# Utility

Tests and observations indicate that:

most people are **unwilling** to pay as much as the expected monetary value

for an uncertain investment.

They are normally ***averse to risk***, and

frequently prefer a sure return of lower monetary value

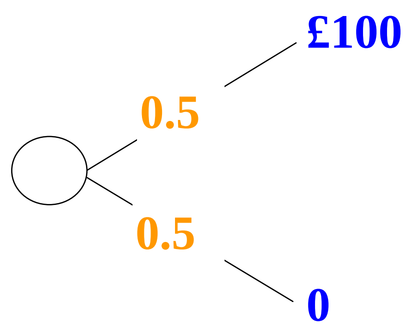
to an uncertain return of higher expected monetary value.

Risk aversion can best be perceived by putting it in a personal context:

Suppose that you were presented with an opportunity

to invest money in a 50:50 proposition

to earn £100 overnight.




The probabilities are equal that you will earn £100 or nothing,

So the **expected value** of the investment is **£50**.

Consequently any investment (payment for the opportunity to play the game) below £50 yields an expected profit.

**What is the most that you would be willing to invest**

**in order to obtain this opportunity?**

In fact, average bids are around **£25**.

As Bernoulli suggested,

***"The pertinent variable to be averaged ...***

***is not the actual monetary worth of the outcomes,***

***but rather the intrinsic worth of their monetary values".***

The intrinsic value or satisfaction

derived from gaining different amounts of money

is measured by an individual's ***utility function***.

Decision makers generally select alternatives according to **expected utility**.

It is precisely because people's utility functions are non-linear that they exhibit

***risk preference***, or,

more commonly, ***risk aversion***.

The utility of an outcome depends upon the decision maker himself.

Most people are willing to gamble small amounts such as 10p

to get a 50:50 chance at a 20p prize.

However, they are usually **not** prepared to make an expected value bid

when the stake is higher

even though the expected value of the return is correspondingly higher

But, in certain circumstances,

some people have utility functions

with increasing returns along part of their range.

They are prepared to act as if

a large prize had a utility

more than R times as large as a prize 1/R times the size.

To them the larger amount, the bonanza, opens up a whole new way of life.

Those who "invest" on the National Lottery are an example of people who exhibit this **risk preference**.

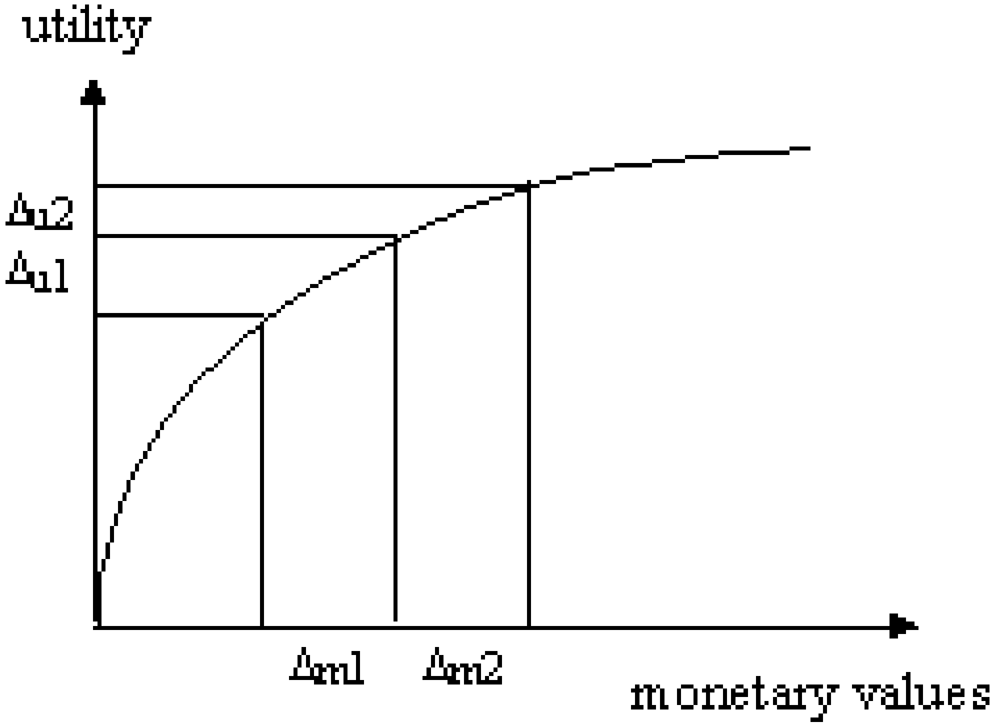
For practical decision-making about large scale problems it is useful to act as if the utility function has decreasing returns.

***Bernoulli's Law of Diminishing Marginal Utility*** describes his observation of such risk aversion:

***any increase in wealth, no matter how insignificant,***

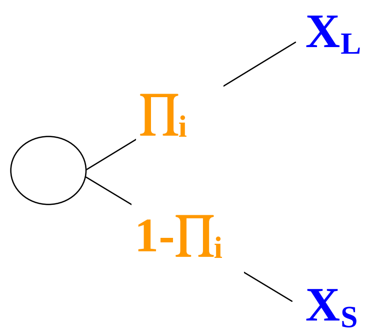
***will always result in an increase in utility***

***which is inversely proportionate to the quantity of goods already possessed.***



A ***standard alternative*** or ***$\Pi_i$ lottery*** has two outcomes,

- a larger one, $X_L$, with probability $\Pi_i$
- a smaller one, $X_S$, with probability 1- $\Pi_i$




In the above game we had a 0.5 lottery for £100 and £0.

The ***certainty monetary equivalent CME*** is

*the maximum amount a person would be willing to pay for a $\Pi_i$ lottery*,

or, equivalently,

*the minimum amount a person owning the opportunity*

*would be willing to sell it for*.

The certainty monetary equivalent is

only rarely identical to the expected monetary value of an opportunity

generally, less than the EMV, since people are risk averse.

In the above game

the certainty monetary equivalent was about £25, even though

the expected monetary value was £50.

## Constructing a Decision-Maker's Utility Function

There are a variety of closely related techniques

for obtaining the utility of outcomes of choice in a probabilistic situation.

The methods generate

a unique number for each possible outcome which can be

used to order all choices according to their desirability to the decision maker.

This ultimate ranking number is derived from a combination of

the measurable rewards,

the probability of the occurrence of each outcome, and

the decision-maker's regard for risk.

To construct a utility function for probabilistic situations

one first selects the reference rewards $X_L$ and $X_S$

that bound the rewards associated with all other alternatives of interest.

For a lottery with prizes $X_L$ and $X_S$ and $\Pi_i = 1$, the CME is $X_L$ .

The monetary sum $X_L$ is assigned the utility value 1,

equal to the probability of the lottery.

For similar reasons, the monetary sum $X_S$ is assigned the utility value 0.

