U \in D$) such that $v \in U$ iff $v' \in U'$. The new state of the game is $(\mathcal{F}, v, \vec{U} \cdot U)$, $(\mathcal{F}', v', \vec{U}' \cdot U')$.
2. Or, for some $a \in I$, Spoiler picks a state $u \in R_a(v)$ (respectively, $u' \in R'_a(v')$) and Duplicator has to reply with state $u' \in R'_a(v')$ (respectively, $u \in R_a(v)$) such that for every i, $u \in U_i$ iff $u' \in U'_i$. The new state of the game is $(\mathcal{F}, u, \vec{U})$, $(\mathcal{F}', u', \vec{U}')$.

As above, if Duplicator cannot match a Spoiler's move, then Spoiler wins the game. Otherwise, Duplicator wins the game. A winning strategy is defined as usual.

By adapting the proof of Theorem 4.15, we can prove the following equivalence between bisimulations and bisimulation games.

Theorem 4.18. *Duplicator has a winning strategy for the bisimulation game starting in $(\mathcal{F}, w)$, $(\mathcal{F}', w')$ iff $(\mathcal{F}, w) \approx (\mathcal{F}', w')$.*

Again, the existence of a winning strategy for Duplicator matches the existence of a bisimulation pair. By Theorem 4.7 and 4.18 we are then able to prove the following preservation result.

Corollary 4.19. *If Duplicator has a winning strategy for the bisimulation game starting in state $(\mathcal{F}, w)$, $(\mathcal{F}', w')$, then for every formula $\varphi \in \mathcal{L}_{sopml}$,*

$$(\mathcal{F}', w') \vDash \varphi \quad \textit{iff} \quad (\mathcal{F}, w) \vDash \varphi$$

We conclude by discussing the two groups of preservation results. Both Theorems 4.3 and 4.7 and Corollaries 4.16 and 4.19 provide results on the preservation of (the universal fragment of) SOPML. However, (bi)simulations define global concepts, as these are defined on the whole state space $W \times W'$ and $D \times D'$; while games are played locally, as at each point in the game the players have only a local view on the frames, centred on a pair of states and finite sequences of sets. Hence, the nature of these two notions is profoundly different. However, they are provably equivalent by Theorems 4.15 and 4.18. We envisage different applications for the two notions. For instance, (bi)simulations are typically used to prove inexpressibility results; while games can be used to show that two frames are not bisimilar, by providing moves for Spoiler to which Duplicator cannot reply. These applications are discussed in the following section.

4.3 Simulations and Expressivity

In this section we explore the expressivity of SOPML, also by using the (bi)simulations and (bi)simulation games introduced in Section 4.1 and 4.2. We focus on some temporal and spatial. In what follows we say that a property P is *expressible* in a language $\mathcal{L}$ and class $\mathcal{K}$ of frames iff for some formula $\phi \in \mathcal{L}$, $\mathcal{K} \vDash \phi$ iff $\mathcal{K}$ has property P. Sometimes we omit either $\mathcal{L}$ or $\mathcal{K}$, whenever these are clear from the context.

First of all, consider Dedekind-completeness of a total order $\leq$, i.e., a total, transitive, and antisymmetric binary relation: a totally ordered set is *Dedekind-complete* if every non-empty subset that has an upper bound, has a least upper bound. We recall that the Dedekind-completeness of the real numbers is not expressible in PML: the proof makes use of a propositional bisimulation between the structure $(\mathbb{R}, \leq)$ of reals and the rationals $(\mathbb{Q}, \leq)$ [12]. On the other hand, in SOPML we can express Dedekind-completeness by means of the following formula, where modal operators $\Box$ and $\Diamond$ are interpreted on the strict linear order $<$, while $\blacksquare\phi$ (respectively, $\blacklozenge\phi$) are shorthands for $\phi \wedge \Box\phi$ (respectively, $\phi \vee \Diamond\phi$). Also, we recall that relation $x \leq y$, which is used for the interpretation of operators $\blacksquare$ and $\blacklozenge$, can be defined as $x < y$ or $x = y$.

$$\forall p\,((\blacklozenge p \wedge \blacklozenge \blacksquare \neg p) \rightarrow \tag{4.3}$$

$$(\blacklozenge(p \wedge \Box\neg p) \vee \tag{4.4}$$

$$\exists q\,(\blacksquare(q \leftrightarrow \blacksquare\neg p) \wedge \tag{4.5}$$

$$\exists s\,(\blacklozenge s \wedge \blacksquare(s \rightarrow q) \wedge \tag{4.6}$$

$$\blacksquare(\neg s \wedge q \rightarrow \blacksquare\neg s) \wedge \blacksquare(s \rightarrow \Box\neg s))))) \tag{4.7}$$

This formula states that (4.3) for every non-empty and upper bounded set p, either (4.4) p has a greatest element, or (4.5) there exists a set q of "strict" upper bounds, (4.6) which includes a non-empty subset s (4.7) that is a singleton and the least upper

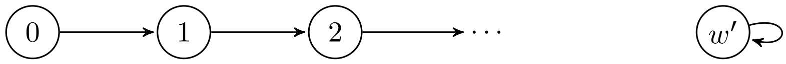


FIGURE 4.2: Frames $\mathcal{G}_1$ and $\mathcal{G}_2$ (the D-components are omitted for clarity).

bound. Thus, the validity of this formula when operators $\Box$ and $\Diamond$ are interpreted on $<$ implies that the corresponding total order $\leq$ is Dedekind-complete.

Intuitively, formula (4.3)-(4.7) fails in $(\mathbb{Q}, \leq)$ since, for instance, the set $\{q \in \mathbb{Q} \mid q < \sqrt{2}\}$ is non-empty and upper bounded, and therefore satisfy (4.3)-(4.5). However, it has no least upper bound $s \in \mathbb{Q}$ to satisfy (4.6) and (4.7).

On the other hand, the identity relation is clearly a simulation between structures $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$ seen as full frames, i.e., $(\mathbb{Q}, \leq) \preceq (\mathbb{R}, \leq)$, and if Dedekind-completeness were expressible as a formula ϕ in universal SOPML, $(\mathbb{R}, \leq) \models \phi$ would imply $(\mathbb{Q}, \leq) \models \phi$, a contradiction. Hence, we immediately obtain the following inexpressibility result.

Lemma 4.20. *Dedekind-completeness is not expressible in the universal fragment $\mathcal{L}_{a-sopml}$.*

As a further example, we prove that neither finiteness nor infinity of the state space W are expressible in boolean frames. This is in line with the situation in PML. Indeed, consider frame $\mathcal{G}_1 = \langle \mathbb{N}, succ, \{\mathbb{N}, \varnothing\} \rangle$ of the naturals with the successor relation and the reflexive-point frame $\mathcal{G}_2 = \langle \{w'\}, \{(w', w')\}, \{\{w'\}, \varnothing\} \rangle$ in Figure 4.2, which are boolean by definition of D_1 and D_2. In particular, the relations ω mapping every natural $n \in \mathbb{N}$ to w', and Ω mapping $\mathbb{N}$ to $\{w'\}$ and the empty set $\varnothing$ to itself, form a bisimulation pair. Equivalently, it is easy to see that Duplicator has a winning strategy in the game starting from state $(\mathcal{G}_1, n)$, $(\mathcal{G}_2, w')$, for every $n \in \mathbb{N}$: Duplicator has only to reply with w' to any $m \in \mathbb{N}$ chosen by Spoiler, and with $\{w'\}$ (respectively, $\varnothing$) whenever Spoiler chooses $\mathbb{N}$ (respectively, $\varnothing$). Thus, $\mathcal{G}_1$ and $\mathcal{G}_2$ validate the same formulas in SOPML. However, $\mathcal{G}_1$ is infinite while $\mathcal{G}_2$ is finite. As consequence, we obtain the following result.

Lemma 4.21. *Neither finiteness nor infinity are expressible in the class of boolean frames.*

To conclude our brief review of expressivity results in SOPML, we show that for the sublanguage of $\mathcal{L}_{sopml}$ without the reflexive and transitive closure operator $\Box^*_A$, finiteness is not even expressible in full frames. For $n \in \mathbb{N}$, let $[n]$ be the set $\{0, \ldots, n\}$, $\mathcal{G}_n$ the frame $\langle [n], succ, 2^{[n]} \rangle$, and $\mathcal{G}_\mathbb{N} = \langle \mathbb{N}, succ, 2^\mathbb{N} \rangle$ the frame isomorphic to the natural numbers. Both $\mathcal{G}_\mathbb{N}$ and each $\mathcal{G}_n$ are full. Let G be the class of all frames $\mathcal{G}_n$, for $n \in \mathbb{N}$, and consider the following result.

Lemma 4.22. *In the fragment of $\mathcal{L}_{sopml}$ without operator $\Box^*_A$, the theory $\mathsf{Th}(G)$ is a subset of $\mathsf{Th}(\mathcal{G}_\mathbb{N})$.*

Hence, if ϕ expressed 'being finite', then it would be valid in G, and hence also in $\mathcal{G}_\mathbb{N}$, a contradiction. Thus, finiteness is not expressible even in the class of full frames.

In this section we made use of (bi)simulations and (bi)simulations games to show that SOPML can express notions, such as Dedekind-completeness, that are not expressible in PML; whereas other properties, such as finiteness, cannot even be expressed in SOPML. Together with the remarks in Section 4.1 on 3-colorability and the existence of Hamiltonian paths, these results provide us with some interesting insight on the application of model-theoretic techniques to the analysis of the expressivity of SOPML.

4.4 Discussion and Related Literature

In this chapter we introduced suitable notions of (bi)simulation and proved that they preserve the satisfaction of (universal) SOPML. Then, we defined game-theoretical counterparts to (bi)simulations and showed that the two approaches are equivalent. This is in marked contrast with [44], which put forward a different, stronger notion of (bi)simulation. We remarked that, while set-theoretical (bi)simulations might be more appropriate to prove inexpressibility results, their game-theoretic counterparts might be better computationally to actually show whether two frames are bisimilar. Finally, we made use of (bi)simulations to obtain some inexpressibility results. Specifically, we showed that being finite and having a Hamiltonian path are not expressible in SOPML; while other properties, viz. Dedekind-completeness and 3-colorability, are actually expressible. We conclude that SOPML can indeed be used as a modelling language for artificial intelligence, particularly for temporal and spatial reasoning, as well as to describe higher-level knowledge of agents, that is, the knowledge agents have about other agents' knowledge and beliefs. In this respect, we reckon that the development of model-theoretic techniques is key for applications.

In our opinion the results presented in this chapter raise a number of interesting questions. We believe that one in particular deserves more attention. The Van Benthem theorem is a well-known result in model theory, stating that modal logic is the bisimulation-invariant fragment of first-order logic [109]. In the light of the notion of bisimulation provided above, it makes sense to ask the same question in the present context: is SOPML the bisimulation-invariant fragment of second-order logic? We leave this problem open for future work.

Part II

Reasoning about Knowledge and Change

Chapter 5

Second-order Propositional Announcement Logic

Modal epistemic logics have historically been focused primarily on the static properties of knowledge [83, 125]. Indeed, in Chapter 3 and 4 we introduced the language of second-order propositional epistemic logic, which is suitable to express static properties of agents in frames. However, nowadays an increasing interest is directed towards the dynamics of knowledge: how is individual knowledge affected by factual change, information exchange, or knowledge updates?

These questions have given rise to temporal epistemic logics [64] and dynamic epistemic logics [122], among others. A particular form of dynamics appearing in epistemic logic deals with truthful public announcements, i.e., publicly observable information that is assumed to be reliable. These occur in many multi-agent scenarios: card games, the muddy children puzzle, security protocols [117]. Public announcements are executed as model refinements on the epistemic state of the agents listening to them. This idea has been formalised into *public announcement logic* (PAL) [71, 100], which extends epistemic logic with formulas of type $[\phi]\psi$, to express that after announcing ϕ publicly, ψ holds.

Once public announcements are introduced, it is legitimate to wonder what remains true after arbitrary announcements (a property known as *preservation*), or what can be known by agents provided some suitable announcement (*knowability*). In this paper we extend the framework of PAL to deal exactly with this sort of issues. We introduce *second-order propositional announcement logic* (SOPAL), which extends PAL with propositional quantification. As a result, the knowability of formula ϕ (by an agent a) becomes intuitively expressible in SOPAL as

$$\phi \to \exists p \langle p \rangle K_a \phi \tag{5.1}$$

that is, if ϕ is true, then after some truthful announcement p, agent a knows that ϕ is true.

The main contributions in this chapter can be summarized as follows. We first introduce the syntax and semantics of SOPAL. Then, we compare SOPAL with *arbitrary public announcement logic* (APAL) [9, 10], an extension of PAL also including arbitrary announcements, and show in which sense SOPAL is strictly more expressive than APAL. We further provide reduction equivalences to eliminate announcements from SOPAL, and thus show that SOPAL is as expressive as *second-order propositional epistemic logic* (SOPEL) introduced in Chapter 3. This result allows us to transfer both the complete axiomatisation and the decidable model checking problem in Chapter 3 for SOPEL to SOPAL. Moreover, we prove that, even if they are equally expressive, SOPAL is exponentially more succinct than SOPEL. Finally, we apply SOPAL to multi-agent game scenarios and specify the dynamic epistemic notions of knowable, preserved, and successful formula.

As a result, we have a powerful logic, with nice computational properties, such as a complete axiomatisation, a decidable model checking problem, and a wide range of interesting applications.

5.1 The Formal Framework

In this section we introduce the syntax and semantics of second-order propositional announcement logic. Also in this chapter we make use of a set AP of atomic propositions and a finite set I of indexes for agents. Formulas in SOPAL are defined as follows.

Definition 5.1 (SOPAL)**.** The formulas in SOPAL are defined in BNF as follows, for $p \in AP$ and $a \in I$:

$$\psi \quad ::= \quad p \mid \neg\psi \mid \psi \to \psi \mid K_a\psi \mid C_A\psi \mid [\psi]\psi \mid \forall p\psi$$

The language $\mathcal{L}_{sopal}$ of SOPAL extends SOPEL with announcement formulas $[\psi]\phi$, whose reading is that "after (truthfully) announcing ψ, ϕ is true". Equivalently, SOPAL can be thought of as an extension of PAL with propositional quantification. As standard, formulas $K_a\psi$ can be introduced as $C_{\{a\}}\psi$. Nonetheless, here we assume $K_a\psi$ as primitive in analogy to SOPML. Further, the dual operator $\langle\psi\rangle$ is defined as $\neg[\psi]\neg$.

Second-order propositional announcement logic extends a number of well-known formalisms. The language $\mathcal{L}_{pal}$ of Public Announcement Logic is obtained by removing inductive construct $\forall p\psi$ in Definition 5.1; language $\mathcal{L}_{el}$ without clause $[\psi]\psi$ as well is epistemic logic, and language $\mathcal{L}_{pl}$ without clauses $K_a\psi$ and $C_A\psi$ as well is propositional logic. Also, language $\mathcal{L}_{sopel}$ obtained by removing clause $[\psi]\psi$ in Definition 5.1 is second-order propositional epistemic logic from Chapter 3; while the language $\mathcal{L}_{qbf}$ of quantified boolean formulas is obtained from $\mathcal{L}_{sopel}$ by removing clause $K_a\psi$ and $C_A\psi$ as well.

In the following we consider for comparison also the language of arbitrary public announcement logic [9, 10], obtained by extending PAL with formulas $\Box\psi$:

$$\psi \quad ::= \quad p \mid \neg\psi \mid \psi \to \psi \mid K_a\psi \mid C_A\psi \mid [\psi]\psi \mid \Box\psi$$

where $\Box\psi$ is here read as "after every truthful announcement, ϕ holds". Hereafter we show that SOPAL is rich enough to express APAL through quantification. We summarize the main (syntactic) language inclusions in the following schema, where languages in boldface, which have already been introduced in Chapter 3, are mentioned for comparison.

$$\begin{array}{ccccccccccc}
 & & & & & & & & & & \mathcal{L}_{apal} \\
 & & & & & & & & & \subseteq & \\
 & & & & & & & & \mathcal{L}_{pal} & & \\
 & & & & & & & \subseteq & & \subseteq & \\
 & & & & & & \mathcal{L}_{\mathbf{el}} & & & & \mathcal{L}_{sopal} \\
 & & & & & \subseteq & & \subseteq & & \subseteq & \\
\mathcal{L}_{\mathbf{ap}} & \subseteq & \mathcal{L}_{\mathbf{pl}} & & & & & & \mathcal{L}_{\mathbf{sopel}} & & \\
 & & & & & \subseteq & & \subseteq & & & \\
 & & & & & & \mathcal{L}_{\mathbf{qbf}} & & & &
\end{array}$$

Example 5.1. *To illustrate the expressive power of SOPAL, we discuss various epistemic notions. In public announcement logic a formula ϕ is said to be* preserved *if ϕ is true after any announcement. In SOPAL we can capture this by requiring that the following formula holds:*

$$\phi \to \forall q[q]\phi \tag{5.2}$$

We informally remark that (5.2) does not hold for Moore's formula $p \wedge \neg K_a p$. However, in SOPAL we can define a suitable restriction of (5.2), concerning epistemic announcements only:

$$\phi \to \forall p[K_a p]\phi \tag{5.3}$$

In Example 5.2 we show that (5.3), differently from (5.2), does hold for Moore's formulas.

Another notion of interest in PAL is knowability: a formula ϕ is knowable *(by agent a) iff after some announcement, a knows ϕ. We remarked that this notion can be stated formally as (5.1). Clearly, Moore's formulas are not knowable. We will discuss and compare preserved, knowable, and other classes of formulas in more detail in Section 5.3.1.*

In the rest of the section we extend to SOPAL the notions of free atom and substitution defined on SOPML in Chapter 3, as well as auxiliary lemmas that are necessary throughout the rest of the chapter.

Definition 5.2 (Free Atoms)**.** The set $fr(\phi)$ of free atoms in a SOPAL formula ϕ is recursively defined as in Definition 3.2, together with the following clause for announcement formulas:

$$fr([\phi]\phi') \quad = \quad fr(\phi) \cup fr(\phi')$$

The set $bnd(\phi)$ of bound atoms in ϕ is standardly introduced as the set of all atoms q appearing in the scope of some quantifier Qq. By renaming bound atoms, here also we assume without loss of generality that for every formula ϕ, sets $fr(\phi)$ and $bnd(\phi)$ are disjoint. We now define when an atom is free for substitution.

Definition 5.3 (Free for ...)**.** A formula ψ is *free* for atom p in ϕ iff p does not appear in ϕ within the scope of any quantifier Qq, whenever q is free in ψ, and if $\phi = [\phi']\phi''$ and $p \in fr(\phi')$, then $\psi \in \mathcal{L}_{qbf}$. Alternatively, we can define whether ψ is *free* for p in ϕ as in Definition 3.3, together with the following clause:

if $\phi = [\phi']\phi''$ then ψ is free for p in ϕ iff it is in ϕ' and ϕ'' and $p \in fr(\phi')$ implies $\psi \in \mathcal{L}_{qbf}$

Finally, we extend the notion of substitution to SOPAL.

Definition 5.4 (Substitution)**.** If ψ is free for p in ϕ, then we inductively define the substitution $\phi[p/\psi]$ as in Definition 3.4, together with the following clause:

$$([\phi]\phi')[p/\psi] \quad = \quad [\phi[p/\psi]]\phi'[p/\psi]$$

Notice that we make use of square brackets [,] for both substitutions and announcement operators, as both usages are standard. The context will disambiguate.

The restriction on substitution can be deemed quite strong, as we allow only for the substitution of quantified boolean formulas in announcements. Intuitively, this is necessary because, while $[p]p$ is valid, substitution $[q \wedge \neg K_a q](q \wedge \neg K_a q)$ is not. Nonetheless, we will see that, also with such restriction, all results mentioned in the introduction are provable.

To interpret SOPAL formulas we make use of the epistemic Kripke frames and models introduced in Chapter 3, where for every agent index $a \in I$, $R_a \subseteq W^2$ is an equivalence relation on the set W of worlds. Nonetheless, for technical reasons, in Section 5.4 we will also consider general frames, whose accessibility relations are not necessarily equivalences. Further, hereafter we consider classes $\mathcal{K}^e_{all}$ of all (epistemic) Kripke frames, $\mathcal{K}^e_{bool}$ of all (epistemic) boolean frames, $\mathcal{K}^e_{modal}$ of all(epistemic) modal frames, and $\mathcal{K}^e_{full}$ of all (epistemic) full frames introduced in Chapter 3, where the accessibility relations are equivalences.

We now define the notion of satisfaction for SOPAL.

Definition 5.5 (Satisfaction)**.** We define whether model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ *satisfies* formula φ at world w, or $(\mathcal{M}, w) \vDash \varphi$, as in Definition 3.7, together with the following clause for annoucement formulas :

$$(\mathcal{M}, w) \vDash [\psi]\psi' \quad \text{iff} \quad (\mathcal{M}, w) \vDash \psi \text{ implies } (\mathcal{M}_{|\psi}, w) \vDash \psi'$$

where the *refinement* $\mathcal{M}_{|\psi} = \langle W_{|\psi}, D_{|\psi}, R_{|\psi}, V_{|\psi} \rangle$ of model $\mathcal{M}$ according to ψ is defined as: *(i)* $W_{|\psi} = \{v \in W \mid (\mathcal{M}, v) \vDash \psi\}$; *(ii)* $D_{|\psi} = \{U_{|\psi} = U \cap W_{|\psi} \mid U \in D\}$; *(iii)* $R_{|\psi,a} = R_a \cap W^2_{|\psi}$; and *(iv)* $V_{|\psi}(p) = V(p) \cap W_{|\psi}$ for every $p \in AP$.

We recall that, given formula $\phi \in \mathcal{L}_{sopal}$, $[\![\phi]\!]_{\mathcal{M}} = \{w \in W \mid (\mathcal{M}, w) \vDash \phi\}$ is the satisfaction set in model $\mathcal{M}$. We omit the subscript $\mathcal{M}$ whenever clear from the context. We then state the following useful extension of Lemma 3.8(2a) on satisfaction sets. In the following we extend function $\widehat{\cdot} : X \to Y$ from language sort symbols to type symbols as follows: $\widehat{ap} = all$; $\widehat{pl} = bool$; $\widehat{el} = modal$; $\widehat{pal} = modal$; and $\widehat{sopal} = full$.

Lemma 5.6. *For every formula $\phi \in \mathcal{L}_x$, for $x = ap$ (resp. pl, el, pal, $sopal$), and for $\mathcal{M} = \langle \mathcal{F}, V \rangle$ with $\mathcal{F} \in \mathcal{K}^e_{\widehat{x}}$, we have that $[\![\phi]\!]_{\mathcal{M}} \in D$.*

We observe that the case of $x = pal$ follows from the fact that PAL is as expressive as epistemic logic [100].

By Lemma 5.6 we can prove the following result, which guarantees that Definition 5.5 is well-defined in the sense that the refinement $\mathcal{M}_{|\phi}$ belongs to the same class as model $\mathcal{M}$.

Lemma 5.7. *If a model $\mathcal{M}$ is boolean (respectively, modal, full), then the model refinement $\mathcal{M}_{|\phi}$ for $\phi \in \mathcal{L}_{sopal}$ is also boolean (respectively, modal, full).*

To conclude, in the following we consider the standard notions of truth and validity introduced in Chapter 3.

Example 5.2. *To illustrate the semantics of SOPAL, we consider the following instance of (5.3) for Moore's formula* $p \wedge \neg K_a p$*:*

$$(p \wedge \neg K_a p) \to \forall q[K_a q](p \wedge \neg K_a p) \tag{5.4}$$

and show that (5.4) is a validity in all frames.

Suppose that $(\mathcal{M}, w) \vDash p \wedge \neg K_a p$*. Then, for some* $w' \in R_a(w)$ *different from* w*,* $(\mathcal{M}, w') \vDash \neg p$*. Also, if* $(\mathcal{M}^q_U, w) \nvDash [K_a q](p \wedge \neg K_a p)$ *for some reinterpretation* $\mathcal{M}^q_U$*, then we have* $((\mathcal{M}^p_U)_{|K_a q}, w) \nvDash p \wedge \neg K_a p$*, that is,* $(\mathcal{M}^p_U, w) \vDash K_a q$ *but* w' *must not appear in* $(\mathcal{M}^p_U)_{|K_a q}$*, i.e.,* $(\mathcal{M}^p_U, w') \nvDash K_a q$*. But then,* $(\mathcal{M}^p_U, w) \nvDash K_a q$ *either. A contradiction.*

Thus, even though Moore's formulas are not preserved under arbitrary announcements, they are indeed preserved by arbitrary epistemic *announcements as in (5.4).*

Example 5.3. *We elaborate on the example of [121, Section 4.3], and consider a simple card game with three players in* $I = \{1, 2, 3\}$*. The cards are identified by their colour: red (r), white (w), and blue (b). In AP we consider atoms* r_i*,* w_i*,* b_i*, for* $i \in I$*, where intuitively* w_1 *denotes that player 1 holds the white card. Also, all players know the cards of the game, and that each player can see his own card, but not that of the other players. The situation where each player is dealt a card can be modeled by the full model* $\mathcal{M}$ *in Fig. 5.1. The state* rwb *in* $\mathcal{M}$ *denotes that player 1 holds red, 2 holds white, and 3 holds blue. We then have for instance*

$$(\mathcal{M}, \mathsf{rwb}) \vDash r_1 \wedge K_1 r_1 \wedge \neg K_2 r_1 \wedge K_1 \neg K_2 r_1$$

i.e., player 1 *holds red, she knows it, but* 2 *does not, and finally,* 1 *knows that* 2 *does not know that* 1 *holds red.*

In general, for every state s*,*

$$(\mathcal{M}, s) \vDash \exists p \left(p \wedge K_i p \wedge \bigwedge_{j \neq i} \neg K_j p \wedge K_i \bigwedge_{j \neq i} \neg K_j p \right)$$

i.e., every player i *knows something that the other players do not know (and she knows that they do not), namely the value of the card that* i *possesses.*

Now suppose player i *announces publicly the card she has. Such an announcement in state* rwb *leads to the updated model* $\mathcal{M}'$ *in Fig. 5.1. Indeed, for* $q_i \in \{r_i, w_i, b_i\}$ *we have*

$$(\mathcal{M}, s) \vDash q_i \to \exists p \langle K_i p \rangle \left(\bigwedge_{j \neq i} K_j q_i \right)$$

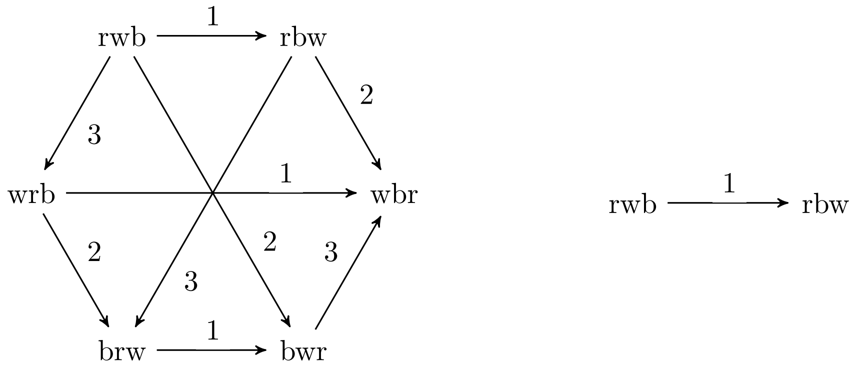


FIGURE 5.1: The full models $\mathcal{M}$ and $\mathcal{M}'$ in Example 5.2 (reflexive edges are omitted for clarity).

that is, there is some proposition (namely, the value $U = R_i(s)$ *of player* i*'s card) that player* i *can truthfully announce, so that any other player knows the value of* i*'s card.*

On the other hand, the mere announcement that player i *knows something is not sufficient to derive the same conclusion, as for every state* $s \in W$, $(\mathcal{M}, s) \vDash \exists p K_i p$, *and therefore* $\mathcal{M}_{|\exists p K_i p} = \mathcal{M}$. *Hence,*

$$(\mathcal{M}, s) \nvDash q_i \to \langle \exists p K_i p \rangle \left(\bigwedge_{j \neq i} K_j q_i \right)$$

Furthermore, the (false) announcement that player i *knows everything implies that the other players know her card:*

$$(\mathcal{M}, s) \vDash q_i \to [\forall p K_i p] \left(\bigwedge_{j \neq i} K_j q_i \right)$$

Indeed, $(\mathcal{M}, s) \nvDash \forall p K_i p$, *and therefore* $(\mathcal{M}, s) \vDash [\forall p K_i p](\bigwedge_{j \neq i} K_j q_i)$ *trivially. However, it is not the case that every truthful announcement pertaining to player* i*'s knowledge entails that the other players know her card:*

$$(\mathcal{M}, s) \nvDash q_i \to \forall p [K_i p] \left(\bigwedge_{j \neq i} K_j q_i \right)$$

as for proposition $U = W$, $(\mathcal{M}^p_U, s') \vDash K_i p$ *for every* $s' \in W$. *But* $((\mathcal{M}^p_U)_{|K_i p}, s) \nvDash \bigwedge_{j \neq i} K_j q_i$, *since* $(\mathcal{M}^p_U)_{|K_i p} = \mathcal{M}^p_U$.

