Probabilistic Program Analysis

Probablistic Abstract Interpretation

Alessandra Di Pierro
University of Verona, Italy
alessandra.dipierro@univr.it

Herbert Wiklicky
Imperial College London, UK
herbert@doc.ic.uk

Approximation and Correctness

Data-flow analyses can be re-formulated in a different scenario where correctness is guaranteed by construction.

Classically, the theory of Abstract Interpretation allows us to

- construct simplified (computable) abstract semantics
- construct approximate solutions
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Notions of Approximation

In order theoretic structures we are looking for
Safe Approximations

$$s^* \sqsubseteq s \quad \text{or} \quad s \sqsubseteq s^*$$

In quantitative, vector space structures we want
Close Approximations

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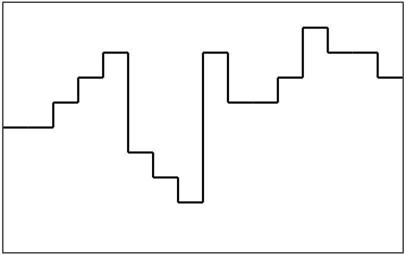

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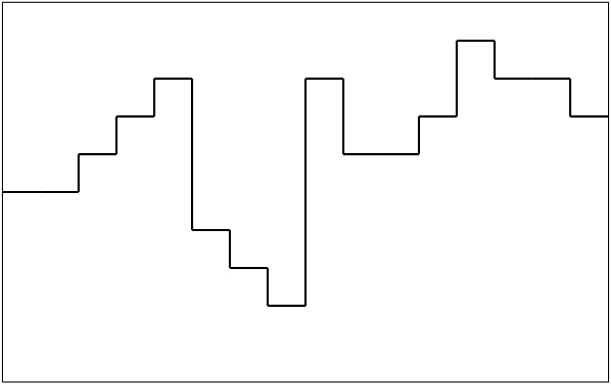

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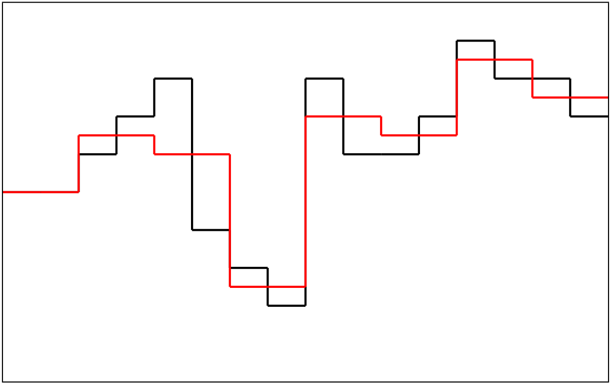

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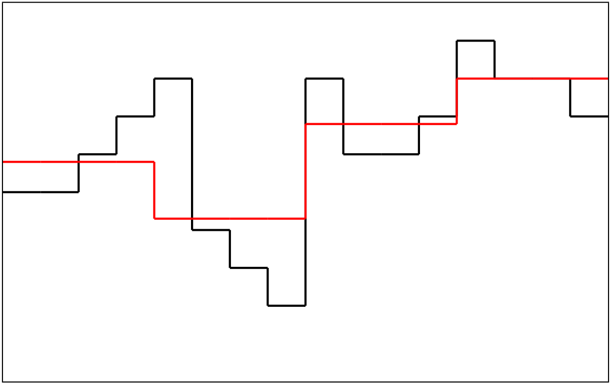

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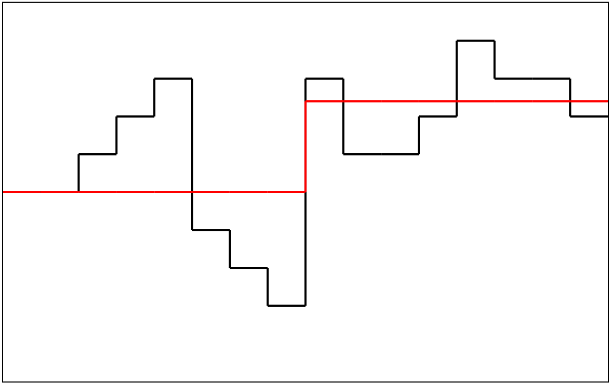