The two points ($X_L$, 1) and ($X_S$, 0) are taken to be

the end points of the decision maker's utility curve.

To obtain intermediate points on the utility curve,

the decision-maker is confronted with a series of $\Pi_i$ lotteries and asked to define his CMEs for each such lottery.

For each $\Pi_i$ lottery

the utility of $CME_i$ is set to the value of probability $\Pi_i$ .

In other words, each point ($CME_i$ , $\Pi_i$ ) is on the utility curve.

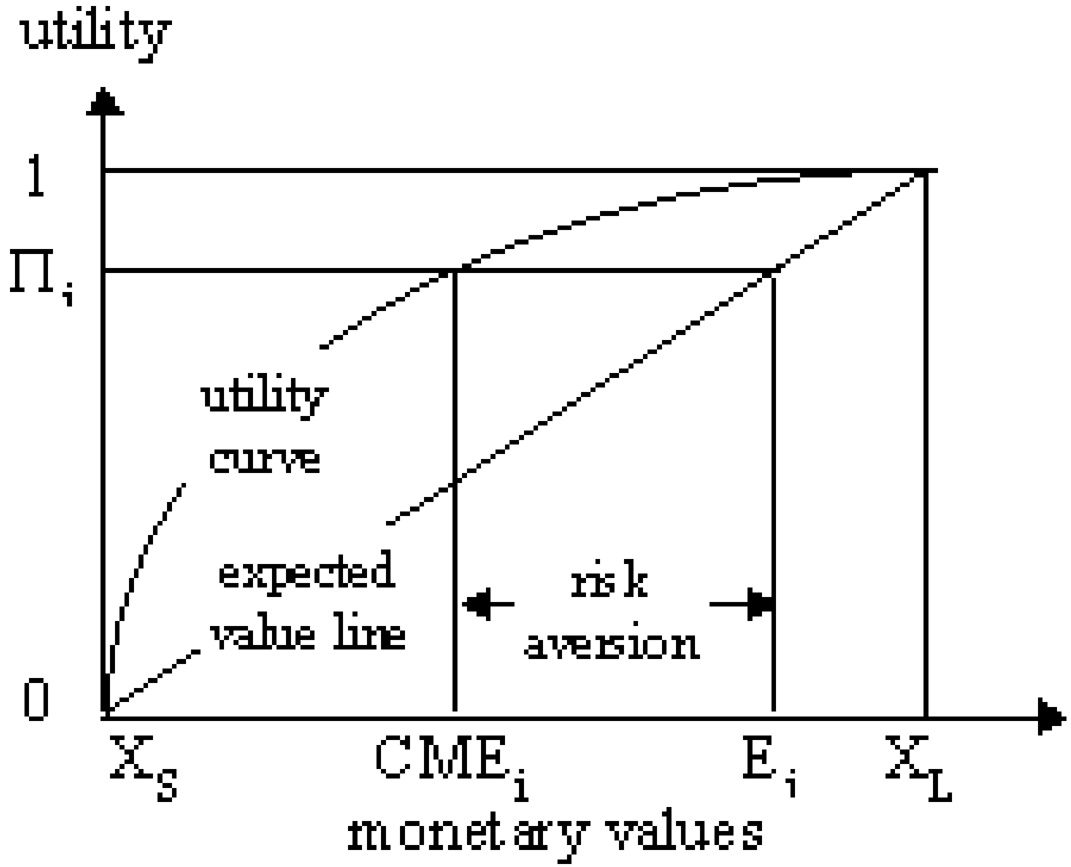
The expected monetary value of any $\Pi_i$ lottery is:

$$\mathbf{E_i = \Pi_i * X_L + (1 - \Pi_i) * X_S} .$$

Each point ($E_i$ , $\Pi_i$ ) lies on the straight line between ($X_L$ , 1) and ($X_S$ , 0).

The extent of risk aversion is measured by the

difference between $CME_i$ and $E_i$ for any $\Pi_i$ lottery or probabilistic situation.



May be hard for a decision-maker

to distinguish his feelings about slightly different probabilistic situations

(involving, say, a probability of 0.75 as against a probability of 0.8).

So it may be more generally useful

to define lotteries in terms of fixed probabilities which

can be visualized readily, and

to vary the prizes instead.

Here is an example of an operational method that

uses only conceptually simple 50:50 gambles,

in a situation where $X_L$ is £80,000 and $X_S$ is -£30,000.

The analyst initially sets

utility(£80,000) = 1, and

utility(-£30,000) = 0.

*ANALYST: What certain outcome would you consider equivalent to a 50:50 gamble on outcomes of £80,000 and -£30,000?*

*DECISION-MAKER: £10,000*

Since the expected utility of the above gamble is

$0.5 \times 1 + 0.5 \times 0 = 0.5$

and the decision-maker is indifferent between

this gamble and

£10,000 for certain,

we deduce that

utility(£10,000) = 0.5 .

*ANALYST:* *What certain outcome would you consider equivalent to a 50:50 gamble on outcomes of £80,000 and £10,000?*

*DECISION-MAKER:* *£25,000*

Since the expected utility of the above gamble is

$$0.5 \times 1 + 0.5 \times 0.5 = 0.75$$

and the decision-maker is indifferent between

this gamble and

£25,000 for certain,

we deduce that

utility(£25,000) = 0.75 .

*ANALYST: What certain outcome would you consider equivalent to a 50:50 gamble on outcomes of £10,000 and -£30,000?*
*etc.*

In this way,

the interval between the best and worst outcomes is bisected repeatedly.

Soon the questioner will have enough information to

plot the decision-maker's utility function.

A freehand curve can then be drawn

to illustrate the decision-maker's utility curve.

## Expected Utility Value Criterion

Many people exhibit decision making behaviour generally in line with the selection of alternatives according to expected utility

**Under a very reasonable set of assumptions, rational decision making should be based on the principle of expected utility.**

One should use the same method as described under the EMV criterion, but use utility values instead of monetary payoffs.

## Assumptions of the utility function

The concept of the utility function and rational choice

rests on fundamental axioms or assumptions.

These utility axioms are

intuitively appealing insofar as they set forth the kind of conditions which

common sense indicates should surround a rational decision process.

These six axioms are

sufficient to define a utility function in a probabilistic context.

Unless one is prepared to define rational decision-making in terms of a contrary set of assumptions, one is forced to accept the principle that

decisions should be based on the principle of maximizing expected utility.

## *Axiom 1.*

A decision-maker can compare and make consistent choices between alternatives.

Faced with a pair of alternatives A and B,

he will always prefer one to the other or be indifferent between them.

## *Axiom 2.*

A decision-maker's choices are transitive.

If he prefers A to B and B to C,

then he will prefer A to C.

If he is indifferent between A and B and indifferent between B and C,

then he will be indifferent between A and C.

