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# Finding Admissible and Preferred Arguments Can be Very Hard

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## Abstract

Bondarenko *et al.* have recently proposed an extension of the argumentation-theoretic semantics of *admissible* and *preferred arguments*, originally proposed for logic programming only, to a number of other nonmonotonic reasoning formalisms. In this paper we analyse the computational complexity of *credulous* and *sceptical* reasoning under the semantics of admissible and preferred arguments for (the propositional variant of) some well-known frameworks for nonmonotonic reasoning, i.e. Theorist, Circumscription and Autoepistemic Logic. While the new semantics have been assumed to mitigate the computational problems of nonmonotonic reasoning under the standard semantics of *stable extensions*, we show that in many cases reasoning under the new semantics is computationally harder than under the standard semantics. In particular, for Autoepistemic Logic, the sceptical reasoning problem under the semantics of preferred arguments is located at the fourth level of the polynomial hierarchy, two levels above the same problem under the standard semantics. In some cases, however, reasoning under the new semantics becomes easier – reducing to reasoning in the monotonic logics underlying the nonmonotonic frameworks.

## 1 Introduction

Bondarenko *et al.* [1997] propose a single abstract framework for nonmonotonic reasoning that can be instantiated to capture many existing logics for nonmonotonic reasoning, in particular Theorist [Poole, 1988], Circumscription [McCarthy, 1980], and Autoepistemic Logic (AEL) [Moore, 1985]. They also propose two new (argumentation-theoretic) semantics for nonmonotonic reasoning, generalising the admissibility semantics [Dung, 1991] and the semantics of preferred extensions [Dung, 1991] or partial stable models [Saccà and Zaniolo, 1990] for logic programming. In this paper we refer to the new semantics as *admissible* and *preferred arguments*, respectively.

The new semantics are more general than the standard semantics of *stable extensions* for nonmonotonic reasoning, since every stable extension is a preferred (and admissible) argument, but not every preferred argument is a stable extension. Moreover, the new semantics are more liberal because for most concrete logics for nonmonotonic reasoning, admissible and preferred arguments are always guaranteed to exist, whereas stable extensions are not. Finally, reasoning under the new semantics appears to be computationally easier than reasoning under the standard semantics [Kowalski and Toni, 1996; Dung *et al.*, 1997].

However, from a complexity-theoretic point of view, it seems unlikely that the new semantics lead to better lower bounds than the standard semantics since all the "sources of complexity" one has in nonmonotonic reasoning are still present under the new semantics. There are potentially exponentially many arguments sanctioned by the semantics. Further, in order to test whether a sentence is entailed by a particular argument one has to reason in the underlying monotonic logic. For this reason, one would expect that reasoning under the new semantics has the same complexity as under the standard semantics, i.e., it is on the second level of the polynomial hierarchy for frameworks with full propositional logic as the underlying logic [Cadoli and Schaerf, 1993]. However, recent results for disjunctive logic programming [Eiter *et al.*, 1998] and default logic [Dimopoulos *et al.*, 1999] show that reasoning under the semantics of preferred extensions can be harder than under the standard semantics, i.e., it is on the third level of the polynomial hierarchy.

In this paper we extend this analysis and provide complexity results for reasoning in the *propositional variants* of Theorist Circumscription, and AEL under the new semantics. In particular, we show that, for AEL, credulous and sceptical reasoning under the admissibility semantics is on the third level of the polynomial hierarchy and sceptical reasoning under the preferability semantics is on the fourth level of the polynomial hierarchy.

The paper is organised as follows. Section 2 summarises relevant features of the abstract framework of [Bondarenko *et al.*, 1997], its semantics and concrete instances. Section 3 introduces the reasoning problems and gives generic upper bounds for credulous and sceptical reasoning, parametric wrt the complexity of the underlying monotonic logics. Section 4 gives then completeness results for Theorist and Circumscription, and Section 5 provides the completeness results for AEL. Section 6 discusses the results and concludes.

## 2 Nonmonotonic Reasoning via Argumentation

Assume a **deductive systems** $(\mathcal{L}, \mathcal{R})$, where $\mathcal{L}$ is some formal language with countably many sentences and $\mathcal{R}$ is a set of inference rules inducing a monotonic derivability notion $\vdash$. Given a theory $T \subseteq \mathcal{L}$ and a formula $\alpha \in \mathcal{L}$, $Th(T) = \{\alpha \in \mathcal{L} \mid T \vdash \alpha\}$ is the deductive closure of $T$. Then, an **abstract (assumption-based) framework** is a triple $\langle T, A, \overline{\phantom{x}} \rangle$, where $T, A \subseteq \mathcal{L}$ and $\overline{\phantom{x}}$ is a mapping from $A$ into $\mathcal{L}$. $T$, the **theory**, is a (possibly incomplete) set of beliefs, formulated in the underlying language, and can be extended by subsets of $A$, the set of **assumptions**. Indeed, an **extension** of an abstract framework $\langle T, A, \overline{\phantom{x}} \rangle$ is a theory $Th(T \cup \Delta)$, with $\Delta \subseteq A$ (sometimes an extension is referred to simply as $T \cup \Delta$ or $\Delta$). Finally, given an assumption $\alpha \in A$, $\overline{\alpha}$ denotes the **contrary** of $\alpha$.

