Matrices Worksheet

This worksheet is designed to help you increase your confidence in handling MATRICES. This worksheet contains both theory and exercises which cover

1. **Introduction**
2. **Order, Addition and subtraction**
3. **Equality**
4. **Multiplication**
5. **Identity and Inverse**

1. Matrices - introduction

A matrix is an array of numbers each element of which gives a single piece of information. Consider the following % marks gained by students in, say, the first half of semester 1:

	module 1	module 2	module 3
Andy	40	58	83
Mary	32	43	73
Bill	56	65	66
Nancy	21	45	34

This can be written as

$$M_1 = \begin{pmatrix} 40 & 58 & 83 \\ 32 & 43 & 73 \\ 56 & 65 & 66 \\ 21 & 45 & 34 \end{pmatrix}$$

The marks for the second half:

	module 1	module 2	module 3
Andy	34	40	67
Mary	56	45	56
Bill	23	57	70
Nancy	78	43	67

can be represented in matrix form as

$$M_2 = \begin{pmatrix} 34 & 40 & 67 \\ 56 & 45 & 56 \\ 23 & 57 & 70 \\ 78 & 43 & 67 \end{pmatrix}$$

Adding the results together, using the matrices, gives

$$M_1 + M_2 = \begin{pmatrix} 40 & 58 & 83 \\ 32 & 43 & 73 \\ 56 & 65 & 66 \\ 21 & 45 & 34 \end{pmatrix} + \begin{pmatrix} 34 & 40 & 67 \\ 56 & 45 & 56 \\ 23 & 57 & 70 \\ 78 & 43 & 67 \end{pmatrix} = \begin{pmatrix} 74 & 98 & 150 \\ 88 & 88 & 129 \\ 79 & 122 & 136 \\ 99 & 88 & 101 \end{pmatrix}$$

This is an example of matrix addition. Often results of this kind are held on a spreadsheet and the addition is very simple. The total gives marks out of 200. To convert back to percentages we need to divide all the values by 2.

$$\text{Final mark} = \frac{1}{2}\begin{pmatrix} 74 & 98 & 150 \\ 88 & 88 & 129 \\ 79 & 122 & 136 \\ 99 & 88 & 101 \end{pmatrix} = \begin{pmatrix} 37 & 49 & 75 \\ 44 & 44 & 64.5 \\ 39.5 & 61 & 78 \\ 49.5 & 44 & 50.5 \end{pmatrix}$$

The final figures would probably be rounded to the nearest whole number.

It may be decided that the second set of marks is more significant than the first and that the marks should be combined taking by taking $0.4M_1 + 0.6M_2$ giving the total (to the nearest whole number) as

$$0.4\begin{pmatrix} 40 & 58 & 83 \\ 32 & 43 & 73 \\ 56 & 65 & 66 \\ 21 & 45 & 34 \end{pmatrix} + 0.6\begin{pmatrix} 34 & 40 & 67 \\ 56 & 43 & 56 \\ 23 & 65 & 70 \\ 78 & 45 & 67 \end{pmatrix} = \begin{pmatrix} 36 & 47 & 73 \\ 46 & 44 & 63 \\ 36 & 60 & 68 \\ 55 & 44 & 54 \end{pmatrix}$$

(This can be done quite easily on a Spreadsheet.)

2. Order, addition and subtraction

These matrices have the same *order* or *size*. Each has 4 rows and 3 columns and are described as 4 by 3 (or 4 × 3) matrices. As each element refers to the mark for a particular person and module it can be seen that matrices can only be added or subtracted if they have the same order (and refer to the same thing).

Examples

If $A=\begin{pmatrix}3 & 0\\ -1 & 4\end{pmatrix}$ $B=\begin{pmatrix}2\\ -3\end{pmatrix}$ $C=\begin{pmatrix}2 & -3\\ 5 & 2\end{pmatrix}$ $D=\begin{pmatrix}3\\ -2\\ 0\end{pmatrix}$

find **(*i*)** $A + B$, **(*ii*)** $A + C$, **(*iii*)** $B + D$, **(*iv*)** $C - A$, **(*v*)** $3D$

(*i*) $A+B=\begin{pmatrix}3 & 0\\ -1 & 4\end{pmatrix}+\begin{pmatrix}2\\ -3\end{pmatrix}$ can't be done (different order)