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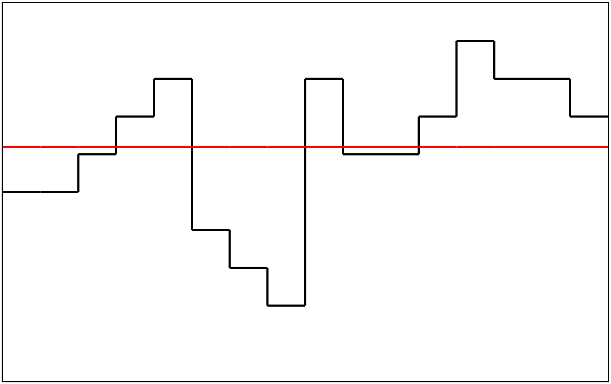

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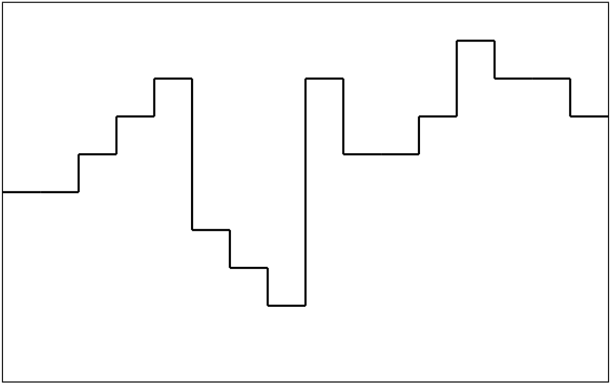

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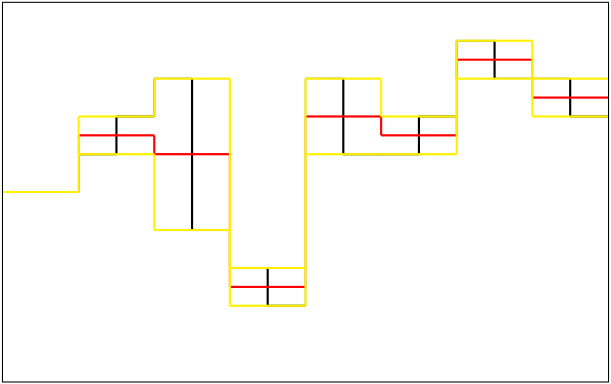

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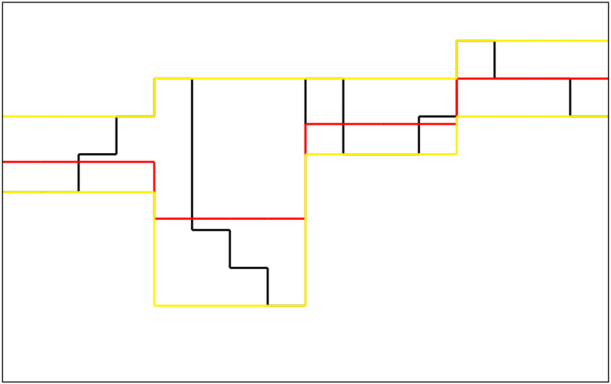

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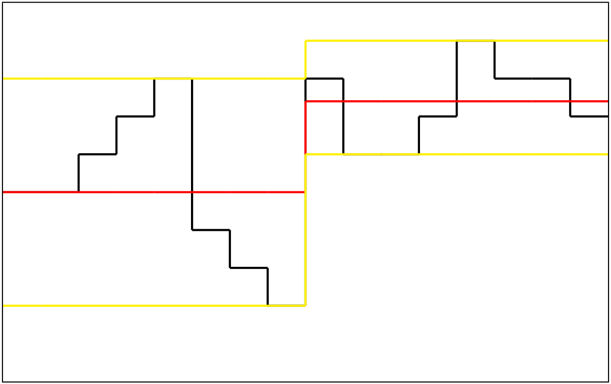

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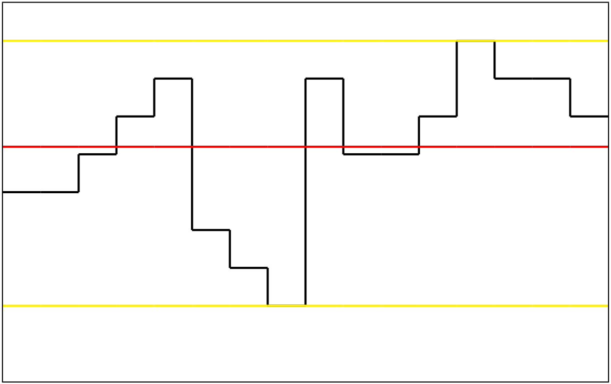

Abstract Interpretation

Some problems may be have too costly solutions or be uncomputable on a concrete space (complete lattice).