*Theorist* [Poole, 1988] can be understood as a framework $\langle T, A, \overline{\phantom{x}} \rangle$ where $T$ and $A$ are both arbitrary sets of sentences of classical (first-order or propositional) logic and the contrary $\overline{\alpha}$ of an assumption $\alpha$ is just its negation $\neg\alpha$. $\vdash$ is ordinary classical provability.

*Circumscription* [McCarthy, 1980] can be understood similarly, except that the assumptions are negations of atomic sentences $\neg p(t)$, for all predicates $p$ which are minimised, and atomic sentences $q(t)$ or their negation $\neg q(t)$, for all predicates $q$ which are fixed.

*Autoepistemic logic* (AEL) [Moore, 1985] has as the underlying language $\mathcal{L}$ a modal logic with a modal operator $L$, but the inference rules in $\mathcal{R}$ are those of classical logic. The assumptions have the form $\neg L\alpha$ or $L\alpha$. The contrary of $\neg L\alpha$ is $\alpha$, and the contrary of $L\alpha$ is $\neg L\alpha$.

Given an abstract framework $\langle T, A, \overline{\phantom{x}} \rangle$ and an assumption set $\Delta \subseteq A$, $\Delta$ **attacks an assumption** $\alpha \in A$ iff $\overline{\alpha} \in Th(T \cup \Delta)$, and $\Delta$ **attacks an assumption set** $\Delta' \subseteq A$ iff $\Delta$ attacks some assumption $\alpha \in \Delta'$.

Given that an assumption set $\Delta \subseteq A$ is **closed** iff $\Delta = A \cap Th(T \cup \Delta)$, the standard semantics of extensions of Theorist [Poole, 1988] and stable expansions of AEL [Moore, 1985] correspond to the "stability" semantics of abstract frameworks, where an assumption set $\Delta \subseteq A$ is **stable** iff

1. $\Delta$ is closed,
2. $\Delta$ does not attack itself, and
3. $\Delta$ attacks each assumption $\alpha \notin \Delta$.

A **stable extension** is an extension $Th(T \cup \Delta)$ for some stable assumption set $\Delta$. The standard semantics of Circumscription [McCarthy, 1980] corresponds to the intersection of all stable extensions of the abstract framework corresponding to Circumscription.

Bondarenko *et al.* [1997] argue that the stability semantics is unnecessarily restrictive, because it insists that an assumption set should take a stand on every issue (assumption). Thus, they define new semantics for the abstract framework by generalising the "admissibility" semantics originally proposed for logic programming by Dung [1991]. An assumption set $\Delta \subseteq A$ is **admissible** iff

1. $\Delta$ is closed,
2. $\Delta$ does not attack itself, and
3. for all closed assumption sets $\Delta' \subseteq A$,
   if $\Delta'$ attacks $\Delta$ then $\Delta$ attacks $\Delta'$.

Maximal (wrt set inclusion) admissible assumption sets are called **preferred**. In this paper we use the following terminology: an **admissible (preferred) argument** is an extension $Th(T \cup \Delta)$ for some admissible (preferred) assumption set $\Delta$. Bondarenko *et al.* show that, in the concrete instance of the abstract framework corresponding to logic programming, preferred arguments correspond to preferred extensions [Dung, 1991] and partial stable models [Saccà and Zaniolo, 1990].

Every stable assumption set/extension is preferred (and thus admissible) [Bondarenko *et al.*, 1997, Theorem 4.6], but not vice versa, in general. However, if the framework is **normal**, i.e., if every maximal closed set that does not attack itself is a stable set, then the preferability and stability semantics coincide [Bondarenko *et al.*, 1997, Theorem 4.8]. Examples of normal frameworks are Theorist and Circumscription.

In order to illustrate the effects of the different semantics, let us consider the following AEL theory:

$$\neg Lp \rightarrow p; \qquad \neg Lq \rightarrow r.$$

This theory has no stable extension, one preferred argument $\{\neg Lq, Lr, L\neg Lq, \ldots\}$, and a number of admissible arguments, e.g., in addition to the preferred argument, the arguments $\emptyset$, $\{\neg Lq\}$, $\{\neg Lq, Lr\}$, $\{\neg Lq, L\neg Lq\}$. If we drop the sentence "$\neg Lp \rightarrow p$," we get the same admissible and preferred arguments. In addition, the preferred argument is also stable.

In the sequel, we will often use the following property [Bondarenko *et al.*, 1997, Theorem 4.8]: Every preferred assumption set is admissible and every admissible assumption set is a subset of some preferred assumption set.