(*ii*) $A+C=\begin{pmatrix}3 & 0\\ -1 & 4\end{pmatrix}+\begin{pmatrix}2 & -3\\ 5 & 2\end{pmatrix}=\begin{pmatrix}5 & -3\\ 4 & 6\end{pmatrix}$

(*iii*) $B + D$ can't be done (different order)

(*iv*) $C-A=\begin{pmatrix}2 & -3\\ 5 & 2\end{pmatrix}-\begin{pmatrix}3 & 0\\ -1 & 4\end{pmatrix}=\begin{pmatrix}-1 & -3\\ 6 & -2\end{pmatrix}$

(*v*) $3D=3\times\begin{pmatrix}3\\ -2\\ 0\end{pmatrix}=\begin{pmatrix}9\\ -6\\ 0\end{pmatrix}$

3. Equality of matrices

Two matrices can only be equal if all the elements in the first matrix are the same as the corresponding elements in the second. This can be seen from the definition.

Example Given $\begin{pmatrix}2 & -3\\ 5 & y\end{pmatrix}-\begin{pmatrix}3 & 0\\ x & 4\end{pmatrix}=\begin{pmatrix}z & -3\\ 6 & -2\end{pmatrix}$ find the values of x, y and z.

Subtracting the matrices on the left we get $\begin{pmatrix}-1 & -3\\ 5-x & y-4\end{pmatrix}=\begin{pmatrix}z & -3\\ 6 & -2\end{pmatrix}$

Equating corresponding elements gives

$$-1=z,\quad 5-x=6\Rightarrow x=-1,\quad y-4=-2\Rightarrow y=2$$

Exercise 1

1. $A=\begin{pmatrix}2 & -2\\ -2 & 2\end{pmatrix}$ $B=\begin{pmatrix}3 & -1\\ 3 & -1\end{pmatrix}$ $C=\begin{pmatrix}3 & 2 & -1\\ 4 & 0 & 1\end{pmatrix}$ $D=\begin{pmatrix}5 & 4 & -1\\ 2 & 3 & 4\end{pmatrix}$

$E=\begin{pmatrix}4 & 1\\ -3 & 2\\ 4 & 5\end{pmatrix}$ $F=\begin{pmatrix}5 & 2\\ -2 & -4\\ -1 & -5\end{pmatrix}$ $G=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & -2\\ 2 & -1 & 3\end{pmatrix}$

Evaluate, where possible, (*i*) $C + D$ (*ii*) $F + 2E$ (*iii*) $A - 2B$ (*iv*) $D - F$ (*v*) $A + B + C$ (*vi*) $F + 2E$ (vii) $2D + 3C$ (viii) $3G$

2. $A=\begin{pmatrix}-1 & 2\\ 1 & -3\end{pmatrix}$ $B=\begin{pmatrix}4 & -5\\ 2 & 1\end{pmatrix}$ $C=\begin{pmatrix}-1 & 2\\ 4 & 1\end{pmatrix}$

Verify that $A + (B + C) = (A + B) + C = (C + A) + B$

3. Find the matrices A, B and C in the following

(i) $2A-\begin{pmatrix}3 & -2\\ 4 & 5\end{pmatrix}=\begin{pmatrix}-5 & 8\\ -2 & 5\end{pmatrix}$ (ii) $\begin{pmatrix}2 & -1\\ 1 & 3\end{pmatrix}+2B=\begin{pmatrix}5 & 4\\ 1 & 2\end{pmatrix}$

(iii) $\begin{pmatrix}1 & 0 & 9\\ -3 & -2 & -5\end{pmatrix}-2C=C+\begin{pmatrix}-2 & -3 & 6\\ 0 & 4 & 4\end{pmatrix}$

4. Multiplication of matrices

Returning to the example of marks it may be decided that the marks in the different modules should not all carry that same weight. For instance marks for modules 1, 2, 3 might be weighted 3, 2, 3. We can represent Andy's marks and the weightings by the following matrices

$$(40\ \ 58\ \ 83) \text{ and } (3\ \ 2\ \ 3)$$

and the total mark would be

$$40 \times 3 + 58 \times 2 + 83 \times 3 = 485$$

This is matrix multiplication which is usually written as

$$\begin{pmatrix}40 & 58 & 83\end{pmatrix}\begin{pmatrix}3\\ 2\\ 3\end{pmatrix}=\begin{pmatrix}40\times 3+58\times 2+83\times 3\end{pmatrix}=\begin{pmatrix}485\end{pmatrix}$$