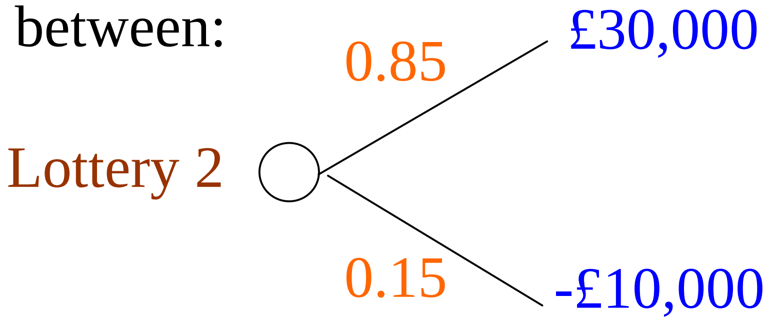
***Axiom 3.***

If two choices are indifferent to the decision-maker,

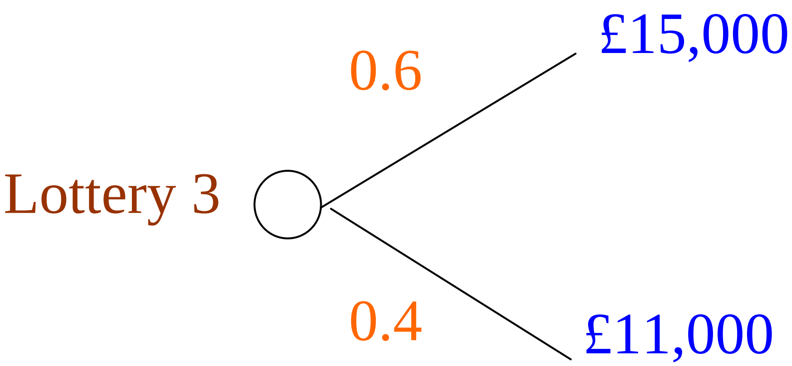
they can be substituted for each other.

For example, if the decision-maker is indifferent between:

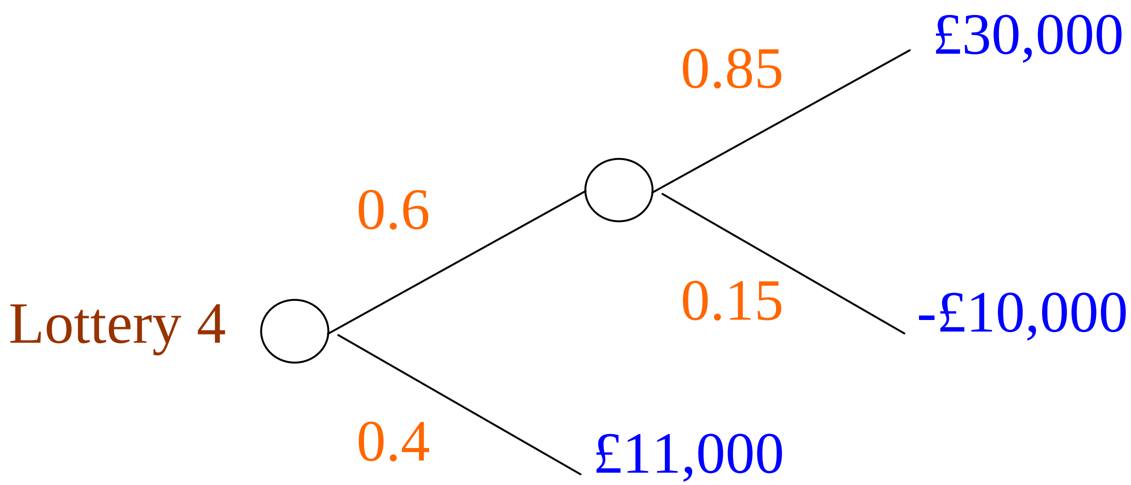

and




then he will also be indifferent between:




and

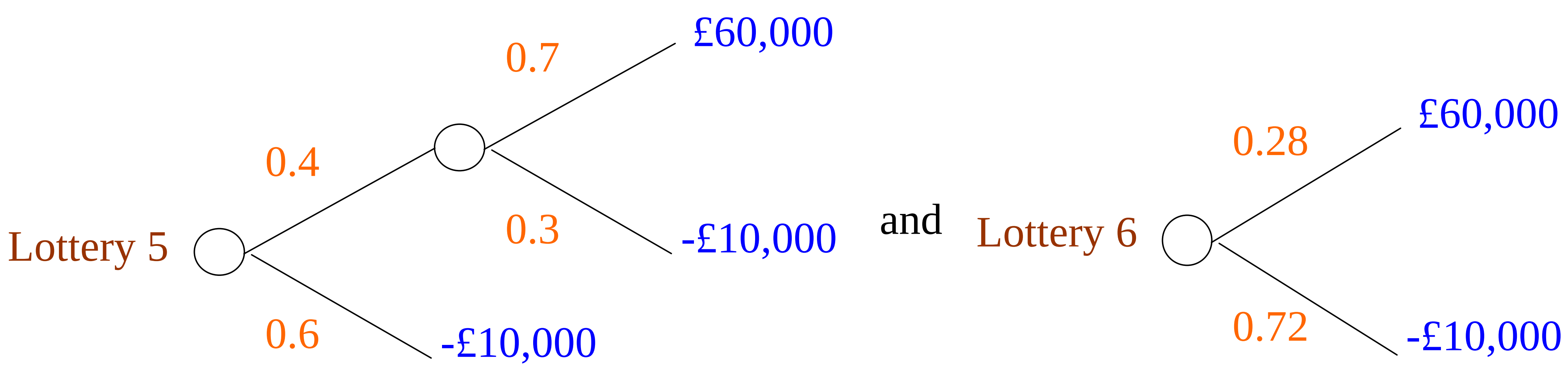


***Axiom 4.***

A decision-maker will be indifferent between

- a compound lottery and
- a simple lottery which offers the same rewards with the same probabilities.

For example, the decision-maker will be indifferent between:



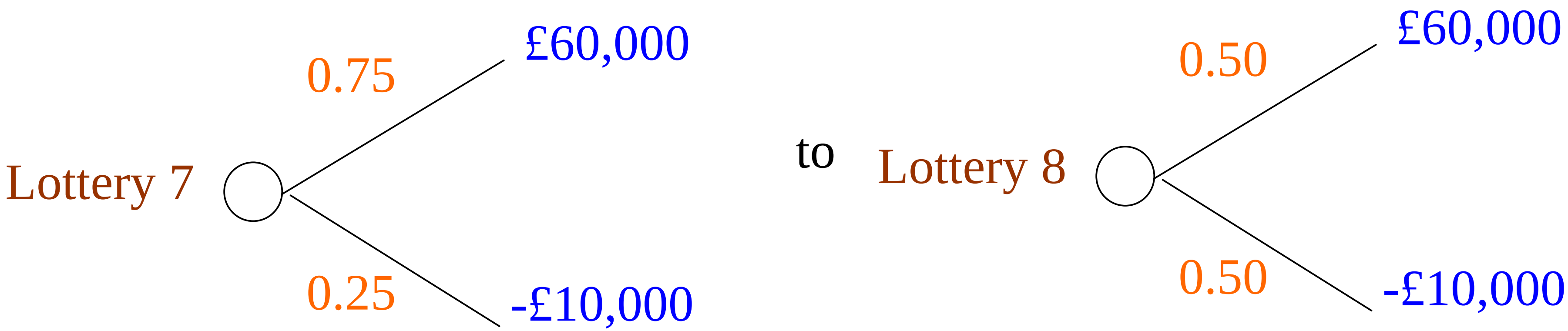
## *Axiom 5.*

If two lotteries A and B lead to the same outcomes, C and D,

the decision-maker will choose the lottery in which

the preferred outcome C has the greater probability.