## 3 Reasoning Problems and Generic Upper Bounds

We will analyse the *computational complexity* of the following reasoning problems for the *propositional variants* of the frameworks for Theorist, Circumscription and AEL under admissibility and preferability semantics:

- the **credulous reasoning problem**, i.e., the problem of deciding for any given sentence $\varphi$ whether $\varphi \in Th(T \cup \Delta)$ for *some* assumption set $\Delta$ sanctioned by the semantics;

- the **sceptical reasoning problem**, i.e., the problem of deciding for any given sentence $\varphi$ whether $\varphi \in Th(T \cup \Delta)$ for *all* assumption sets $\Delta$ sanctioned by the semantics.

The sentence $\varphi$ is any (variable-free) modal sentence in the AEL case, and any formula in propositional logic in the Theorist and Circumscription cases.

Instead of the sceptical reasoning problem, we will often consider its complementary problem, i.e.

- the **co-sceptical reasoning problem**, i.e, the problem of deciding for any given sentence $\varphi$ whether $\varphi \notin Th(T \cup \Delta)$ for *some* assumption set $\Delta$ sanctioned by the semantics.

In addition, we will consider a sub-problem of all these problems, namely:

- the **assumption set verification problem**, i.e., the problem of deciding whether a given assumption set $\Delta$ is sanctioned by the semantics.

The computational complexity [1] of the above problems for all frameworks and semantics we consider is located at the lower end of the *polynomial hierarchy*. This is an infinite hierarchy of complexity classes above NP defined by using *oracle machines*, i.e. Turing machines that are allowed to call a subroutine—the *oracle*—deciding some fixed problem in constant time. Let $\mathcal{C}$ be a class of decision problems. Then, for any class $\mathcal{X}$ defined by resource bounds, $\mathcal{X}^{\mathcal{C}}$ denotes the class of problems decidable on a Turing machine with an oracle for a problem in $\mathcal{C}$ and a resource bound given by $\mathcal{X}$. Based on these notions, the sets $\Delta_k^p$, $\Sigma_k^p$, and $\Pi_k^p$ are defined as follows:

$$\Delta_0^p = \Sigma_0^p = \Pi_0^p = \mathsf{P}$$

$$\Delta_{k+1}^p = \mathsf{P}^{\Sigma_k^p}, \quad \Sigma_{k+1}^p = \mathsf{NP}^{\Sigma_k^p}, \quad \Pi_{k+1}^p = \text{co-}\mathsf{NP}^{\Sigma_k^p}.$$

The "canonical" complete problems are SAT for $\Sigma_1^p$=NP and $k$-QBF for $\Sigma_k^p$ ($k > 1$), where $k$-QBF is the problem of deciding whether the quantified boolean formula

$$\underbrace{\exists\vec{p}\,\forall\vec{q}\ldots}_{k \text{ alternating quantifiers starting with } \exists}\Phi(\vec{p},\vec{q},\ldots).$$

is true. The complementary problem, denoted by co-$k$-QBF, is complete for $\Pi_k^p$.

All problems in the polynomial hierarchy can be solved in polynomial time iff P = NP. Further, all these problems can be solved by worst-case exponential time algorithms. Thus, the polynomial hierarchy might not seem too meaningful. However, different levels of the hierarchy differ considerably in practice, e.g. methods working for moderately sized instances of NP-complete problems do not work for $\Sigma_2^p$-complete problems.

The complexity of the problems we are interested in has been extensively studied for existing logics for nonmonotonic reasoning under the standard, stability semantics [Cadoli and Schaerf, 1993; Gottlob, 1992; Niemelä, 1990; Marek and Truszczynski, 1993]. In particular, the credulous and sceptical reasoning problems are located at the second level of the polynomial hierarchy for the formalisms we are interested in, i.e., sceptical reasoning is $\Pi_2^p$-complete and credulous reasoning is $\Sigma_2^p$-complete for all of Theorist, Circumscription and AEL.

In this section we prove a number of *generic* upper bounds for reasoning under the admissibility and preferability semantics that are parametric on the complexity of the *derivability problem* in the underlying monotonic logic. For all of Theorist, Circumscription and AEL, the underlying logic

[1]For the following, we assume that the reader is familiar with the basic concepts of complexity theory, i.e., the complexity classes P, NP and co-NP, and the notion of completeness with respect to log-space reductions. Good textbooks covering these notions are [Garey and Johnson, 1979] and [Papadimitriou, 1994].

is classical propositional logic, hence the derivability problem is in co-NP.

In order to decide the credulous and co-sceptical reasoning problems, one can apply the following "*guess & check*" algorithm:

(i) guess an assumption set,

(ii) verify that it is sanctioned by the semantics, and

(iii) verify that the formula under consideration is derivable from the assumption set and the monotonic theory or not derivable from it, respectively.

From that it follows that credulous reasoning and co-sceptical reasoning are in the complexity class $\mathsf{NP}^{\mathcal{C}}$, provided reasoning in the underlying logic is in $\mathcal{C}$ and the verification that an assumption set is sanctioned by the semantics can be done with polynomially many calls to a $\mathcal{C}$-oracle. For the stability semantics, we need indeed only polynomially many $\mathcal{C}$-oracle calls in order to verify that the assumption set $\Delta$ is not self-attacking and that it is closed and attacks all assumptions $\alpha \notin \Delta$. For the admissibility and preferability semantics, instead, the verification step does not appear to be that easy.