By comparing the formulas above, we clearly see that quantifying inside or outside (epistemic) announcements allows us to express subtle differences in SOPAL.

5.2 Comparison with APAL

In this section we compare SOPAL with APAL, whose original motivation also included the ability to express arbitrary announcements in PAL. The main result of this section is that SOPAL is capable of capturing APAL at the frame level, while the two logics are incomparable at the model level. But first we state an extension of Lemma 3.8 containing auxiliary results that will be routinely applied throughout the paper.

Lemma 5.8. *Let q and ψ be free for p in ϕ.*

1. *For $x = ap$ (resp. pl, el, pal, $sopal$), $\psi \in \mathcal{L}_x$, and $\mathcal{M} \in \mathcal{K}^e_{\widehat{x}}$, $(\mathcal{M}^p_{[\![\psi]\!]_\mathcal{M}}, w) \models \phi$ iff $(\mathcal{M}, w) \models \phi[p/\psi]$*
2. *If $p \in fr(\phi)$ implies $\psi \in \mathcal{L}_{qbf}$, then $(\mathcal{M}^p_{[\![\psi]\!]})_{|\phi} = (\mathcal{M}_{|\phi[p/\psi]})^p_{[\![\psi]\!]}$*
3. *If $V(fr(\phi)) = V'(fr(\phi))$ then $(\mathcal{M}, w) \models \phi$ iff $(\mathcal{M}', w) \models \phi$*
4. *If $V(fr(\psi)) = V'(fr(\psi))$ then $\mathcal{M}_\psi = \mathcal{M}'_\psi$*

According to Lemma 5.8(1), the syntactic notion of substitution $\phi[p/\psi]$ corresponds to the semantic concept of reinterpretation $\mathcal{M}^p_{[\![\psi]\!]}$; while Lemma 5.8(2) specifies the interaction between substitution, reinterpretation and model refinement, namely the refinement $(\mathcal{M}^p_{[\![\psi]\!]})_{|\phi}$ of a reinterpreted model is equal to the reinterpretation $(\mathcal{M}_{|\phi[p/\psi]})^p_{[\![\psi]\!]}$ of the model refined by the substituted formula $\phi[p/\psi]$, provided that $\psi \in \mathcal{L}_{qbf}$ whenever $p \in fr(\phi)$. Moreover, by Lemma 5.8(3-4) models built on the same frame and agreeing on the interpretation of free atoms, also satisfy the same formulas, and their model refinements are equal. These results, which show that quantification in SOPAL is "well-behaved", will be extensively used hereafter.

To compare SOPAL and APAL we recall the clause for interpreting the operator $\Box$ [9]:

$$(\mathcal{M}, w) \models \Box\psi \quad \text{iff} \quad \text{for all } \phi \in \mathcal{L}_{el}, (\mathcal{M}, w) \models [\phi]\psi \tag{5.5}$$

We now prove that, according to (5.5), APAL can be captured within SOPAL in the following sense.

Definition 5.9. Given a class $\mathcal{K}$ of frames, a logic L' is

- *at least as m-expressive as* logic L, or $L \leq_m L'$, iff for any $\phi \in L$, for some $\phi' \in L'$, for any model $\mathcal{M}$ in $\mathcal{K}$,

$$(\mathcal{M}, w) \models \phi \quad \text{iff} \quad (\mathcal{M}, w) \models \phi'$$

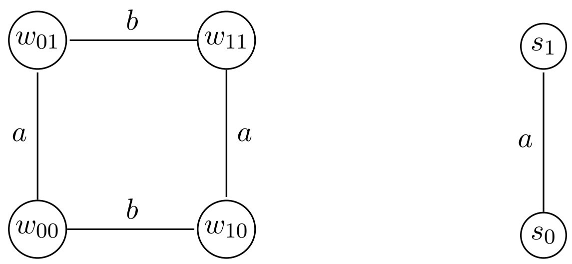


FIGURE 5.2: The full models $\mathcal{M}$ and $\mathcal{M}'$ (reflexive edges are omitted for clarity).

- *at least as f-expressive as* logic L, or $L \leq_f L'$, iff for any $\phi \in L$, for some $\phi' \in L'$, for any frame $\mathcal{F}$ in $\mathcal{K}$,

$$(\mathcal{F}, w) \vDash \phi \quad \text{iff} \quad (\mathcal{F}, w) \vDash \phi'$$

Clearly, each relation $\leq$ is a partial order, and we write $L = L'$ iff $L \leq L'$ and $L' \leq L$, and $L < L'$ iff $L \leq L'$ and $L \neq L'$. Also, $L \leq_m L'$ implies $L \leq_f L'$, but the converse is not true in general.

To investigate the relation between SOPAL and APAL, we start with some preliminary results. First of all, we test the intuition that the operator $\Box$ can be expressed by quantification and announcements.

Lemma 5.10. *Let $\mathcal{M}$ be an epistemic model, then*

$$(\mathcal{M}, w) \vDash \forall p[p]\phi \quad \textit{implies that} \quad (\mathcal{M}, w) \vDash \Box\phi \tag{5.6}$$

However, the converse of (5.6) does not always hold. Consider the full model $\mathcal{M}$ in Fig. 5.2. Formally, we have that $\mathcal{M} = \langle W, R, D, V \rangle$ with $W = \{w_{00}, w_{01}, w_{10}, w_{11}\}$; $R_a = \{(w_{ij}, w_{i'j'}) \mid i = i'\}$; $R_b = \{(w_{ij}, w_{i'j'}) \mid j = j'\}$; $D = 2^W$; and $V(q) = \{w_{ij} \mid j = 0\}$ for every $q \in AP$. We can check that, for every $\psi \in \mathcal{L}_{el}$, $[\![\psi]\!]$ is equal to either W, or $\varnothing$, or $\{w_{ij} \mid j = 0\}$, or $\{w_{ij} \mid j = 1\}$. As a consequence, for every $\psi \in \mathcal{L}_{el}$, $(\mathcal{M}, w_{i0}) \vDash [\psi](K_a q \to K_b K_a q)$, that is, $(\mathcal{M}, w_{i0}) \vDash \Box(K_a q \to K_b K_a q)$. However, for $U = \{w_{00}, w_{01}, w_{10}\}$ we obtain that $(\mathcal{M}^p_U, w_{10}) \nvDash [p](K_a q \to K_b K_a q)$, i.e., $(\mathcal{M}, w_{10}) \nvDash \forall p[p](K_a q \to K_b K_a q)$.

Actually, clause (5.5) for APAL preserves bisimilarity of structures, while Definition 5.5 for SOPAL does not. To see this, consider the full model $\mathcal{M}'$ in Fig. 5.2. We remark without proof that the pointed models $(\mathcal{M}, w_{10})$ and $(\mathcal{M}', s_0)$ are bisimilar [47], and satisfy the same formulas in PAL, and consequently, in APAL. However, we noticed that $(\mathcal{M}, w_{10}) \nvDash \forall p[p](K_a q \to K_b K_a q)$, while it is easy to check that $(\mathcal{M}', s_0) \vDash \forall p[p](K_a q \to K_b K_a q)$.

Incidentally, models $\mathcal{M}$ and $\mathcal{M}'$ above prove that SOPAL is no less expressive than APAL at the level of models. Indeed, if SOPAL $\leq_m$ APAL, then for $\phi = \forall p[p](K_a q \rightarrow K_b K_a q)$ in SOPAL there exists a corresponding ϕ' in APAL. However, $(\mathcal{M}, w_{10}) \nvDash \phi$ implies $(\mathcal{M}, w_{10}) \nvDash \phi'$, which implies $(\mathcal{M}', s_0) \nvDash \phi'$ by bisimulation, which finally implies $(\mathcal{M}', s_0) \nvDash \phi$. A contradiction.

Lemma 5.11. *In class $\mathcal{K}^e_{full}$ of full frames, SOPAL $\nleq_m$ APAL.*

We remark that Definition 5.5 is discussed in [9], but discarded exactly on the ground that it does not preserve bisimulations. Bisimulation-preserving quantification is analysed in [67] and the resulting logic is proved as expressive as epistemic logic. Here we maintain that in second-order propositional modal logics a stronger notion of bisimulation is needed, which takes into account also quantification, as discussed in Chapter 4. Also, clause (5.5) has other issues, in particular, it is not analytic (more below).

Even though $\forall p[p]\psi$ is not equivalent to $\Box\psi$ at the level of models, the two formulas are provably equivalent at the level of frames, under a cardinality assumption.

Consider the following translation $\tau : \mathcal{L}_{apal} \rightarrow \mathcal{L}_{sopal}$ from APAL to SOPAL:

$$
\begin{array}{lcl}
\tau(p) & = & p \\
\tau(\neg\psi) & = & \neg\tau(\psi) \\
\tau(\psi \rightarrow \psi') & = & \tau(\psi) \rightarrow \tau(\psi') \\
\tau(K_a\psi) & = & K_a\tau(\psi) \\
\tau(C_A\psi) & = & C_A\tau(\psi) \\
\tau([\psi]\psi') & = & [\psi]\tau(\psi') \\
\tau(\Box\psi') & = & \forall p[p]\tau(\psi')
\end{array}
$$

where p does not appear free in ψ'.

We can now prove the following result.

Lemma 5.12. *In the class of epistemic frames where $|D|$ is enumerable,*

$$\vDash \phi \quad \textit{iff} \quad \vDash \tau(\phi)$$

As a result, whenever the domain D of propositions is enumerable, APAL can be captured within SOPAL at the frame level, by means of translation τ. Specifically, the arbitrary announcement operator $\Box$ can be expressed by quantification and standard announcements. As a corollary of Lemma 5.12, we have the following result.

Corollary 5.13. *In the class of epistemic frames where $|D|$ is enumerable, APAL $\leq_f$ SOPAL.*

We now show that the converse of Corollary 5.13 does not hold in general. Specifically, consider set $I = \{a\}$ of agents, and the reflexive and full frames $\mathcal{F}_1 = \langle \{w\}, \{(w,w)\}, 2^{W_1} \rangle$ and $\mathcal{F}_2 = \langle \{v, v'\}, W_2^2, 2^{W_2} \rangle$. For $\phi = \exists p(p \wedge M_a \neg p)$, notice that $(\mathcal{F}_1, w) \nvDash \phi$, while $(\mathcal{F}_2, s) \vDash \phi$ for every $s \in W_2$. Suppose that there exists ϕ' corresponding to ϕ in the sense of Definition 5.9 for $\leq_f$. Then, $(\mathcal{F}_1, w) \nvDash \phi'$, while $(\mathcal{F}_2, s) \vDash \phi'$. Moreover, for every APAL formula ψ, if $(\mathcal{M}_1, w) \nvDash \psi$ for some model $\mathcal{M}_1$ on $\mathcal{F}_1$, then there exists a bisimilar model $\mathcal{M}_2$ on $\mathcal{F}_2$ such that $(\mathcal{M}_2, s) \nvDash \psi$. Hence, $(\mathcal{F}_2, s) \vDash \psi$ implies $(\mathcal{F}_1, w) \vDash \psi$. In particular, $(\mathcal{F}_2, s) \vDash \phi'$ implies $(\mathcal{F}_1, w) \vDash \phi'$. A contradiction.

Lemma 5.14. *In class $\mathcal{K}^e_{full}$ of full frames, SOPAL $\nleq_f$ APAL.*

As an immediate consequence of Corollary 5.13 and Lemma 5.14 we obtain the following.

Corollary 5.15. *In class of epistemic frames where $|D|$ is enumerable, APAL $<_f$ SOPAL.*

Finally, we remarked above that clause (5.5) is discussed in [9], but dismissed on the ground that it does not preserve bisimulations. On the other hand, the APAL semantics is not analytic in the sense that Lemma 5.8(3) fails: models that agree on the interpretation of free atoms, may differ in the satisfaction of formulas. Consider again model $\mathcal{M}$ in Fig. 5.2 and formula $\varphi = \Box(K_a q \to K_b K_a q)$. Then, define model $\mathcal{M}''$ such that for every $U \subseteq W$, $V(p_U) = U$ for some $p_U \neq q$. Clearly, V and V'' agree on the only free variable q in φ. However, $(\mathcal{M}, w_{00}) \vDash \varphi$ as noticed above, while $(\mathcal{M}'', w_{00}) \nvDash \varphi$. In particular, for $U = \{w_{00}, w_{01}, w_{10}\}$, $(\mathcal{M}'', w_{00}) \nvDash [p_U](K_a q \to K_b K_a q)$. Therefore, in APAL the satisfaction of formulas does not depend on values assigned to free variables only, but, if the formula contains an operator $\Box$, on the interpretation of all variables in AP. The example above also entails the following result.

Lemma 5.16. *In class $\mathcal{K}^e_{full}$ of full frames, APAL $\nleq_m$ SOPAL.*

To summarize the main results proved in this section, SOPAL and APAL are incomparable at the model level, that is, we have both APAL $\nleq_m$ SOPAL and SOPAL $\nleq_m$ APAL; while the former is strictly more expressive than the latter at the frame level, i.e., APAL $<_f$ SOPAL.

5.3 Expressivity

In this section we explore the expressivity of SOPAL in the various classes of Kripke frames, starting with the properties of quantifiers. The main result of this section is that SOPAL is as expressive as second-order propositional epistemic logic introduced in Chapter 3.

Lemma 5.17. *In SOPAL we have the following validities, for $x \in \{ap, pl, el, pal, sopal\}$:*

$$\mathcal{K}^e_{\widehat{x}} \vDash \forall p \phi \to \phi[p/\psi] \qquad \textit{for every } \psi \in \mathcal{L}_x \tag{5.7}$$

where ψ is free for p in ϕ.

For every class $\mathcal{K}$ of frames,

$$\mathcal{K} \vDash \psi \to \phi \quad \textit{implies} \quad \mathcal{K} \vDash \psi \to \forall p \phi \tag{5.8}$$

where p does not appear free in ψ.

We remark that the proof of Lemma 5.17 makes essential use of Lemma 5.8.

By Lemma 5.17 we can see that SOPAL satisfies the standard principles of quantification: the axioms $\mathbf{Ex}_x$ (5.7) for exemplification and **Gen** (5.8) for generalisation are indeed valid in SOPAL.

Furthermore, it is of utmost interest to study the interactions between quantification and public announcements in SOPAL. In this respect, we obtain the following key result.

Lemma 5.18. *The following validities hold in all classes of frames.*

$$[\psi]\forall p\phi \leftrightarrow \psi \to \forall p[\psi]\phi \tag{5.9}$$

$$\langle\psi\rangle\exists p\phi \leftrightarrow \psi \wedge \exists p\langle\psi\rangle\phi \tag{5.10}$$

$$[\psi]\,\exists p\phi \leftrightarrow \psi \to \exists p[\psi]\phi \tag{5.11}$$

$$\langle\psi\rangle\forall p\phi \leftrightarrow \psi \wedge \forall p\langle\psi\rangle\phi \tag{5.12}$$

where p does not appear in ψ (without loss of generality bound variables can always be renamed).

We recall that SOPEL is obtained by removing clause $[\psi]\psi$ from Definition 5.1. From Lemma 5.18 and the standard reduction axioms of public announcement logic [100], we immediately derive the following expressivity result.

Theorem 5.19. *SOPAL is as expressive as SOPEL.*

This result is extremely relevant, as it allows to apply to SOPAL the model theory and techniques developed for SOPEL in Part I. As an example, the truth preserving bisimulations introduced in Chapter 4 for second-order propositional modal logic apply to SOPAL as well. Further consequences of Theorem 5.19 regard the decidability of model checking SOPAL and its axiomatisation.

Corollary 5.20.

- *The model checking problem for SOPAL is PSPACE-hard.*
- *SOPAL has sound and complete axiomatisations with respect to classes K_{all} of all frame, K_{bool} of boolean frames, and K_{full} of full frames, obtained by adding validities (5.9)-(5.12) and the reduction axioms of PAL to the corresponding axiomatisations of SOPEL in Chapter 3.*

Thus, the fact that SOPAL and SOPEL are equally expressive allows us to transfer to the former many useful results proved in Chapter 3 and 4 about the latter. However, equal expressivity does not mean that SOPAL and SOPEL are the same, as it will become apparent in Section 5.4.

5.3.1 Knowability

In this section we analyse the notions of preservation and knowability introduced in Example 5.1, and present successfulness. Such concepts are of interest to understand the epistemic capabilities of agents in response to different types of public announcements.

We start by introducing the *positive* fragment $\mathcal{L}^+_{sopal}$ of SOPAL inductively defined as

$$\psi \quad ::= \quad p \mid \neg p \mid \psi \wedge \psi \mid \psi \vee \psi \mid K_a\psi \mid C_A\psi \mid [\neg\psi]\psi \mid \forall p\psi$$

As anticipated in Example 5.1, *preserved formulas* keep their truth under arbitrary announcements. Given a class $\mathcal{K}$, they are defined semantically as those $\phi \in \mathcal{L}_{sopal}$ for which $\mathcal{K} \vDash \phi \to \forall p[p]\phi$. We immediately extend the following result proved in [9] for APAL.

Lemma 5.21. *Positive formulas are preserved in $\mathcal{K}^e_{all}$.*

As an immediate consequence of Lemma 5.21, positive formulas are preserved in every class of frames.

In connection with preserved formulas, in Example 5.1 we introduced the formulas preserved under arbitrary *epistemic* announcements (in a class $\mathcal{K}$) as those formulas ϕ for which $\mathcal{K} \vDash \phi \to \forall p[K_a p]\phi$. In Example 5.2 we remarked that Moore's formulas are not preserved under arbitrary announcements, but they are for epistemic announcements. Obviously, positive formulas are also preserved epistemically. So, it would be of interest to characterize exactly the class of formulas preserved under arbitrary epistemic announcements, but this is beyond the scope of the present contribution.

Another semantic notion of interest when dealing with public announcements is that of *success*. Formally, a formula ϕ is successful in class $\mathcal{K}$ of frames iff $[\phi]\phi$ is valid in $\mathcal{K}$. Now let $\phi \in \mathcal{L}_x$, for $x \in \{ap, pl, el, pal, sopal\}$, be a preserved formula. In particula, $\phi \to \forall p[p]\phi$ is a validity in the corresponding class $\mathcal{K}^e_{\widehat{x}}$ of frames, and therefore $\mathcal{K}^e_{\widehat{x}} \vDash \phi \to [\phi]\phi$ by exemplification $\mathbf{Ex}_x$, i.e., $\mathcal{K}^e_{\widehat{x}} \vDash [\phi]\phi$. Hence, we obtain the following result.

Lemma 5.22. *For $x \in \{ap, pl, el, pal, sopal\}$, every formula $\phi \in \mathcal{L}_x$ preserved in the corresponding class $\mathcal{K}^e_{\widehat{x}}$ of frames is also successful in $\mathcal{K}^e_{\widehat{x}}$.*

Finally, we recall that for a given class $\mathcal{K}$ of frames, *knowable* formulas as those formulas ϕ for which, for any agent $a \in Ag$, $\mathcal{K} \vDash \phi \to \exists p\langle p\rangle K_a\phi$. Now observe that, for a preserved formula $\phi \in \mathcal{L}_x$, $\mathcal{K}^e_{\widehat{x}} \vDash \phi \to \forall p[p]\phi$ implies that $\phi \to [\phi]\phi$ is a validity in the class $\mathcal{K}^e_{\widehat{x}}$ of frames, and therefore $\mathcal{K}^e_{\widehat{x}} \vDash \phi \to \langle\phi\rangle K_a\phi$. Finally, $\mathcal{K}^e_{\widehat{x}} \vDash \phi \to \exists p\langle p\rangle K_a\phi$ by $\mathbf{Ex}_x$. As a result, the following lemma holds.

Lemma 5.23. *Positive formulas are knowable in $\mathcal{K}^e_{all}$ (always knowable). Formulas preserved (resp. successful) in the corresponding class $\mathcal{K}^e_{\widehat{x}}$ of frames are also knowable in $\mathcal{K}^e_{\widehat{x}}$.*

We clearly see that SOPAL allows for a fine-grained analysis of the dynamic epistemic notions of preservation, successfulness, and knowability.

5.4 Succinctness of SOPAL

The fact that SOPAL and SOPEL are equally expressive does not necessarily mean that they are 'the same'. Indeed, we now argue that SOPAL is more succinct than SOPEL, in the sense described below. We will sketch the argument using techniques from [68], where it was proven that PAL is exponentionally more succinct than epistemic logic. For the following we define the length $|\phi|$ of a formula $\phi \in \mathcal{L}_{sopal}$ as standard [47].

Definition 5.24 (Succinctness)**.** Given two logics L_1 and L_2 that are equally expressive on a class $\mathcal{K}$ of frames, we say that L_1 is exponentially more succinct than L_2 on $\mathcal{K}$, written $L_1 \preceq_{\mathcal{K}}^{exp} L_2$, iff there are sequences $\varphi_{n\in\mathbb{N}} = \varphi_1, \varphi_2, \ldots$ of formulas in L_1 and $\psi_{n\in\mathbb{N}} = \psi_1, \psi_2, \ldots$ in L_2 and a polynomial function f such that, for all $n \in \mathbb{N}$,

1. $|\varphi_n| \leq f(n)$;
2. $|\psi_n| > 2^n$;
3. ψ_n is the shortest formula in L_2 equivalent to φ_n in $\mathcal{K}$.

In stating the main result below we also consider the class $\mathcal{K}_{all}$ of frames with arbitrary accessibility relations.

Theorem 5.25.

- *SOPAL* $\preceq_{\mathcal{K}_{all}}^{exp}$ *SOPEL* *if* $|I| \geq 2$
- *SOPAL* $\preceq_{\mathcal{K}_{all}^{e}}^{exp}$ *SOPEL* *if* $|I| \geq 4$

To prove Theorem 5.25 consider the following sequences $\varphi_{n\in\mathbb{N}}$ and $\psi_{n\in\mathbb{N}}$:

$$
\begin{aligned}
\varphi_0 &= \top \\
\varphi_{n+1} &= \langle \varphi_n \rangle (M_a p \vee M_b q) \\
\psi_0 &= \top \\
\psi_n &= M_a(\psi_{n-1} \wedge p) \vee M_b(\psi_{n-1} \wedge q)
\end{aligned}
$$

It is easy to see that $|\varphi_i| \leq i \cdot 6$ and $|\psi_i| \geq 2^i$. Using PAL equivalences, we also have that φ_i and ψ_i are equivalent, for all i. So the first two items for succinctness are easily checked, what remains to establish is, that even when we allow for quantification, there are no formulas $\beta_i \in \mathcal{L}_{sopel}$ shorter than $\psi_i \in \mathcal{L}_{el}$ equivalent to $\varphi_i \in \mathcal{L}_{pal}$.

For propositional epistemic logic, the technique that [68] uses to prove that $\psi_i \in \mathcal{L}_{el}$ is the shortest formula equivalent to $\varphi_i \in \mathcal{L}_{pal}$ is that of *formula size games*. We now extend such games to deal with quantification.

Definition 5.26 (Formula Size Game)**.** The rules of the one-person formula size game (FSG) for Spoiler are the following. The game is played on a tree, where each node is labeled with a pair $\langle \mathbb{M} \circ \mathbb{N} \rangle$ such that $\mathbb{M}$ and $\mathbb{N}$ are finite sets of finite pointed models. At each step of the game, a node is labeled with one of the symbols from the set $\Sigma = \{\top, \bot, p, \neg, \vee, \wedge, M_a, K_a, \overline{C}_A, C_A, \exists p, \forall p\}$ and either it is closed or at most two new nodes are added. Let a node $\langle \mathbb{M} \circ \mathbb{N} \rangle$ be given. Spoiler can make the following moves at this node:

$\top$-move This can be played only if $\mathbb{N} = \varnothing$. When Spoiler plays this move, the node is closed and labeled with $\top$.

atomic-move Spoiler chooses an atom p such that every pointed model in $\mathbb{M}$ satisfies p, and no pointed model in $\mathbb{N}$ does. After this move, this node is closed and labeled with p.

not-move Spoiler labels the node with symbol $\neg$ and adds one new node denoted $\langle \mathbb{N} \circ \mathbb{M} \rangle$ as a successor to $\langle \mathbb{M} \circ \mathbb{N} \rangle$.

or-move Spoiler labels the node with symbol $\vee$ and splits $\mathbb{M}$ in two sets $\mathbb{M} = \mathbb{M}_1 \cup \mathbb{M}_2$. Two new nodes are added to the tree as successors to $\langle \mathbb{M} \circ \mathbb{N} \rangle$, i.e., $\langle \mathbb{M}_1 \circ \mathbb{N} \rangle$ and $\langle \mathbb{M}_2 \circ \mathbb{N} \rangle$.

and-move Spoiler labels the node with the symbol $\wedge$ and splits $\mathbb{N}$ in two sets $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2$. Two new nodes are added to the tree as successors to $\langle \mathbb{M} \circ \mathbb{N} \rangle$, namely $\langle \mathbb{M} \circ \mathbb{N}_1 \rangle$ and $\langle \mathbb{M} \circ \mathbb{N}_2 \rangle$.

M_a-move Spoiler labels the node with symbol M_a and for each pointed model $(\mathcal{M}, w) \in \mathbb{M}$, he chooses a pointed model $(\mathcal{M}, w')$ such that $R_a(w, w')$. All such choices are collected in $\mathbb{M}_1$. A set of models $\mathbb{N}_1$ is then constructed as follows. For each pointed model $(\mathcal{N}, v) \in \mathbb{N}$, add to $\mathbb{N}_1$ all pointed models $(\mathcal{N}, v')$ such that $R'_a(v, v')$. If for some pointed model $(\mathcal{N}, v)$, world v does not have an R'_a-successor, nothing is added to $\mathbb{N}_1$ for $(\mathcal{N}, v)$. A new node $\langle \mathbb{M}_1 \circ \mathbb{N}_1 \rangle$ is added as a successor to $\langle \mathbb{M} \circ \mathbb{N} \rangle$.