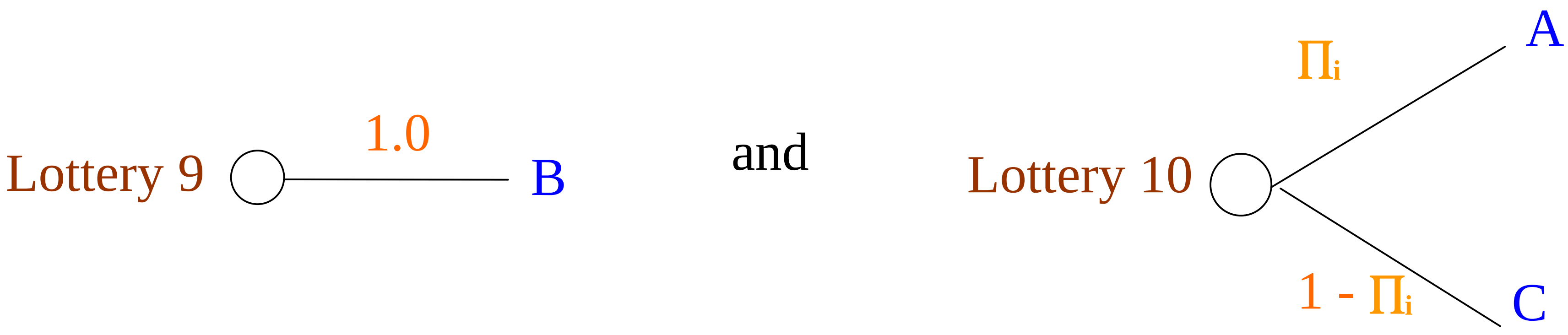
For example, the decision-maker will prefer:



## *Axiom 6.*

If the decision-maker prefers A to B and B to C, then

there is a $\Pi_i$ lottery with prizes A and C which is indifferent to B:



## Example

A business woman who is organizing an exhibition in a provincial town has to choose between two venues: the Luxuria Hotel and the Maxima Centre.

To simplify her problem, she decides to estimate her potential profit at these locations on the basis of two scenarios: high attendance and low attendance at the exhibition.

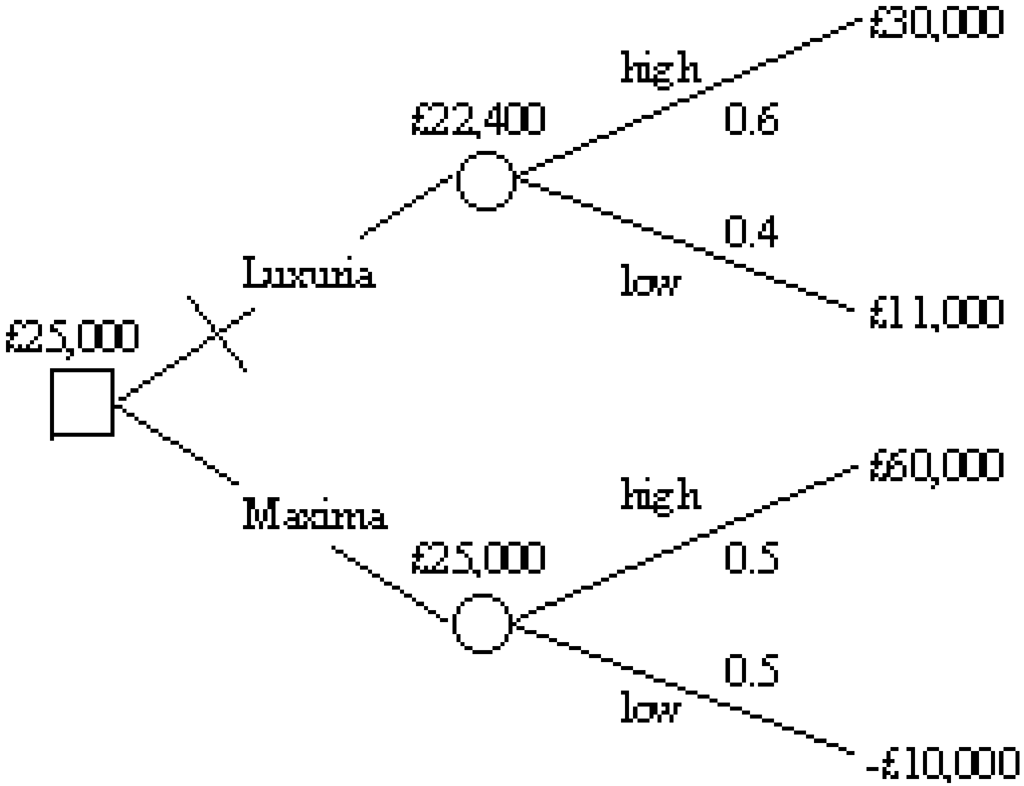
If she chooses the Luxuria Hotel, she reckons that she has a 60% chance of achieving a high attendance and hence a profit of £30,000 (after taking into account the costs of advertising, hiring the venue, etc.).

There is, however, a 40% chance that attendance will be low, in which case her profit will be just £11,000.

If she chooses the Maxima Centre, she reckons she has a 50% chance of high attendance, leading to a profit of £60,000, and a 50% chance of low attendance leading to a loss of £10,000.

The business woman's expected profit is £22,400 if she chooses the Luxuria Hotel and £25,000 if she chooses the Maxima Centre.

By the EMV criterion she should choose the Maxima Centre, but this is the riskier option, offering high rewards if things go well but losses if things go badly.



After having been asked a series of questions about her indifference between various hypothetical lotteries,

we have managed to identify the following points on her utility curve:

| **Monetary sum** | **Utility** |
|---|---|
| £60,000 | 1.0 |
| £30,000 | 0.85 |
| £11,000 | 0.60 |
| -£10,000 | 0 |

These results are now applied to the decision tree by

replacing the monetary values with their utilities.