Find abstract descriptions on which computations are easier; then relate the concrete and abstract solutions.

Definition

Let $\mathcal{C} = (\mathcal{C}, \leq)$ and $\mathcal{D} = (\mathcal{D}, \sqsubseteq)$ be two partially ordered set. If there are two functions $\alpha : \mathcal{C} \to \mathcal{D}$ and $\gamma : \mathcal{D} \to \mathcal{C}$ such that for all $c \in \mathcal{C}$ and all $d \in \mathcal{D}$:

$$c \leq_{\mathcal{C}} \gamma(d) \text{ iff } \alpha(c) \sqsubseteq d,$$

then $(\mathcal{C}, \alpha, \gamma, \mathcal{D})$ form a Galois connection.

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(i) $\alpha \circ \gamma$ is reductive i.e. $\forall d \in \mathcal{D}$, $\alpha \circ \gamma(d) \leq_{\mathcal{D}} d$,

(ii) $\gamma \circ \alpha$ is extensive i.e. $\forall c \in \mathcal{C}$, $c \leq_{\mathcal{C}} \gamma \circ \alpha(c)$.

Proposition

Let $(\mathcal{C}, \alpha, \gamma, \mathcal{D})$ be a Galois connection. Then α and γ are quasi-inverse, i.e.

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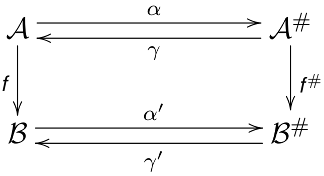

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Induced semantics:

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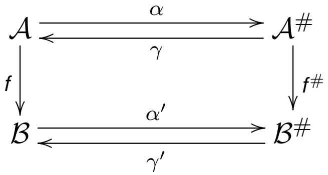

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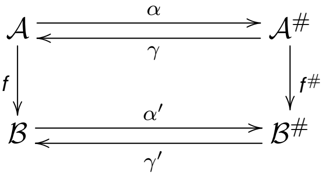

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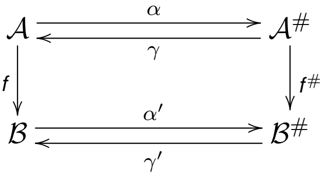

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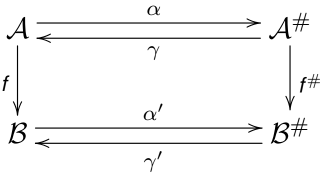

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Probabilistic Abstraction Domains

A probabilistic domain is essentially a vector space which represents the distributions $Dist(S)$ on the state space S of a probabilistic transition system, i.e. for finite state spaces

$$\mathcal{V}(S) = \{ (v_s)_{s \in S} \mid v_s \in \mathbb{R} \}.$$

In the finite setting we can identify $\mathcal{V}(S)$ with the Hilbert space $\ell^2(S)$.

The notion of *norm* is essential for our treatment; we will consider normed vector spaces.

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Norm and Operator Norm

A norm on a vector space $\mathcal{V}$ is a map $\|.\| : \mathcal{V} \mapsto \mathbb{R}$ such that for all $v, w \in \mathcal{V}$ and $c \in \mathbb{C}$:

- $\|v\| \geq 0$,
- $\|v\| = 0 \Leftrightarrow v = o$,
- $\|cv\| = |c|\|v\|$,
- $\|v + w\| \leq \|v\| + \|w\|$,

with $o \in \mathcal{V}$ the zero vector.

We can always use a norm to define a metric topology on a vector space via the distance function $d(v, w) = \|v - w\|$.

$$\|\mathbf{M}\| = \sup_{v \in \mathcal{V}} \frac{\|\mathbf{M}(v)\|}{\|v\|} = \sup_{\|v\|=1} \|\mathbf{M}(v)\|.$$

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Generalised Inverse

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be two finite-dimensional vector spaces and $\mathbf{A} : \mathcal{C} \to \mathcal{D}$ a linear map. Then the linear map $\mathbf{A}^\dagger = \mathbf{G} : \mathcal{D} \to \mathcal{C}$ is the Moore-Penrose pseudo-inverse of $\mathbf{A}$ iff

(i) $\mathbf{A} \circ \mathbf{G} = \mathbf{P}_A$,

(ii) $\mathbf{G} \circ \mathbf{A} = \mathbf{P}_G$,

where $\mathbf{P}_A$ and $\mathbf{P}_G$ denote orthogonal projections onto the ranges of $\mathbf{A}$ and $\mathbf{G}$.