K_a-move Spoiler labels the node with symbol K_a and for each pointed model $(\mathcal{N}, v) \in \mathbb{N}$, he chooses a pointed model $(\mathcal{N}, v')$ such that $R'_a(v, v')$. All those choices are collected in $\mathbb{N}_1$. A set of models $\mathbb{M}_1$ is then constructed as follows. For each pointed model $(\mathcal{M}, w) \in \mathbb{M}$, add to $\mathbb{M}_1$ all pointed models $(\mathcal{M}, w')$ such that $R_a(w, w')$. If for some pointed model $(\mathcal{M}, w)$, world w does not have an R_a-successor, nothing is added to $\mathbb{M}_1$ for $(\mathcal{M}, w)$. A new node $\langle \mathbb{M}_1 \circ \mathbb{N}_1 \rangle$ is added as a successor to $\langle \mathbb{M} \circ \mathbb{N} \rangle$.

$\overline{C}_A$-move Spoiler labels the node with symbol $\overline{C}_A$ and for each pointed model $(\mathcal{M}, w) \in \mathbb{M}$, he chooses a pointed model $(\mathcal{M}, w')$ such that $R^*_A(w, w')$. All such choices are collected in $\mathbb{M}_1$. A set of models $\mathbb{N}_1$ is then constructed as follows. For each pointed model $(\mathcal{N}, v) \in \mathbb{N}$, add to $\mathbb{N}_1$ all pointed models $(\mathcal{N}, v')$ such that $R'^*_A(v, v')$. If for some pointed model $(\mathcal{N}, v)$, world v does not have a reachable successor, nothing is added to $\mathbb{N}_1$ for $(\mathcal{N}, v)$. A new node $\langle \mathbb{M}_1 \circ \mathbb{N}_1 \rangle$ is added as a successor to $\langle \mathbb{M} \circ \mathbb{N} \rangle$.

C_A-move Spoiler labels the node with symbol C_A and for each pointed model $(\mathcal{N}, v) \in \mathbb{N}$, he chooses a pointed model $(\mathcal{N}, v')$ such that $R'^*_A(v, v')$. All such choices are collected in $\mathbb{N}_1$. A set of models $\mathbb{M}_1$ is then constructed as follows. For each pointed model $(\mathcal{M}, w) \in \mathbb{M}$, add to $\mathbb{M}_1$ all pointed models $(\mathcal{M}, w')$ such that $R_A^*(w, w')$. If for some pointed model $(\mathcal{M}, w)$, world w does not have a reachable successor, nothing is added to $\mathbb{M}_1$ for $(\mathcal{M}, w)$. A new node $\langle \mathbb{M}_1 \circ \mathbb{N}_1 \rangle$ is added as a successor to $\langle \mathbb{M} \circ \mathbb{N} \rangle$.

$\exists p$-move Spoiler labels the node with symbol $\exists p$ and, for each $(\mathcal{M}, w) \in \mathbb{M}$, she chooses a set $U \in D$ and replaces $(\mathcal{M}, w)$ with $(\mathcal{M}_U^p, w)$. All these choices are collected in $\mathbb{M}_1$. A set $\mathbb{N}_1$ is then constructed as follows: for each $(\mathcal{N}, v) \in \mathbb{N}$ and any $U' \in D'$, add $(\mathcal{N}_{U'}^p, v)$ to $\mathbb{N}_1$.

$\forall p$-move Spoiler labels the node with symbol $\forall p$ and, for each $(\mathcal{N}, v) \in \mathbb{N}$, she chooses a set $U' \in D'$ and replaces $(\mathcal{N}, v)$ with $(\mathcal{N}_{U'}^p, v)$. All these choices are collected in $\mathbb{N}_1$. A set $\mathbb{M}_1$ is then constructed as follows: for each $(\mathcal{M}, w) \in \mathbb{M}$ and any $U \in D$, add $(\mathcal{M}_U^p, w)$ to $\mathbb{M}_1$.

Notice that the moves for $\bot$, **and**, K_a, C_A, and $\forall p$ are derived from the moves for their dual operators in the following senses: Spoiler acts on $\mathbb{N}$, instead of $\mathbb{M}$. Moves **and** and **or** are called *splitting* moves, while K_a-, M_a-, C_A, and $\overline{C}_A$-moves are called *agent* moves.

Definition 5.27. Spoiler wins the FSG starting at $\langle \mathbb{M} \circ \mathbb{N} \rangle$ in n moves iff there is a game tree T with root $\langle \mathbb{M} \circ \mathbb{N} \rangle$ and precisely n nodes such that every leaf of T is closed. Otherwise, Spoiler loses the game in n moves.

We are now in a position to prove the following result, which extends Theorem 1 in [68] with the case to deal with quantification.

Theorem 5.28. *Spoiler can win the FSG starting at $\langle \mathbb{M} \circ \mathbb{N} \rangle$ in less than k moves iff there is a SOPEL formula $\psi \in \mathcal{L}_{sopel}$ such that $\mathbb{M} \vDash \psi$, $\mathbb{N} \vDash \neg\psi$, and $|\psi| < k$.*

Example 5.4. *Consider Fig. 5.3. This is a game tree for pair $\langle \mathbb{M}, \mathbb{N} \rangle$ with $\mathbb{M} = \{(\mathcal{M}, w)\}$ and $\mathbb{N} = \{(\mathcal{N}, w)\}$, respectively depicted on the left and on the right of the root of the tree. Designated points of the models are black dots, non-designated points are open dots. Leaves are closed nodes and are depicted with thick perimeters. We further assume that in $\mathbb{M}$ and $\mathbb{N}$ all atoms are true in all worlds, and there is only one agent a. Notice that the two initial models are bisimilar, and hence have the same epistemic theory. This implies that the FSG starting in $\langle \mathbb{M} \circ \mathbb{N} \rangle$ can only be won if an $\exists p$ or $\forall p$ move is played. Note that the game displayed 'corresponds' to the formula $\exists p(M_a \neg p \wedge p)$.*

In light of Theorem 5.28, if for every $n \in \mathbb{N}$ we can find classes $\mathbb{M}_n$ and $\mathbb{N}_n$ of pointed models such that the following holds:

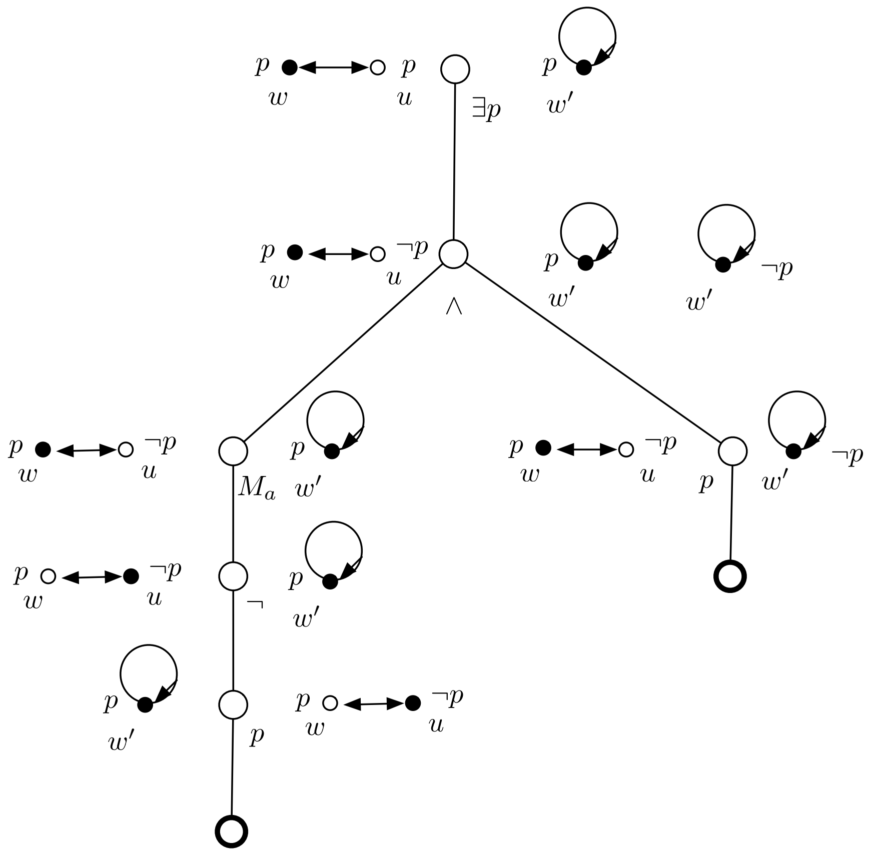


FIGURE 5.3: The game tree from Example 5.4.

1. $\mathbb{M}_n \vDash \psi_n$ and $\mathbb{N}_n \vDash \neg\psi_n$;

2. it takes Spoiler at least 2^n moves to win the FSG starting in $\langle \mathbb{M}_n \circ \mathbb{N}_n \rangle$;

then we have shown that also item 3 of Definition 5.24 holds for the three step proof, thus settling that SOPAL $\preceq^{exp}_{\mathcal{K}_{all}}$ SOPEL:

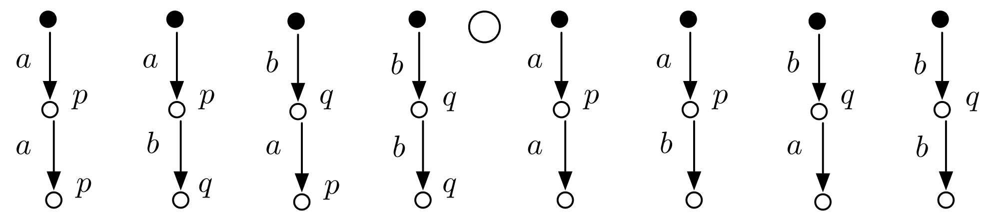


FIGURE 5.4: The starting node $\langle \mathbb{M}_2 \circ \mathbb{N}_2 \rangle$ for the formula size game.

We show the two items above, first for $n = 2$ and the class $\mathcal{K}_{all}$ of all frames. Consider $\psi_2 = M_a(p \wedge (M_a p \vee M_b q)) \vee M_b(q \wedge (M_a p \wedge M_b q))$. We let $\mathbb{M}_2 = \{(\mathcal{M}_{aa}, \epsilon), (\mathcal{M}_{ab}, \epsilon), (\mathcal{M}_{ba}, \epsilon), \mathcal{M}_{bb}, \epsilon)\}$ and $\mathbb{N}_2 = \{\mathcal{N}_{aa}, \epsilon), (\mathcal{N}_{ab}, \epsilon), \mathcal{N}_{ba}, \epsilon), \mathcal{N}_{bb}, \epsilon)\}$ as depicted in

Fig. 5.4. We leave it to the reader to check that $\mathbb{M}_2 \vDash \psi_2$ while $\mathbb{N}_2 \vDash \neg\psi_2$. Let us call the worlds in $\mathcal{M}_{ab}$, for instance, ϵ, w_a, and w_{ab}, and ϵ, v_a, v_{aa} those in $\mathcal{N}_{aa}$. Note that the models in $\mathbb{N}_2$ are similar to those in $\mathbb{M}_2$, the only difference being that in the final point, no atom is true. For Spoiler to exploit this (let us initially not consider quantifier moves), she has to take care that for instance there is a path in the game tree that has $(\mathcal{M}_{aa}, w_{aa})$ at the left of the node, and $(\mathcal{N}_{aa}, v_{aa})$ on the right, after which he can play an atom-move and close that branch. Model pairs of type $(\mathcal{M}_{aa}, \epsilon)$ and $(\mathcal{N}_{aa}, \epsilon)$ are called *diverging pairs*. Note that every diverging pair will generate at least one branch in the game tree that takes exactly two agent moves and one atom move. Also, when an agent a-move is played at a certain node, there can be no models at that node in which the first transition is for agent b: otherwise Spoiler cannot make the move. So, whenever there are models on one side of a node that have an a-transition first together with models that have a b-transition first, Spoiler needs to play a splitting move. Then, it is clear that for every diverging pair, there is a terminal node in the game tree that only contains that pair, and which is closed by an atom-move. This explains that there are at least 4 moves needed to win the game starting in $\langle \mathbb{M}_2 \circ \mathbb{N}_2 \rangle$, and at least 2^n moves for a game starting in $\langle \mathbb{M}_n \circ \mathbb{N}_n \rangle$.

So, can Spoiler do any better now that she has quantifier moves available? No, she cannot: note that Spoiler's task is to find a 'difference' between the models on the left and those on the right, so in particular he needs to demonstrate a difference between the models that make up a diverging pair. Note that the difference between the models in such pairs is n steps away from ϵ (forcing Spoiler to take n agent-moves), and this difference is then between the truth of one atom, p or q. If Spoiler plays a quantifier move, he runs the risk of two models of a diverging pair becoming identical (the valuations could become the same) in which case Spoiler looses the game: having a model $(\mathcal{M}, w)$ appearing both on the left and right of a node in the game is a losing position! In our example, if Spoiler plays an $\exists r$-move, the effect is that there are still the same number of diverging pairs, which completely agree on the valuation of r. If Spoiler plays an $\exists p$-move in a node within a model $\mathcal{M}_{xa}$ (for $x \in \{a, b\}$), then as an effect of this move there will be a model $\mathcal{N}_{xa}$ with a valuation such that the resulting $\mathcal{M}'_{xa}$ and $\mathcal{N}'_{xa}$ will be identical models! (for instance, if the valuation in $\mathcal{M}_{aa}$ was changed by Spoiler such that p were to be false in w_a and true in w_{aa}, then this model would become identical to $\mathcal{N}_{aa}$ with the same valuation.) It should be clear that any node in the game with two identical models, one on each side of the node, is a losing position for Spoiler. In sum, by playing a quantifier move, Spoiler cannot improve, but possibly worsen his chances of winning the game in k moves.

So, the key idea in [68] can be summarised as follows: pairs $\langle (\mathcal{M}_{aa}, \epsilon), (\mathcal{N}_{aa}, \epsilon) \rangle$ and $\langle (\mathcal{M}_{ba}, \epsilon), (\mathcal{N}_{ba}, \epsilon) \rangle$ are called diverging pairs, because Spoiler cannot keep them in the same branch of the game tree in order to win the game. So the number of diverging pairs in the starting node of the game is an indicator for the number of splitting moves that Spoiler needs to play. We then argued that by playing quantifier moves, Spoiler does not change the number of diverging pairs.

For the class $\mathcal{K}^{e}_{full}$ of full (epistemic) frames, we use the models and formulas as presented in [68]. We now assume $I = \{a, b, c, d\}$ and introduce four atoms $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$. Define $\varphi_n \in \mathcal{L}_{sopal}$ and $\psi_n \in \mathcal{L}_{sopel}$ as follows:

$$\begin{array}{rcl} \varphi_1 & = & M_c(\mathbf{c} \wedge M_d(\mathbf{d} \wedge (M_a\mathbf{a} \vee M_b\mathbf{b}))) \\ \varphi_{n+1} & = & \langle \varphi_n \rangle \varphi_1 \end{array}$$

and

$$\begin{array}{rcl} \psi_1 & = & M_c(\mathbf{c} \wedge M_d(\mathbf{d} \wedge (M_a\mathbf{a} \vee M_b\mathbf{b}))) \\ \psi_{n+1} & = & \psi_n \wedge M_c(\psi_n \wedge \mathbf{c} \wedge M_d(\psi_n \wedge \mathbf{d} \wedge (M_a(\psi_n \wedge \mathbf{a}) \vee M_b(\psi_n \wedge \mathbf{b})))) \end{array}$$

Then, every φ_n and ψ_n are equivalent (see [68, Proposition 5]). For the sets of models $\mathbb{E}_n$ and $\mathbb{F}_n$, we refer to [68, Definition 20, Fig. 7 and 8]. Note that here, in the starting node of the game, the designated points in the models are $\grave{x}_n$ and $\acute{x}_n$.

Now take, for example, the models $(E_{cda}, \grave{x}_1)$, which will be on the left of the FSG for $n = 1$, while $(E_{cda}, \acute{x}_1)$ appears on the right. The only difference between the two models is the designated point. But note that in the first model, there is a path $c - d - a$ to a world labeled with the atom $\mathbf{a}$, while in the second model, there is no such path (starting in $\acute{x}_1$). So, the pairs $\langle (E_{cda}, \grave{x}_1), (E_{cda}, \acute{x}_1) \rangle$ and $\langle (E_{cdb}, \grave{x}_1), (E_{cdb}, \acute{x}_1) \rangle$ is a diverging pair of models, which Spoiler has to split before winning the game. Can Spoiler get rid of a diverging pair by playing a quantifier move? She can't, as we now argue, the reason being that the models on the left and the right of each node are based on the same frame. Suppose that Spoiler plays a $\exists\mathbf{a}$-move, changing the model $(E_{cda}, \grave{x}_1)$ to some model with an assignment V' that possibly changes the interpretation of $\mathbf{a}$. This will have the effect that on the right of the node, we will have an assignment V' for $(E_{cda}, \acute{x}_1)$ which is exactly the same. It can be seen that the two new pointed models $(E'_{cda}, \grave{x}_1)$ and $(E'_{cda}, \grave{x}_1)$, which will appear each on a side of the next node in the game tree, mean that this is a losing position for Spoiler: they both verify the same formulas, and every quantifier move in one of them can be mimicked in the other.

As a result of the discussion above, we conclude that SOPAL is exponentially more succinct than SOPEL.

5.5 Discussion and Related Literature

In this chapter we introduced second-order propositional announcement logic: a logic to reason about arbitrary announcements in multi-agent contexts. We presented the language of SOPAL, which extends public announcement logic by means of propositional quantification, or equivalently, enriches SOPEL with public announcement operators. We endowed SOPAL with a semantics in terms of multi-agent Kripke frames and models. We illustrated the expressivity of SOPAL by analysing relevant notions in knowledge reasoning and representation, such as preservation under arbitrary (epistemic) announcements, knowability, and successfulness. Further, we compared SOPAL to arbitrary public announcement logic, a language whose motivation and aim is similar to SOPAL's, but which follows a completely different philosophy. To compare SOPAL and APAL we provided two notions of order between logics. Then, we proved that, while SOPAL and APAL are uncomparable at the model level, the former is strictly more expressive than the latter at the frame level. Furthermore, we analysed the set of validities in SOPAL and provided reduction equivalences for quantified formulas, thus showing that SOPAL is exactly as expressive as SOPEL. As a consequence of the results in Chapter 3, SOPAL has a decidable model checking problem as well as sound and complete axiomatisations. Moreover, SOPAL is preserved by the (bi)simulation relations introduced in Chapter 4. Announcements make a difference nonetheless. Indeed, SOPAL is exponentially more succinct than SOPEL. We conclude that SOPAL is a succinct, rich logic, strictly more expressive than previous proposals in the area, but still with nice computational properties.

This chapter draws inspiration from a well-established tradition in knowledge reasoning and representation: dynamic epistemic logic [122], including public announcement logic [71, 100]. This line of research is well-studied, with a rich literature. Hence, we only discuss the contributions most closely related to the present work. Arbitrary public announcement logic has been introduced in [9, 10], with the aim of capturing arbitrary announcements. We share the same motivation, but the formal analysis through propositional quantification is novel. In particular, differently from APAL, SOPAL is not preserved by standard propositional (bi)simulations. To deal with this issue, in Chapter 4 we introduced novel notions of (bi)simulation that indeed preserve SOPEL, and therefore SOPAL by reduction. On the other hand, differently from APAL, SOPAL is analytic in the sense of Lemma 5.8(3): the truth value of a formula depends only on the value of atoms appearing therein. Further, quantification (on bisimilar models) has been analysed in [67]. However, the resulting logic is as expressive as epistemic logic, and therefore strictly weaker than SOPAL.

Along this line of research it is of great interest to analyse further SOPAL in multi-agent contexts, for instance, agents performing announcements: which announcements can an agent perform based on her knowledge? How do such announcements modify the epistemic state of other agents (including knowability and preservation)? How is the proposed framework to be modified to accommodate private communication? In this direction contributions such as group announcement logic [3] are certainly relevant. We partially address some of these questions in the following chapter.

Chapter 6

Global and Local Announcements

In this chapter we take a break from propositional quantification and return to a purely propositional setting. Specifically, we take inspiration from the state-of-the-art in public announcement logic (PAL) and introduce a logic for global as well as local announcements. Public announcement logic has two key features. First, announcements are *public*, in the sense that all agents equally observe the new information, and are (commonly) aware of all agents equally observing the information. Second, announcements are *global*, that is, although for truthful public announcements the truth of the announced formula in the actual state is a precondition, how the new information is processed does not depend on the actual state but rather on the model (i.e., public announcements are model transformers).

In the proposed language and framework we carefully distinguish the two, independent features of *publicity* and *globality*, which are packed together in the announcement operator $[\phi]$, and relax them both. Hence, by weakening publicity, we allow formulas to be announced to a proper subset A of the set I of all agents. Then, only the agents in A partake of the new information contained in the announcement. Further, by weakening globality, we distinguish between local announcements, whose meaning depend on the actual state, and global announcements that depend on general features of the model.

Thus, the language of global and local announcement logic (GLAL) contains two modalities $[\phi]^+_A$ and $[\phi]^-_A$, for the global and local announcement of formula ϕ respectively, each of them indexed to a coalition A of agents. Further, we provide a semantics in terms of model updates that reflects the intuitions illustrated above. Most interestingly, we are able to provide a unified account of global and local announcements, in which the difference between the two depends on a subtle distinction in the model update.

The rest of the chapter is structured as follows. In Section 6.1 we introduce the syntax and semantics of GLAL and provide two examples to illustrate the formalism. In Section 6.2 we analyse the expressivity of GLAL and state the main result of the paper: differently from PAL, GLAL cannot be reduced to epistemic logic as it is strictly more expressive. We discuss the meaning and relevance of these results in Section 6.3, where we also point to directions for future research.

6.1 The Logic of Global and Local Announcements

In this section we introduce the syntax and semantics of GLAL. We warn that the term 'announcement' is used here with a different meaning with respect to public announcement logic. As discussed in the introduction, the announcements of PAL appear here as global announcements to all agents. Hence, our notion of announcement is more general as it also covers local announcements and announcement to only a selected subset of all agents. These distinctions will be clear once the appropriate semantics is introduced.

Given sets AP of atomic propositions and I of indexes for agents in Chapter 3, we introduce the syntax of GLAL as follows.

Definition 6.1 (GLAL)**.** The formulas in GLAL are defined in BNF as

$$\psi \quad ::= \quad p \mid \neg\psi \mid \psi \wedge \psi \mid K_a \mid C_A\psi \mid [\psi]^+_A\psi \mid [\psi]^-_A\psi$$

where $p \in AP$ and $A \subseteq I$.

The language $\mathcal{L}_{glal}$ of GLAL extends the language $\mathcal{L}_{el}$ of epistemic logic with global announcement formulas $[\psi]^+_A\phi$, whose reading is that "after globally announcing ψ to the agents in A, ϕ is true", as well as local announcements $[\psi]^-_A\phi$, whose intuitive meaning is that "after locally announcing ψ to the agents in A, ϕ is true". We will illustrate and discuss, using our semantics, the different interpretations of operators $[\psi]^+_A$ and $[\psi]^-_A$, particularly with respect to the logic of semi-private announcements [13, 71, 112, 114]. The dual operators $\langle\psi\rangle^-_A$ and $\langle\psi\rangle^+_A$ can be defined as $\neg[\psi]^-_A\neg$ and $\neg[\psi]^+_A\neg$ respectively. As usual, the "everybody knows" formula $E_A\phi$ is a shorthand for $\bigwedge_{a\in A} K_a\phi$, and we omit subscript A from $E_A\phi$ and $C_A\phi$ whenever A is the grand coalition I, then simply write $E\phi$ and $C\phi$.

Global and local announcement logic extends a number of well-known formalisms. We mentioned that the language $\mathcal{L}_{el}$ without clauses $[\psi]^-_A\psi$ and $[\psi]^+_A\psi$ is epistemic logic (with common knowledge), and language $\mathcal{L}_{pl}$ without clauses $K_a\psi$ and $C_A\psi$ as well is propositional logic. These (syntactic) language inclusions, also with respect to the languages introduced in Chapter 3-5, can be summarised as follows:

$$
\begin{array}{ccccccccc}
 & & & & & & \mathcal{L}_{glal} & & \\
 & & & & & \rotatebox{60}{$\subsetneq$} & & & \\
 & & & & \mathcal{L}_{\mathbf{el}} & \subseteq & \mathcal{L}_{\mathbf{pal}} & \subseteq & \mathcal{L}_{\mathbf{apal}} \\
\mathcal{L}_{\mathbf{ap}} & \subseteq & \mathcal{L}_{\mathbf{pl}} & \rotatebox{30}{$\subsetneq$} & & \rotatebox{60}{$\subsetneq$} & & \rotatebox{60}{$\subsetneq$} & \\
 & & & \rotatebox{-30}{$\subsetneq$} & \mathcal{L}_{\mathbf{qbf}} & \subseteq & \mathcal{L}_{\mathbf{sopel}} & \subseteq & \mathcal{L}_{\mathbf{sopal}}
\end{array}
$$

We interpret formulas in GLAL on propositional Kripke frames and models, obtained by removing the interpretation domain D from Definition 3.5. Then, we define appropriate update mechanisms for global and local announcements.

Definition 6.2 (Frame)**.** A Kripke frame is a tuple $\mathcal{F} = \langle W, \{R_a\}_{a \in I} \rangle$ where

- W is a set of *possible worlds*;
- for every agent index $a \in I$, $R_a \subseteq 2^{W \times W}$ is an *equivalence relation* on W.

As customary in epistemic logic [47, 95], for every agent $a \in I$, R_a is the corresponding *indistinguishability* relation between worlds in W. We recall that, whenever A is a singleton $\{a\}$, we have that $R^*_A(w) = \{w' \mid R_{\{a\}}(w, w')\} = R_a(w)$ is the equivalence class of $w \in W$, and R^*_A can be represented as the set $\mathcal{E}_A = \{R^*_A(w) \mid w \in W\}$ of its equivalence classes.

To assign a meaning to formulas in GLAL we introduce *assignments* as functions $V : AP \to 2^W$. A *(Kripke) model* is then defined as a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$. Finally, we introduce the notion of satisfaction in a model.