**Theorem 1** *For frameworks with an underlying monotonic logic with a derivability problem in $\mathcal{C}$, the* assumption set verification problem *is*

1. *in $\mathsf{P}^{\mathcal{C}}$ under the stability semantics,*
2. *in $\mathsf{co\text{-}NP}^{\mathcal{C}}$ under the admissibility semantics,*
3. *in $\mathsf{co\text{-}NP}^{\mathsf{NP}^{\mathcal{C}}}$ under the preferability semantics.*

**Proof:** Claim (1) follows from the argument above that polynomially many $\mathcal{C}$-oracle calls are sufficient to verify that an assumption set is stable.

In order to prove claim (2), we specify the following nondeterministic, polynomial-time algorithm that uses a $\mathcal{C}$-oracle:

1. Check if $\Delta$ is closed, using $|A - \Delta|$ $\mathcal{C}$-oracle calls. If not, succeed.
2. Check if $\Delta$ is self-attacking using $|\Delta|$ $\mathcal{C}$-oracle calls. If so, succeed.
3. Guess a set $\Delta' \subseteq A$.
4. Verify that $\Delta'$ is closed, using $|A - \Delta'|$ $\mathcal{C}$-oracle calls.
5. Verify that $\Delta'$ attacks $\Delta$ using $|\Delta|$ $\mathcal{C}$-oracle calls.
6. Verify that $\Delta$ does not attack $\Delta'$ using $|\Delta'|$ $\mathcal{C}$-oracle calls.

Obviously, the algorithm succeeds iff $\Delta$ is not admissible, i.e., it solves the complement of the assumption set verification problem.

Claim (3) is proved by the following nondeterministic, polynomial-time algorithm that uses an $\mathsf{NP}^{\mathcal{C}}$-oracle:

1. Check if $\Delta$ is admissible, which, by claim (2), can be done by one call to an $\mathsf{NP}^{\mathcal{C}}$-oracle. If it is not, succeed.
2. Guess a set $\Delta' \supset \Delta$.
3. Check if $\Delta'$ is admissible. If it is, succeed. Otherwise fail.

Obviously, this algorithm decides the complement of the assumption set verification problem under the preferability semantics, demonstrating that this problem is in $\mathsf{co\text{-}NP}^{\mathsf{NP}^{\mathcal{C}}}$.
■

Furthermore, there does not appear to be a way around the difficulty of the assumption set verification problem in the general case. For some special frameworks, however, the problem can be solved more efficiently. In particular, this is the case for normal frameworks, at least under the preferability semantics. Indeed, since preferred assumption sets coincide with stable assumption sets for these frameworks, the result for the stability semantics in Theorem 1 applies.

**Corollary 2** *For* normal *frameworks with an underlying monotonic logic with a derivability problem in $\mathcal{C}$, the* assumption set verification problem *under the preferability semantics is in $\mathsf{P}^{\mathcal{C}}$.*

We could now apply the *guess & check* algorithm described above for deriving upper bounds for the reasoning problems. However, the following considerations (sometimes) allow to obtain better upper bounds for the reasoning problems than the ones obtained by using the *guess & check* algorithm in conjunction with Theorem 1.

First, since any preferred assumption set is admissible and any admissible set is a subset of a preferred set, the following result holds.

**Proposition 3** *Credulous reasoning under the admissibility semantics is equivalent to credulous reasoning under the preferability semantics.*

Thus, credulous reasoning under the admissibility semantics has the same upper bound as credulous reasoning under the preferability semantics. This implies that, for normal frameworks, credulous reasoning under admissibility and preferability semantics has the same upper bound as credulous reasoning under the stability semantics.

In addition, co-sceptical and sceptical reasoning under the admissibility semantics is often much simpler than suggested by the upper bounds in Theorem 1 for the respective assumption set verification problem. We call an abstract framework $\langle T, A, \overline{\phantom{x}} \rangle$ **simple** iff there is no admissible set if $T$ is inconsistent and otherwise there exists a *least* admissible set $\Delta_l = A \cap Th(T)$.

**Proposition 4** *For* simple *frameworks with an underlying monotonic logic with derivability problem in $\mathcal{C}$, the sceptical reasoning problem under the admissibility semantics is in $\mathcal{C}$.*

Putting all the above results together and applying the *guess & check* algorithm leads to the next theorem, specifying upper bounds for all the frameworks and reasoning problems of interest.