For Mary the total would be given by

$$\begin{pmatrix}32 & 43 & 73\end{pmatrix}\begin{pmatrix}3\\2\\3\end{pmatrix} = \begin{pmatrix}32\times3+43\times2+73\times3\end{pmatrix} = \begin{pmatrix}401\end{pmatrix}$$

The two sets of results can be combined as

$$\begin{pmatrix}40 & 58 & 83\\32 & 43 & 73\end{pmatrix}\begin{pmatrix}3\\2\\3\end{pmatrix} = \begin{pmatrix}40\times3+58\times2+83\times3\\32\times3+43\times2+73\times3\end{pmatrix} = \begin{pmatrix}485\\401\end{pmatrix}$$

and the whole group (of four students) as

$$\begin{pmatrix}40 & 58 & 83\\32 & 43 & 73\\56 & 65 & 66\\21 & 45 & 34\end{pmatrix}\begin{pmatrix}3\\2\\3\end{pmatrix} = \begin{pmatrix}40\times3+58\times2+83\times3\\32\times3+43\times2+73\times3\\56\times3+65\times2+66\times3\\21\times3+45\times2+34\times3\end{pmatrix} = \begin{pmatrix}485\\401\\496\\255\end{pmatrix}$$

Someone suggested that the weighting should be (2 3 2);

Show that this gives $\begin{pmatrix}420\\339\\439\\245\end{pmatrix}$

both 'weightings' can be put together, which enables simple comparisons to be made:-

$$\begin{pmatrix}40 & 58 & 83\\32 & 43 & 73\\56 & 65 & 66\\21 & 45 & 34\end{pmatrix}\begin{pmatrix}3 & 2\\2 & 3\\3 & 3\end{pmatrix} = \begin{pmatrix}485 & 420\\401 & 339\\496 & 439\\255 & 245\end{pmatrix}$$

Examples

Given $A=\begin{pmatrix}3 & 0\\-1 & 4\end{pmatrix}$ $B=\begin{pmatrix}2\\-3\end{pmatrix}$ $C=\begin{pmatrix}2 & -3\\5 & 2\end{pmatrix}$ $D=\begin{pmatrix}3\\-2\\0\end{pmatrix}$

Evaluate, where possible **(*i*)** *AB* **(*ii*)** *AC* **(*iii*)** *CA* **(*iv*)** *DA*

(i) $AB=\begin{pmatrix}3 & 0\\-1 & 4\end{pmatrix}\begin{pmatrix}2\\-3\end{pmatrix}=\begin{pmatrix}3\times2+0\times(-3)\\(-1)\times2+4\times(-3)\end{pmatrix}=\begin{pmatrix}6\\-14\end{pmatrix}$

(ii) $AC=\begin{pmatrix}3 & 0\\-1 & 4\end{pmatrix}\begin{pmatrix}2 & -3\\5 & 2\end{pmatrix}=\begin{pmatrix}3\times2+0\times5 & 3\times(-3)+0\times2\\(-1)\times2+4\times5 & (-1)\times(-3)+4\times2\end{pmatrix}=\begin{pmatrix}6 & -9\\18 & 11\end{pmatrix}$

(iii) $CA=\begin{pmatrix}2 & -3\\5 & 2\end{pmatrix}\begin{pmatrix}3 & 0\\-1 & 4\end{pmatrix}=\begin{pmatrix}2\times3+(-3)\times(-1) & 2\times0+(-3)\times4\\5\times3+2\times(-1) & 5\times0+2\times4\end{pmatrix}=\begin{pmatrix}9 & -12\\13 & 8\end{pmatrix}$

(iv) $=\begin{pmatrix}3\\-2\\0\end{pmatrix}\begin{pmatrix}3 & 0\\-1 & 4\end{pmatrix}$ – impossible

From this we can see that to multiply two matrices, the right hand matrix has to have the same number of columns as the left hand has rows. We also notice that from above *A* is a (2 × 2) matrix and *B* is a (2 × 1) matrix, and the product *AB* is a (2 × 1) matrix and *BA* does not exist. Also if *M* is a 3 × 2 matrix and *N* is a 2 × 1 matrix then *MN* exists and is a 3 × 1 matrix. In general if *M* is a *p* × *q* matrix and N is a *q* × *r* matrix then MN exists and will be an *p* × *r* matrix.