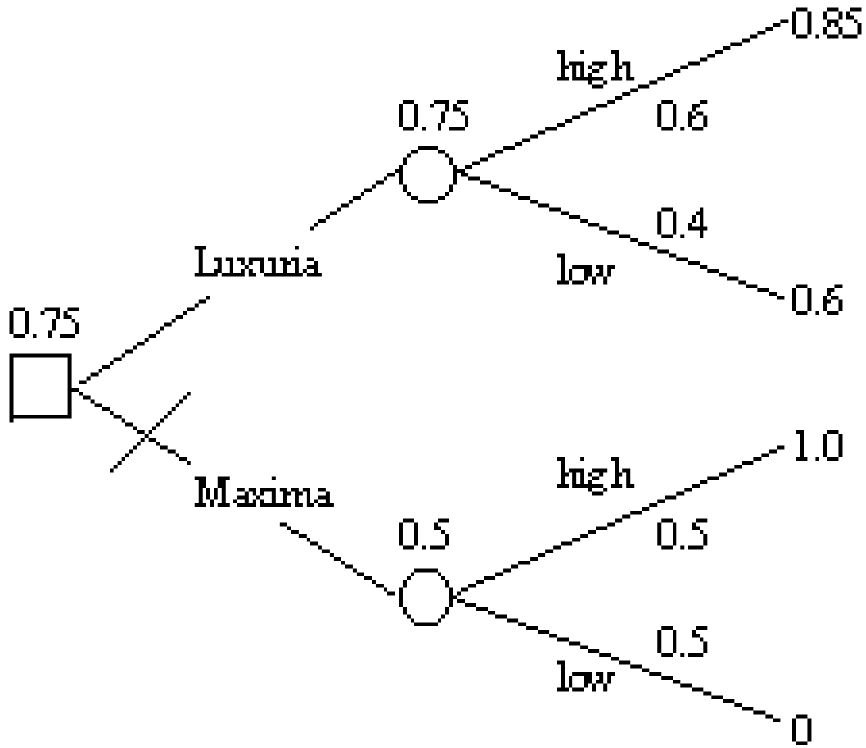
By treating these utilities in the same way as the monetary values

we are able to identify the course of action which

leads to the highest expected utility.

Choosing the Luxuria Hotel gives an expected utility of 0.75 .

the Maxima Centre gives an expected utility of 0.5 .



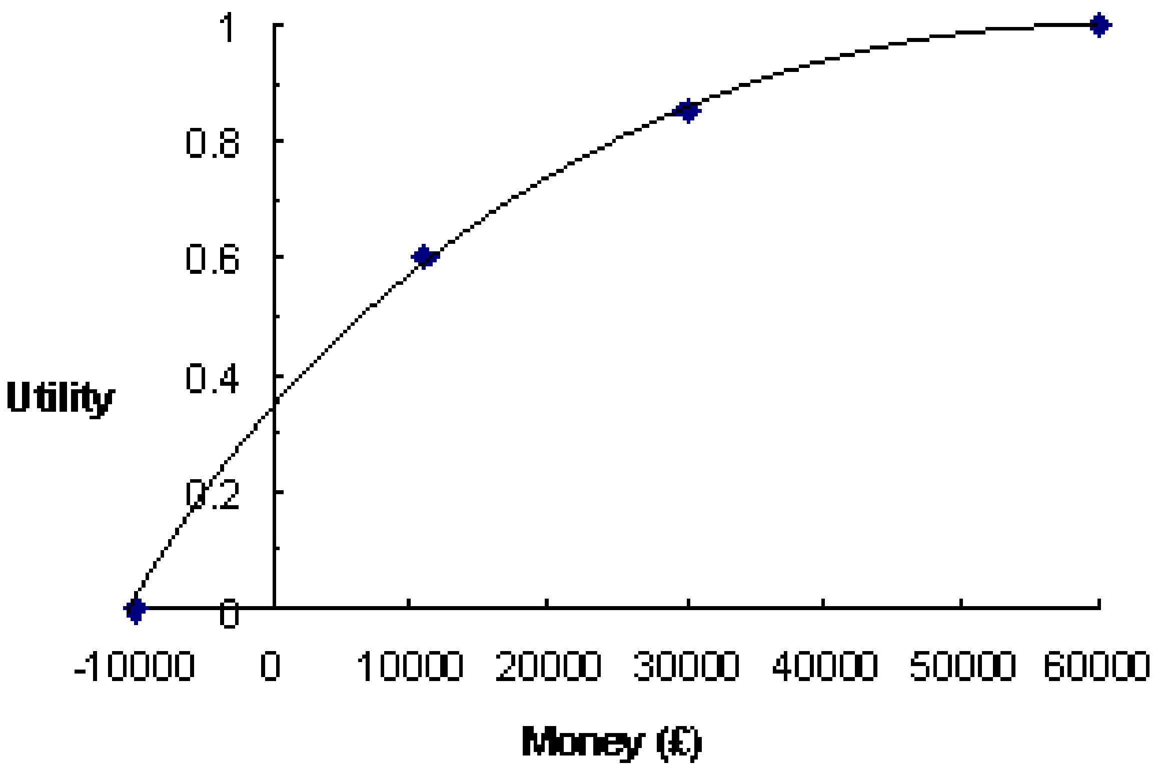

So she should choose the Luxuria Hotel as the venue for her exhibition.

The Maxima Centre is too risky.

## Various types of Utility Functions

If we plot the business woman’s utility function,

we obtain the concave curve of someone who is ***risk averse***.



Suppose that the business woman had assets of £30,000 and

were offered a gamble that would give her a

50% chance of doubling her money and a

50% chance of losing it all.

Her current assets have a utility of 0.85.

If she gambles she has a

50% chance of increasing her assets, so raising their utility to 1.0, and a

50% chance of ending with assets with a utility of about 0.35 .