**Theorem 5** *Upper bounds for the different reasoning problems are as specified in the following table:*

| *Upper bounds for credulous reasoning* | | | |
|---|---|---|---|
| | *Stability* | *Admissibility* | *Preferability* |
| *general* | $\mathsf{NP}^{\mathcal{C}}$ | $\mathsf{NP}^{\mathsf{NP}^{\mathcal{C}}}$ | $\mathsf{NP}^{\mathsf{NP}^{\mathcal{C}}}$ |
| *normal* | $\mathsf{NP}^{\mathcal{C}}$ | $\mathsf{NP}^{\mathcal{C}}$ | $\mathsf{NP}^{\mathcal{C}}$ |
| *simple* | $\mathsf{NP}^{\mathcal{C}}$ | $\mathsf{NP}^{\mathsf{NP}^{\mathcal{C}}}$ | $\mathsf{NP}^{\mathsf{NP}^{\mathcal{C}}}$ |
| *Upper bounds for sceptical reasoning* | | | |
| | *Stability* | *Admissibility* | *Preferability* |
| *general* | co-$\mathsf{NP}^{\mathcal{C}}$ | co-$\mathsf{NP}^{\mathsf{NP}^{\mathcal{C}}}$ | co-$\mathsf{NP}^{\mathsf{NP}^{\mathsf{NP}^{\mathcal{C}}}}$ |
| *normal* | co-$\mathsf{NP}^{\mathcal{C}}$ | co-$\mathsf{NP}^{\mathsf{NP}^{\mathcal{C}}}$ | co-$\mathsf{NP}^{\mathcal{C}}$ |
| *simple* | co-$\mathsf{NP}^{\mathcal{C}}$ | $\mathcal{C}$ | co-$\mathsf{NP}^{\mathsf{NP}^{\mathsf{NP}^{\mathcal{C}}}}$ |

**Proof:** The columns for the *stability* semantics follow because the credulous and co-sceptical reasoning problems can be decided by an $\mathsf{NP}$ guessing step of an assumption set, followed by the verification that the assumption set is stable, which by Theorem 1 can be solved by a call to a $\mathsf{P}^{\mathcal{C}}$ oracle, followed in turn by the verification that the desired sentence is derivable or not derivable, respectively, from the theory augmented by the guessed assumption set, which can be solved by a call to a $\mathcal{C}$-oracle. This means that $\mathsf{NP}^{\mathsf{P}^{\mathcal{C}}} = \mathsf{NP}^{\mathcal{C}}$ is an upper bound for the credulous and co-sceptical reasoning problems under the stability semantics.

The results for the admissibility semantics in the *general* case follow by the same kind of argument. The entry for credulous reasoning under the preferability semantics in the general case follows from Proposition 3 and the entry for the admissibility semantics in the same table. The entry for sceptical reasoning under the preferability semantics in the general case follows again by using a *guess & check* algorithm and Theorem 1.

The results for *normal* frameworks are justified as follows. Credulous reasoning under the admissibility and preferability semantics can be shown to be in $\mathsf{NP}^{\mathcal{C}}$ by using the *guess & check* algorithm and applying Proposition 3 and Corollary 2. Since the same applies for co-sceptical reasoning under the preferability semantics, membership in co-$\mathsf{NP}^{\mathcal{C}}$ follows. Finally, the upper bound for sceptical reasoning under the admissibility semantics is the same as the upper bound for the same problem in the general case.

Finally, the results for *simple* frameworks are the general results with the exception of sceptical reasoning under the admissibility semantics where the upper bound is the one given by Proposition 4. ■

As shown in Theorem 5 and already remarked above, sceptical reasoning under the admissibility semantics can be sometimes simpler than sceptical reasoning under the stability semantics. However, in all such cases the problem reduces to deriving monotonic conclusions from the theory $T$ alone, ignoring the assumptions completely. In other words, in these cases, sceptical nonmonotonic reasoning reduces to monotonic reasoning and is thus trivialised.

## 4 Simple, Normal Frameworks: Theorist and Circumscription

The frameworks for Theorist and Circumscription are *normal* (see [Bondarenko *et al.*, 1997]) and *simple*, as shown below.

**Lemma 6** *The frameworks for Theorist and Circumscription are simple.*

**Proof:** Circumscription is a special instance of Theorist. Thus, we only need to prove the theorem for Theorist.

If the given Theorist theory $T$ is inconsistent then the corresponding framework admits no admissible argument, as any closed assumption set attacks itself.

Assume that $T$ is consistent. Then, we only need to prove that $\Delta = Th(T) \cap A$ attacks every closed assumption set $\Delta'$ which attacks $\Delta$. Now, if $\Delta = \emptyset$, then there is no set $\Delta'$ that attacks $\Delta$. If $\Delta \neq \emptyset$, then $\Delta'$ attacks $\Delta$ iff $T \cup \Delta'$ is inconsistent and, as $\Delta'$ is closed, $\Delta' = A$. Thus, necessarily $\Delta$ attacks $\Delta'$. ■

For both frameworks, the credulous and sceptical reasoning problems reach the respective minimal upper bounds specified in Theorem 5. Indeed, because of Proposition 3 and the fact that the frameworks are normal, credulous reasoning under the admissibility and preferability semantics is identical to credulous reasoning under the standard, stability semantics, leading to the result that the complexity of credulous reasoning under all three semantics is identical.

**Corollary 7** *Credulous reasoning in Theorist and Circumscription under the admissibility and preferability semantics is $\Sigma_2^p$-complete.*

Similarly, for normal frameworks, sceptical reasoning under the preferability semantics is identical to sceptical reasoning under the stability semantics, from which the next result follows immediately.