Note also if we have matrices *M* and *N* and *MN* exists then the element in, say, the 3rd row and 2nd column of *MN* comes from multiplying the 3rd row of *M* with the 2nd column of *N*.

Exercise 2

Find the following products where they exist

1. $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}-1\\3\end{pmatrix}$
2. $\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}0 & 3\\2 & -2\end{pmatrix}$
3. $\begin{pmatrix}-1 & -3\\2 & 0\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix}$
4. $\begin{pmatrix}1 & -2 & 2\end{pmatrix}\begin{pmatrix}2 & 4 & -2\\0 & -5 & 3\end{pmatrix}$
5. $\begin{pmatrix}-3 & -1\\-1 & -2\end{pmatrix}\begin{pmatrix}2 & 2\\3 & 0\\4 & -4\end{pmatrix}$
6. $\begin{pmatrix}2 & 1 & 3\\1 & 4 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 3\\-2 & -3\end{pmatrix}$
7. $\begin{pmatrix}1 & 0\\0 & 3\\-2 & -3\end{pmatrix}\begin{pmatrix}2 & 1 & 3\\1 & 4 & 1\end{pmatrix}$
8. $\begin{pmatrix}3\\-2\\4\end{pmatrix}\begin{pmatrix}2 & 0 & -3 & 5\end{pmatrix}$

9. Given the following matrices

$$A=\begin{pmatrix}2 & -1\\ -2 & 2\end{pmatrix}\quad B=\begin{pmatrix}3 & -1\\ -1 & 3\end{pmatrix}\quad C=\begin{pmatrix}3 & 2 & -1\\ 4 & 0 & 1\end{pmatrix}\quad D=\begin{pmatrix}4 & 1\\ -3 & 2\\ 4 & 5\end{pmatrix}$$

$$E=\begin{pmatrix}5 & 4 & -1\\ 2 & 3 & 4\end{pmatrix}\quad F=\begin{pmatrix}5 & 2\\ -2 & -4\\ -1 & -5\end{pmatrix}\quad G=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & -2\\ 2 & -1 & 3\end{pmatrix}\quad H=(5)$$

Find, where possible

$(i)\ AB$ $(ii)\ BA$ $(iii)\ BC$ $(iv)\ FB$ $(v)\ BF$ $(vi)\ EG$
$(vii)\ GB$ $(viii)\ GD$ $(ix)\ EH$ $(x)\ CF$ $(xi)\ DE$

10. Given $A=\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}$ $B=\begin{pmatrix}5 & -2\\ -2 & 1\end{pmatrix}$ $C=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

find $(i)\ AB$ $(ii)\ BA$ $(iii)\ AC$ $(iv)\ CB$ $(v)\ BC$

5. Identity and Inverse matrices

The matrix I such that $MI = M = MI$ is called the **identity matrix.** (see question 9 above) For MI and IM to be equal then both M and I must be square matrices.

The 2 × 2 identity matrix is $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ because

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \text{and} \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

The 3 × 3 identity matrix is $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

Matrices such as A and B in question 9 above are called **inverse matrices.**

We see that $AB = BA = I$ so $A = B^{-1}$ and $B = A^{-1}$

Finding inverse 2 × 2 matrices:

Given $M=\begin{pmatrix}3 & 1\\ -1 & 1\end{pmatrix}$ Let $M^{-1}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

Then $M \times M^{-1}=\begin{pmatrix}3 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

hence $\begin{pmatrix}3a+c & 3b+d\\ -a+c & -b+d\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

If two matrices are equal then corresponding elements are equal hence

$$3a+c=1 \quad 3b+d=0$$
$$-a+c=0 \quad -b+d=1$$

solving gives $a=\frac{1}{4},\quad b=-\frac{1}{4}\quad c=\frac{1}{4}\quad d=\frac{3}{4}$

hence $M^{-1}=\begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ \frac{1}{4} & \frac{3}{4}\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix}$

Similarly if $N=\begin{pmatrix}5 & 3\\ -6 & -2\end{pmatrix}$ then it can be shown that $N^{-1}=\frac{1}{8}\begin{pmatrix}-2 & -3\\ 6 & 5\end{pmatrix}$ in both cases, if you compare M with M^{-1} and N with N^{-1} you will notice that you appear to be able to get from one to the other by swapping the elements on the leading diagonal and changing the sign of the other two. But what about the fraction?