Definition 6.3 (Satisfaction)**.** We define the *satisfaction set* $[[\varphi]]_{\mathcal{M}} \subseteq W$ of formula φ in model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ as follows:

$$
\begin{array}{lcl}
[[p]]_{\mathcal{M}} & = & V(p) \\
[[\neg\psi]]_{\mathcal{M}} & = & W \setminus [[\psi]]_{\mathcal{M}} \\
[[\psi \wedge \psi']]_{\mathcal{M}} & = & [[\psi]]_{\mathcal{M}} \cap [[\psi']]_{\mathcal{M}} \\
[[C_A\psi]]_{\mathcal{M}} & = & \{w \in W \mid \text{for all } w' \in R^*_A(w), w' \in [[\psi]]_{\mathcal{M}}\} \\
[[[\psi]^-_A\psi']]_{\mathcal{M}} & = & \{w \in W \mid \text{if } w \in [[\psi]]_{\mathcal{M}} \text{ then } w \in [[\psi']]_{\mathcal{M}^-_{(w,\psi,A)}}\} \\
[[[\psi]^+_A\psi']]_{\mathcal{M}} & = & \{w \in W \mid \text{if } w \in [[\psi]]_{\mathcal{M}} \text{ then } w \in [[\psi']]_{\mathcal{M}^+_{(w,\psi,A)}}\}
\end{array}
$$

where *refinements* $\mathcal{M}^-_{(w,\psi,A)} = \langle W^-, \{R^-_a\}_{a \in I}, V^- \rangle$ and $\mathcal{M}^+_{(w,\psi,A)} = \langle W^+, \{R^+_a\}_{a \in I}, V^+ \rangle$ of model $\mathcal{M}$ with respect to world w, formula ψ, and coalition A, are defined so that

- $W^- = W^+ = W$ and $V^- = V^+ = V$;
- for every agent $b \notin A$, $R_b^- = R_b^+ = R_b$; while for $a \in A$,

$$R_a^-(v) = \begin{cases} R_a(v) \cap [[\psi]]_{\mathcal{M}} & \text{if } v \in R_a(w) \cap [[\psi]]_{\mathcal{M}} \\ R_a(v) \cap [[\neg\psi]]_{\mathcal{M}} & \text{if } v \in R_a(w) \cap [[\neg\psi]]_{\mathcal{M}} \\ R_a(v) & \text{otherwise} \end{cases}$$

$$R_a^+(v) = \begin{cases} R_a(v) \cap [[\psi]]_{\mathcal{M}} & \text{if } v \in R_A^*(w) \cap [[\psi]]_{\mathcal{M}} \\ R_a(v) \cap [[\neg\psi]]_{\mathcal{M}} & \text{if } v \in R_A^*(w) \cap [[\neg\psi]]_{\mathcal{M}} \\ R_a(v) & \text{otherwise} \end{cases}$$

By Definition 6.3 the refinement $\mathcal{M}^-_{(w,\psi,A)}$ only affects worlds that are accessible by each agent in A separately, while $\mathcal{M}^+_{(w,\psi,A)}$ involves all worlds reachable through relation R_A^*. In all these worlds the accessibility relation is updated according to whether the world in question satisfies the announcement, that is, the announcement refines the equivalence class of each such world. In Example 6.1 we illustrate the differences between the two types of refinement. Notice that in the case of single agents, the refinements $\mathcal{M}^-_{(w,\psi,a)}$ and $\mathcal{M}^+_{(w,\psi,a)}$ coincide, hence we omit superscripts $-$ and $+$ from single-agent refinements and modalities. Indeed, globally or locally announcing a fact to a single agent is tantamount, as she is the only one to witness the announcement. For instance, an announcement to a single agent by telephone or in a face-to-face meeting have the same outcome (as long as we are not concerned with issues pertaining to other agents being aware of the communication act.) In such a case, model refinement $\mathcal{M}_{(w,\psi,a)}$ can be interpreted as "in $R_a(w)$, agent a learns whether ψ". As a consequence, formula $[\psi]_a\phi$ then becomes: if ψ holds and a learns whether ψ, then ϕ holds as well. Also, the updated set $\mathcal{E}'_a$ of equivalence classes in $\mathcal{M}_{(w,\psi,a)}$ can be shown to be equal to $(\mathcal{E}_a \setminus \{R_a(w)\}) \cup \{R_a(w) \cap [[\psi]], R_a(w) \cap [[\neg\psi]]\}$.

We introduce standard notions of truth and validity. A formula ϕ is *satisfied* at w, or $(\mathcal{M}, w) \vDash \phi$, iff $w \in [[\phi]]_{\mathcal{M}}$; ϕ is *true* at w, or $(\mathcal{F}, w) \vDash \phi$, iff $(\langle \mathcal{F}, V\rangle, w) \vDash \phi$ for every assignment V; ϕ is *valid* in a frame $\mathcal{F}$, or $\mathcal{F} \vDash \phi$, iff $(\mathcal{F}, w) \vDash \phi$ for every world w in $\mathcal{F}$; ϕ is *valid* in a class $\mathcal{K}$ of frames, or $\mathcal{K} \vDash \phi$, iff $\mathcal{F} \vDash \phi$ for every $\mathcal{F} \in \mathcal{K}$. We often omit the subscript $\mathcal{M}$ from the satisfaction set $[[\psi]]_{\mathcal{M}}$ whenever clear by the context.

We now prove that our model refinements in Definition 6.3 are well-defined as both R_a^- and R_a^+ are actually equivalence relations.

Lemma 6.4. *Let $\mathcal{M}$ be a model with refinements $\mathcal{M}^-_{(w,\psi,A)}$ and $\mathcal{M}^+_{(w,\psi,A)}$. For every agent $a \in I$, if R_a is an equivalence relation, then also R_a^- and R_a^+ are such.*

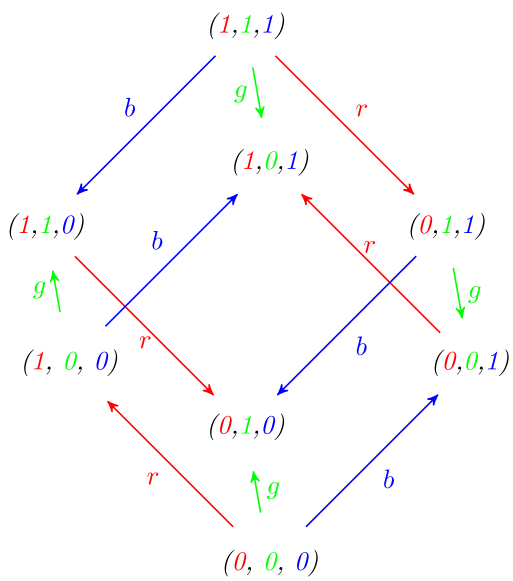
We observe that the semantics of global and local announcements given in Definition 6.3 differs from semi-private announcements in [13, 71, 112, 114]. Specifically, after announcing semi-privately ϕ to coalition A, all agents in A know ϕ, and the agents in $I \setminus A$ know that all agents in A know whether ϕ. None of the above is the case here in general. Nonetheless, the framework provided above is flexible enough to accommodate a further model refinement $\mathcal{M}^{sp}_{(w,\psi,a)}$ suitable for semi-private announcement, according to which $W^{sp} = W$, $V^{sp} = V$, and for $a \in A$,[1]

$$R^{sp}_a(v) = \begin{cases} R_a(v) \cap [\![\psi]\!]_{\mathcal{M}} & \text{if } v \in R^*_I(w) \cap [\![\psi]\!]_{\mathcal{M}} \\ R_a(v) \cap [\![\neg\psi]\!]_{\mathcal{M}} & \text{if } v \in R^*_I(w) \cap [\![\neg\psi]\!]_{\mathcal{M}} \\ R_a(v) & \text{otherwise} \end{cases}$$

We do not provide a detailed discussion of semi-private announcements here, as it is beyond the scope of this work. We only remark the adaptability of the present account.

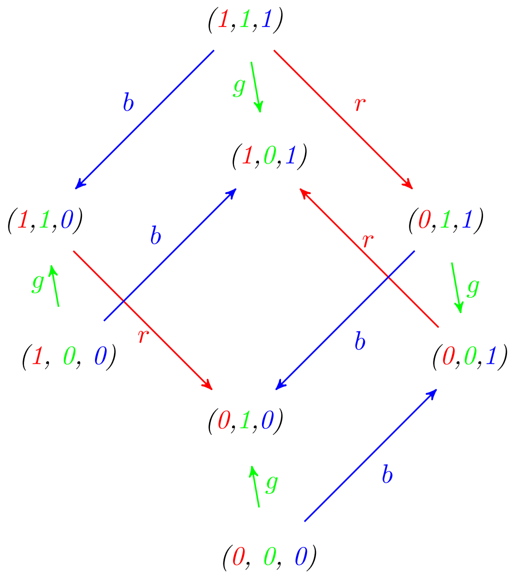
We now illustrate the difference between global and local announcements by means of two examples drawn from the literature on multi-agent systems and dynamic logic [64, 122]. Hereafter we often represent a set as some sequence of its elements.

Example 6.1. *We first consider the well-known puzzle of muddy children. We assume familiarity with this scenario and refer to [64, 117] for a detailed presentation. The initial model $\mathcal{M}$ for 3 children (red, blue, and green), where no child knows whether she is muddy, can be represented as follows:*



[1]The standard way to define this semantics is as $R^{sp}_a = R_a$ for $a \notin A$, whereas $R^{sp}_a = R_a \cap ([\![\psi]\!]^2_{\mathcal{M}} \cup [\![\neg\psi]\!]^2_{\mathcal{M}})$ for $a \in A$. In the submodel generated by the actual state the result is identical to the semantics here defined.

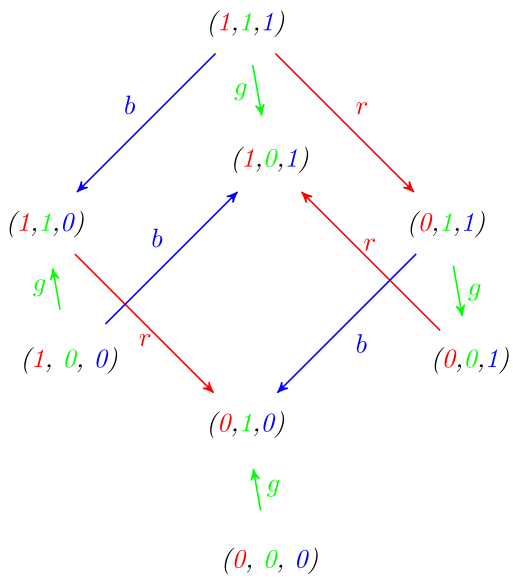
Now suppose that only red is muddy, i.e., the actual world is $(1,0,0)$*, and the father* locally *announces to red and blue that at least one child is muddy, that is, formula* $\alpha := m_r \vee m_b \vee m_g$ *is true. The refined model* $\mathcal{M}^-_{(100,\alpha,rb)}$ *is then given as follows:*



Notice that only the indistinguishability relation for red is updated, as in all worlds that blue considers possible from $(1,0,0)$*, formula* α *is indeed true. Hence, after the father's local announcement, in* $(1,0,0)$ *all children know that at least one child is muddy, i.e.,* $(1,0,0) \vDash [\alpha]^-_{rb} E\alpha$*. Moreover, red learns that she is muddy, i.e.,* $(1,0,0) \vDash [\alpha]^-_{rb} K_r m_r$*. This is in line with the classic version of the muddy children puzzle.*

On the other hand, the father's local announcement is not enough to make α *common knowledge for red and blue, that is,* $(1,0,0) \nvDash [\alpha]^-_{rb} C_{rb}\alpha$*, as blue considers possible that red considers possible that blue considers possible that no child is muddy, that is,* $(1,0,0) \vDash [\alpha]^-_{rb} M_b M_r M_b \neg\alpha$ *via epistemic path* $(1,0,0) \sim_b (1,0,1) \sim_r (0,0,1) \sim_b (0,0,0)$*. This is in contrast with the classic version of the muddy children puzzle with public announcements, where the father global and public announcement entail common knowledge of* α*.*

Now suppose that at the beginning, again in world $(1,0,0)$*, the father* globally *announces to red and blue that at least one child is muddy. The refined model* $\mathcal{M}^+_{(100,\alpha,rb)}$ *is indeed different from* $\mathcal{M}^-_{(100,\alpha,rb)}$*, as shown below.*



Specifically, in $\mathcal{M}^+_{(100,\alpha,rb)}$ *the indistinguishability relations are refined for both red and blue, and as a result, after the father's global announcement, in* $(1,0,0)$ *red and blue have common knowledge that at least one child is muddy:* $(1,0,0) \vDash [\alpha]^+_{rb} C_{rb}\alpha$. *However, also in this case the father's global announcement is not enough to make* α *common knowledge amongst all children, that is,* $(1,0,0) \nvDash [\alpha]^+_{rb} C\alpha$, *as* $(1,0,0) \vDash [\alpha]^+_{rb} M_g M_r M_g \neg\alpha$ *via epistemic path* $(1,0,0) \sim_g (1,1,0) \sim_r (0,1,0) \sim_g (0,0,0)$.

Example 6.2. *We now consider a scenario of coordinated attack. General* a *and* b *are planning to jointly attack the enemy, but each of them will attack only if the other is also attacking, also none of them is sure about whether the other will actually attack. As customary in such scenarios [64], we suppose that they attack simultaneously or not at all, and they actually attack simultaneously iff they reach common knowledge of the fact that they are both attacking, that is,* $C(att_a \wedge att_b)$. *This scenario can be represented intuitively as model* $\mathcal{N}$ *in Fig. 6.1.*

Further, we model communication between the two generals as an exchange of messages $att_a \to att_b$ *from* a *to* b, *and* $att_b \to att_a$ *from* b *to* a, *that is, each general communicates that he is attacking only if the other is also attacking. Since communication is between individuals, here we can omit the distinction between global and local announcements.*

Then we can check that, even though both generals intend to attack in world (att_a, att_b), *they will never attain common knowledge of this fact, independently from the number of*

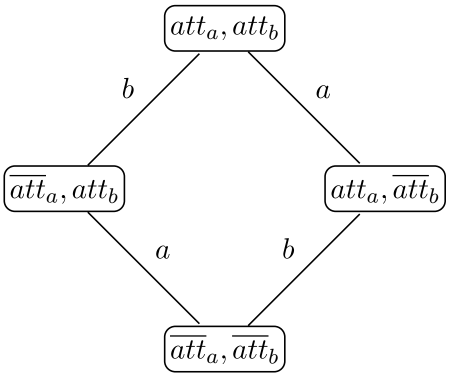


FIGURE 6.1: The model $\mathcal{N}$ for the coordinated attack scenario (reflexive edges are omitted for clarity).

messages they exchange. Specifically, in $\mathcal{N}$ *we have*

$$
\begin{aligned}
(att_a, att_b) &\not\models [att_a \to att_b]_b C(att_a \wedge att_b)\\
(att_a, att_b) &\not\models [att_b \to att_a]_a C(att_a \wedge att_b)
\end{aligned}
$$

Indeed, we can verify that $\mathcal{N}_{((att_a,att_b),att_a \to att_b,b)} = \mathcal{N}_{((att_a,att_b),att_b \to att_a,a)} = \mathcal{N}$. *As a result, no matter how many messages general* a *and* b *exchange, common knowledge that they are both attacking will never be attained. Hence, as a particular instance we have*

$$(att_a, att_b) \not\models [att_a \to att_b]_b[att_b \to att_a]_a[att_a \to att_b]_b C(att_a \wedge att_b)$$

Then, we might think of stronger messages to be exchanged by the generals, for instance, each general might communicate that the other general attacking is a sufficient, rather than necessary, condition for his own attack, that is, general a *sends message* $att_b \to att_a$ *to* b*, and* b *sends message* $att_a \to att_b$ *to* a*. Now, we can see that the following exchange is sufficient to achieve common knowledge:*

$$(att_a, att_b) \models [att_b \to att_a]_b[att_a \to att_b]_a C(att_a \wedge att_b)$$

On the other hand, this protocol is not robust against deviant behaviours. Indeed, each general is uncertain as to whether the other general is attacking. Hence, for instance, general a *considers world* $(att_a, \overline{att}_b)$*, where we have the following:*

$$(att_a, \overline{att}_b) \not\models [att_b \to att_a]_b(att_a \wedge att_b)$$

and as a consequence,

$$(att_a, att_b) \not\models K_a[att_b \to att_a]_b[att_a \to att_b]_a C(att_a \wedge att_b)$$

and similarly for general b.

Thus, even though the protocol is correct, no general knows this fact, and if we consider knowledge as a prerequisite for action, no general will actually follow the protocol.

Hence, we investigate a protocol that is robust against deviant behaviours and consider messages $K_a att_b \to att_a$ *from a to b, and* $K_b att_a \to att_b$ *from b to a, which weaken the second pair of messages by requiring knowledge of the fact that the other agent is attacking as sufficient condition for attacking. Then, we can check that*

$$\begin{aligned}(att_a, att_b) &\models K_a[K_a att_b \to att_a]_b K_b(att_a \wedge att_b)\\ (att_a, att_b) &\models K_b[K_b att_a \to att_b]_a K_a(att_a \wedge att_b)\end{aligned}$$

In particular, we have the following

$$(att_a, att_b) \models C[K_a att_b \to att_a]_b[K_b att_a \to att_b]_a C(att_a \wedge att_b)$$

that is, both generals have common knowledge that if they tell each other that they are attacking if they know that the other general is attacking as well, then they will be able to coordinate for an attack.

These examples are intended to illustrate the formal features of GLAL to represent global and local communication, as well as message exchanges in coordination problems. In particular, GLAL allows to express local communication that cannot be captured in PAL. In the following section we analyse the expressivity of GLAL and provide a formal proof of the fact that it is strictly more powerful than PAL.

6.2 Expressivity and Validities

This section is devoted to explore the expressive power of GLAL through its validities. The main result is that GLAL, differently from PAL, is not reducible to epistemic logic, and therefore strictly more expressive than both.

As a preliminary result, we show that after announcing (truthfully) a propositional formula ϕ to the agents in A, they know ϕ.

Lemma 6.5. *For every propositional formula $\phi \in \mathcal{L}_{pl}$,*

$$\vDash [\phi]^-_A E_A \phi \tag{6.1}$$

$$\vDash [\phi]^+_A C_A \phi \tag{6.2}$$

According to Lemma 6.5, if a propositional formula ϕ is announced locally, then all agents involved in the announcement know ϕ; while if ϕ is announced globally, then it also becomes common knowledge amongst those agents. Note that global announcements imply group knowledge in particular: $\vDash [\phi]^+_A E_A \phi$, but local announcements normally do not entail common knowledge. For instance, in the muddy children puzzle above, we had that $(1,0,0) \nvDash [\alpha]^-_{rb} C_{rb} \alpha$. As a consequence, the notions of global and local announcement as described by operators $[\phi]^-_A$ and $[\phi]^+_A$ are indeed different.

Lemma 6.5 does not hold for general formulas $\phi \in \mathcal{L}_{glal}$. As a counterexample, take a model $\mathcal{M}$ with $W = \{w, v\}$, $R_a = W^2$, and p only true in w. Then we have $(\mathcal{M}, w) \vDash (p \wedge \neg K_a p)$, but a (either global or local) truthful announcement of $p \wedge \neg K_a p$ to agent a does entail that a knows p: in the refined model $\mathcal{M}_{(w, p \wedge \neg K_a p, a)}$ we have $R^-_a = R^+_a = \{(w, w), (v, v)\}$, and therefore $(\mathcal{M}, w) \nvDash [p \wedge \neg K_a p]_a (p \wedge \neg K_a p)$.

Further, GLAL is provably at least as expressive as public announcement logic. Indeed, we show that the global announcement modality $[\phi]^+_I$ for the grand coalition mimicks operator $[\phi]$ from PAL. We refer to [100] for a formal presentation of PAL. Here we only recall the satisfaction clause for $[\phi]$-formulas.

$$[\![[\psi]\psi']\!]_{\mathcal{M}} \quad = \quad \{w \in W \mid \text{if } w \in [\![\psi]\!]_{\mathcal{M}} \text{ then } w \in [\![\psi']\!]_{\mathcal{M}_\psi}\}$$

where the *refinement* $\mathcal{M}_\psi = \langle W_\psi, \{R_{\psi,a}\}_{a \in}, V_\psi \rangle$ of model $\mathcal{M}$ with respect to formula ψ is defined as *(i)* $W_\psi = W \cap [\![\psi]\!]_{\mathcal{M}}$; *(ii)* for every agent $a \in I$, $R_{\psi,a} = R_a \cap ([\![\psi]\!]_{\mathcal{M}} \times [\![\psi]\!]_{\mathcal{M}})$; and *(iii)* for every $p \in AP$, $V_\psi(p) = V(p) \cap [\![\psi]\!]_{\mathcal{M}}$. Intuitively, $\mathcal{M}_\psi$ is the restriction of $\mathcal{M}$ to the worlds satisfying ψ.

Now consider the mapping $tr : \mathcal{L}_{pal} \to \mathcal{L}_{glal}$ recursively defined as follows:

$$\begin{array}{lcl}
tr(p) & = & p \\
tr(\neg\phi) & = & \neg tr(\phi) \\
tr(\phi \wedge \phi') & = & tr(\phi) \wedge tr(\phi') \\
tr(K_a \phi) & = & K_a tr(\phi) \\
tr(C_A \phi) & = & C_A tr(\phi) \\
tr([\phi]\phi') & = & [tr(\phi)]^+_I tr(\phi')
\end{array}$$

Then, we are able to prove the following equivalence result.

Proposition 6.6. *For all formulas ψ in PAL,*

$$(\mathcal{M}, w) \models \psi \quad \textit{iff} \quad (\mathcal{M}, w) \models tr(\psi)$$

An immediate corollary of Proposition 6.6 is that public announcements in PAL really correspond in GLAL to global announcements to all agents.

Corollary 6.7. *For all formulas ϕ, ψ in PAL,*

$$(\mathcal{M}, w) \models [\phi]^{+}_{I}\psi \quad \textit{iff} \quad (\mathcal{M}, w) \models [\phi]\psi$$

In agreement with intuition, by Definition 6.7 public announcements in PAL are really global announcements to all agents in GLAL. As a consequence, GLAL is at least as expressive as PAL.

Next we prove that GLAL is strictly more expressive than epistemic logic. Since epistemic logic and PAL are equally expressive [100], it immediately follows that GLAL is strictly more expressive than PAL as well. The result is shown by providing two bisimilar models, that therefore satisfy the same epistemic formulas, but satisfy different formulas in GLAL. To this end, consider models $\mathcal{M} = \langle W, R_a, R_b, V \rangle$ and $\mathcal{M}' = \langle W', R'_a, R'_b, V' \rangle$, depicted in Fig. 6.2, such that

- $W = \{v_e, v_o\}$;
- $R_a = R_b = W^2$;
- $V(p) = \{v_e\}$;
- W' is the set $\mathbb{Z}$ of the integers;
- for all $n \in \mathbb{Z}$, $R'_a(n, n)$, $R'_b(n, n)$, $R'_a(2n, 2n + 1)$, and $R'_b(2n + 1, 2n)$;
- $V'(p) = \{n \in \mathbb{Z} \mid n \text{ is even}\}$.

Define a relation B such that $B(v_e, n)$ iff n is even, and $B(v_o, n)$ iff n is odd. It is easy to check that the B is a bisimulation between pointed models $(\mathcal{M}, v_e)$ and $(\mathcal{M}', 0)$. In particular, the same epistemic formulas are satisfied at states v_e and 0. However, for $Kw_a\phi := K_a\phi \vee K_a\neg\phi$, we can check that $(\mathcal{M}, v_e) \models [p]_a K_b Kw_a p$; while $(\mathcal{M}', 0) \not\models [p]_a K_b Kw_a p$. Hence, GLAL is capable of distinguishing between models that satisfy the same epistemic formulas, and we obtain the following result.

Theorem 6.8. *GLAL is strictly more expressive than epistemic logic.*

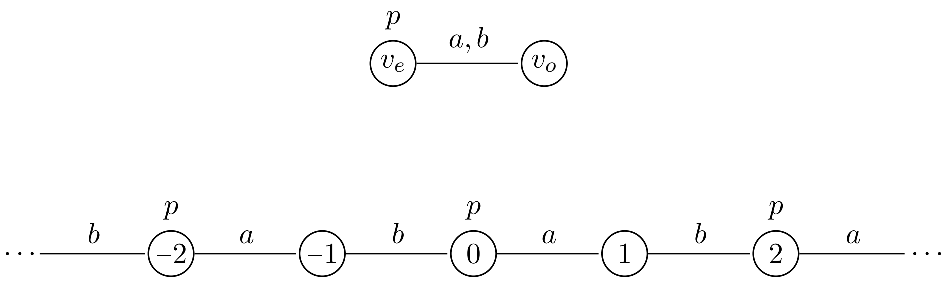


FIGURE 6.2: Models $\mathcal{M}$ and $\mathcal{M}'$ for the proof of Theorem 6.8 (reflexive edges are omitted for clarity).

By Theorem 6.8 and the equi-expressivity of epistemic logic and PAL [100], we immediately obtain the following result.

Corollary 6.9. *GLAL is strictly more expressive than PAL.*

By the proof of Theorem 6.8 we can see that not even announcements to single agents are reducible to epistemic formulas. Hence, it looks as if the reducibility of PAL to epistemic logic is a fortuitous accident. Also, the same proof points out that a more robust notion of bisimulation is needed to preserve formulas in GLAL.

A further consequence of Theorem 6.8 is that, differently from PAL, there is no set of validities in GLAL to rewrite any announcement in terms of simpler formulas. Nonetheless we state without proof the validity of the following equivalences.

Lemma 6.10. *The following formulas are validities in GLAL:*

$$\begin{aligned} [\phi]^-_A p \leftrightarrow \phi \rightarrow p \qquad & [\phi]^+_A p \leftrightarrow \phi \rightarrow p \\ [\phi]^-_A \neg\psi \leftrightarrow \phi \rightarrow \neg[\phi]^-_A\psi \qquad & [\phi]^+_A \neg\psi \leftrightarrow \phi \rightarrow \neg[\phi]^+_A\psi \\ [\phi]^-_A (\psi \wedge \psi') \leftrightarrow [\phi]^-_A\psi \wedge [\phi]^-_A\psi' \qquad & [\phi]^+_A (\psi \wedge \psi') \leftrightarrow [\phi]^+_A\psi \wedge [\phi]^+_A\psi' \end{aligned}$$

Thus, both announcement operators commute with propositional connectives.

Moreover, the announcement and common knowledge operators commute if they both refer to the same agent or the same coalition.

Lemma 6.11. *The following are GLAL validities:*

$$[\phi]^+_A C_A\psi \quad \leftrightarrow \quad \phi \rightarrow C_A[\phi]^+_A\psi \tag{6.3}$$

$$[\phi]_a K_a\psi \quad \leftrightarrow \quad \phi \rightarrow K_a[\phi]_a\psi \tag{6.4}$$

On the other hand, (6.3) and (6.4) do not hold for arbitrary agents and coalitions. As an example of this, consider model $\mathcal{M}'$ in Fig. 6.2: we remarked that $(\mathcal{M}', 0) \not\models$

$[p]_a K_b K w_a p$, but we have $(\mathcal{M}', 0) \vDash K_b [p]_a K w_a p$. Hence, $(\phi \to K_b [\phi]_a \psi) \to [\phi]_a K_b \psi$ do not hold in general, and we can find counterexamples for the opposite direction as well.

As regards nested announcements, again we have that operators commute iff they are indexed by the same coalitions

Lemma 6.12. *The following formulas are validities in GLAL:*

$$[\phi]^-_A [\phi']^-_A \psi \quad \leftrightarrow \quad [\phi \wedge [\phi]^-_A \phi']^-_A \psi \tag{6.5}$$

$$[\phi]^+_A [\phi']^+_A \psi \quad \leftrightarrow \quad [\phi \wedge [\phi]^+_A \phi']^+_A \psi \tag{6.6}$$

Also for formulas (6.5) and (6.6) we can check that they do not hold for arbitrary coalitions and agents. As an example, consider model $\mathcal{M}$ in Fig. 6.2: we have that $(\mathcal{M}, v_e) \vDash [p]_a [K_b K w_a]_b K_b K w_a$, but $(\mathcal{M}, v_e) \nvDash [p \wedge [p]_a K_b K w_a]_b K_b K w_a$. In particular, $[\phi]_a [\phi']_b \psi \leftrightarrow [\phi \wedge [\phi]_a \phi']_b \psi$ fails for agents $a \neq b$.

Given that operators $[\phi]^-_A$ and $[\phi]^+_A$ are not reducible, it is of interest to investigate what kind of modalities they are, specifically what modal principles their semantics validates. First, it is easy to see that both axiom **K** and rule **Nec** of necessitation are valid:

$$[\phi]^-_A(\psi \to \psi') \to ([\phi]^-_A \psi \to [\phi]^-_A \psi') \qquad [\phi]^+_A (\psi \to \psi') \to ([\phi]^+_A \psi \to [\phi]^+_A \psi')$$

$$\psi \Rightarrow [\phi]^-_A \psi \qquad \psi \Rightarrow [\phi]^+_A \psi$$

On the other hand, all axioms **T**, **4** and **B** fail. As regards **T**, if ϕ is false, then $[\phi]_a \psi$ holds for any formula ψ, but it does not follow that ψ holds whenever it is false itself. As to axiom **4**, notice that in the muddy children puzzle a child not stepping forward is tantamount to globally stating that she does not know whether she is muddy, or $[no\_step] := [\bigwedge_{a \in I} \neg K w_a m_a]^+_I$. Hence, after the father's announcement, in state $(1,1,0)$ we have that no child knows whether she is muddy after the first round, that is, $(1,1,0) \vDash [no\_step] \bigwedge_{a \in I} \neg K w_a m_a$. However, at the second round all muddy children know that they are muddy: $(1,1,0) \vDash [no\_step][no\_step] \bigwedge_{a \in I} (m_a \to K w_a m_a)$. In particular, $(1,1,0) \nvDash [no\_step][no\_step] \bigwedge_{a \in I} \neg K w_a m_a$, thus invalidating **4**. As regards **B**, consider again model $\mathcal{M}$ appearing after Lemma 6.5. We have that $(\mathcal{M}, w) \vDash p \wedge \neg K_a p$, and if p is announced to a, then p still holds but is no longer the case that a does not know p, that is, $(\mathcal{M}, w) \vDash p \wedge \neg K_a p$ but $(\mathcal{M}, w) \nvDash [p]_a \langle p \rangle_a (p \wedge \neg K_a p)$.