**Corollary 8** *Sceptical reasoning in Theorist and Circumscription under the preferability semantics is $\Pi_2^p$-complete.*

Finally, sceptical reasoning under the admissibility semantics is trivial as Theorist and Circumscription are simple frameworks and thus sceptical reasoning reduces to classical derivability.

**Corollary 9** *Sceptical reasoning in Theorist and Circumscription under the admissibility semantics is* co-NP-*complete.*

In other words, for Theorist and Circumscription, we either get the same results under the new semantics as under the stability semantics or we get trivial results.

## 5 General Frameworks: Autoepistemic Logic

AEL is neither normal nor simple. In addition, it satisfies no other property that might help to reduce the computational complexity that is suggested by the upper bound results in Section 3. (In particular, AEL is not *flat*, see [Bondarenko *et al.*, 1997; Dimopoulos *et al.*, 1999]). We prove that these upper bounds are tight for AEL. This implies that, in AEL, even under the admissibility semantics sceptical reasoning is harder than under the standard semantics – a phenomenon we do not have with simple, normal, or flat frameworks.

**Theorem 10** *In AEL, credulous reasoning under the admissibility semantics is $\Sigma_3^p$-complete and sceptical reasoning under the admissibility semantics is $\Pi_3^p$-complete.*

**Proof:** Membership for credulous and sceptical reasoning follows from Theorem 5.

Hardness for credulous reasoning will be proven by a reduction from 3-QBF. Assume the following quantified boolean formula: $\exists p_1,\ldots,p_n\, \forall q_1,\ldots,q_m\, \exists r_1,\ldots,r_k\, \Phi$, with $\Phi$ a formula in 3CNF over the propositional variables $p_1,\ldots,p_n,\ q_1,\ldots,q_m,\ r_1,\ldots,r_k$. We construct an AEL theory $T$ such that the given quantified boolean formula is true iff a particular sentence $F$ is contained in some admissible argument of $T$.

The language of $T$ contains atoms $p_1,\ldots,p_n,\ q_1,\ldots,q_m$, and $r_1,\ldots,r_k$ as well as atoms $t_1,\ldots,t_n$, intuitively holding if a truth value for the variables $p_1,\ldots,p_n$, has been chosen. Finally, there is an atom $s$ that is used to inhibit that any admissible argument can choose truth values for the $q_j$. $T$ consists of the following sentences:

$$
\begin{array}{rcl}
\neg L\neg p_i & \rightarrow & p_i \wedge t_i, \\
\neg L p_i & \rightarrow & \neg p_i \wedge t_i, \\
\neg L\neg\Phi, & & \\
\neg L\neg q_j & \rightarrow & q_j \wedge s \wedge \neg Ls, \\
\neg L q_j & \rightarrow & \neg q_j \wedge s \wedge \neg Ls,
\end{array}
$$

for each $i = 1,\ldots,n$, $j = 1,\ldots,m$.

We prove that there exists an admissible extension containing $F = \bigwedge t_i$ iff the 3-QBF formula is true.

Let us consider the case that the above $T$ has an admissible extension $Th(T \cup \Delta)$ containing $\bigwedge t_i$. This implies that for each $p_i$ either $\neg L\neg p_i$ or $\neg Lp_i$ is part of the assumption set $\Delta$. Further $\neg L\neg\Phi$ must be part of $\Delta$ in order for $\Delta$ to be closed. None of the assumptions $\neg L\neg q_i$ or $\neg Lq_i$ can be part of $\Delta$, however, as otherwise $\Delta$ would attack itself.

Let $\Delta^\star$ be an assumption set attacking the admissible $\Delta$ above, and assume that $\Delta^\star$ attacks $\neg L\neg\Phi$ in $\Delta$. Then $\Delta^\star$ together with the theory $T$ would allow for the derivation of $\neg\Phi$. If $\Delta^\star$ makes choices for the $p_i$ in conflict with the ones made by $\Delta$, then $\Delta$ automatically attacks $\Delta^\star$. If $\Delta^\star$ makes the same choices for the $p_i$ as $\Delta$, then $\Delta^\star$ must include assumptions from the set $\{\neg L\neg q_j, \neg Lq_j\}_{j=1,\ldots,m}$. However, no such assumption can be counter-attacked, as otherwise $\Delta$ would be self-attacking and thus non-admissible. Therefore, for the given choices on the $p_i$'s in $\Delta$, no choices for the $q_j$'s exist that make $\neg\Phi$ true. In other words, for the given choice of the $p_i$'s in $\Delta$, and for all choices of the truth values for the $q_j$'s, there exists an assignment of truth values to the $r_l$'s that make $\Phi$ true, which implies that the 3-QBF formula must be true.

Conversely, if there is no admissible extension containing $\bigwedge t_i$, regardless of our choices for the $p_i$'s, there is always an attack on $\neg L\neg\Phi$, which by the arguments presented above implies that the 3-QBF formula cannot be true.

Hence, this is a log-space reduction from 3-QBF to credulous reasoning under the admissibility semantics in AEL, which proves the first claim in the theorem.