Least Squares Solutions

Definition

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then $\mathbf{u} \in \mathbb{R}^n$ is called a least squares solution to $\mathbf{Ax} = \mathbf{b}$ if

$$\|\mathbf{Au} - \mathbf{b}\| \leq \|\mathbf{Av} - \mathbf{b}\|, \text{ for all } \mathbf{v} \in \mathbb{R}^n.$$

Theorem

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then $\mathbf{A}^\dagger \mathbf{b}$ is the minimal least squares solution to $\mathbf{Ax} = \mathbf{b}$.

Orthogonal Projections

Corollary

Let $\mathbf{P}$ *be a orthogonal projection on a finite dimensional vector space* $\mathcal{V}$. *Then for any* $\mathbf{x} \in \mathcal{V}$, $\mathbf{Px}$ *is the unique closest vector in* $\mathcal{V}$ *to* $\mathbf{x}$ *wrt the Euclidean norm.*

Extraction Functions

An extraction function $\eta : C \mapsto D$ is a mapping from a set of values to their descriptions in D.

It is easy to show that

Proposition

Given an extraction function $\eta : \mathcal{C} \mapsto \mathcal{D}$, the quadruple $(\mathcal{P}(\mathcal{C}), \alpha_\eta, \gamma_\eta, \mathcal{P}(\mathcal{D}))$ is a Galois connection with α_η and γ_η defined by:

$$\alpha_\eta(C') = \{\eta(c) \mid c \in C'\} \text{ and } \gamma_\eta(D') = \{v \mid \eta(v) \in D'\}$$

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Vector Space Lifting

Free vector space construction on a set S:

$$\mathcal{V}(S) = \{\sum x_s s \mid x_s \in \mathbb{R}, s \in S\}$$

An obvious way to lift an extraction function to a linear map between vector spaces is to construct the free vector spaces on $\mathcal{C}$ and $\mathcal{D}$ and define:

Vector Space lifting: $\vec{\alpha} : \mathcal{V}(\mathcal{C}) \to \mathcal{V}(\mathcal{D})$

$$\vec{\alpha}(p_1 \cdot \vec{c}_1 + p_2 \cdot \vec{c}_2 + \ldots) = p_i \cdot \eta(c_1) + p_2 \cdot \eta(c_2) \ldots$$

Support Set: **supp** : $\mathcal{V}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$

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Vector Space lifting: $\vec{\alpha} : \mathcal{V}(\mathcal{C}) \to \mathcal{V}(\mathcal{D})$

$$\vec{\alpha}(p_1 \cdot \vec{c}_1 + p_2 \cdot \vec{c}_2 + \ldots) = p_i \cdot \eta(c_1) + p_2 \cdot \eta(c_2) \ldots$$

Support Set: $\textbf{supp} : \mathcal{V}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$

$$\textbf{supp}(\vec{x}) = \left\{ c_i \mid \langle c_i, p_i \rangle \in \vec{x} \text{ and } p_i \neq 0 \right\}$$

Relation with Classical Abstractions

Lemma

Let $\vec{\alpha}$ be a probabilistic abstraction function and let $\vec{\gamma}$ be its Moore-Penrose pseudo-inverse.

Then $\vec{\gamma} \circ \vec{\alpha}$ is extensive with respect to the inclusion on the support sets of vectors in $\mathcal{V}(\mathcal{C})$, i.e. $\forall \vec{x} \in \mathcal{V}(\mathcal{C})$,

$$\mathbf{supp}(\vec{x}) \subseteq \mathbf{supp}(\vec{\gamma} \circ \vec{\alpha}(\vec{x})).$$

Analogously we can show that $\vec{\alpha} \circ \vec{\gamma}$ is reductive. Therefore,

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$(\vec{\alpha}, \vec{\gamma})$ form a Galois connection wrt the support sets of $\mathcal{V}(\mathcal{C})$ and $\mathcal{V}(\mathcal{D})$, ordered by inclusion.

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Examples of Lifted Abstractions

Parity Abstraction operator on $\mathcal{V}(\{1,\ldots,n\})$ (with n even):

$$\mathbf{A}_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \qquad \mathbf{A}_p^\dagger = \begin{pmatrix} \frac{2}{n} & 0 & \frac{2}{n} & 0 & \ldots & 0 \\ 0 & \frac{2}{n} & 0 & \frac{2}{n} & \ldots & \frac{2}{n} \end{pmatrix}$$

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Example: Function Approximation (ctd.)