6.3 Discussion and Related Literature

In this chapter we introduced a unified account to formalise both global and local announcements in GLAL, an extension of public announcement logic that is strictly more expressive. The key feature of the semantics of GLAL is that the refinement of the indistinguishability relations is defined in the same way for public and private announcements, i.e., as the refinement of the equivalence classes to the worlds satisfying or not the given announcement. Then, the crucial difference between global and local announcements is the domain of application of such updates: for local announcements the updates are refined to worlds accessible in one step through the indistinguishability relation of each given agent; while in global announcements we consider all worlds epistemically reachable. In Example 6.1 and 6.2 we showed how these formal notions capture our intuitions about global and local announcements.

The present framework can be extended in several directions Firstly, since differently from PAL announcements are not necessarily broadcast to all agents (so that only one such announcement can be broadcast at any given time), we can think about global and local announcements communicated simultaneously and introduce formulas $([\phi]_A \circ [\phi']_B)\psi$ with the intended meaning that if ϕ is (truthfully) announced to coalition A and *simultaneously* ϕ' is announced to coalition B, then ψ holds. This is of interest to model synchronous communication. Particular care has to be taken in defining the semantics of operator $[\phi]_A \circ [\phi']_B$ whenever the intersection of coalitions A and B is non-empty. Secondly, as the receiver of the announcement can be a subset $A \subseteq I$ of the set of all agents, we can think that the announcement originates from some other coalition B and introduce GLAL operators $[\phi]_{B,A}$ indexed to both A and B. Such an extension would provide a finer-grained analysis of scenario such as the coordinated attack in Example 6.2. Indeed, if we assume reliable communication channels, the truth of $[\phi]_{B,A}\psi$ does imply that A knows ϕ, but also that B knows that A knows ϕ, and so on. On the other hand, if communication is not reliable, a different, weaker semantics has to be taken into account.

Related to GLAL, public announcement logics have witnessed a steady growth and a wealth of contributions in recent years [11, 100, 111, 115, 118], thus making virtually impossible to give an exhaustive account of this research area. Here we mention the references most closely related to the present work, as well as some surveys on PAL. In [71] a logic of fully private announcements was proposed, while [13, 71, 112, 114] put forward logics of semi-private announcements, which relax the publicity assumption of PAL in various directions. Such private or semi-private announcements have also been modelled as action models [11]. Differently from our proposal, in semi-private announcements the agents that do not observe the announcement of ϕ learn at least that the other agents have learnt whether ϕ. No such assumption holds in the present context. On the othe hand, in fully private announcements the other agents learn nothing at all about the agents learning ϕ (which is typically interpreted as the other agents not even being aware of the announcement having taken place). This is also different from our setting, wherein these other agents learn something about ϕ with respect to the actual state.

Modal logics based on model transformation have also been proposed in [5, 6, 65, 110]. These accounts share the aspect of locality (dependence of the model transforming operation on the actual state) that also characterizes our approach. However, differently from our proposal, these are very expressive formalisms (typically undecidable, or non-axiomatizable) that allow to add or remove individual pairs of states from an agent's accessibility relation; thus operating on a purely semantic level. On the contrary, in GLAL the model transformation is determined by the announced formula, so that only pairs satisfying a condition relative to this formula can be removed. This gives less opportunity to separate non-bisimilar states. Although we did not prove such theoretical results here, our operators may therefore be more promising to obtain decidability and axiomatizability.

Chapter 7

Conclusions

In this work we aimed at developing formal methods to represent and reason about individual and group knowledge in multi-agent contexts, particularly by means of propositional quantification. In Chapter 3 we introduced second-order propositional modal logic and analysed its features to express local properties in Kripke frames. Specifically, we compared SOPML to the language of local properties in modal logic [119–121], and showed that the former is as expressive as the latter. Furthermore, we considered second-order propositional epistemic logic, an epistemic version of SOPML, and observed that this language is suitable to represent higher-order knowledge of agents and coalitions, namely the knowledge agents have of other agents' and coalitions' knowledge, including truthfulness of knowledge, inclusion of one agent's knowledge in another's, etc. As an example, we showed that SOPEL is capable of capturing comparative epistemic logic [121]. In the same chapter we presented theoretical results that are crucial to assess the computational properties of the formal framework. Specifically, we proved that for the classes of all frames and propositional frames, we are able to provide sound and complete axiomatisations with respect to all normal modalities. As regards the class of full frames, we gave an axiomatisation only for the normal modality **S5**, the standard logic for knowledge. We highlighted the essential use of the common knowledge operator in obtaining such a result. Moreover, we remarked that no complete axiomatisation exists for weaker modalities, while we left the same question for modal frames as an open problem.

To assess the expressive power of SOPML, in Chapter 4 we introduced (bi)simulation relations, which are well-known to be a useful tool in modal logic [47]. Then, we proved that (bi)simulations preserve the interpretation of formulas in (the universal fragment of) SOPML. We also defined (bi)simulation games played by Spoiler and Duplicator, and proved that the existence of a winning strategy for Duplicator is tantamount to the existence of a (bi)simulation. These results have then been applied to analyse the expressivity of SOPML in reasoning about spatial and temporal properties. In particular, we showed that while 3-colorability and Dedekind-completeness are expressible in SOPML, having a Hamiltonian path or being (in)finite are not. Results along this line are key to assess the expressive power of SOPML.

In Part II we moved to analyse the dynamics of knowledge, by extending the framework of SOPEL with announcement operators, in the spirit of public announcement logic [100]. The motivation for doing so comes from arbitrary public announcement logic, in which implicit quantification is used to express epistemic properties such as knowability, preservation, and successfulness of announcements. In Chapter 5 we introduced the syntax and semantics of SOPAL, compared it thoroughly with APAL, and showed that the former is more expressive than the latter at the frame level. Moreover, we proved that SOPAL is as expressive as SOPEL, but exponentially more succint.

Finally, in Chapter 6 we relaxed the assumptions of globality and publicity of public announcement logic and introduced the logic of global and local announcements. Technically, GLAL includes announcement operators for both global and local announcements, indexed to coalitions of agents. This language is shown to be suitable to describe multi-agent scenarios such as variants of the muddy children puzzle, as well as coordinated attacks. Differently from PAL, we proved that GLAL is strictly more expressive than epistemic logic.

To conclude, in this work we introduced a wealth of expressive epistemic langugages enjoying different features: quantified v. purely propositional, static v. dynamic. We applied them to represent a number of multi-agents scenarios, and analysed their theoretical properties. We reckon that a lot is left to be done in the area of knowledge representation, particularly with respect to agent communication. We leave these topics for future developments.

Chapter 8

Perspectives

We conclude this work by discussing some perspectives for future developments in the areas of knowledge representation and reasoning, as well as logics for multi-agent systems. In particular, our research efforts in the coming years will be directed to the realisation of the SVeDaS project, funded by the *Jeunes Chercheuses/Jeunes Chercheurs* scheme of the ANR. The SVeDaS project is intended to leverage on the techniques and results obtained thus far in the specification and verification of multi-agent systems, and to extend and to apply these to the novel class of data-aware systems. In Chapter 1 we saw that data-aware systems have emerged in the last decade as the leading paradigm to analyse use cases in which data play an essential role in the system's execution [46, 60, 86]. Data-aware systems are focused on the combined perspective of *data models* and *system processes*. Data are visible and accessible to agents, possibly in a controlled way through some permission restrictions; they directly account for the system's evolution and can be exhibited explicitly in the system's specification. These considerations apply to auctioning processes as well: one original tenet of the SVeDaS is to model auction-based mechanisms as data-aware systems. For the effective deployment of DaS, including auction in e-markets, verification and validation methodologies are essential. The SVeDaS project takes inspiration from the state-of-the-art in the application of formal methods to data-aware systems, and aims at developing a tailored methodology for their modelling, analysis and verification, then to apply these techniques to the formal certification of auctions. However, the enhanced expressivity provided by DaS comes at a computational price. In particular, we identify two main shortcomings in the present state-of-the-art.

1. Most of the literature on DaS [60, 61, 86, 97] focuses almost exclusively on the data model of business processes, while neglecting the software agents implementing the service infrastructure. These software agents might have only a partial view of the

relevant data, or, in the terminology of multi-agent systems, they have *imperfect information* of the global state of the system [127]. This (lack of) information shapes the capabilities of the software agents to interact and bring about change, which in turn has an impact on the overall behaviour and performance of the data-aware system. Thus, modelling both the information state and the strategic abilities of agents operating on DaS are key to describe and predict the evolution of the system.

2. The actual deployment of DaS in concrete, safety- and security-critical scenarios demands for the development of automated verification techniques, including by model checking [8, 55]. However, while formal methods are relatively well-understood in plain process-based modelling, the presence of data makes the typical verification questions much harder to answer and only partially explored. Notably, the common assumption of a possibly infinite data domain in the underlying system leads to an infinite state-space and undecidability of the corresponding model checking problem in the general case. Hence, the verification of DaS is highly non-trivial and it cannot be immediately handled by standard techniques for finite-state systems. Data-driven, tailored solutions need to be developed and deployed in an up-to-date model checking tool.

These are among the main challenges faced by the application of the data-centric paradigm in modelling concrete multi-agent systems. In the SVeDaS project we envisage to tackle these issues by

(i) developing an agent-oriented approach to DaS modelling, in order to account also for the imperfect information in the system;

(ii) investigating verification techniques based on (bi)simulations and abstraction for contexts of imperfect information;

(iii) implementing the relevant techniques in a model checking tool to certify auctioning mechanisms represented as data-aware systems.

The overall proposed solution is enbodied in a series of (partial) objectives that can be summarized as follows:

1. To introduce agent-based, computationally-grounded models for data-aware systems, that are capable of expressing rich business workflows, including auction-based mechanisms in e-markets (e.g., English, Dutch, and Vickrey auctions, real-time bidding). On this point we published preliminaries results for specific types of auctions [14, 35, 42].

2. To explore logic-based formal languages for the specification of strategic behaviours of autonomous agents (including robustness against malicious behaviours, as well as manipulability and collusion in auctions) pertaining to business processes and agents operating on them. Rich specification languages for knowledge representation and reasoning, such as those considered in this work, are of utmost importance with respect to this task. We envisage to make use of our results in [43–45] to express strategic reasoning of agents in auction-based mechanisms.

3. To analyse the formal properties of these data-aware models, particularly the issues concerning formal verification by model checking in a setting with imperfect information. This task will benefit from results on MAS verification in contexts of imperfect information we obtained in [15, 18, 37].

4. To find classes of data-aware systems and expressive language fragments relevant for auction-driven applications, which have a decidable model checking problem and possibly are also amenable to practical verification. Of particular interest for this objective are the notions of *uniformity* and *boundedness* developed within the ACSI project. However, the restrictions imposed by these conditions needs to be overcome, as we attempted to do in [17, 19].

5. To develop the SVeDaS model checker for the verification and validation of data-aware systems in multi-agent scenarios. This tool will leverage on model checking methodologies and techniques we explored in [20, 25, 41].

6. To evaluate the performance of the SVeDaS tool in the influential auctioning mechanisms mentioned above, and to release it as open-source software.

We anticipate that the research envisaged in the SVeDaS project will impact on the application of formal methods to data-aware systems in general, and auction-based mechanisms in particular. The certification against deviant behaviours guaranteed by formal verification will contribute to design more robust and reliable auctions in e-commerce. Society as a whole will benefit from the findings and results of the SVeDaS project.

Appendix A

Proofs

A.1 Chapter 3

Lemma 3.8. All proofs are by induction on the structure of ϕ.

1. If $\phi = p$, then $fr(\phi) = \{p\}$ and $(\mathcal{M}, w) \vDash \phi$ iff $w \in V(p) = V'(p)$, iff $(\mathcal{M}', w) \vDash \phi$.

   The inductive cases for the propositional connectives are immediate.

   If $\phi = \Box_a \psi$, then $(\mathcal{M}, w) \vDash \phi$ iff for all $w' \in R_a(w)$, $(\mathcal{M}, w') \vDash \psi$. Since $fr(\phi) = fr(\psi)$, V and V' coincide on $fr(\psi)$ as well, and by induction hypothesis for all $w' \in R_a(w)$, $(\mathcal{M}', w') \vDash \psi$, i.e., $(\mathcal{M}', w) \vDash \phi$. The case for $\phi = \Box^*_A \psi$ is similar.

   If $\phi = \forall p \psi$, then $(\mathcal{M}, w) \vDash \phi$ iff for any $U \in D$, $(\mathcal{M}^p_U, w) \vDash \psi$. Since $fr(\phi) = fr(\psi) \setminus \{p\}$, V^p_U and V'^p_U coincide on $fr(\psi)$, and by induction hypothesis $(\mathcal{M}'^p_U, w) \vDash \psi$. By the arbitrariness of U, this is the case iff $(\mathcal{M}', w) \vDash \phi$.

2a The case for $x = ap$ is immediate, as assignments are functions in D. Hence, $V(p) \in D$ for every $p \in AP$.

The case for $x = pl$, follows from identities $[\![\neg\psi]\!] = W \setminus [\![\psi]\!]$, $[\![\psi \wedge \psi']\!] = [\![\psi]\!] \cap [\![\psi']\!]$, $[\![\psi \vee \psi']\!] = [\![\psi]\!] \cup [\![\psi']\!]$ and the fact that D is a boolean algebra.

As for $x = ml$, notice that $[\![\Box_a \psi]\!] = [a]([\![\psi]\!])$, $[\![\Box^*_A \psi]\!] = [A]^*([\![\psi]\!])$, and D is a boolean algebra closed under operators $[a]$ and $[A]^*$.

The case for $x = sopml$, is immediate, as $[\![\psi]\!] \subseteq W$ for every $\psi \in \mathcal{L}_{sopml}$.

2b Let us first consider $x = ap$. If ϕ is an atom r, $(\mathcal{M}^p_{V(q)}, w) \vDash \phi$ iff $w \in V^p_{V(q)}(r)$, iff $w \in V(r)$ whenever $r \neq p$ or $w \in V(q)$ for $r = p$. In both cases $(\mathcal{M}, w) \vDash \phi[p/q]$.

The inductive cases for propositional connectives and modal operators are immediate, as these simply commute with substitution.

If $\phi = \forall r\varphi$ for $r \neq p$, then $(\mathcal{M}^p_{V(q)}, w) \vDash \phi$ iff for every $U \in D$, $((\mathcal{M}^p_{V(q)})^r_U, w) \vDash \varphi$. Since $r \neq p$ and q is free for p in ϕ, $q \neq r$ and assignment $(V^p_{V(q)})^r_U$ is equal to $(V^r_U)^p_{V^r_U(q)}$. As a consequence, we obtain $((\mathcal{M}^r_U)^p_{V^r_U(q)}, w) \vDash \varphi$, i.e., $(\mathcal{M}^r_U, w) \vDash \varphi[p/q]$ by induction hypothesis. But this means that $(\mathcal{M}, w) \vDash \forall r(\varphi[p/q]) = (\forall r\varphi)[p/q]$.

As regards cases $x = pl, ml, sopml$, we make use of (1). We only prove the inductive step for $\phi = \forall r\varphi$, with $r \neq p$, the other cases being similar to the case for $x = ap$ above. Observe that $(\mathcal{M}^p_{[\![\psi]\!]}, w) \vDash \phi$ iff for every $U \in D$, $((\mathcal{M}^p_{[\![\psi]\!]})^r_U, w) \vDash \varphi$. Since $r \neq p$ and ψ is free for p in ϕ, $r \notin fr(\psi)$, and by (1) above, $[\![\psi]\!]_{\mathcal{M}} = [\![\psi]\!]_{\mathcal{M}^r_U}$. Therefore assignment $(V^p_{[\![\psi]\!]})^r_U$ is equal to $(V^r_U)^p_{[\![\psi]\!]_{\mathcal{M}^r_U}}$. Hence, we obtain $((\mathcal{M}^r_U)^p_{[\![\psi]\!]_{\mathcal{M}^r_U}}, w) \vDash \varphi$, i.e., $(\mathcal{M}^r_U, w) \vDash \varphi[p/\psi]$ by induction hypothesis. But this means that $(\mathcal{M}, w) \vDash \forall r(\varphi[p/\psi]) = (\forall r\varphi)[p/\psi]$.

□

Proposition 3.10. The proof for full frames is immediate, as for every $U \subseteq W_w$, $U \subseteq W$ and then $U \in D$. Hence, $U_w = U \in D_w$.

The proof for boolean frames follows from the following identities:

$$\begin{aligned} U_w \cap U'_w &= (U \cap U')_w \\ U_w \cup U'_w &= (U \cup U')_w \\ \smallsetminus(U_w) &= (\smallsetminus U)_w \end{aligned}$$

As for modal frames, we remark that

$$[a](U_w) = ([a]U)_w$$

(observe that here $[a]$ denotes two different operations, the former on $\mathcal{M}_w$ and the latter on $\mathcal{M}$.) Indeed, $v \in [a](U_w)$ iff $R_{w,a}(v) \subseteq U_w$. Since, $R_{w,a}(v) = R_a(v) \cap W^2_w$, this is the case iff v is reachable from w and for every $v' \in R_a(v)$, if v' is reachable from w then $v' \in U$, iff v is reachable from w and for every $v' \in R_a(v)$, $v' \in U$, iff $v \in ([a]U)_w$. The proof for operator $[A]^*$ is similar.

Finally, since reflexivity, symmetry, and transitivity are all universal properties, they are preserved under taking subsets, and therefore, if R_a satisfies any of them, then for any $W_w \subseteq W$, the relation $R_{w,a} = R_a \cap W^2_w$ also satisfies the same property. In particular, the generated submodel of an epistemic model with equivalence relations is still a model with equivalence relations. □

Lemma 3.11. The proof is by induction on the structure of ϕ. For $\phi = p$, $(\mathcal{M}, v) \models \phi$ iff $v \in V(p)$, iff $v \in V_w(p) = V(p) \cap W_w$, iff $(\mathcal{M}_w, v) \models \phi$. The cases for propositional connectives are immediate. As to $\phi = \Box_a \psi$, if $(\mathcal{M}, v) \not\models \phi$ then for some $v' \in R_a(v)$, $(\mathcal{M}, v') \not\models \psi$. In particular, v' is reachable from v and therefore from w, thus $v' \in W_w$. Hence, the induction hypothesis holds for $v' \in R_{w,a}(v)$ and $(\mathcal{M}_w, v') \not\models \psi$, that is, $(\mathcal{M}_w, v) \not\models \phi$. The case for $\phi = \Box^*_A \psi$ is similar. Finally, for $\phi = \forall p \psi$, if $(\mathcal{M}, v) \not\models \phi$ then for some $U \in D$, $(\mathcal{M}^p_U, v) \not\models \psi$. Consider $U_w = U \cap W_w$. We have that for every $q \in AP$, $(V^p_U)_w(q) = V^p_U(q) \cap W_w = V^p_{U_w}(q) \cap W_w = (V_w)^p_{U_w}(q)$, and by induction hypothesis $((\mathcal{M}_w)^p_{U_w}, v) \not\models \psi$. That is, $(\mathcal{M}_w, v) \not\models \phi$. □

Corollary 3.12. Obviously, $\mathsf{Th}(\mathcal{K}^e_y) \subseteq \mathsf{Th}(\mathcal{K}^e_y \cap \mathcal{K}_{univ})$. As to the converse, suppose that $(\mathcal{F}, V, w) \not\models \phi$ for some $\mathcal{F} \in \mathcal{K}^e_y$, and consider the generated submodel $\mathcal{M}_w = \langle \mathcal{F}_w, V_w \rangle$. Clearly, $\mathcal{F}_w \in \mathcal{K}_{univ}$ is universal. Moreover, by Proposition 3.10 $\mathcal{F}_w$ belongs to the relevant class $\mathcal{K}^e_y$ of frames, and by Lemma 3.11 we have that $(\mathcal{M}_w, w) \not\models \phi$ for $\mathcal{F}_w \in \mathcal{K}_{univ} \cap \mathcal{K}^e_y$. □

Theorem 3.16. We prove the theorem in general, but first we show what it means for the following specific case: $\theta(a, b, p) = \Box_a p \to \Box_b p$, $\boxdot(a, b) = Sup(a, b)$, and $\Theta(a, b, x) = \forall y (R_b(x, y) \to R_a(x, y))$, also written as $R_b(x) \subseteq R_a(x)$.

$\Leftarrow$ Since $(\mathcal{M}, w) \models Sup(a, b)$, we have that $R_b(x) \subseteq R_a(x)$ holds in $(\mathcal{M}, w)$, and hence in $(\mathcal{F}, w)$. Since $\Box_a p \to \Box_b p$ locally defines $R_b(x) \subseteq R_a(x)$, we have $(\mathcal{F}, w) \models (\Box_a p \to \Box_b p)$, and in particular $(\mathcal{F}, w) \models \forall p (\Box_a p \to \Box_b p)$. Since $\forall p (\Box_a p \to \Box_b p)$ is a sentence, we obtain $(\mathcal{M}, w) \models \forall p (\Box_a p \to \Box_b p)$. $\Rightarrow$ Now suppose that $(\mathcal{M}, w) \not\models Sup(a, b)$. Then $(\mathcal{F}, w) \not\models Sup(a, b)$. Since $\Box_a p \to \Box_b p$ locally defines $R_b(x) \subseteq R_a(x)$, we know that $(\mathcal{F}, w) \not\models \Box_a p \to \Box_b p$, and since $\mathcal{F}$ is full, for some assignment V', we have $(\mathcal{F}, V', w) \not\models \Box_a p \to \Box_b p$, that is, $(\mathcal{M}, w) \not\models \forall p (\Box_a p \to \Box_b p)$.

As for the general case: $\Leftarrow$ Since $(\mathcal{M}, w) \models \boxdot(\vec{a})$, we have that $\Theta(\vec{a}, x)$ holds in $(\mathcal{M}, w)$, and hence in $(\mathcal{F}, w)$ (note that $\Theta \in \mathcal{L}^1_{fo}$ only talks about what is accessible from what). Since $\theta(\vec{a}, \vec{p})$ locally defines $\Theta(\vec{a}, x)$, we have $(\mathcal{F}, w) \models \theta(\vec{a}, \vec{p})$, and in particular $(\mathcal{F}, w) \models \forall \vec{p} \theta(\vec{a}, \vec{p})$. Since $\forall \vec{p} \theta(\vec{a}, \vec{p})$ is a sentence, $(\mathcal{M}, w) \models \forall p \theta(\vec{a}, \vec{p})$. $\Rightarrow$ Suppose that $(\mathcal{M}, w) \not\models \boxdot(\vec{a})$. Then, $(\mathcal{F}, w) \not\models \boxdot(\vec{a})$, and therefore $(\mathcal{F}, w) \not\models \Theta(\vec{a}, x)$. Since $\theta(\vec{a}, \vec{p})$ locally defines $\Theta(\vec{a}, x)$, we know that $(\mathcal{F}, w) \not\models \theta(\vec{a}, \vec{p})$, and since $\mathcal{F}$ is full, for some assignment V', we have $(\mathcal{F}, V', w) \not\models \theta(\vec{a}, \vec{p})$, that is, $(\mathcal{M}, w) \not\models \forall \vec{p} \theta(\vec{a}, \vec{p})$. □

Lemma 3.17.

1. If $\mathcal{F}$ is full and irreflexive, that is, for all $w \in W$, $\neg R_a(w, w)$, then clearly $(\mathcal{F}, w) \models \exists p (\Box_a p \wedge \neg p)$ for all $w \in W$, by considering assignment $V(p) = R_a(w)$ for which $w \notin V(p)$. As to the converse, suppose that $\mathcal{F} \models \exists p (\Box_a p \wedge \neg p)$.

Hence, for every model $\mathcal{M}$ based on $\mathcal{F}$ and $w \in W$, $(\mathcal{M}, w) \vDash \exists p(\Box_a p \wedge \neg p)$, i.e., $(\mathcal{M}, w) \nvDash \forall p(\Box_a p \rightarrow p)$. However, by Lemma 3.19 below, this is the case iff $R_a(w, w)$ does not hold. As a result, $\mathcal{F}$ is irreflexive.

2. Define ψ_i as $(p_i \wedge \bigwedge_{j \leq n, j \neq i} \neg p_j))$: ψ_i says that of all atoms $p_1, \ldots, p_n$, only p_i is true. If w has at least n R_a-successors, it is easy to find a valuation such that each one of the ψ_i's becomes true in one of those successors. Conversely, if w has less than n R_a-successors, it is easy to see that we cannot make all the ψ_i's true at the same time for all of w's successors.

3. Suppose that $\mathcal{F}$ is full and anti-symmetric. Further, for any world w, consider V_w such that $V_w(p) = \{w\}$. We then have $(\mathcal{F}, V_w, w) \vDash (p \wedge \forall q(\Diamond_a(q \wedge \Diamond_a p) \rightarrow q))$. To see this, take any subset $U \subseteq W$ and suppose $(\mathcal{F}, (V_w)^q_U, w) \vDash \Diamond_a(q \wedge \Diamond_a p)$. This means that for some $v \in R_a(w)$, $(\mathcal{F}, (V_w)^q_U, v) \vDash q \wedge \Diamond_a p$ holds. Since $\Diamond_a p$ is true in v and w is the only p-world, we have $R_a(v, w)$. Further, R_a is anti-symmetric, and therefore $w = v$. Hence, $(\mathcal{F}, (V_w)^q_U, w) \vDash q$. Conversely, suppose that $\mathcal{F}$ is full but not anti-symmetric. That is, for some $w, v \in W$, we have $R_a(w, v)$ and $R(v, w)$, but $w \neq v$. We show that φ_3 is false in $(\mathcal{F}, w)$, i.e., $(\mathcal{F}, w) \vDash \forall p(p \rightarrow \exists q(\Diamond_a(q \wedge \Diamond_a p) \wedge \neg q))$. To prove this, let V be such that $(\mathcal{F}, V, w) \vDash p$. Then, for $V^q_{\{v\}}$ we have $(\mathcal{F}, V^q_{\{v\}}, w) \vDash \Diamond_a(q \wedge \Diamond_a p) \wedge \neg q$.

4. Suppose that $\mathcal{F}$ is full and that R_a and R_b have an empty intersection. Then, $(\mathcal{F}, V^p_{R_a(w)}, w) \vDash \Box_a p \wedge \Box_b \neg p$, which shows that $(\mathcal{F}, w) \vDash \exists p(\Box_a p \wedge \Box_b \neg p)$. Conversely, suppose that Θ_4 does not hold for w, that is, for some $v \in W$, we have both $R_a(w, v)$ and $R_b(w, v)$. It follows immediately that $\exists p(\Box_a p \wedge \Box_b \neg p)$ is then false in w.