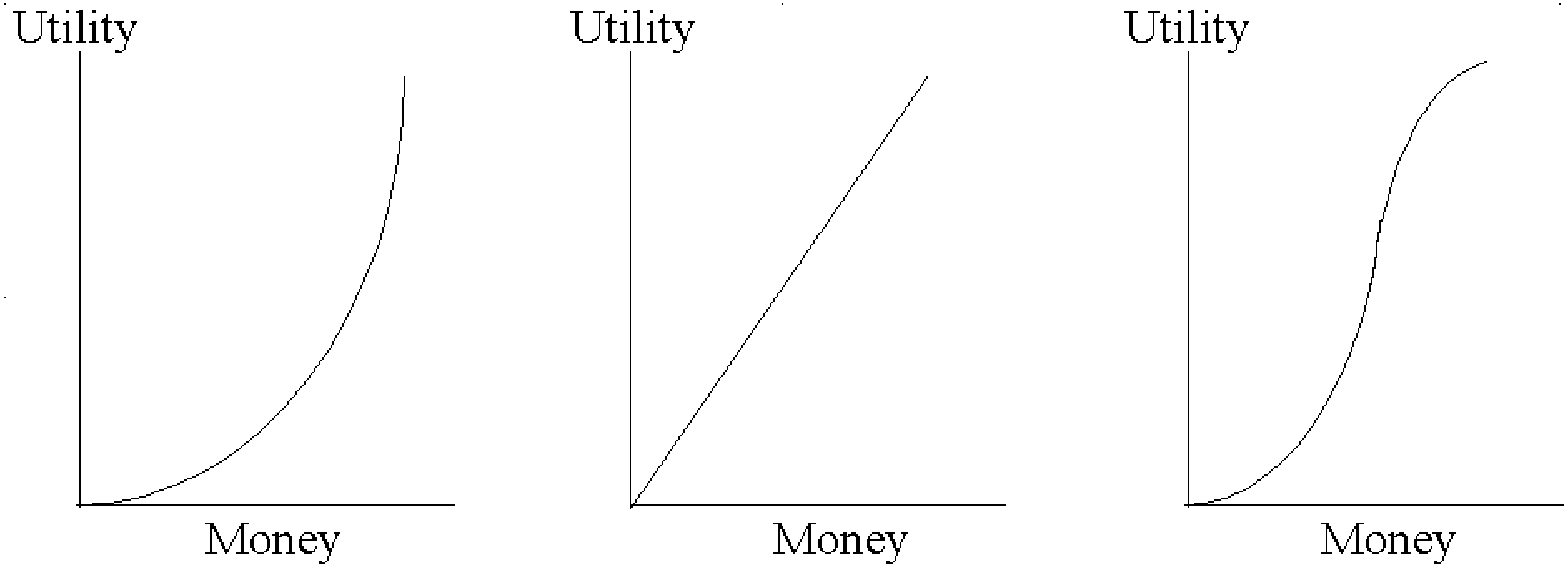
Hence the expected utility of the gamble is about 0.675, which is

less than the utility of her current assets.

The increase in utility that will occur if she wins is

far less than the loss in utility she will suffer if she loses.

Other typical utility curves are shown below:




The left utility curve indicates a ***risk-seeking*** attitude.

A risk-seeker would have accepted the above gamble.

The linear utility function in the centre demonstrates a ***risk-neutral*** attitude.

If a person's utility function looks like this then

the EMV criterion will represent their preferences.

The utility curve on the right indicates both a risk-seeking attitude and risk aversion, depending on the current level of assets.

## Utility Functions for Non-Monetary Attributes

Utility functions can be defined for attributes other than money.

## Example

A drug company is hoping to develop a new product.

If it succeeds with its existing research methods it estimates that there is a
0.4 probability that the drug will take 6 years to develop and a
0.6 probability that the development will take 4 years.

However, recently a "short-cut" method has been proposed which might lead to significant reductions in the development time

The company, which has limited resources available for research, has to decide whether to take a risk and switch completely to the proposed new method.

The head of research estimates that, if the new approach is adopted, there is a

- 0.2 probability that development will take a year, a
- 0.4 probability that it will take 2 years and a
- 0.4 probability that the approach will not work and,
  because of the time wasted,
  it will take 8 years to develop the product.

Adopting the new approach is risky, and so
we need to derive utilities for the utility times.

The worst development time is 8 years, so u(8 years) = 0.0
the best time is 1 year, so u(1 year) = 1.0

After being asked a series of questions, the head of research is able to say that

he is indifferent between

a development time of 2 years and

engaging in a lottery which will give him a

- 0.95 probability of a 1-year development and a
- 0.05 probability of an 8-year development time.

Thus:

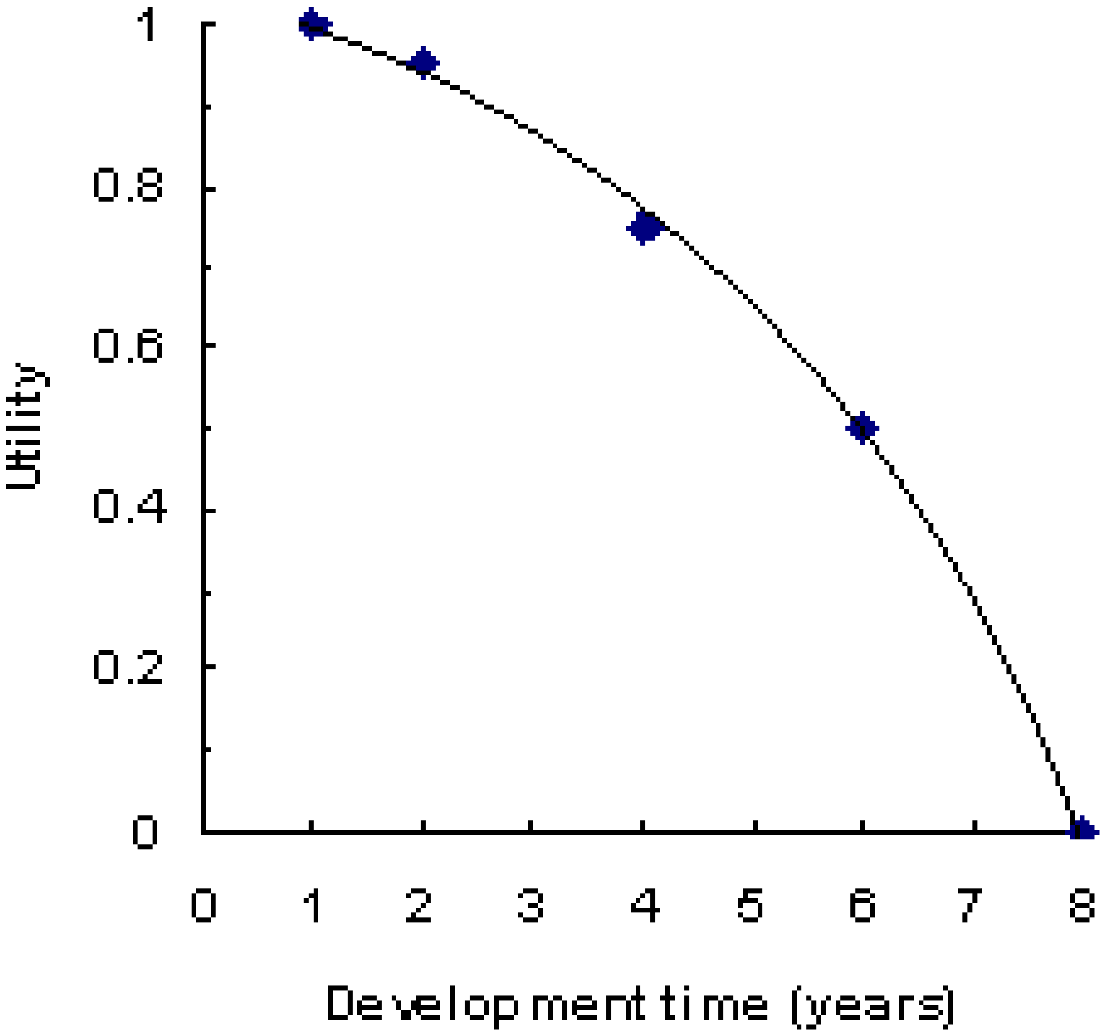
$$\begin{aligned} u(2\text{ years}) &= 0.95 \times u(1\text{ year}) + 0.05 \times u(8\text{ years}) \\ &= 0.95 \times 1.0 + 0.05 \times 0 \\ &= 0.95 \end{aligned}$$