The 4 and 8 above are called the determinant of the matrix, possibly because it determines if the matrix has an inverse or not. If the determinant of matrix A is zero then, as we cannot divide by zero, A^{-1} does not exist.

In the first example we have $M=\begin{pmatrix}3 & 1\\ -1 & 1\end{pmatrix}$ and [3 × 1] - [1 × (-1)] = 4

In the second example $N=\begin{pmatrix}5 & 3\\ -6 & -2\end{pmatrix}$ and [5 × (-2)] - [3 × (-6)] = 8

If $P=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ the determinant of P, written as |P| or $\det P=\begin{vmatrix}a & b\\ c & d\end{vmatrix}$ is $ad-bc$

and $P^{-1} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

Check by multiplication that that $P \times P^{-1} = I$

Note that the inverse only exists if $ad - bc \neq 0$

Exercise 3

Find the inverses of the following matrices where they exist.

$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 3 & 2 \\ -2 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 2 & -5 \\ -6 & 15 \end{pmatrix} \quad E = \begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}$

6. Inverse of a 3 × 3 matrix

To find the inverse of a 3 × 3 matrix requires a different technique.

Given $M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

The determinant of M, written as $|M|$ is defined as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix}$$

We also define co-factors. A co-factor is the determinant of the matrix left when row m and column n are deleted from the matrix M and with the appropriate sign. The sign is $(-1)^{m+n}$. This is easier to deal with than it sounds as the signs alternate along rows and columns as shown

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

This gives

$A_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad A_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \quad A_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \quad A_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$ etc.

For $M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad M^{-1} = \frac{1}{|M|}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$

where A_{mn} is the co-factor of a_{mn}

Note first <u>row</u> of M is a_{11} a_{12} a_{13} - first <u>column</u> of M^{-1} is A_{11} A_{12} A_{13}
The rows and columns are transposed.

***Example*:** Find the inverse of $M = \begin{pmatrix} 1 & 4 & -1 \\ 0 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

The minors are

$A_{11} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5, \quad A_{12} = -\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} = 3, \quad A_{13} = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -2$

$A_{21} = -\begin{vmatrix} 4 & -1 \\ 3 & 2 \end{vmatrix} = -11, \quad A_{22} = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 3, \quad A_{23} = -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1$

$A_{31} = \begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix} = 14, \quad A_{32} = -\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} = -3, \quad A_{33} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2$

$$|M| = \begin{vmatrix} 1 & 4 & -1 \\ 0 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} = 1(4-9) - 4(0-3) - 1(0-2) = 9$$

Hence $M^{-1} = \frac{1}{9}\begin{pmatrix} -5 & -11 & 14 \\ 3 & 3 & -3 \\ -2 & 1 & 2 \end{pmatrix}$

Check $MM^{-1} = \begin{pmatrix} 1 & 4 & -1 \\ 0 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \times \frac{1}{9}\begin{pmatrix} -5 & -11 & 14 \\ 3 & 3 & -3 \\ -2 & 1 & 2 \end{pmatrix}$

$$= \frac{1}{9}\begin{pmatrix} -5+12+2 & -11+12-1 & 14-12-2 \\ 0-6+6 & 0+6+3 & 0-6+6 \\ -5+9-4 & -11+9+2 & 14-9+4 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Exercise 4

Find the inverses of the following

$$A = \begin{pmatrix} -5 & 10 & 8 \\ 4 & -7 & -6 \\ -3 & 6 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & -3 \end{pmatrix} \quad C = \begin{pmatrix} 1 & -2 & -1 \\ -3 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} \quad D = \begin{pmatrix} -1 & -3 & 2 \\ 0 & -2 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$