5. Suppose that Θ_5 holds in $\mathcal{F}$, and let V and w be such that $(\mathcal{F}, V, w) \vDash \Box_c p$. It is easy to check that $(\mathcal{F}, V^q_{R_a(w)}, w) \vDash \Box_a q \wedge \Box_b(q \rightarrow p)$. In words, if we modify V in such a way that q becomes true in exactly w's a-successors, then for every b-successor of w that satisfies q (note that this successor must then also be an a-successor), p must be true. Conversely, suppose that Θ_5 does not hold, i.e., for some $w, v \in W$, we have $R_a(w, v)$ and $R_b(w, v)$, but not $R_c(w, v)$. We now show that $(\mathcal{F}, w) \vDash \neg\varphi_5 = \exists p(\Box_c p \wedge \forall q(\Box_a q \rightarrow \Diamond_b(q \wedge \neg p)))$. The assignment V such that $V(p) = R_c(w)$ is a witness for this: if p is exactly true in the c-successors of w, then it is false in v, so whenever $\Box_a q$ is true in w, we have that $q \wedge \neg p$ holds in v, and hence $\Diamond_b(q \wedge \neg p)$ holds in w.

$\square$

Lemma 3.19. The proof is by induction on the structure of ψ. As regards the base of induction for $\psi = p$, $(\mathcal{F}, \rho) \vDash ST_x(\psi)$ iff $\rho(x) \in \rho(P)$, iff $w \in V(p)$, iff $(\mathcal{M}, w) \vDash$

p. The inductive steps for propositional connectives are immediate. For $\psi = \Box_a \phi$, $(\mathcal{M}, w) \vDash \psi$ iff for all $w' \in R_a(w)$, $(\mathcal{M}, w') \vDash \phi$, that is, $(\mathcal{F}, \rho') \vDash ST_y(\phi)$ for $\rho'(y) = w'$ by induction hypothesis. In particular, for all $w' \in R_a(w)$, $(\mathcal{F}, \rho^y_{w'}) \vDash ST_y(\phi)$, i.e., $(\mathcal{F}, \rho) \vDash \forall y(R_a(x, y) \to ST_y(\phi)) = ST_x(\psi)$. The case for $\psi = \Box^*_A \phi$ is similar. Finally, for $\psi = \forall p \phi$, $(\mathcal{M}, w) \vDash \psi$ iff for all $U \in D$, $(\mathcal{M}^p_U, w) \vDash \phi$, that is, $(\mathcal{F}, \rho') \vDash ST_x(\phi)$ by induction hypothesis, for ρ' that coincides with ρ but $\rho'(P) = U$. However, this means that $(\mathcal{F}, \rho^P_U) \vDash ST_x(\phi)$, i.e., $(\mathcal{F}, \rho) \vDash \forall P(ST_x(\phi)) = ST_x(\psi)$. $\Box$

Theorem 3.22. As customary, the axioms of each logic $\mathbf{L}_x$ are shown to be validities in the corresponding class $\mathcal{K}^{\tau(\mathbf{L}_x)}_{\widehat{x}}$ of frames, and that inference rules preserve validity in $\mathcal{K}^{\tau(\mathbf{L}_x)}_{\widehat{x}}$. Specifically, axioms **Prop**, **K**, **C1**, **C2**, **MP**, and **Nec** are valid in any frame, in which operators $\Box^*_A$ are interpreted on the reflexive and transitive closure of $\bigcup_{a \in A} R_a$. The validity of axioms **T**, **4**, and **B** in specific classes of frames follows as in standard propositional modal logics [47]. The validity of axioms $\mathbf{Ex}_x$ in each corresponding class of frames follows by Lemma 3.8(2); while the validity of **Gen** follows by Lemma 3.8(1). We provide a proof for $\mathbf{Ex}_{sopml}$: suppose that $(\mathcal{M}, w) \vDash \forall p \phi$, that is, for every $U \in D$, $(\mathcal{M}^p_U, w) \vDash \phi$. By Lemma 3.8(2a), $[\![\psi]\!] \in D$, hence in particular $(\mathcal{M}^p_{[\![\psi]\!]}, w) \vDash \phi$. Then, by Lemma 3.8(2b), $(\mathcal{M}, w) \vDash \phi[p/\psi]$. As regards **Gen**, suppose that $(\mathcal{M}, w) \vDash \phi$ and $p \notin fr(\phi)$. In particular, for every $U \in D$, $V(fr(\phi)) = V^p_U(fr(\phi))$, and by Lemma 3.8(1), we have $(\mathcal{M}^p_U, w) \vDash \phi$ as well. By **MP** we obtain that $(\mathcal{M}^p_U, w) \vDash \psi$, and since U is arbitrary, $(\mathcal{M}, w) \vDash \forall p \psi$.

Moreover, the Barcan formula **BF** is valid as in any frame all worlds have the same domain of quantification, namely $D \subseteq 2^W$. Indeed, $(\mathcal{M}, w) \vDash \forall p \, \Box_a \, \phi$ iff for all $U \in D$, $(\mathcal{M}^p_U, w) \vDash \Box_a \phi$, iff for every $w' \in R_a(w)$, $(\mathcal{M}^p_U, w') \vDash \phi$. But this means that for every $w' \in R_a(w)$, $(\mathcal{M}, w') \vDash \forall p \phi$, that is, $(\mathcal{M}, w) \vDash \Box_a \forall p \phi$. The case for $\Box^*_A$ is similar.

As for axiom **At**, note this only appears in $\mathbf{K}_{sopml}$ so we can assume that $\mathcal{F} \in \mathcal{K}_{full}$. For every $w \in W$, $\{w\}$ is an atom in $D = 2^W$. That is, *(i)* $(\mathcal{M}^p_{\{w\}}, w) \vDash p$; *(ii)* for all $U \in D$, $((\mathcal{M}^p_{\{w\}})^q_U, w) \vDash q \to \Box^*(p \to q)$, since $w \in U$ implies $\{w\} \subseteq U$; and *(iii)* for all $U \in D$, $a \in I$, if $((\mathcal{M}^p_{\{w\}})^r_U, w) \vDash \Diamond_a r$ then it is indeed the case that for every $w' \in W$, if $w' = w$ then $((\mathcal{M}^p_{\{w\}})^r_U, w') \vDash \Diamond_a r$. Hence, $(\mathcal{M}, w) \vDash$ **At** whenever $\mathcal{M}$ is based on a full frame. $\Box$

Lemma 3.26. Let $\theta_0, \theta_1, \ldots$ be an enumeration of the formulas in Σ. We define by induction a sequence $\Phi_0, \Phi_1, \ldots$ of subsets of Σ as follows.

$$\begin{array}{rcl} \Phi_0 & = & \Delta \\ \Phi_{n+1} & = & \begin{cases} \Phi_n \cup \{\theta_n\} & \text{if } \Phi_n \cup \{\theta_n\} \text{ is consistent;} \\ \Phi_n \cup \{\sim \theta_n\} & \text{otherwise.} \end{cases} \end{array}$$

Note that $\sim \theta_n \in \Sigma$, since Σ is closed under single negation. Also note that $\vdash_{\mathbf{L}_x} \neg\theta_n \leftrightarrow \sim \theta_n$. Now we prove by induction on n that every Φ_n is consistent. First of all, $\Phi_0 = \Delta$ is consistent by hypothesis. As to the inductive step, suppose that Φ_n is consistent, we consider the various cases. If $\Phi_{n+1} = \Phi_n \cup \{\theta_n\}$, then $\Phi_n \cup \{\theta_n\} = \Phi_{n+1}$ has to be consistent by construction. Further, $\Phi_{n+1} = \Phi_n \cup \{\sim \theta_n\}$ only if $\Phi_n \cup \{\theta_n\}$ is not consistent. Indeed, if Φ_n is consistent, $\Phi_n \cup \{\theta_n\}$ and $\Phi_n \cup \{\sim \theta_n\}$ cannot be both inconsistent, since otherwise for some $\varphi_0, \ldots, \varphi_m, \varphi'_0, \ldots, \varphi'_{m'} \in \Phi_n$,

$$\begin{array}{l} \vdash \bigwedge_{i \le m} \varphi_i \to \neg\theta_n \\ \vdash \bigwedge_{i \le m'} \varphi'_i \to \neg\neg\theta_n \end{array}$$

and by propositional reasoning,

$$\vdash \bigwedge_{i \le m} \varphi_i \wedge \bigwedge_{i \le m'} \varphi'_i \to (\neg\theta_n \wedge \neg\neg\theta_n)$$

that is, Φ_n itself is inconsistent. Hence, $\Phi_n \cup \{\sim \theta_n\} = \Phi_{n+1}$ is indeed consistent.

Finally, let $\Phi = \bigcup_{n \in \mathbb{N}} \Phi_n$: Φ is a consistent subset of Σ as each Φ_n is so, and it extends Δ. Moreover, Φ is Σ-maximal by construction. $\square$

Lemma 3.27. Let Y be an infinite denumerable set of new atoms. We define an infinite sequence of tuples

$$\Upsilon_i = \langle \Gamma_i^{\mathsf{pre}}, \Omega_i^{\mathsf{pre}}, \Gamma_i, \Omega_i, Q_i, Y_i \rangle$$

in such a way that each Υ_i has the following properties:

1. Γ_i^{pre} and Γ_i are consistent;
2. Ω_i^{pre} and Ω_i are closed under negation;
3. Γ_i is well-defined and Ω_i-maximal.

For the base case, define $\Gamma_0^{\mathsf{pre}} = \Omega_0^{\mathsf{pre}} = \varnothing$, $\Gamma_i = \Phi$, $\Omega_0 = \Sigma$, $Q_0 = \varnothing$, and $Y_0 = Y$. It is then easy to see that claims 1-3 above all hold for $i = 0$.

Now suppose $\Upsilon_i = \langle \Gamma_i^{\mathsf{pre}}, \Omega_i^{\mathsf{pre}}, \Gamma_i, \Omega_i, Q_i, Y_i \rangle$ has been defined. Let $\exists p_0\theta_0, \exists p_1\theta_1, \ldots$ be an enumeration of the existential formulas in Γ_i. Since Y_i is denumerable, we can write it as $Y_i = \{q_n \mid \exists p_n\theta_n \in \Gamma_i\} \cup \{r_0, r_1, \ldots\}$. Then, define Υ_{i+1} as follows:

$\Gamma_{i+1}^{\mathsf{pre}}$: $\Gamma_i \cup \bigcup_{n\in\mathbb{N}}\{\theta_n[p_n/q_n]\}$

$\Omega_{i+1}^{\mathsf{pre}}$: $\Omega_i \cup \bigcup_{n\in\mathbb{N}} Sub^{\neg}(\theta_n[p_n/q_n])$

Ω_{i+1}: $\Omega_{i+1}^{\mathsf{pre}} \cup \bigcup\{Sub^{\neg}(\theta'[p'/q]) \mid \forall p'\theta' \in \Gamma_{i+1}^{\mathsf{pre}}$ and $q \in Q_{i+1}\}$

Γ_{i+1}: a Ω_{i+1}-maximal extension of $\Gamma_{i+1}^{\mathsf{pre}}$

Q_{i+1}: $Q_i \cup \{q_0, q_1, \ldots\}$

Y_{i+1}: $\{r_0, r_1, \ldots\}$

We verify that items 1-3 do apply to Υ_{i+1}, given that they hold for Υ_i.

1. For $\Gamma_{i+1}^{\mathsf{pre}} = \Gamma_i \cup \bigcup_{n\in\mathbb{N}}\{\theta_n[p_n/q_n]\}$, suppose for reduction that $\Gamma_{i+1}^{\mathsf{pre}}$ is not consistent, this means that for some $\varphi_0, \ldots, \varphi_k, \exists p_0\theta_0, \ldots, \exists p_m\theta_m \in \Gamma_i$,

$$\vdash \bigwedge_{i\leq k} \varphi_i \wedge \bigwedge_{i\leq m} \exists p_i\theta_i \rightarrow \bigvee_{i\leq m} \neg\theta_i[p_i/q_i]$$

Since no q_i appears in any φ_i nor in any $\exists p_i\phi_i$, by repeated applications of **Gen** we obtain that

$$\vdash \bigwedge_{i\leq k} \varphi_i \wedge \bigwedge_{i\leq m} \exists p_i\theta_i \rightarrow \forall q_0 \ldots q_m(\bigvee_{i\leq m} \neg\theta_i[p_i/q_i])$$

Further, each q_i appears in $\theta_i[p_i/q_i]$ only, hence we can distribute universal quantification on disjunction:

$$\vdash \bigwedge_{i\leq k} \varphi_i \wedge \bigwedge_{i\leq m} \exists p_i\theta_i \rightarrow \bigvee_{i\leq m} \forall q_i\neg\theta_i[p_i/q_i]$$

Finally, by renaming bound variables and propositional reasoning we obtain,

$$\vdash \bigwedge_{i\leq k} \varphi_i \wedge \bigwedge_{i\leq m} \exists p_i\theta_i \rightarrow \left(\bigwedge_{i\leq m} \exists q_i\theta_i[p_i/q_i] \wedge \bigvee_{i\leq m} \neg\exists q_i\theta_i[p_i/q_i]\right)$$

and this contradicts the consistency of Γ_i.

As regards Γ_{i+1}, it is consistent by Lemma 3.26.

2. It is clear that if a set X is closed under single negation, then $X \cup Sub^{\neg}(\alpha)$ is also closed under single negation, for any formula α.

3. Since Ω_{i+1} is closed under single negation and $\Gamma^{\mathsf{pre}}_{i+1} \subseteq \Omega_{i+1}$ is consistent, we can apply Lemma 3.26 to conclude that Γ_{i+1} is well-defined.

Finally, define $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$, $Q = \bigcup_{n \in \mathbb{N}} Q_n$ and $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$. Then Γ is a consistent set of formulas over $AP \cup Q$, and $\Gamma \supseteq \Phi$. Γ is also Ω-complete: if $\omega \in \Omega$, then this formula was introduced at some Ω_i, and since Γ_i is Ω_i-maximal, we have that exactly one of ω and $\sim \omega$ is in $\Gamma_i \subseteq \Gamma$. We finally claim that Γ is Q-rich and Q-universal. As to richness, observe that every existential formula $\exists p \theta$ introduced at level i in Γ_i is taken care of at level $i+1$ through formula $\theta[p/q]$ for some witness $q \in Q$. As to universality, suppose that $q \in Q$ and $\forall p' \theta' \in \Gamma$. In particular, assume that $\forall p' \theta'$ was introduced at some Γ_k, while q was added at step k'. Let $i = max\{k, k'\}$, then at step $i+1$ we can verify that $\theta'[p'/q]$ is added to Ω_{i+1}. Moreover, by axiom $\mathbf{Ex}_{ap}$ and the maximality of Γ_{i+1}, we see that $\theta'[p'/q]$ belongs to Γ_{i+1}, and therefore to Γ. Thus, Γ is Ω-saturated. □

Lemma 3.30. The proof is by induction on the length of ψ. As to the base case for $\psi = p$, by definition of satisfaction, $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \vDash p$ iff $w \in V(p)$, iff $p \in w$.

For $\psi = \neg\chi$, $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \vDash \psi$ iff $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \nvDash \chi$, iff by induction hypothesis $\chi \notin w$. Since w is maximal in Ω, this is the case iff $\psi \in w$.

For $\psi = \chi \to \chi'$, $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \vDash \psi$ iff $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \nvDash \chi$ or $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \vDash \chi'$. By induction hypothesis this is the case iff $\chi \notin w$ or $\chi' \in w$; in both cases we have that $\psi \in w$, as w is maximal in Ω.

Suppose that $\psi = \forall p \chi$. $\Leftarrow$ Let $\psi \in w$. Since w is Q-universal, we have that $\chi[p/q] \in w$ for every $q \in Q$. By induction hypothesis $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \vDash \chi[p/q]$. Now consider the set $V(q) = \{w \in W \mid q \in w\} = U_q \in D$. By Lemma 3.8(2b), $((\mathcal{M}_{\mathbf{L}_{ap}})^p_{U_q}, w) \vDash \chi$, and by the arbitrariness of variant $V^p_{U_q}$ we obtain that $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \vDash \psi$. $\Rightarrow$ Assume that $\psi \notin w$. Since w is maximal in Ω, $\exists p \neg\chi \in w$, and w is Q-rich, so $\neg\chi[p/q] \in w$ for some atom $q \in Q$. Then, by induction hypothesis, $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \nvDash \chi[p/q]$, and by Lemma 3.8(2b), $((\mathcal{M}_{\mathbf{L}_{ap}})^p_{V(q)}, w) \nvDash \chi$. In particular, for $U_q = V(q) = \{v \in W \mid q \in v\} \in D$, $((\mathcal{M}_{\mathbf{L}_{ap}})^p_{U_q}, w) \nvDash \chi$, i.e., $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \nvDash \psi$.

Suppose that $\psi = \Box_a \chi$. $\Leftarrow$ Assume that $\psi \in w$ and $v \in R_a(w)$. By definition of R_a, $\chi \in v$; therefore by induction hypothesis $(\mathcal{M}_{\mathbf{L}_{ap}}, v) \vDash \chi$. Thus, $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \vDash \psi$. $\Rightarrow$ Assume that $\psi \notin w$ and consider set $\{\phi \mid \Box_a \phi \in w\} \cup \{\neg\chi\}$. This set is consistent, for if not, then for some $\phi_1, \dots, \phi_n \in \{\phi \mid \Box_a \phi \in w\}$, $\vdash \bigwedge \phi \to \chi$. Then, by axiom $\mathbf{K}$, $\vdash \bigwedge \Box_a \phi \to \Box_a \chi$ and since $\bigwedge \Box_a \phi \in w$, also $\Box_a \chi \in w$ against hypothesis. Apply Lemma 3.27

for $\Delta = \{\phi \mid \Box_a\phi \in w\} \cup \{\neg\chi\}$ and obtain a saturated set $v \in W$. In particular, $v \in R_a(w)$ by construction. By induction hypothesis $(\mathcal{M}_{\mathbf{L}_{ap}}, v) \nvDash \chi$. Since $v \in R_a(w)$, we have that $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \nvDash \psi$.

Suppose that $\psi = \Box^*_A\chi$. $\Leftarrow$ if $\psi \in w$ then we show by induction on k that if v is reachable from w in k steps, then both χ and ψ belong to v. For $k = 1$, observe that by axiom **C1**, $\psi \in w$ implies $\Box_a(\chi \wedge \Box^*\chi) \in w$ for every $a \in A$, as w is maximal in $Sub^\neg(\phi)$. So, if v is reachable in one step (i.e., $R_a(w, v)$ for some $a \in A$), then $\chi \wedge \Box^*_A\chi \in v$, that is, both χ and ψ belong to v. As for the inductive step, suppose that v is reachable from w in k steps and v' is reachable from v in one step. By the induction hypothesis, both χ and $\Box^*_A\chi$ belong to v. Similarly to the base case, we can show that $\chi, \Box^*_A\chi \in v'$. As a result, χ belongs to v for all states v that are reachable from w and by induction hypothesis, $(\mathcal{M}_{\mathbf{L}_{ap}}, v) \vDash \chi$. Thus, we obtain that $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \vDash \psi$.

$\Rightarrow$ Assume that $(\mathcal{M}_{\mathbf{L}_{ap}}, w) \vDash \psi$ and consider the finite set $Sub^\neg(\psi) \cap w$, which can be described by the conjunction ψ_w of all its formulas. Also, consider the set $U = \{v \in W \mid (\mathcal{M}_{\mathbf{L}_{ap}}, v) \vDash \psi\}$ of worlds satisfying ψ: in general U is infinite; however, since $Sub^\neg(\psi)$ is finite, there are only finitely many different ψ_v. Hence, the disjunction ψ_U of all such ψ_v is well-defined.

Next we prove that for every $v \in U$ and $a \in A$, $\vdash \psi_v \to \Box_a\chi$. Indeed, $(\mathcal{M}_{\mathbf{L}_{ap}}, v) \vDash \Box^*_A\chi$ implies $(\mathcal{M}_{\mathbf{L}_{ap}}, v) \vDash \Box_a\chi$ in particular. Then, we use an argument similar to the case for $\psi = \Box_a\chi$. Indeed, if $(\mathcal{M}_{\mathbf{L}_{ap}}, v) \vDash \Box_a\chi$ then the set $\{\phi \mid \Box_a\phi \in v\} \cup \{\neg\chi\}$ is not consistent, for otherwise, by Lemma 3.27 we would have a saturated extension v' such that $R_a(v, v')$ by construction and $(\mathcal{M}_{\mathbf{L}_{ap}}, v') \vDash \neg\chi$ by induction hypothesis. But then $(\mathcal{M}_{\mathbf{L}_{ap}}, v) \nvDash \Box_a\chi$, against hypothesis. As a consequence, for some finite set $\{\phi_1, \ldots, \phi_k\} \subseteq \{\phi \mid \Box_a\phi \in v\}$,

$$\vdash \bigwedge_{i \le k} \phi_i \to \chi$$

and by axioms **K** and **Nec**,

$$\vdash \bigwedge_{i \le k} \Box_a\phi_i \to \Box_a\chi$$

However, not all $\Box_a\phi_i$ necessarily belong to $Sub^\neg(\psi) \cap v$. However, all $\Box_a\phi_i$ have been introduced either to witness an existential formula or to exemplify a universal one. In the former case, by axiom $\mathbf{Ex}_{ap}$ we obtain

$$\vdash \bigwedge_{i \le k} \exists p_i \, \Box_a \, \phi_i \to \Box_a\chi$$

In the latter, by rule **Gen** we have

$$\vdash \forall \vec{q} \left(\bigwedge_{i \leq k} \exists p_i \, \Box_a \, \phi_i \rightarrow \Box_a \chi \right)$$

where $\vec{q}$ are all the free atoms appearing in $\exists p_1 \phi_1, \ldots, \exists p_k \phi_k$, but not in χ.

Finally, for an appropriate substitution $[\vec{q}/\vec{p}]$ of atoms $\vec{q}$ with atoms $\vec{p}$ appearing in χ, we can rename the bounded atoms $\forall \vec{q}$ with atoms belonging to $Sub^{\neg}(\psi) \cap v$, so that all $\exists p_i \, \Box_a \, \phi_i[\vec{q}/\vec{p}]$ belong to $Sub^{\neg}(\psi)$ and appear in v. Then, again by axiom $\mathbf{Ex}_{ap}$,

$$\vdash \bigwedge_{i \leq k} \exists p_i \, \Box_a \, \phi_i[\vec{q}/\vec{p}] \rightarrow \Box_a \chi \tag{A.1}$$

and therefore,

$$\vdash \psi_v \rightarrow \Box_a \chi \tag{A.2}$$

Furthermore, we show that for $v \in U$ and $v' \in \overline{U}$, $\vdash \psi_v \rightarrow \Box_a \neg \psi_{v'}$ for every $a \in A$. Indeed, by definition of U we have that $(\mathcal{M}_{\mathbf{L}_{ap}}, v) \vDash \Box^*_A \chi$, while $(\mathcal{M}_{\mathbf{L}_{ap}}, v') \nvDash \Box^*_A \chi$. As a consequence, v' is not reachable from v and in particular it is not the case that $R_a(v, v')$. By definition of R_a, we have that $\{\phi \mid \Box_a \phi \in v\} \nsubseteq v'$. So, for some formula θ, $\Box_a \theta \in v$ but $\theta \notin v'$. Since $\theta \notin v'$, we have that

$$\vdash \theta \rightarrow \neg \psi_{v'}$$

Again it is not necessarily the case that $\theta \in Sub^{\neg}(\psi) \cap v$. Nonetheless, by reasoning as above, by axioms **Gen**, **K**, and **Nec** we can derive

$$\vdash \exists r \, \Box_a \, \theta[\vec{q}/\vec{p}] \rightarrow \Box_a \neg \psi_{v'}$$

for an appropriate substitution $[\vec{q}/\vec{p}]$ such that $\exists r \, \Box_a \, \theta[\vec{q}/\vec{p}]$ appears in $Sub^{\neg}(\psi) \cap v$. Hence, we obtain

$$\vdash \psi_v \rightarrow \Box_a \neg \psi_{v'} \tag{A.3}$$

Now, from formulas (A.2) and (A.3) we derive that

$$\vdash \psi_v \rightarrow \Box_a \left(\chi \wedge \bigwedge_{v' \in \overline{U}} \neg \psi_{v'} \right)$$

Since $\vdash \psi_U \leftrightarrow \bigwedge_{v' \in \overline{U}} \neg \psi_{v'}$, we obtain

$$\vdash \psi_v \rightarrow \Box_a \left(\chi \wedge \neg \psi_U \right)$$

and by definition of ψ_U we conclude that

$$\vdash \psi_U \to \bigwedge_{a \in A} \Box_a(\chi \wedge \psi_U)$$

Then, by applying axiom **C2** we have

$$\vdash \psi_U \to \Box_A^* \chi$$

In particular, since $\vdash \psi_w \to \psi_U$ we obtain

$$\vdash \psi_w \to \psi$$

Finally, since w is maximal, we derive that $\psi \in w$. $\Box$

Lemma 3.31. Also in the present case we let Y be an infinite denumerable set of new atoms, and define an infinite sequence of tuples

$$\Upsilon_i = \langle \Gamma_i^{\mathsf{pre}}, \Omega_i^{\mathsf{pre}}, \Gamma_i, \Omega_i, Q_i, Y_i \rangle$$

such that each Υ_i satisfies conditions 1-3 in Lemma 3.27.

The base case for $i = 0$ is given as in Lemma 3.27, and again properties 1-3 hold. As to the inductive step, let $\exists p_0 \theta_0, \exists p_1 \theta_1, \ldots$ be an enumeration of the existential formulas in Γ_i, and define Υ_{i+1} as in Lemma 3.27, but for Ω_{i+1}, which goes as follows:

Ω_{i+1}: $\Omega_{i+1}^{\mathsf{pre}} \cup \bigcup \{ Sub^{\neg}(\theta'[p'/\psi]) \mid \forall p' \theta' \in \Gamma_{i+1}^{\mathsf{pre}}$ and $\psi \in \mathcal{L}_{pl}$ is a propositional formula over $Q_{i+1} \}$

We can verify that 1-3 hold for Υ_{i+1}, provided that they hold for Υ_i. In particular, Ω_i is closed under negation; and Γ_i is well-defined and Ω_i-maximal by Lemma 3.26.