By a similar process we find that

$u(4\text{ years}) = 0.75$ and

$u(6\text{ years}) = 0.5$

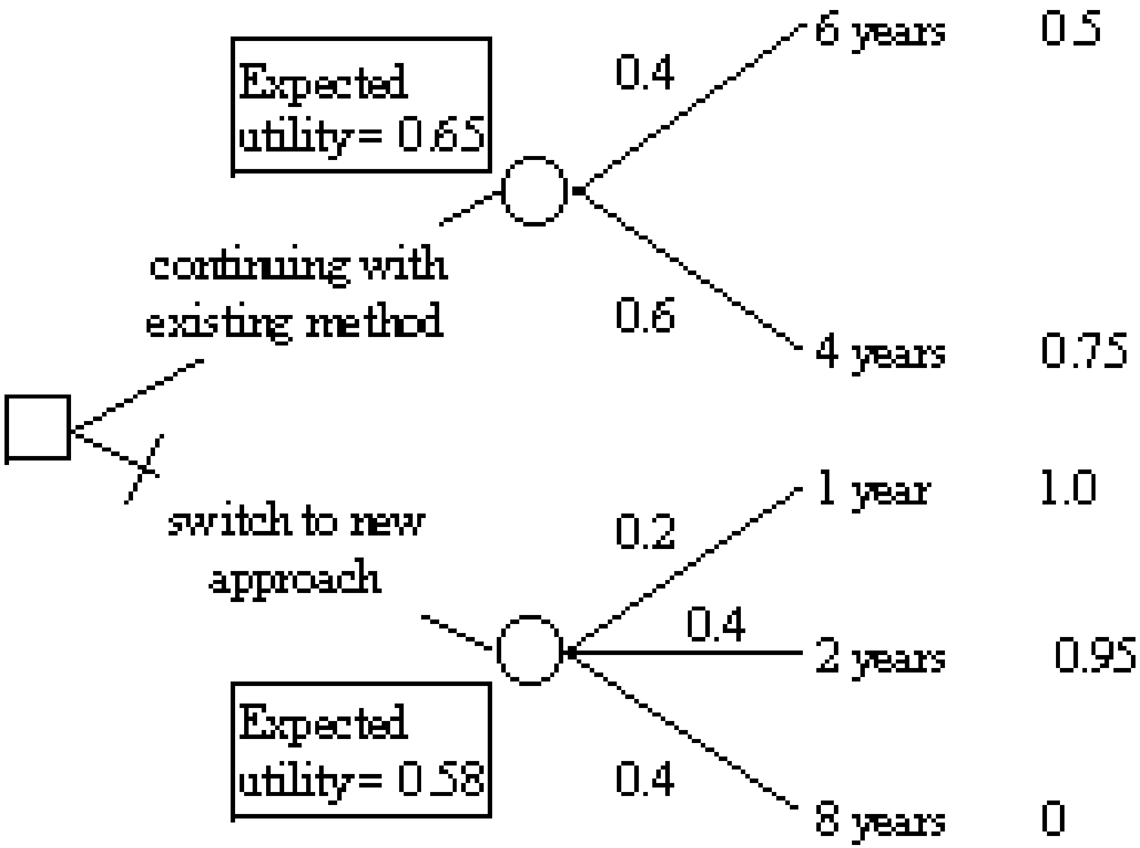
The utility function has a concave shape indicating risk aversion.



The utilities are shown in this decision tree.

It can be seen that

continuing with the existing method gives the highest expected utility.



It is also possible

to derive utility functions

for attributes which are not easily measured in numerical terms.

Consider the choice of design for a chemical plant.

Design A may have

a small probability of failure which may lead to

pollution of the local environment.

Design B may also carry

a small probability of failure which would not lead to pollution but

would cause damage to some expensive equipment.

If a decision-maker ranks the possible outcomes from best to worst as:

(i) no failure,
(ii) equipment damage, and
(iii) pollution,

then

u(no failure) = 1 and

u(pollution) = 0.

The value of u(equipment damage) could then be determined by posing questions such as:

*Which would you prefer:*

*(1) A design which was certain at some stage to fail, causing equipment damage; or*

*(2) A design which had a 90% chance of not failing and a 10% chance of failing and causing pollution?*

Once a point of indifference was established, u(equipment damage) could be derived.

A similar application in the electronics industry involves designs of electronic circuits for cardiac pacemakers.

The designs carry a risk of particular malfunctions and the utilities relate to outcomes such as

“pacemaker not functioning at all”,

“pacemaker working too fast”,

“pacemaker working too slowly” and

“pacemaker functioning OK”.