In order to prove the second claim, we consider the theory $T' = T \cup \{L \bigwedge_i t_i\}$, where $T$ is the theory constructed from

$\Phi$. Any admissible set $\Delta$ must contain the assumptions $\neg L\neg\Phi$ and $L\bigwedge_i t_i$ in order for $\Delta$ to be closed. Furthermore, any admissible extension of $T'$ must contain $\bigwedge_i t_i$ because otherwise it is attacked by $\neg L\bigwedge_i t_i$ without having a counter-attack. From this fact and the above observations it follows that $T'$ has an admissible extension iff the 3-QBF formula is true. Given that if no admissible extension exists all co-sceptical queries will be answered negatively, the above is equivalent to the fact that $\neg\bigwedge_i t_i$ is not a sceptical consequence of $T'$ iff the 3-QBF formula is true, i.e., the construction is a log-space reduction from 3-QBF to co-sceptical reasoning under the admissibility semantics. From that and the upper bound in Theorem 5, the second claim follows. ∎

Applying Proposition 3, we get the following corollary.

**Corollary 11** *Credulous reasoning in AEL under the preferability semantics is $\Sigma_3^p$-complete.*

Sceptical reasoning under the preferability semantics is even harder, since for each candidate preferred argument we have to verify that none of its supersets is admissible, resulting in a reasoning problem on the fourth level of the polynomial hierarchy.

**Theorem 12** *Sceptical reasoning in AEL under the preferability semantics is $\Pi_4^p$-complete.*

**Proof:** Membership follows from Theorem 5.

Hardness will be proven by reducing 4-QBF to co-sceptical reasoning. Assume the following quantified boolean formula: $\exists p_1,\ldots,p_n\ \forall q_1,\ldots,q_m\ \exists r_1,\ldots,r_k\ \forall s_1,\ldots,s_o\ \Phi$, with $\Phi$ a formula in 3CNF over the propositional variables $p_1,\ldots,p_n$, $q_1,\ldots,q_m$, $r_1,\ldots,r_k$, and $s_1,\ldots,s_l$. We construct an AEL theory $T$ such that the given quantified boolean formula is true iff a particular sentence $F$ is not contained in some preferred argument of $T$.

The language of $T$ contains atoms $p_1,\ldots,p_n$, $q_1,\ldots,q_m$, $r_1,\ldots,r_k$ and $s_1,\ldots,s_o$ as well as atoms $t_1,\ldots,t_m$, the latter intuitively holding if a truth value for the variables $q_1,\ldots,q_m$ has been chosen. Finally, we have atoms $v$, used to block the truth assignment to the $q_j$, and $w$, used to prohibit any choices on assumptions $\{\neg L\neg r_h, \neg Lr_h\}$.

$T$ consists of the following sentences:

$$
\begin{array}{rcl}
\neg L\neg p_i & \rightarrow & p_i, \\
\neg Lp_i & \rightarrow & \neg p_i, \\
\neg L\neg q_j \wedge \neg Lv & \rightarrow & q_j \wedge t_j, \\
\neg Lq_j \wedge \neg Lv & \rightarrow & \neg q_j \wedge t_j, \\
\neg Lt_j & \rightarrow & v, \\
\neg L\neg r_h & \rightarrow & r_h \wedge w \wedge \neg Lw, \\
\neg Lr_h & \rightarrow & \neg r_h \wedge w \wedge \neg Lw, \\
\Phi & \rightarrow & v,
\end{array}
$$

for each $i = 1,\ldots,n$, $j = 1,\ldots,m$, $h = 1,\ldots,k$.

We prove that there exists a preferred extension not containing $F = \bigwedge t_i$ iff the 4-QBF formula is true.

Let us construct one such preferred extension. First, note that any assumption set containing non-conflicting assumptions from the set $\{\neg L\neg p_i, \neg Lp_i\}$ is an admissible set. Let $\Delta$ be a maximal such set.

Secondly, the set $\Delta$ cannot be expanded using assumptions from $\{\neg L\neg r_h, \neg Lr_h\}$, because these assumptions lead to an immediate self-attack.

Thirdly, it is obvious that $\Delta$ can be expanded (in a non-trivial way) only by adding the assumption $\neg Lv$ and assumptions from the set $\{\neg L\neg q_j, \neg Lq_j\}$. Let us call this expanded set $\Delta'$. Such a set $\Delta'$ is only admissible if we make choices for all $q_j$'s because otherwise $\Delta'$ can be attacked by some $\{\neg Lt_j\}$ (using $\neg Lt_j \rightarrow v$) and $\Delta'$ cannot counter-attack such an attack. Moreover, such a set $\Delta'$ is only admissible if every attack against $\neg Lv$ can be counter-attacked by $\Delta'$. The only attack $\Delta^\star$ against $\neg Lv$ which cannot be counter-attacked by $\Delta'$ would consists of all assumptions in $\Delta'$ and assumptions from $\{\neg L\neg r_h, \neg Lr_h\}$ such that $\Phi$ is true. Thus, assuming that $\Delta'$ is admissible means that no such $\Delta^\star$ exists, i.e. for all possible choices for the $r_h$'s, $\Phi$ is not derivable, i.e., there is always a truth assignment to the $s_k$'s that make $\neg\Phi$ true. This means that $\Delta$ cannot be expanded to $\Delta'$ by assumptions from $\{\neg L\neg q_j, \neg Lq_j\}$ and by $\neg Lv$ if, under the truth assignment to the $p_i$'s corresponding to the assumptions in $\Delta$, and for all truth assignments to the $q_j$'s, there exists a truth assignment to the $r_h$'s that makes $\Phi$ true. In other words, if there exists a preferred assumption set $\Delta'$ such that $Th(T \cup \Delta')$ does not contain $\bigwedge t_j$, then the given 4-QBF formula is true.