Finally, define $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$, $Q = \bigcup_{n \in \mathbb{N}} Q_n$ and $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$. Then Γ is a consistent set of formulas over $AP \cup Q$, and $\Gamma \supseteq \Phi$ is also Ω-complete and Q-rich by construction. As to Q-universality, suppose that $\psi \in \mathcal{L}_{pl}$ is propositional formula over Q and $\forall p' \theta' \in \Gamma$. In particular, assume that $\forall p' \theta'$ was introduced at some Γ_k, while ψ was added at step k'. Let $i = max\{k, k'\}$, then at step $i + 1$, we can verify that $\theta'[p'/\psi]$ is added to Ω_{i+1}. Moreover, by axiom $\mathbf{Ex}_{pl}$ and the maximality of Γ_{i+1}, we see that $\theta'[p'/\psi]$ belongs to Γ_{i+1}, and therefore to Γ. Thus, Γ is Ω-saturated. $\Box$

Lemma 3.33. We have to prove that domain D is closed under boolean operations. Let U_ϕ and $U_{\phi'}$ be sets in D, we show that $U_\phi \cap U_{\phi'} = U_{\phi\wedge\phi'} \in D$. Clearly, $w \in U_\phi \cap U_{\phi'}$ iff $\phi \in w$ and $\phi' \in w$, and by maximality, this is the case iff $\phi \wedge \phi' \in w$ as well. Closure under disjunction is proved similarly. As to taking complement, we show that $W \setminus U_\phi = U_{\neg\phi} \in D$. Again, $w \notin U_\phi$ iff $\phi \notin w$, and by maximality, this is the case iff $\neg\phi \in w$. □

Lemma 3.34. To prove the truth lemma for $\mathbf{L}_{pl}$ we have to modify the induction hypothesis, so that the proof is also on the modal depth of formulas. Then, for every boolean formula φ of modal depth 0, the proof goes as in Lemma 3.30. Hence, we have that $U_\varphi = [\![\varphi]\!]_{\mathcal{M}} = \{w \in W \mid (\mathcal{M}_{\mathbf{L}_{ap}}, w) \vDash \varphi\} \in D$. Then, for $\psi = \forall p\chi$, if $\psi \in w$ then by axiom $\mathbf{Ex}_{pl}$ we have that $\chi[p/\varphi] \in w$. By induction hypothesis, $(\mathcal{M}, w) \vDash \chi[p/\varphi]$. Now consider the set $U_\varphi = [\![\varphi]\!]_{\mathcal{M}} \in D$. By Lemma 3.8(2b), it is the case that $(\mathcal{M}^p_{U_\varphi}, w) \vDash \chi$, and by the arbitrariness of variant $V^p_{U_\varphi}$ we obtain that $(\mathcal{M}, w) \vDash \forall p\chi$. As to the implication from left to right, the proof is the same as in Lemma 3.30, as each w is maximal and rich. Finally, the case for modal operators also goes as in Lemma 3.30. Here the crucial remark is that, since we assume a canonical representation for propositional formulas (e.g., conjunctive normal form), the restriction of any world w in the canonical model to formulas in $\mathit{Sub}^\neg(\psi)$ is indeed finite and can be described by the conjunction ψ_w. □

Lemma 3.37. Since $\mathsf{Th}(\mathcal{K}^e_{bool,at,com}) \subseteq \mathsf{Th}(\mathcal{K}^e_{full})$, we prove the converse inclusion. To do so, suppose that $(\mathcal{M}, w) \nvDash \phi$ for some models $\mathcal{M} = \langle \mathcal{F}, V \rangle$, based on frame $\mathcal{F} \in \mathcal{K}^e_{bool,at,com}$, and $w \in W$. The proof of Theorem 3.36 makes use of a function f from D to the power set 2^A of the set $A \subseteq D$ of atoms such that for $U \in D$, $f(U) = \{U' \in A \mid U' \subseteq U\}$. Now, define $\mathcal{M}' = \langle A, 2^A, R', V' \rangle$ where *(i)* $R'_a(U, U')$ iff $R_a(u, u')$ for some $u \in U$ and $u' \in U'$, and *(ii)* $V'(p) = f(V(p))$. Notice that R'_a is well-defined by the definition of atomicity. Indeed, if $R'_a(U, U')$ then $R_a(u, u')$ for some $u \in U$ and $u' \in U'$. Then, by atomicity, if $v \in U$ then $R_a(v, v')$ for some $v' \in U'$ as well. Hence, the definition of R'_a is independent from the particular witnesses u, u'. Also, R'_a is an equivalence relation, whenever R_a is.

By induction on the structure of ψ we prove that for every formula $\psi \in \mathcal{L}_{sopml}$, $(\mathcal{M}, w) \vDash \psi$ iff $(\mathcal{M}', U) \vDash \psi$, where U is any atom containing w. The base case for $\psi \in AP$ is immediate as $(\mathcal{M}, w) \vDash \psi$, iff $w \in V(p)$, iff $U \in f(V(p))$, iff $(\mathcal{M}', U) \vDash \psi$. The inductive cases for propositional connectives are immediate. For $\psi = \Box_a\theta$, $(\mathcal{M}, w) \nvDash \psi$ iff for some $w' \in R_a(w)$, $(\mathcal{M}, w') \nvDash \theta$. By induction hypothesis we have $(\mathcal{M}', U') \nvDash \theta$ for atom $U' \in A$ such that $w' \in U'$. Also, $R'_a(U, U')$ by definition, and therefore $(\mathcal{M}', U) \nvDash \psi$. As to the other direction, if $(\mathcal{M}', U) \nvDash \psi$ then for some $U' \in A$, $R'_a(U, U')$ and $(\mathcal{M}', U') \nvDash \theta$, and therefore $(\mathcal{M}, w') \nvDash \theta$ for $w' \in U'$ by induction hypothesis. Finally, by definition of atomicity $R_a(w, w')$, i.e., $(\mathcal{M}, w) \nvDash \psi$. The case for $\psi = \Box^*_A\theta$ is dealt with similarly.

Specifically, $(\mathcal{M}, w) \vDash \psi$ iff for all w-reachable states w', $(\mathcal{M}, w') \vDash \theta$, iff for all U-reachable states U', $(\mathcal{M}', U') \vDash \theta$, iff $(\mathcal{M}, U) \vDash \psi$. We can prove that for every w-reachable states w' and atom U' containing w', $(\mathcal{M}, w') \vDash \theta$ iff $(\mathcal{M}', U') \vDash \theta$, by induction on the length of the path from w to w', by using the case for $\Box_a$-formulas in the inductive step. Finally, for $\psi = \forall p \theta$, $(\mathcal{M}, w) \nvDash \psi$ iff for some $T \in D$, $(\mathcal{M}^p_T, w) \nvDash \theta$. By induction hypothesis we have $(\mathcal{M}'^p_{f(T)}, U) \nvDash \theta$ for $f(T) \in 2^A$, and therefore $(\mathcal{M}', U) \nvDash \psi$. On the other hand, if $(\mathcal{M}', U) \nvDash \psi$ then for some $T' \in 2^A$, $(\mathcal{M}'^p_{T'}, U) \nvDash \theta$. Clearly, there exists some set $T \in D$ that contains all world corresponding to atoms in T' and $f(T) = T'$. Hence, by induction hypothesis $(\mathcal{M}^p_T, w) \nvDash \theta$, i.e., $(\mathcal{M}, w) \nvDash \psi$. As a result, we have that $(\mathcal{M}', U) \nvDash \phi$, that is, $\mathsf{Th}(\mathcal{K}^e_{full}) \subseteq \mathsf{Th}(\mathcal{K}^e_{bool,at,com})$ $\Box$

Lemma 3.38. Clearly, $\mathsf{Th}(\mathcal{K}^e_{bool,at}) \subseteq \mathsf{Th}(\mathcal{K}^e_{bool,at,com})$. So, suppose that $(\mathcal{M}, w) \nvDash \phi$ for some atomic boolean frame $\mathcal{F}$. Then, consider its completion $\mathcal{F}^+$ and model $\mathcal{M}^+$ such that $V^+(p) = \{U \in D \mid U \subseteq V(p) \text{ is an atom}\}$. The submodels $\mathcal{M}_u$ and $\mathcal{M}^+_U$ generated respectively by u and corresponding atom U are isomorphic by the isomorphism that maps each world to its atom. Hence, by applying Lemma 3.11 we obtain that $(\mathcal{M}, u) \nvDash \phi$ iff $(\mathcal{M}_u, u) \nvDash \phi$, iff $(\mathcal{M}^+_U, U) \nvDash \phi$, iff $(\mathcal{M}^+, U) \nvDash \phi$. Thus, $\mathsf{Th}(\mathcal{K}^e_{bool,at,com}) \subseteq \mathsf{Th}(\mathcal{K}^e_{bool,at})$. $\Box$

Lemma 3.40. The proof that $\mathcal{M}_{\mathbf{S5}_{sopml}}$ is boolean goes as in Lemma 3.33. As regards atomicity, if $\mathbf{At} \in w$, then $(p_n \wedge \forall q(q \to C(p_n \to q)) \wedge \bigwedge_{a \in I} \forall r(M_a r \to C(p_n \to M_a r))) \in w$, for some witness $p_n \in AP \cup Y$. Hence, for $U_n = \{v \in W \mid p_n \in v\}$, we have that $w \in U_n$ and for all $U_\psi \in D$ containing w, $U_n \subseteq U_\psi$. That is, U_n is minimal according to set inclusion. Further, suppose that $R_a(w, v)$ for some $v \in W$ and let $U_\psi \in D$ be the atom containing v. Then, $M_a \psi \in w$ by maximality and consistency, and for every $w' \in U_n$, $M_a \psi \in w'$ as well. Therefore, $v' \in R_a(w')$ for some $v' \in U_\psi$. $\Box$

Theorem 3.43. Soundness follows immediately by Lemma 3.19, as every frame that satisfies condition $\forall x \forall \vec{P} ST_x(\psi) = \forall x ST_x(\forall \vec{p} \psi)$, also validates $\forall \vec{p} \psi$. As to completeness, if $\forall \vec{p} \psi$ is an axiom, then it appears in every state of the canonical model $\mathcal{M}$, and by the truth lemma, $\mathcal{M}$ validates $\forall \vec{p} \psi$. Finally, by another application of Lemma 3.19, $\mathcal{F}$ validates $\forall x \forall \vec{P} ST_x(\psi)$. $\Box$

Theorem 3.45. As regards hardness, we reduce satisfiability of quantified boolean formulas to SOPML model checking. Given a formula $\phi \in \mathcal{L}_{qbf}$, consider frame $\mathcal{F} = \langle \{w\}, (w, w), \{\{w\}, \varnothing\} \rangle$ and an arbitrary assignment V, and define $\mathcal{M} = \langle \mathcal{F}, V \rangle$. Then, we have that ϕ is satisfiable iff $(\mathcal{M}, w) \vDash \exists \vec{p} \phi$, iff $\mathcal{M} \vDash \exists \vec{p} \phi$, where $\vec{p}$ are all the atoms in ϕ. Hence, model checking SOPML is PSPACE-hard.

As regards completeness, by combining the algorithms for modal logic and quantified boolean formulas we check that model checking formulas in SOPML takes polynomial space in the size of the formula and exponential time in the size of the model. Specifically, Algorithm 1 takes as input a formula $\phi \in \mathcal{L}_{sopml}$ and a finite model $\mathcal{M}$, and returns the set $[\![\phi]\!]_{\mathcal{M}} \subseteq W$ of worlds satisfying ϕ in $\mathcal{M}$ in exponential time. Then, the model checking algorithm returns a positive answer iff $[\![\phi]\!]_{\mathcal{M}} = W$. In particular, the case of modal operators is dealt with by computing pre-images of sets according to the accessibility relation, which can be done in polynomial time, while for propositional quantification we have to consider all reinterpretations $\mathcal{M}^p_U$. This is where the exponential blow-up comes from. In order to obtain a (deterministic) algorithm in PSPACE, we describe a non-deterministic algorithm in PSPACE. The result follows from NPSPACE = PSPACE. Specifically, we suppose w.l.o.g. that ϕ contains only existential quantifiers and we deal with them by guessing an assignment satisfying the immediate subformula. Since the other cases can be treated in polynomial time as in Algorithm 1, the overall complexity of the procedure is in NPSPACE. □

A.2 Chapter 4

Theorem 4.3. Since $w \preceq w'$, there is a simulation pair (σ, Σ) such that $\sigma(w, w')$. Fix this σ. We prove by induction on φ that if $(\mathcal{F}, V, w) \nvDash \varphi$ for some assignment V, then $(\mathcal{F}', \Sigma(V), w') \nvDash \varphi$, where $\Sigma(V)$ is any assignment such that for every $p \in AP$, $(\Sigma(V))(p) = U'$ with $\Sigma(V(p), U')$. We write $\Sigma(V)(p)$ for $(\Sigma(V))(p)$. By clause *(i)* of Definition 4.1, $\Sigma(V)(p) \in D'$.

For $\varphi = p$, $(\mathcal{F}, V, w) \nvDash \varphi$ iff $w \notin V(p) \in D$. By clause *(ii)*.2 in Definition 4.1, $w \notin V(p) \in D$ iff $w' \notin \Sigma(V)(p) \in D'$, that is, $(\mathcal{F}', \Sigma(V), w') \nvDash \varphi$.

For $\varphi = \neg p$, $(\mathcal{F}, V, w) \nvDash \varphi$ iff $w \in V(p) \in D$. Again by clause *(ii)*.2 in Definition 4.1, $w \in V(p) \in D$ iff $w' \in \Sigma(V) \in D'$, that is, $(\mathcal{F}', \Sigma(V), w') \nvDash \varphi$.

The inductive cases for propositional connectives are immediate.

For $\varphi = \Box_a \psi$, $(\mathcal{F}, V, w) \nvDash \varphi$ iff for some $v \in R_a(w)$, $(\mathcal{F}, V, v) \nvDash \psi$. By clause *(ii)*.1, for some $v' \in R'_a(w')$, $\sigma(v, v')$. In particular, $(\mathcal{F}', \Sigma(V), v') \nvDash \psi$ by induction hypothesis. That is, $(\mathcal{F}', \Sigma(V), w') \nvDash \varphi$. The case for $\varphi = \Box^*_A \psi$ is similar.

For $\varphi = \forall p \psi$, $(\mathcal{F}, V, w) \nvDash \varphi$ iff for some $U \in D$, $(\mathcal{F}, V^p_U, w) \nvDash \psi$. By induction hypothesis, $(\mathcal{F}', \Sigma(V^p_U), w') \nvDash \psi$. By condition *(i)* in Definition 4.1, for $U \in D$, $\Sigma(U, U')$ for some $U' \in D'$. In particular, we have that $\Sigma(V^p_U) = \Sigma(V)^p_{U'}$ and therefore $(\mathcal{F}', \Sigma(V)^p_{U'}, w') \nvDash \psi$ for $U' \in D'$, that is, $(\mathcal{F}', \Sigma(V), w') \nvDash \varphi$. □

Corollary 4.4. Suppose that $\mathcal{F} \not\models \varphi$, that is, $(\mathcal{F}, w) \not\models \varphi$ for some $w \in W$. Since $\mathcal{F} \leq \mathcal{F}'$, for some $w' \in W'$, $w \leq w'$. Therefore, by Lemma 4.3 we obtain that $(\mathcal{F}', w') \not\models \varphi$, that is, $\mathcal{F}' \not\models \varphi$. $\square$

Theorem 4.7. We prove the implication from right to left, the opposite direction being symmetric. If $w \approx w'$ then $\omega(w, w')$ holds for some bisimulation pair (ω, Ω). As above, we show by induction on φ that if $(\mathcal{F}, V, w) \not\models \varphi$ for some assignment V, then $(\mathcal{F}', \Omega(V), w') \not\models \varphi$, where $\Omega(V)$ is any assignment such that for every $p \in AP$, $(\Omega(V))(p) = U'$ with $\Omega(V(p), U')$. Since ω is a simulation relation in particular, the base cases for $\varphi = p$ and $\varphi = \neg p$ are as in Theorem 4.3, as well as the inductive cases for propositional connectives and $\varphi = \Box_a \psi$, $\varphi = \Box^*_A \psi$, and $\varphi = \forall p \psi$.

For $\varphi = \Diamond_a \psi$, $(\mathcal{F}', \Omega(V), w') \models \varphi$ iff for some $v' \in R'_a(w')$, $(\mathcal{F}', \Omega(V), v') \models \psi$. By bisimulation, for some $v \in R_a(w)$, $\omega(v, v')$. In particular, $(\mathcal{F}, V, v) \models \psi$ by induction hypothesis. That is, $(\mathcal{F}, V, w) \models \varphi$. The case for $\varphi = \Diamond^*_A \psi$ is similar.

For $\varphi = \exists p \psi$, $(\mathcal{F}', \Omega(V), w') \models \varphi$ iff for some $U' \in D'$, $(\mathcal{F}', (\Omega(V))^p_{U'}, w') \models \psi$. Now consider $U \in D$ such that $\Omega(U, U') \in D$. In particular, assignments $(\Omega(V))^p_{U'}$ and $\Omega(V^p_U)$ coincides. Hence, $(\mathcal{F}', \Omega(V^p_U), w') \models \psi$, and by induction hypothesis, $(\mathcal{F}, V^p_U, w) \models \psi$ for $U \in D$, that is, $(\mathcal{F}, V, w) \models \varphi$. $\square$

Lemma 4.12. We show that the pair $(\mapsto, \mapsto)$ of mappings satisfies Definition 4.1. As to condition *(i)*, if $U \in D$ then $U \mapsto [U]$ for $[U] \in D^A$. Next, for *(i)*.1 suppose that $R_a(w, v)$. Then, for $[v] \in W^A$ we have that $R^A_a([w], [v])$ and $v \mapsto [v]$. Finally, as to *(ii)*.2, if $w \mapsto [w]$ and $w \in U$, then clearly $[w] \in [U]$. On the other hand, if $[w] \in [U]$ then for some $v \in [w]$, $v \in U$. However, $v \in [w]$ implies that $v \sim w$. In particular, $w \in U$ by the constraint on $\sim$. $\square$

Theorem 4.15. As to the $\Leftarrow$-direction, we show that if (σ, Σ) is a simulation pair such that $\sigma(v, v')$, then Duplicator can always reply to any Spoiler's move in state $(\mathcal{F}, v, \vec{U})$, $(\mathcal{F}', v', \vec{U}')$ with $\Sigma(U_i, U'_i)$. Moreover, the new state $(\mathcal{F}, u, \vec{T})$, $(\mathcal{F}', u', \vec{T}')$ is such that $\sigma(u, u')$ and $\Sigma(T_i, T'_i)$. As a consequence, Duplicator has a winning strategy and the result follows. As regards move (1) in Definition 4.14, suppose that Spoiler picks a set $U \in D$. Then Duplicator can reply with $U' \in D'$ such that $\Sigma(U, U')$: by *(i)* in Definition 4.1 such U' always exists. Moreover, by *(ii)*.2, $v \in U$ iff $v' \in U'$. On the other hand, if Spoiler chooses $u \in R_a(v)$, then by *(ii)*.1 Duplicator can reply with some $u' \in R'_a(v')$ such that $\sigma(u, u')$. In particular, for every i, $u \in U_i$ iff $u' \in U'_i$, as $\Sigma(U_i, U'_i)$ holds. In both cases the new state $(\mathcal{F}, u, \vec{T})$, $(\mathcal{F}', u', \vec{T}')$ is such that $\sigma(u, u')$ and $\Sigma(T_i, T'_i)$ as required.

As to the $\Rightarrow$-direction, we show that relations $\sigma \subseteq W \times W'$ and $\Sigma \subseteq D \times D'$ defined as: *$\sigma(v,v')$ and $\Sigma(U_i,U_i')$ hold iff Duplicator has a winning strategy at state $(\mathcal{F},v,\vec{U})$, $(\mathcal{F}',v',\vec{U}')$* form a simulation pair. As regards condition *(i)* in Definition 4.1, consider any winning state $(\mathcal{F},v,\vec{U})$, $(\mathcal{F}',v',\vec{U}')$ in the game starting in $(\mathcal{F},w)$, $(\mathcal{F}',w')$. Spoiler can play any $U \in D$, but then Duplicator has to reply with some $U' \in D'$ so that he has a winning strategy in the resulting state $(\mathcal{F},v,\vec{T})$, $(\mathcal{F}',v',\vec{T}')$. In particular, by the definition of Σ, we have $\Sigma(U,U')$. Further, for condition *(ii)*.1, if $\sigma(v,v')$ then $(\mathcal{F},v,\vec{U})$, $(\mathcal{F}',v',\vec{U}')$ is a winning state for some $\vec{U},\vec{U}'$ with $\Sigma(U_i,U_i')$. Then, for every $u \in R_a(v)$, Duplicator can reply with $u' \in R_a'(v')$ (otherwise, Duplicator has no winning strategy.) Moreover, Duplicator has a winning strategy in the resulting state $(\mathcal{F},u,\vec{U})$, $(\mathcal{F}',u',\vec{U}')$. Hence, we have $\sigma(u,u')$ by definition of σ. Finally, for condition *(ii)*.2, if $\sigma(v,v')$ then again $(\mathcal{F},v,\vec{U})$, $(\mathcal{F}',v',\vec{U}')$ is a winning state for some $\vec{U}$ and $\vec{U}'$. Further, if $\Sigma(T,T')$ then $(\mathcal{F},u,\vec{T})$, $(\mathcal{F}',u',\vec{T}')$ is a winning state for some $u \in W$ and $u' \in W'$ such that $\sigma(u,u')$. Now we analyse the following two cases: if couple (v,v') has appeared before (u,u') in the game then, when introducing sets U and U', condition $v \in U$ iff $v' \in U'$ has to be satisfied by (1) in Definition 4.14. On the other hand, if couple (u,u') has appeared first, when introducing (v,v') condition $v \in U$ iff $v' \in U'$ has to be satisfied by (2). Finally, since Duplicator has a winning strategy for game $(\mathcal{F},w)$, $(\mathcal{F}',w')$, it is the case that w' simulates w. □

Theorem 4.18. The proof follows the one for Theorem 4.15, once we notice that the behaviour of Spoiler and Duplicator on frames $\mathcal{F}$ and $\mathcal{F}'$ is completely symmetric. As regards the $\Leftarrow$-direction, if (ω,Ω) is a bisimulation pair such that $\omega(v,v')$, then Duplicator can always reply to any Spoiler's move in state $(\mathcal{F},v,\vec{U})$, $(\mathcal{F}',v',\vec{U}')$ with $\Omega(U_i,U_i')$. As to the $\Rightarrow$-direction, the relations $\omega \subseteq W \times W'$ and $\Omega \subseteq D \times D'$ defined as: *$\omega(v,v')$ and $\Omega(U_i,U_i')$ hold iff Duplicator has a winning strategy at state $(\mathcal{F},v,\vec{U})$, $(\mathcal{F}',v',\vec{U}')$* form a bisimulation pair. □

Lemma 6.5. Consider $\phi \notin \mathsf{Th}(\mathcal{G}_{\mathbb{N}})$ with finite modal depth $k \in \mathbb{N}$, where the modal depth is defined as in the propositional case, as the maximum embedding of modal operators [47]. We can assume without loss of generality that $(\mathcal{G}_{\mathbb{N}},V,0) \not\models \phi$ for some assignment V, then we prove that $(\mathcal{G}_k,V',0) \not\models \phi$, where assignment V' is such that $V'(p) = V(p) \cap [k] \in D'$ for every $p \in AP$. Now we prove that if ψ is subformula of ϕ of modal depth $n \leq k$, then $(\mathcal{G}_{\mathbb{N}},V,k-n) \models \psi$ iff $(\mathcal{G}_k,V',k-n) \models \psi$.

We start with the case for $n = 0$. If ψ is an atom p, then $(\mathcal{G}_{\mathbb{N}},V,k) \models \psi$ iff $k \in V(p)$, iff $k \in V'(p)$, iff $(\mathcal{G}_k,V',k) \models \psi$. The inductive cases for propositional connectives are immediate. Finally, if $\psi = \exists p\chi$, then $(\mathcal{G}_{\mathbb{N}},V,k) \models \psi$ implies that for some $U \in D$,

$(\mathcal{G}_{\mathbb{N}}, V_U^p, k) \vDash \psi$. Consider $U' = U \cap [k] \in D'$. In particular, $(V_U^p)' = V_{U'}'^p$. By induction hypothesis $(\mathcal{G}_k, V_{U'}'^p, k) \vDash \chi$, that is, $(\mathcal{G}_k, V', k) \vDash \psi$.

As for the inductive step, we again consider the subformulas ψ of ϕ. If ψ is an atom p, then $(\mathcal{G}_{\mathbb{N}}, V, k-(n+1)) \vDash \psi$ iff $k-(n+1) \in V(p)$, iff $k-(n+1) \in V'(p)$, iff $(\mathcal{G}_k, V', k-(n+1)) \vDash \psi$. The inductive cases for propositional connectives are immediate. If $\psi = \exists p \chi$, then $(\mathcal{G}_{\mathbb{N}}, V, k-(n+1)) \vDash \psi$ implies that for some $U \in D$, $(\mathcal{G}_{\mathbb{N}}, V_U^p, k-(n+1)) \vDash \chi$. Again consider, $U' = U \cap [k] \in D'$. By induction hypothesis $(\mathcal{G}_k, V_{U'}'^p, k-(n+1)) \vDash \chi$, that is, $(\mathcal{G}_k, V', k-(n+1)) \vDash \psi$. Finally, if $\psi = \Diamond \chi$, then $(\mathcal{G}_{\mathbb{N}}, V, k-(n+1)) \vDash \psi$ implies that $(\mathcal{G}_{\mathbb{N}}, V, k-n)) \vDash \chi$. By induction hypothesis $(\mathcal{G}_k, V', k-n)) \vDash \chi$, that is, $(\mathcal{G}_k, V', k-(n+1)) \vDash \psi$.

As a consequence, $(\mathcal{G}_{\mathbb{N}}, V, 0) \nvDash \phi$ implies $(\mathcal{G}_k, V', 0) \nvDash \phi$, i.e., $\phi \notin \mathsf{Th}(G)$. □

A.3 Chapter 5

Lemma 5.6. The case for $x = pl$ and $y = bool$ follows from equalities $[\![\neg\psi]\!] = \setminus[\![\psi]\!]$, $[\![\psi \wedge \psi']\!] = [\![\psi]\!] \cap [\![\psi']\!]$, $[\![\psi \vee \psi']\!] = [\![\psi]\!] \cup [\![\psi']\!]$ and the fact that D is a boolean algebra.

For $x = el$ and $y = el$, notice that $[\![K_a\psi]\!] = [a]([\![\psi]\!])$, $[\![C_A\psi]\!] = [A]([\![\psi]\!])$, and D is a boolean algebra with operators $[a]$ and $[A]$.

The case of $x = pal$ and $y = el$ follows since PAL is as expressive as epistemic logic [100]; while the case for $x = sopal$ and $y = full$ is trivial. □

Lemma 5.7. The proof for full frames is immediate, as for all $U \subseteq W_{|\phi}$, $U \subseteq W$ and then $U \in D$. Hence, $U_{|\phi} = U \in D_{|\phi}$.

The proof for boolean frames follows from the identities below, for $\star \in \{\cap, \cup\}$:

$$
\begin{array}{rcl}
U_{|\phi} \star U'_{|\phi} & = & (U \star U')_{|\phi} \\
\setminus(U_{|\phi}) & = & (\setminus U)_{|\phi}
\end{array}
$$

As for epistemic frames, we remark that

$$
\begin{array}{rcl}
[a](U_{|\phi}) & = & ([a](\setminus[\![\phi]\!] \cup U))_{|\phi} \\
[A](U_{|\phi}) & = & ([A](\setminus[\![\phi]\!] \cup U))_{|\phi}
\end{array}
$$

Observe that here $[a]$ and $[A]$ denotes two different operations, the former on $\mathcal{M}_{|\phi}$ and the latter on $\mathcal{M}$. Then, $w \in [a](U_{|\phi})$ iff $R_{|\phi,a}(w) \subseteq U_{|\phi}$. Since, $R_{|\phi,a}(w) = R_a(w) \cap W_{|\phi}^2$, this is the case iff $w \in [\![\phi]\!]$ and for every $w' \in R_a(w)$, $w' \in [\![\phi]\!]$ implies $w' \in U$, iff $w \in [\![\phi]\!]$ and $R_a(w) \subseteq (\setminus[\![\phi]\!] \cup U)$, iff $w \in ([a](\setminus[\![\phi]\!] \cup U))_\phi$. Finally, notice that $[a](\setminus[\![\phi]\!] \cup U) \in D$, as $[\![\phi]\!] \in D$ for every $\phi \in \mathcal{L}_{el}$ by Lemma 5.6 and D is a boolean algebra with operators. Hence, $[a](U_{|\phi}) = ([a](\setminus[\![\phi]\!] \cup U))_\phi \in D_{|\phi}$. The proof for operator $[A]$ is similar. □

Lemma 5.8. All proofs are by induction on the structure of ϕ. Notice that the proofs of Lemmas 5.8(1) and 5.8(2) (resp. Lemmas 5.8(3) and 5.8(4)) make use of each other. This circularity is safe nonetheless, as in each step formulas of strictly smaller length are considered.