Concrete and abstract domain are step-functions on $[a, b]$.
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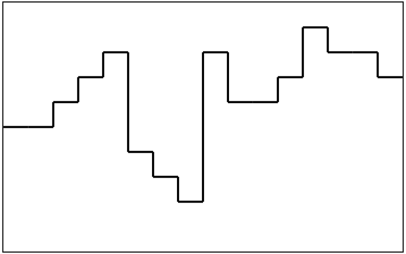

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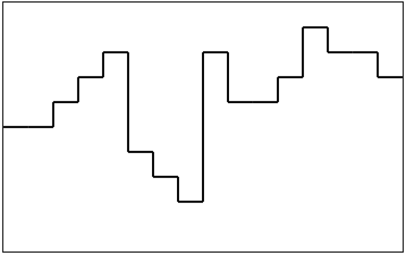

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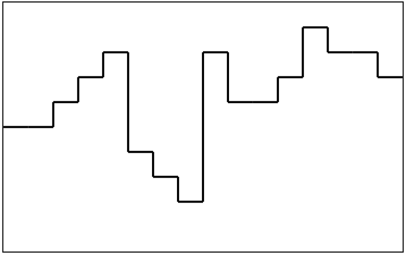

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$$\begin{pmatrix} 5 & 5 & 6 & 7 & 8 & 4 & 3 & 2 & 8 & 6 & 6 & 7 & 9 & 8 & 8 & 7 \end{pmatrix}$$

Example: Abstraction Matrices

$$\mathbf{A}_8 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
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\end{pmatrix}$$

Compute the abstractions of f as $f\mathbf{A}_j$.

In a similar way we can also compute the over- and

Example: Abstraction Matrices

$$\mathbf{G}_8 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

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Approximation Estimates

Compute the *least square error* as

$$\|f - f\mathbf{AG}\|.$$

$$\begin{aligned}
\|f - f\mathbf{A}_8\mathbf{G}_8\| &= 3.5355 \\
\|f - f\mathbf{A}_4\mathbf{G}_4\| &= 5.3151 \\
\|f - f\mathbf{A}_2\mathbf{G}_2\| &= 5.9896 \\
\|f - f\mathbf{A}_1\mathbf{G}_1\| &= 7.6444
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Concrete Semantics (LOS)

$$\mathbf{T}(P) = \sum_{\langle i,p_{ij},j\rangle \in flow(P)} p_{ij} \cdot \mathbf{T}(\ell_i, \ell_j),$$

where

$$\mathbf{T}(\ell_i, \ell_j) = \mathbf{N} \otimes \mathbf{E}(\ell_i, \ell_j),$$

with **N** an operator representing a state update while the second factor realises the transfer of control from label ℓ_i to label ℓ_j.

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Moore-Penrose Pseudo-Inverse of a Tensor Product is:

$$(\mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \ldots \otimes \mathbf{A}_n)^\dagger = \mathbf{A}_1^\dagger \otimes \mathbf{A}_2^\dagger \otimes \ldots \otimes \mathbf{A}_n^\dagger$$

Via linearity we can construct $\mathbf{T}^\#$ in the same way as $\mathbf{T}$, i.e

$$\mathbf{T}^\#(P) = \sum_{\langle i, p_{ij}, j\rangle \in \mathcal{F}(P)} p_{ij} \cdot \mathbf{T}^\#(\ell_i, \ell_j)$$

with local abstraction of individual variables:

$$\mathbf{T}^\#(\ell_i, \ell_j) = (\mathbf{A}_1^\dagger \mathbf{N}_{i1} \mathbf{A}_1) \otimes (\mathbf{A}_2^\dagger \mathbf{N}_{i2} \mathbf{A}_2) \otimes \ldots \otimes (\mathbf{A}_v^\dagger \mathbf{N}_{iv} \mathbf{A}_v) \otimes \mathbf{M}_{ij}$$

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Argument

$$
\begin{aligned}
\mathbf{T}^{\#} &= \mathbf{A}^{\dagger}\mathbf{T}\mathbf{A} \\
&= \mathbf{A}^{\dagger}\Big(\sum_{i,j}\mathbf{T}(i,j)\Big)\mathbf{A} \\
&= \sum_{i,j}\mathbf{A}^{\dagger}\mathbf{T}(i,j)\mathbf{A} \\
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Parity Analysis

Determine at each program point whether a variable is *even* or *odd*.