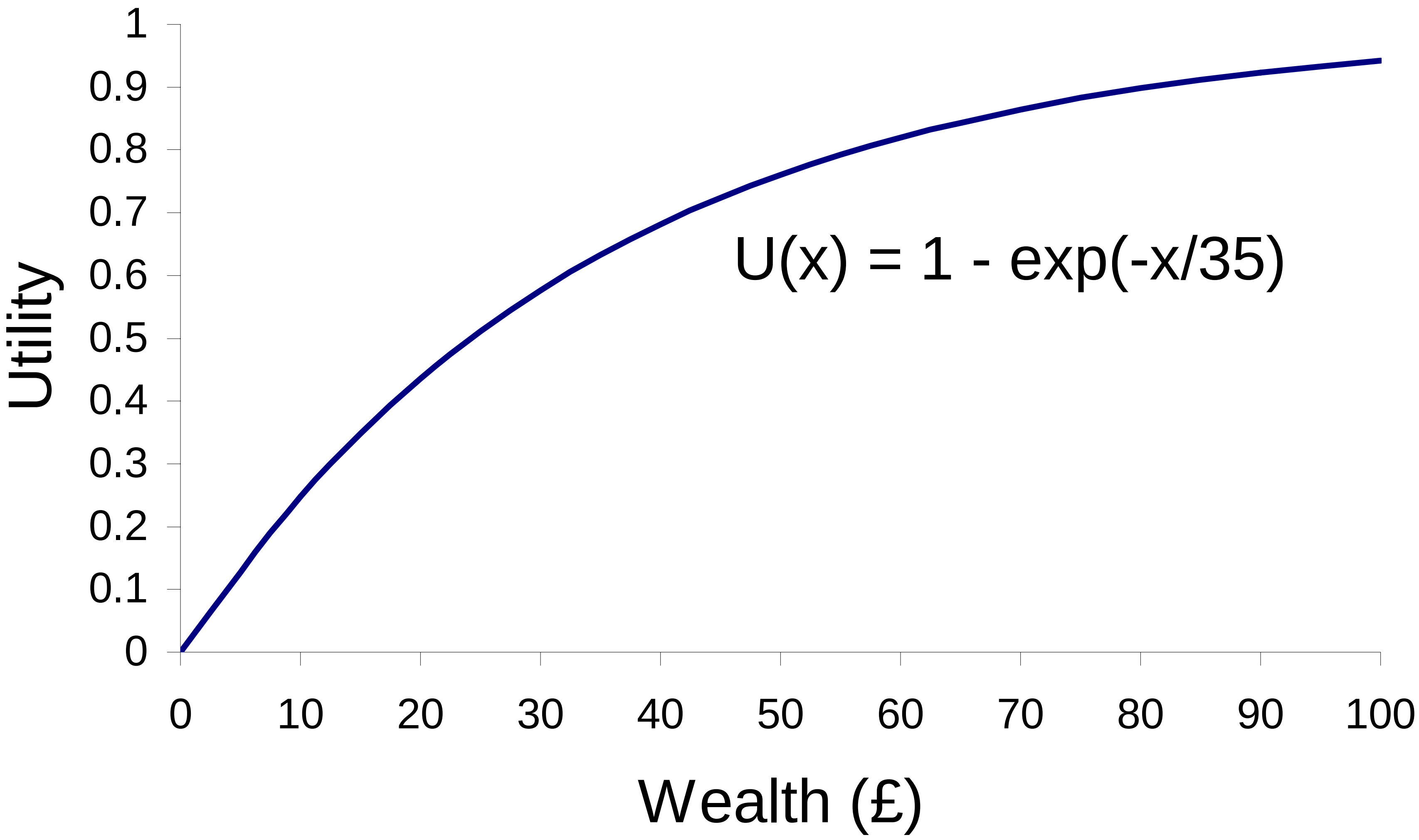
# Risk Tolerance

The process of assessing a utility function subjectively involves
asking the decision maker quite a few questions
about certainty equivalents to various lotteries,
in order to plot sufficient points on the curve to fit a smooth line.

If we are willing to assume that the utility function
follows a well-defined mathematical curve, then
all we need to do is to
estimate the parameter(s) involved, which
generally involves less work.

For example, consider the exponential utility function:

$$U(x) = 1 - e^{-x/R}$$



The function is concave and can be used to represent risk-averse behaviour.

- U(x) tends to 1 as x becomes large
- U(0) = 0
- U(x) < 0  for x < 0 (being in debt).

The parameter R (equal to £35 in the above diagram)

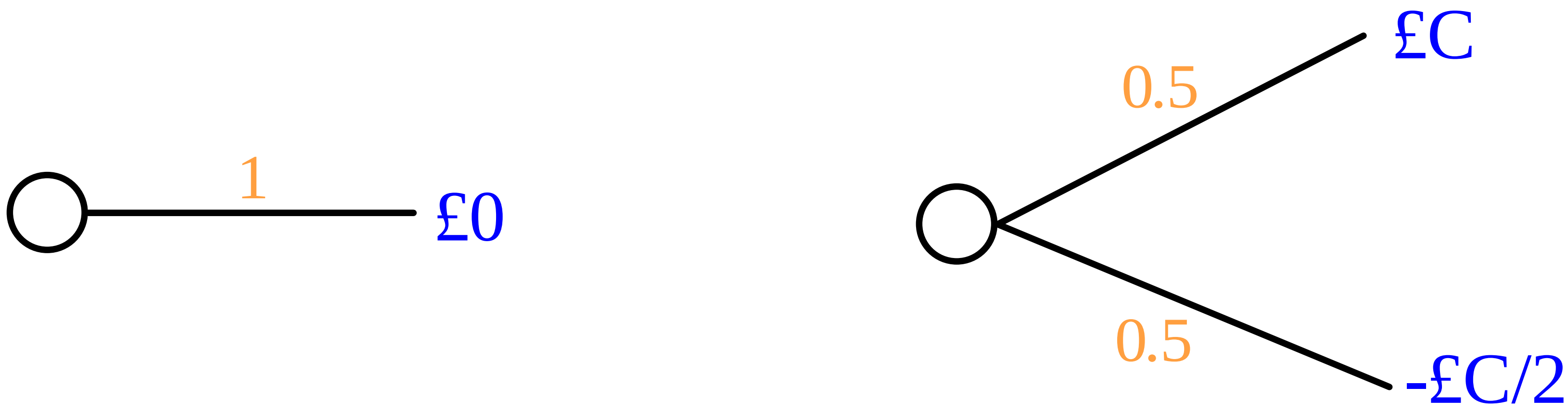
determines how risk averse a person is, and is called the **risk tolerance**.

A large value of R implies a flatter curve and high risk tolerance.

A low value implies a highly curved function and low risk tolerance.

R can be approximated relatively easily.

Ask the decision maker to consider the following choice:




For which sum of money £C is the decision maker indifferent between:

- the 50-50 gamble with prizes £C and -£C/2
- simply breaking even

This amount £C is

approximately equal to the

value of the decision maker's risk tolerance £R.

Once you know R (approximately)

you can plug the value into the exponential function and

can then calculate utility values for different amounts of money.

Once you know the expected utility of a lottery

you can find the certainty monetary equivalent by

inverting the utility function and using natural logarithms.

Within any given problem the utility function can be

- re-scaled by multiplying by a constant
- shifted by moving the origin

$$U(x) = S*(1 - e^{-(x-T)/R})$$

The exponential utility function

implies an attitude towards risk called **constant risk aversion**.

No matter how much wealth a decision maker had

a particular gamble would be viewed in the same way - in other words

the risk premium (EMV minus CME) would be the same.

The trouble is that most people's risk tolerance is not constant, but

increases (and risk aversion decreases) with the amount of money they have.

Despite this reservation,

the exponential utility function can be a useful "quick-and-dirty" method of modelling someone's utility function.

If a decision maker is facing a choice between two alternatives and the right decision, based on expected utilities is quite clear cut, then this utility function may be sufficient.

If the choice is not so clear, then a more careful assessment of the utility function may be called for.

**Increasing risk tolerance** (and so decreasing risk aversion) implies that
the more money a person has,
the less nervous they are about a particular bet and so
the smaller its risk premium.

The logarithmic utility curve:

$$U(x) = \log_e(x)$$

corresponds to risk preference that exhibits increasing risk tolerance.

This function can also be re-scaled and shifted.

Examples of (re-scaled) logarithmic and exponential utility curves:

Notice how similar they are.

It does **not** take a large change in the utility curve's shape to alter the nature of the individual's risk attitude.