We first prove (1) for $x = el$. The other cases follow similarly by Lemma 5.6. For $\phi = r$, $(\mathcal{M}^p_{[\![\psi]\!]}, w) \vDash \phi$ iff $w \in V^p_{[\![\psi]\!]}(r)$, iff $w \in V(r)$ whenever $r \neq p$ or $w \in [\![\psi]\!]$ for $r = p$. In both cases $(\mathcal{M}, w) \vDash \phi[p/\psi]$.

The inductive cases for the propositional connectives and epistemic operators are straightforward, as these simply commute with substitution.

If $\phi = [\varphi]\varphi'$ then $(\mathcal{M}^p_{[\![\psi]\!]}, w) \vDash \phi$ iff $(\mathcal{M}^p_{[\![\psi]\!]}, w) \vDash \varphi$ implies $((\mathcal{M}^p_{[\![\psi]\!]})_{|\varphi}, w) \vDash \varphi'$. By induction hypothesis, this is the case iff $(\mathcal{M}, w) \vDash \varphi[p/\psi]$ implies $((\mathcal{M}^p_{[\![\psi]\!]})_{|\varphi}, w) \vDash \varphi'$. By (2) we have that $((\mathcal{M}^p_{[\![\psi]\!]})_{|\varphi}, w) \vDash \varphi'$ iff $((\mathcal{M}_{|\varphi[p/\psi]})^p_{[\![\psi]\!]}, w) \vDash \varphi'$, as ψ is free for p in ϕ. And again by induction hypothesis we have that $(\mathcal{M}, w) \vDash \varphi[p/\psi]$ implies $(\mathcal{M}_{|\varphi[p/\psi]}, w) \vDash \varphi'[p/\psi]$, that is, $(\mathcal{M}, w) \vDash [\varphi[p/\psi]](\varphi'[p/\psi]) = ([\varphi]\varphi')[p/\psi]$.

If $\phi = \forall r\varphi$ for $r \neq p$, then $(\mathcal{M}^p_{[\![\psi]\!]}, w) \vDash \phi$ iff for any $U \in D$, $((\mathcal{M}^p_{[\![\psi]\!]})^r_U, w) \vDash \varphi$. Since $r \neq p$ and ψ is free for p in φ, the assignment $(V^p_{[\![\psi]\!]})^r_U$ coincides with $(V^r_U)^p_{[\![\psi]\!]}$. As a consequence, we obtain $((\mathcal{M}^r_U)^p_{[\![\psi]\!]}, w) \vDash \varphi$, i.e., $(\mathcal{M}^r_U, w) \vDash \varphi[p/\psi]$ by induction hypothesis. But this means that $(\mathcal{M}, w) \vDash \forall r(\varphi[p/\psi]) = (\forall r\varphi)[p/\psi]$. This completes the proof for (1).

As for (2), by (1) $(\mathcal{M}^p_{[\![\psi]\!]}, w) \vDash \phi$ iff $(\mathcal{M}, w) \vDash \phi[p/\psi]$. Hence, $W_{|\phi}$ in $(\mathcal{M}^p_{[\![\psi]\!]})_{|\phi}$ is equal to $W_{|\phi[p/\psi]}$ in $(\mathcal{M}_{|\phi[p/\psi]})^p_{[\![\psi]\!]}$. Similarly for components $R_{|\phi}$ and $D_{|\phi}$ in $(\mathcal{M}^p_{[\![\psi]\!]})_{|\phi}$. Finally, we have to prove that $(V^p_{[\![\psi]\!]})_{|\phi}(r) = (V_{|\phi[p/\psi]})^p_{[\![\psi]\!]}(r)$, for every $r \in AP$, under the restriction that $p \in fr(\phi)$ implies $\psi \in \mathcal{L}_{qbf}$. If $p \in fr(\phi)$ then from $(\mathcal{M}^p_{[\![\psi]\!]}, w) \vDash \phi$ iff $(\mathcal{M}, w) \vDash \phi[p/\psi]$ follows that $(V^p_{[\![\psi]\!]})_{|\phi}(r) = V_{|\phi[p/\psi]}(r) = (V_{|\phi[p/\psi]})^p_{[\![\psi]\!]}(r)$ for $r \neq p$. For $r = p$, notice that for every $\psi' \in \mathcal{L}_{qbf}$, $((\mathcal{M}^p_{[\![\psi]\!]})_{|\phi}, w) \vDash \psi'$ iff $(\mathcal{M}_{|\phi[p/\psi]}, w) \vDash \psi'$. In particular, $(V^p_{[\![\psi]\!]})_{|\phi}(p) = [\![\psi]\!]_{|\phi} = [\![\psi]\!]_{|\phi[p/\psi]} = (V_{|\phi[p/\psi]})^p_{[\![\psi]\!]}(p)$. On the other hand, if $p \notin fr(\phi)$ then we have to prove that $(V^p_{[\![\psi]\!]})_{|\phi}(r) = (V_{|\phi})^p_{[\![\psi]\!]}(r)$, for every $r \in AP$. For $r \neq p$, we have that $(V^p_{[\![\psi]\!]})_{|\phi}(r) = V_{|\phi}(r) = (V_{|\phi})^p_{[\![\psi]\!]}(r)$. For $r = p$, $(V^p_{[\![\psi]\!]})_{|\phi}(p) = ([\![\psi]\!]_{\mathcal{M}})_{|\phi} = (V_{|\phi})^p_{[\![\psi]\!]}(p)$. This completes the proof for (2).

As regards (3), if $\phi = p$, then $fr(\phi) = \{p\}$ and $(\mathcal{M}, w) \vDash \phi$ iff $w \in V(p) = V'(p)$, iff $(\mathcal{M}', w) \vDash \phi$.

The inductive cases for the propositional connectives and epistemic operators are straightforward.

For $\phi = [\psi]\psi'$, $(\mathcal{M}, w) \vDash \phi$ iff $(\mathcal{M}, w) \vDash \psi$ implies $(\mathcal{M}_{|\phi}, w) \vDash \psi'$, iff $(\mathcal{M}', w) \vDash \psi$ implies $(\mathcal{M}'_{|\phi}, w) \vDash \psi'$ by induction hypothesis and (4). That is, $(\mathcal{M}', w) \vDash \phi$.

If $\phi = \forall p\psi$, then $(\mathcal{M}, w) \vDash \phi$ iff for every $U \in D$, $(\mathcal{M}^p_U, w) \vDash \psi$. Since $fr(\phi) = fr(\psi) \setminus \{p\}$, $V^p_U(fr(\psi)) = V'^p_U(fr(\psi))$ and by induction hypothesis $(\mathcal{M}'^p_U, w) \vDash \psi$, that is, $(\mathcal{M}', w) \vDash \phi$. This completes the proof for (3).

As for (4), by (3zx) $(\mathcal{M}, w) \vDash \psi$ iff $(\mathcal{M}', w) \vDash \psi$. Hence $W_{|\psi} = W'_{|\psi}$, $R_{|\psi} = R'_{|\psi}$, $D_{|\psi} = D'_{|\psi}$, and $V_{|\psi} = V'_{|\psi}$. □

Lemma 5.10. Indeed, if $(\mathcal{M}, w) \nvDash [\psi]\phi$ for some $\psi \in \mathcal{L}_{el}$, then in particular, $[\![\psi]\!] \in D$ by Lemma 5.6 and for $U = [\![\psi]\!]$, $(\mathcal{M}^p_U, w) \nvDash [p]\phi$. That is, $(\mathcal{M}, w) \nvDash \forall p[p]\phi$. □

Lemma 5.12. The $\Leftarrow$ direction follows from (5.6) above. As for the $\Rightarrow$ direction, suppose that for some model $\mathcal{M}$ and state w, $(\mathcal{M}, w) \nvDash \tau(\phi)$. Consider now a model $\mathcal{M}'$ such that $\mathcal{F}' = \mathcal{F}$ and V' coincides with V on all atoms appearing in ϕ. Further, for every $U \in D$ take $q_U \in AP$ not appearing in ϕ and let $V'(q_U) = U$. By Lemma 5.8(3), $(\mathcal{M}, w) \nvDash \tau(\phi)$ implies $(\mathcal{M}', w) \nvDash \tau(\phi)$ (the assignment $V'(q)$ for all atoms not appearing in ϕ and not assigned to a set U is uninfluential, let it be W.)

Hereafter we write $\mathcal{N} \subseteq \mathcal{M}$ to express that $\mathcal{N}$ is a submodel of $\mathcal{M}$, i.e., $W_\mathcal{N} \subseteq W$; $D_\mathcal{N} = \{U \cap W_\mathcal{N} \mid U \in D\}$; $R_{\mathcal{N},a} = R_a \cap W^2_\mathcal{N}$; and $V_\mathcal{N}(p) = V(p) \cap W_\mathcal{N}$ for every $p \in AP$. We can now prove the following auxiliary result: for every submodel $\mathcal{N}'$ of $\mathcal{M}'$ and subformula ψ of ϕ,

$$(\mathcal{N}', w) \vDash \psi \quad \text{iff} \quad (\mathcal{N}', w) \vDash \tau(\psi) \tag{A.4}$$

For ψ being an atom, a boolean combination of formulas, an epistemic formula or a PAL formula, this is indeed clear. So consider ϕ of the form $\Box\psi$, with the claim proven for ψ. For the implication from left to right,

$$\begin{aligned}
(\mathcal{N}', w) \nvDash \tau(\Box\psi) &\Rightarrow (\mathcal{N}', w) \nvDash \forall p[p]\tau(\psi) \\
&\Rightarrow \text{ for some } U \in D, (\mathcal{N}'^p_U, w) \nvDash [p]\tau(\psi) \\
&\Rightarrow \text{ for some } U \in D, (\mathcal{N}'^p_U, w) \vDash p \text{ and } ((\mathcal{N}'^p_U)_{|p}, w) \nvDash \tau(\psi) \\
&\Rightarrow \text{ for some } q_U \in \mathcal{L}_{el}, (\mathcal{N}'^p_{[\![q_U]\!]}, w) \vDash p \text{ and } ((\mathcal{N}'^p_{[\![q_U]\!]})_{|p}, w) \nvDash \tau(\psi) \\
&\Rightarrow \text{ for some } q_U \in \mathcal{L}_{el}, (\mathcal{N}', w) \vDash p[p/q_U] \text{ and } ((\mathcal{N}'_{|q_U})^p_{[\![q_U]\!]}, w) \nvDash \tau(\psi) \\
&\Rightarrow \text{ for some } q_U \in \mathcal{L}_{el}, (\mathcal{N}', w) \vDash q_U \text{ and } (\mathcal{N}'_{|q_U}, w) \nvDash (\tau(\psi))[p/q_U] \\
&\Rightarrow \text{ for some } q_U \in \mathcal{L}_{el}, (\mathcal{N}', w) \vDash q_U \text{ and } (\mathcal{N}'_{|q_U}, w) \nvDash \tau(\psi) \\
&\Rightarrow \text{ for some } q_U \in \mathcal{L}_{el}, (\mathcal{N}', w) \vDash q_U \text{ and } (\mathcal{N}'_{|q_U}, w) \nvDash \psi \\
&\Rightarrow \text{ for some } q_U \in \mathcal{L}_{el}, (\mathcal{N}', w) \nvDash [q_U]\psi \\
&\Rightarrow (\mathcal{N}', w) \nvDash \Box\psi
\end{aligned}$$

As to the implication from right to left,

$$\begin{aligned}
(\mathcal{N}', w) \nvDash \Box\psi &\Rightarrow \text{ for some } \varphi \in \mathcal{L}_{el}, (\mathcal{N}', w) \nvDash [\varphi]\psi \\
&\Rightarrow \text{ for } [\![\varphi]\!] \in D, (\mathcal{N}', w) \vDash \varphi \text{ and } (\mathcal{N}'_{|\varphi}, w) \nvDash \psi \\
&\Rightarrow \text{ for } [\![\varphi]\!] \in D, (\mathcal{N}', w) \vDash p[p/\varphi] \text{ and } (\mathcal{N}'_{|\varphi}, w) \nvDash \tau(\psi) \\
&\Rightarrow \text{ for } [\![\varphi]\!] \in D, (\mathcal{N}'^p_{[\![\varphi]\!]}, w) \vDash p \text{ and } ((\mathcal{N}'^p_{[\![\varphi]\!]})_{|p}, w) \nvDash \tau(\psi) \\
&\Rightarrow \text{ for } [\![\varphi]\!] \in D, (\mathcal{N}'^p_{[\![\varphi]\!]}, w) \nvDash [p]\tau(\psi) \\
&\Rightarrow (\mathcal{N}', w) \nvDash \forall p[p]\tau(\psi)
\end{aligned}$$

Notice that the deductions above make essential use of Lemmas 5.8(1-2).

Finally, by (A.4) $(\mathcal{M}', w) \nvDash \tau(\phi)$ implies $(\mathcal{M}', w) \nvDash \phi$ as required. □

Lemma 5.16. If APAL $\leq_m$ SOPAL, then for $\varphi = \Box(K_a q \to K_b K_a q)$ in APAL there exists a corresponding φ' in SOPAL. However, $(\mathcal{M}, w_{00}) \vDash \varphi$ implies $(\mathcal{M}, w_{00}) \vDash \varphi'$, which implies $(\mathcal{M}'', w_{00}) \vDash \varphi'$ by Lemma 5.8.4, which implies $(\mathcal{M}'', w_{00}) \vDash \varphi$. A contradiction. □

Lemma 5.17. We prove statement (5.7). For $x \in \{ap, pl, el, pal, sopal\}$, if $(\mathcal{M}, w) \vDash \forall p\phi$ then for every $U \in D$, $(\mathcal{M}^p_U, w) \vDash \phi$. In particular, $(\mathcal{M}^p_{[\![\psi]\!]}, w) \vDash \phi$ as by Lemma 5.6, $[\![\psi]\!] \in D$ whenever $\psi \in \mathcal{L}_x$. By Lemma 5.8(1), $(\mathcal{M}^p_{[\![\psi]\!]}, w) \vDash \phi$ implies $(\mathcal{M}, w) \vDash \phi[p/\psi]$.

As to (5.8), suppose that $(\mathcal{M}, w) \vDash \psi$. Since p does not appear free in ψ, for every $U \in D$, assignment V^p_U coincides with V on $fr(\psi)$. By Lemma 5.8(3) we have that

$(\mathcal{M}^p_U, w) \vDash \psi$ and by hypothesis, $(\mathcal{M}^p_U, w) \vDash \phi$. Hence, for every $U \in D$, $(\mathcal{M}^p_U, w) \vDash \phi$, that is, $(\mathcal{M}, w) \vDash \forall p \phi$. □

Lemma 5.18. As regards (5.9) observe that,

$$\begin{array}{lll} (\mathcal{M}, w) \vDash [\psi]\forall p\phi & \text{iff} & (\mathcal{M}, w) \vDash \psi \text{ implies } (\mathcal{M}_{|\psi}, w) \vDash \forall p\phi \\ & \text{iff} & (\mathcal{M}, w) \vDash \psi \text{ implies for all } U' \in D_{|\psi}, ((\mathcal{M}_{|\psi})^p_{U'}, w) \vDash \phi \end{array}$$

Now, if $U \in D$ then $U' = U \cap W_{|\psi} \in D_{|\psi}$. On the other hand, if $U' \in D_{|\psi}$ then for some $U \in D$, $U' = U \cap W_{|\psi}$. In particular $(V_{|\psi})^p_{U'} = (V^p_U)_{|\psi}$, as p does not appear free in ψ. Hence,

$$\begin{array}{lll} (\mathcal{M}, w) \vDash [\psi]\forall p\phi & \text{iff} & (\mathcal{M}, w) \vDash \psi \text{ implies for all } U \in D, ((\mathcal{M}^p_U)_{|\psi}, w) \vDash \phi \\ & \text{iff} & (\mathcal{M}, w) \vDash \psi \text{ implies } (\mathcal{M}, w) \vDash \forall p[\psi]\phi \\ & \text{iff} & (\mathcal{M}, w) \vDash \psi \to \forall p[\psi]\phi \end{array}$$

The other equivalences are proved similarly. In particular, formulas (5.10) and (5.12) are dual of (5.9) and (5.11) respectively. □

Lemma 5.21. We show that for every model $\mathcal{M}$, $\mathcal{M}'$, $\mathcal{M}''$ with $\mathcal{M}'' \subseteq \mathcal{M}' \subseteq \mathcal{M}$, $s \in W''$, and positive formula $\phi \in \mathcal{L}^+_{sopal}$, $(\mathcal{M}', s) \vDash \phi$ implies $(\mathcal{M}'', s) \vDash \phi$. The inductive cases for $\phi \neq \forall p\psi$ follow seamlessly as in [116]. As for $\phi = \forall p\psi$, Consider $U'' \in W''$ s.t. $(\mathcal{M}''^p_{U''}, s) \vDash \psi$. Clearly, $\mathcal{M}''^p_{U''} \subseteq \mathcal{M}'^p_{U'}$ for $U' \in D'$ such that $U'' = U' \cap W''$. Moreover, hypothesis $(\mathcal{M}', s) \vDash \forall p\psi$ implies $(\mathcal{M}'^p_{U'}, s) \vDash \psi$, and by induction hypothesis it follows that $(\mathcal{M}''^p_{U''}, s) \vDash \psi$. Since U'' is arbitrary, $(\mathcal{M}'', s) \vDash \forall p\psi$.

As a result, positive formulas are preserved under taking submodels. In particular, they are preserved by the model refinement of public announcement operators. □

Theorem 5.28. We briefly sketch the case for quantifiers, for the other cases we refer to [68, Theorem 1]. The 'if' direction is by induction on the formula, so assume $\varphi = \exists p\psi$, with the claim proven for ψ with $|\psi| = n - 1$. That is, suppose that φ has size $n < k$ and that $\mathbb{M} \vDash \exists p\psi$ while $\mathbb{N} \vDash \neg\exists p\psi$. Spoiler plays the $\exists p$-move: since $\mathbb{M} \vDash \exists p\psi$, for every model $(\mathcal{M}, w) \in \mathbb{M}$, Spoiler can choose some $U \in D$ such that $(\mathcal{M}^p_U, w) \vDash \psi$. Collecting all pointed models thus obtained in $\mathbb{M}_1$, we have $\mathbb{M}_1 \vDash \psi$. Since $\mathbb{N} \vDash \neg\exists p\psi$, if we put all models $(\mathcal{N}, v)$ in a set $\mathbb{N}_1$, we have $\mathbb{N}_1 \vDash \neg\psi$. We know that Spoiler can win the sub-game starting in $\langle \mathbb{M}_1 \circ \mathbb{N}_1 \rangle$ in $n - 1$ moves, which in turn ensures he wins the game starting in $\langle \mathbb{M} \circ \mathbb{N} \rangle$ in n moves.

For the 'only-if' direction, if Spoiler has won the FSG starting at $\langle \mathbb{M} \circ \mathbb{N} \rangle$ in $n < k$ moves, then the resulting closed game tree is a parse tree of a formula φ of length n such that $\mathbb{M} \models \varphi$ and $\mathbb{N} \models \neg\varphi$. To see this, we label the nodes of the tree with formulas, starting with the leaves. In particular, if a node has a label $\exists p$ and its successor is labeled with ψ, then the current node is labeled with $\exists p \psi$. One can verify that for each node $\langle \mathbb{A} \circ \mathbb{B} \rangle$, the formula labelling the node is true in $\mathbb{A}$, and false in $\mathbb{B}$. Hence, the game tree is a parse tree for the formula labelling the root. $\square$

A.4 Chapter 6

Lemma 6.4. We consider the properties of reflexivity, symmetry, and transitivity of equivalence relations. The case for $a \notin A$ is immediate, as $R^-_a = R^+_a = R_a$.

Reflexivity of R^-_a: suppose that $v \in W' = W$. Clearly, $R_a(v, v)$ holds, and in the definition of R^-_a, if $v \in R_a(w) \cap [[\psi]]$ then $v \in R_a(v) \cap [[\psi]]$. Hence, $R^-_a(v, v)$ holds. The case for $v \in R_a(w) \cap [[\neg\psi]]$ is symmetric. The case for R^+_a is similar.

Symmetry of R^-_a: suppose that $R^-_a(v_1, v_2)$ holds. Clearly, $R_a(v_1, v_2)$ as $R^-_a \subseteq R_a$, and $R_a(v_2, v_1)$ as R_a is symmetric. Now notice that $R^-_a(v_1, v_2)$ implies that both v_1 and v_2 satisfy ψ, or they both satisfy $\neg\psi$. In both cases $R^-_a(v_2, v_1)$ holds as well. Also in this case, the proof for R^+_a is similar.

Transitivity of R^-_a: suppose that $R^-_a(v_1, v_2)$ and $R^-_a(v_2, v_3)$. Clearly, $R_a(v_1, v_2)$ and $R_a(v_2, v_3)$ as $R^-_a \subseteq R_a$, and therefore $R_a(v_1, v_3)$ by transitivity. Moreover, if $R^-_a(v_1, v_2)$ and $R^-_a(v_2, v_3)$, then all v_1, v_2 and v_3 satisfy ψ, or they all satisfy $\neg\psi$. In both cases $R^-_a(v_1, v_3)$ holds. The case for R^+_a is similar. $\square$

Lemma 6.5. We prove (6.1) for a propositional formula ϕ. Suppose that $(\mathcal{M}, w) \models \phi$ but $(\mathcal{M}^-_{(w,\phi,A)}, w) \not\models E_A\phi$ to obtain a contradiction, that is, $(\mathcal{M}^-_{(w,\phi,A)}, w') \not\models \phi$ for some $a \in A$ and $w' \in R^-_a(w)$. In particular, this means that $(\mathcal{M}, w') \not\models \phi$, as ϕ is propositional. Hence, $w' \neq w$ (as ϕ is true in w) and $w' \in R_a(w) \supseteq R^-_a(w)$. But then $w' \notin R_a(w) \cap [[\phi]]$, against the hypothesis that $R^-_a(w, w')$. Therefore, it is the case that $(\mathcal{M}^-_{(w,\phi,A)}, w) \models E_A\phi$. The proof for (6.2) follows a similar line. $\square$

Proposition 6.6. The only non-trivial case is for $\psi = [\phi]\phi'$. In particular, we show that for every $w \in W$, refinement $\mathcal{M}^+_{(w,\phi,I)}$ satisfies the same formulas in PAL as refinement $\mathcal{M}_\phi$. The key remark here is that worlds that are not reachable from w via relation R^*_I do not account for the truth value of formulas at w. Specifically, in refinement $\mathcal{M}^+_{(w,\phi,I)}$, any state w' is reachable from w via $(R^+)^*_I$ iff w' is reachable from w in $\mathcal{M}_\phi$. Also, in both refinements the indistinguishability relations and assignments are restricted to $[\![\phi]\!]_\mathcal{M}$. As a result, the two models satisfy the same announcement formulas at w. □

Lemma 6.11. We prove (6.3) as (6.4) is the special case for $A = \{a\}$. By Definition 6.3, $(\mathcal{M}, w) \models [\phi]^+_A C_A\psi$ iff $(\mathcal{M}, w) \models \phi$ implies that for every $w' \in R'^*_A(w)$, $(\mathcal{M}^+_{(w,\phi,A)}, w') \models \psi$. Now notice that for every $w' \in R'^*_A(w) = R^*_A(w) \cap [\![\phi]\!]_\mathcal{M}$, the refinements $\mathcal{M}^+_{(w,\phi,A)}$ and $\mathcal{M}^+_{(w',\phi,A)}$ are equal. Hence, $(\mathcal{M}, w) \models [\phi]^+_A C_A\psi$ iff $(\mathcal{M}, w) \models \phi$ implies that for every $w' \in R^*_A(w) \cap [\![\phi]\!]_\mathcal{M}$, $(\mathcal{M}^+_{(w',\phi,A)}, w') \models \psi$, that is, $(\mathcal{M}, w) \models \phi$ implies that for every $w' \in R^*_A(w)$, $(\mathcal{M}, w') \models [\phi]^+_A\psi$, i.e., $(\mathcal{M}, w) \models \phi \to C_A[\phi]^+_A\psi$. □

Lemma 6.12. We prove (6.5). Suppose that $(\mathcal{M}, w) \models [\phi]^-_A [\phi']^-_A \psi$, that is, if $(\mathcal{M}, w) \models \phi$ and $(\mathcal{M}^-_{(w,\phi,A)}, w) \models \phi'$, then $((\mathcal{M}^-_{(w,\phi,A)})^-_{(w,\phi',A)}, w) \models \psi$. We have to show that this is equivalent to $(\mathcal{M}, w) \models \left[\phi \wedge [\phi]^-_A \phi'\right]^-_A \psi$, that is, if $(\mathcal{M}, w) \models \phi$ and $(\mathcal{M}^-_{(w,\phi,A)}, w) \models \phi'$, then $(\mathcal{M}^-_{(w,\phi\wedge[\phi]^-_A\psi',A)}, w) \models \psi$. Hence, it is enough to prove that $((\mathcal{M}^-_{(w,\phi,A)})^-_{(w,\phi',A)}, w) \models \psi$ iff $(\mathcal{M}^-_{(w,\phi\wedge[\phi]^-_A\psi',A)}, w) \models \psi$. In particular, refinements $(\mathcal{M}^-_{(w,\phi,A)})^-_{(w,\phi',A)}$ and $\mathcal{M}^-_{(w,\phi\wedge[\phi]^-_A\psi',A)}$ are identical. To see this we remark that in refinement $(\mathcal{M}^-_{(w,\phi,A)})^-_{(w,\phi',A)}$, for every $a \in A$,

$$R^-_a(v) = \begin{cases} R_a(v) \cap [\![\phi]\!]_\mathcal{M} \cap [\![\phi']\!]_{\mathcal{M}_{(w,\phi,A)}} & \text{if } v \in R_a(w) \cap [\![\phi]\!]_\mathcal{M} \cap [\![\phi']\!]_{\mathcal{M}_{(w,\phi,A)}} \\ R_a(v) \cap [\![\phi]\!]_\mathcal{M} \cap [\![\neg\phi']\!]_{\mathcal{M}_{(w,\phi,A)}} & \text{if } v \in R_a(w) \cap [\![\phi]\!]_\mathcal{M} \cap [\![\neg\phi']\!]_{\mathcal{M}_{(w,\phi,A)}} \\ R_a(v) \cap [\![\neg\phi]\!]_\mathcal{M} \cap [\![\phi']\!]_{\mathcal{M}_{(w,\phi,A)}} & \text{if } v \in R_a(w) \cap [\![\neg\phi]\!]_\mathcal{M} \cap [\![\phi']\!]_{\mathcal{M}_{(w,\phi,A)}} \\ R_a(v) \cap [\![\neg\phi]\!]_\mathcal{M} \cap [\![\neg\phi']\!]_{\mathcal{M}_{(w,\phi,A)}} & \text{if } v \in R_a(w) \cap [\![\neg\phi]\!]_\mathcal{M} \cap [\![\neg\phi']\!]_{\mathcal{M}_{(w,\phi,A)}} \\ R_a(v) & \text{otherwise} \end{cases}$$

which is tantamount to the following in model $\mathcal{M}^-_{(w,\phi\wedge[\phi]^-_A\phi',A)}$:

$$R^-_a(v) = \begin{cases} R_a(v) \cap [\![\phi \wedge [\phi]_A\phi']\!]_\mathcal{M} & \text{if } v \in R_a(w) \cap [\![\phi \wedge [\phi]_A\phi']\!]_\mathcal{M} \\ R_a(v) \cap [\![\neg(\phi \wedge [\phi]_A\phi')]\!]_\mathcal{M} & \text{if } v \in R_a(w) \cap [\![\neg(\phi \wedge [\phi]_A\phi')]\!]_\mathcal{M} \\ R_a(v) & \text{otherwise} \end{cases}$$

Hence, the two models are identical and (6.5) holds. □

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