Parity Abstraction operator on $\mathcal{V}(\{0, \ldots, n\})$ (with n even):

$$\mathbf{A}_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \qquad \mathbf{A}^\dagger = \begin{pmatrix} \frac{2}{n} & 0 & \frac{2}{n} & 0 & \ldots & 0 \\ 0 & \frac{2}{n} & 0 & \frac{2}{n} & \ldots & \frac{2}{n} \end{pmatrix}$$

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Example

$[m \leftarrow i]^1$;
while $[n > 1]^2$ **do**
    $[m \leftarrow m \times n]^3$;
    $[n \leftarrow n - 1]^4$
od
$[\textbf{stop}]^5$

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5: **od**
6: $[\textbf{stop}]^5$

$$
\begin{aligned}
\mathbf{T} \;=\;& \mathbf{U}(\mathtt{m} \leftarrow i) \otimes \mathbf{E}(1,2) \\
+\;& \mathbf{P}(n > 1) \otimes \mathbf{E}(2,3) \\
+\;& \mathbf{P}(n \leq 1) \otimes \mathbf{E}(2,5) \\
+\;& \mathbf{U}(\mathtt{m} \leftarrow m \times n) \otimes \mathbf{E}(3,4) \\
+\;& \mathbf{U}(\mathtt{n} \leftarrow n - 1) \otimes \mathbf{E}(4,2) \\
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+\;& \mathbf{I}^\# \otimes \mathbf{E}(5,5)
\end{aligned}
$$

Abstract Semantics

Abstraction: $\mathbf{A} = \mathbf{A}_p \otimes \mathbf{I}$, i.e. m abstract (parity) but n concrete.

$$
\begin{aligned}
\mathbf{T}^{\#} &= \mathbf{U}^{\#}(m \leftarrow 1) \otimes \mathbf{E}(1,2) \\
&+ \mathbf{P}^{\#}(n > 1) \otimes \mathbf{E}(2,3) \\
&+ \mathbf{P}^{\#}(n \leq 1) \otimes \mathbf{E}(2,5) \\
&+ \mathbf{U}^{\#}(m \leftarrow m \times n) \otimes \mathbf{E}(3,4) \\
&+ \mathbf{U}^{\#}(n \leftarrow n - 1) \otimes \mathbf{E}(4,2) \\
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\end{aligned}
$$

Abstract Semantics

$$\mathbf{U}^{\#}(m \leftarrow 1) =$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & \dots & 1 \end{pmatrix}$$

Abstract Semantics

$$\mathbf{U}^{\#}(n \leftarrow n-1) =$$

$$= \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Abstract Semantics

$$\mathbf{P}^{\#}(n > 1) =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Abstract Semantics

$$\mathbf{P}^{\#}(n \leq 1) =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Abstract Semantics

$$\mathbf{U}^{\#}(m \leftarrow m \times n) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \ddots \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \ddots \end{pmatrix}$$

Implementation

Implementation of concrete and abstract semantics of Factorial using **octave**. Ranges: $n \in \{1, \ldots, d\}$ and $m \in \{1, \ldots, d!\}$.

d	$\dim(\mathbf{T}(F))$	$\dim(\mathbf{T}^{\#}(F))$
2	45	30
3	140	40
4	625	50
5	3630	60
6	25235	70
7	201640	80
8	1814445	90
9	18144050	100

Using uniform initial distributions $\mathbf{d_0}$ for n and m.

Implementation

Implementation of concrete and abstract semantics of Factorial using **octave**. Ranges: $n \in \{1, \ldots, d\}$ and $m \in \{1, \ldots, d!\}$.

d	$\dim(\mathbf{T}(F))$	$\dim(\mathbf{T}^{\#}(F))$
2	45	30
3	140	40
4	625	50
5	3630	60
6	25235	70
7	201640	80
8	1814445	90
9	18144050	100

Using uniform initial distributions $\mathbf{d_0}$ for n and m.