Conversely, let us assume the 4-QBF formula is true. Let $\Delta$ be an assumption set containing the assumptions from $\{\neg L\neg p_i, \neg Lp_i\}$ corresponding to a truth assignment to the $p_i$'s rendering $\forall q_1,\ldots,q_m\exists r_1,\ldots,r_k\forall s_1,\ldots,s_o\Phi$ true. $\Delta$ is admissible. Moreover, any extension $\Delta'$ of $\Delta$ by assumptions from $\{\neg L\neg q_j, \neg Lq_j\}$ together with $\neg Lv$ would not be admissible. Indeed, for any such $\Delta'$ there exists a value assignment to the $r_h$'s which makes $\Phi$ true, corresponding to an assumption set with choices from $\{\neg L\neg r_h, \neg Lr_h\}$ which, together with $\Delta'$, would attack $\Delta'$ without being counter-attacked by $\Delta'$. For this reason, there exists a preferred extension not containing both $\neg Lv$ and choices from $\{\neg L\neg q_j, \neg Lq_j\}$, and hence this preferred extension does not contain $\bigwedge t_j$.

Hence, this is a log-space reduction from 4-QBF to co-sceptical reasoning under the preferability semantics in

AEL, which proves the claim. ■

## 6 Conclusion and Discussion

We have analysed the computational complexity of credulous and sceptical reasoning under the admissibility and preferability semantics for (the propositional variant of) the nonmonotonic frameworks of Theorist, Circumscription and Autoepistemic Logic. Table 1 summarises the results.

Table 1: Complexity results

| Framework | Property | Admissibility | | Preferability | |
|---|---|---|---|---|---|
| | | cred. | scept. | cred. | scept. |
| Theorist | simple & | $\Sigma_2^p$ | co-NP | $\Sigma_2^p$ | $\Pi_2^p$ |
| Circum. | normal | $\Sigma_2^p$ | co-NP | $\Sigma_2^p$ | $\Pi_2^p$ |
| AEL | general | $\Sigma_3^p$ | $\Pi_3^p$ | $\Sigma_3^p$ | $\Pi_4^p$ |

These results imply that, for the *simple* and *normal* frameworks of Theorist and Circumscription, credulous reasoning is as hard under the new semantics as it is under the standard, stability semantics [2], whereas sceptical reasoning is simpler under the new semantics than it is under the standard semantics, but it is a trivial form of nonmonotonic reasoning as it amounts to reasoning in classical logic, the monotonic logic underlying both Theorist and Circumscription. Moreover, for the "general" framework of Autoepistemic logic, reasoning under the new semantics is considerably harder than under the standard semantics, as credulous reasoning under both new semantics is located *one level higher* in the polynomial hierarchy than credulous reasoning under the standard semantics, and sceptical reasoning under the admissibility and preferability semantics is located *one and two level higher* (respectively) in the polynomial hierarchy than sceptical reasoning under the standard semantics.

These results suggest that neither of the new semantics is appropriate for reasoning in general frameworks, that the admissibility semantics is not appropriate for sceptical reasoning in simple and normal frameworks, and that, in all other cases, no gain is obtained by reasoning under the new semantics with respect to reasoning under the standard semantics. These conclusions confirm those drawn from earlier results for logic programming and default logic in [Eiter *et al.*, 1998; Dimopoulos *et al.*, 1999].

These results seem to contradict the expectation, put forwards in [Bondarenko *et al.*, 1997; Dung *et al.*, 1997], that the new, argumentation-theoretic semantics lead to computationally less involved nonmonotonic reasoning processes than the standard semantics. However, they do not contradict the expectation that in practice credulous reasoning under the admissibility semantics is often easier than credulous reasoning under the stability semantics, especially if the frameworks employed are *flat* (e.g. logic programming and default logic). Indeed, [Kowalski and Toni, 1996] envisages practical and legal reasoning applications for the argumentation-theoretic semantics, where credulous reasoning under the admissibility semantics in flat frameworks is well-suited, as unilateral arguments are put forwards and defended against all counterarguments, in a credulous manner.

Future work include identifying, by means of empirical investigations, the practical impact of the complexity results proven in this paper as well as in [Dimopoulos *et al.*, 1999] for the envisaged applications of the semantics here investigated.

[2]See Section 3 for a summary of the results for the standard semantics.

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