Scalablity

The abstract probabilities for *m* being **even** or **odd** when we execute the abstract program for various *d* values are:

d	**even**	**odd**
10	0.81818	0.18182
100	0.98019	0.019802
1000	0.99800	0.0019980
10000	0.99980	0.00019998

Live Variable Analysis

$[\textbf{skip}]^1$
if $[odd(y)]^2$ **then**
    $[x \leftarrow x + 1]^3$
else
    $[y \leftarrow y + 1]^4$
fi
$[y \leftarrow y + 1]^5$

Classical Analysis: $\text{LV}_{entry}(2) = \{x, y\}$

Probabilistic Analysis:

Live Variable Analysis

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Classical Analysis: $\mathrm{LV}_{entry}(2) = \{x, y\}$

Probabilistic Analysis: $\mathrm{LV}_{entry}(2) = \{\langle x, \frac{1}{2}\rangle, \langle y, 1\rangle\}$

Live Variable Analysis

$[y \leftarrow 2 \times x]^1$
if $[odd(y)]^2$ **then**
    $[x \leftarrow x + 1]^3$
else
    $[y \leftarrow y + 1]^4$
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$[y \leftarrow y + 1]^5$

Classical Analysis: $\mathrm{LV}_{entry}(2) = \{x, y\}$

Probabilistic Analysis: $\mathrm{LV}_{entry}(2) =$

Live Variable Analysis

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Classical Analysis: $\mathrm{LV}_{entry}(2) = \{x, y\}$

Probabilistic Analysis: $\mathrm{LV}_{entry}(2) = \{\langle y, 1\rangle\}$

Program “Transformation”

$[y \leftarrow 2 \times x]^1$
if $[odd(y)]^2$ **then**
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$[y \leftarrow y + 1]^5$

$[y \leftarrow 2 \times x]^1$
[**choose**]^2
    $p_\top : [x \leftarrow x + 1]^3$
or
    $p_\bot : [y \leftarrow y + 1]^4$
$[y \leftarrow y + 1]^5$

Determine branching probabilities in a first-phase analysis and utilise this information to perform the actual analysis:

$$p^\top = \mathbf{A}^\dagger \cdot \mathbf{P}(b = \mathbf{true}) \cdot \mathbf{A} \text{ and } p^\bot = \mathbf{A}^\dagger \cdot \mathbf{P}(b = \mathbf{false}) \cdot \mathbf{A}$$

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Syntax of pWhile with Pointers

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\begin{array}{rcl}
S & ::= & [\textbf{skip}]^{\ell} \\
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$$p ::= *^{r}\mathrm{x} \text{ with } \mathrm{x} \in \textbf{Var} \qquad e ::= a \mid b \mid l$$

$$a ::= n \mid p \mid a_1 \odot a_2 \qquad l ::= \mathrm{NIL} \mid p \mid \&p$$

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Example

if $[(z_0 \text{ mod } 2 = 0)]^1$ **then**
$[x \leftarrow \&z_1]^2;\ [y \leftarrow \&z_2]^3$
else
$[x \leftarrow \&z_2]^4;\ [y \leftarrow \&z_1]^5$
fi
$[\textbf{stop}]^6$

$[\textbf{choose}]^1$
$\frac{1}{2} : ([x \leftarrow \&z_1]^2;\ [y \leftarrow \&z_2]^3)$
or
$\frac{1}{2} : ([x \leftarrow \&z_2]^4;\ [y \leftarrow \&z_1]^5)$
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Example

if $[(\mathtt{z}_0 \bmod 2 = 0)]^1$ **then**
$[\mathtt{x} \leftarrow \& \mathtt{z}_1]^2; \; [\mathtt{y} \leftarrow \& \mathtt{z}_2]^3$
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Test Operators and Filters

Select a certain value $c \in \textbf{Value}$:

$$(\mathbf{P}(c))_{ij} = \begin{cases} 1 & \text{if } i = c = j \\ 0 & \text{otherwise.} \end{cases}$$

Test Operators and Filters

Select a certain value $c \in \mathbf{Value}$:

$$(\mathbf{P}(c))_{ij} = \begin{cases} 1 & \text{if } i = c = j \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{P}(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Test Operators and Filters

Select a certain value $c \in \textbf{Value}$:

$$(\mathbf{P}(c))_{ij} = \begin{cases} 1 & \text{if } i = c = j \\ 0 & \text{otherwise.} \end{cases}$$

Select a certain classical state $\sigma \in \textbf{State}$:

$$\mathbf{P}(\sigma) = \bigotimes_{i=1}^{v} \mathbf{P}(\sigma(\mathrm{x}_i))$$

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$$\mathbf{P}(\sigma) = \bigotimes_{i=1}^{v} \mathbf{P}(\sigma(\mathrm{x}_i))$$

$$\mathbf{P}(\sigma(\mathrm{x}_1 \mapsto 2, \mathrm{x}_2 \mapsto 4)) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Test Operators and Filters

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Select states where expression $e = a \mid b \mid l$ evaluates to c:

$$\mathbf{P}(e = c) = \sum_{\mathcal{E}(e)\sigma = c} \mathbf{P}(\sigma)$$

Selection via Projections

Filtering out *relevant* configurations, i.e. only those which fulfill a certain condition. Use diagonal matrix **P**:

$$(\mathbf{P})_{ii} = \begin{cases} 1 & \text{if condition holds for } c_i \in \textbf{Value} \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix}^t \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ d_2 \\ d_3 \\ 0 \\ d_5 \\ 0 \end{pmatrix}^t$$

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if $[(z_0 \text{ mod } 2 = 0)]^1$ **then**
$[x \leftarrow \& z_1]^2; \; [y \leftarrow \& z_2]^3$
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$[\textbf{stop}]^6$

$\textbf{Var} = \{x, y, z_0, z_1, z_2\}$

$$\mathbf{P}(z_0 \text{ mod } 2 \neq 0) = \mathbf{I} \otimes \mathbf{I} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \otimes \mathbf{I} \otimes \mathbf{I}$$

Example

if $[(\mathtt{z}_0 \bmod 2 = 0)]^1$ **then**
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Update Operators

For all initial values change to constant $c \in \mathbf{Value}$:

$$(\mathbf{U}(c))_{ij} = \begin{cases} 1 & \text{if } j = c \\ 0 & \text{otherwise.} \end{cases}$$

Set value of variable $\mathtt{x}_k \in \mathbf{Var}$ to constant $c \in \mathbf{Value}$:

$$\mathbf{U}(\mathtt{x}_k \leftarrow c) = \left(\bigotimes_{i=1}^{k-1} \mathbf{I} \right) \otimes \mathbf{U}(c) \otimes \left(\bigotimes_{i=k+1}^{v} \mathbf{I} \right)$$

Set variable $\mathtt{x}_k \in \mathbf{Var}$ to value given by expression $e = a \mid b \mid l$:

$$\mathbf{U}(\mathtt{x}_k \leftarrow e) = \sum_c \mathbf{P}(e = c)\mathbf{U}(\mathtt{x}_k \leftarrow c)$$

Update Operators

For all initial values change to constant $c \in \textbf{Value}$:

$$(\mathbf{U}(c))_{ij} = \begin{cases} 1 & \text{if } j = c \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{U}(3) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Set value of variable $\mathtt{x}_k \in \textbf{Var}$ to constant $c \in \textbf{Value}$:

$$\mathbf{U}(\mathtt{x}_k \leftarrow c) = \left(\bigotimes_{i=1}^{k-1} \mathbf{I} \right) \otimes \mathbf{U}(c) \otimes \left(\bigotimes_{i=k+1}^{v} \mathbf{I} \right)$$

Set variable $\mathtt{x}_k \in \textbf{Var}$ to value given by expression $e = a \mid b \mid l$:

Update Operators

For all initial values change to constant $c \in \textbf{Value}$:

$$(\mathbf{U}(c))_{ij} = \begin{cases} 1 & \text{if } j = c \\ 0 & \text{otherwise.} \end{cases}$$

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$$\mathbf{U}(\mathtt{x}_k \leftarrow e) = \sum_c \mathbf{P}(e = c)\mathbf{U}(\mathtt{x}_k \leftarrow c)$$

Update Operators

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Update for Pointers

For an assignment with a pointer on the l.h.s. we need to determine recursevly the actual variable p is pointing to:

$$\mathbf{U}(*^r \mathrm{x}_k \leftarrow e) = \sum_{\mathrm{x}_i} \mathbf{P}(\mathrm{x}_k = \&\mathrm{x}_i)\mathbf{U}(*^{r-1}\mathrm{x}_i \leftarrow e)$$

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Note that we always get eventually to the base case, i.e. where p refers to a concrete variable $\mathtt{x}_k$ and thus the update operator $\mathbf{U}(\mathtt{x}_k \leftarrow e)$ from before.

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For a pointer of second order with $\mathrm{x}_2 \to \mathrm{x}_1 \to \mathrm{x}_0$ we get:

$$\begin{aligned}
\mathbf{U}(**\mathrm{x}_2 \leftarrow 4) &= \sum_{\mathrm{x}_i} \mathbf{P}(\mathrm{x}_2 = \&\mathrm{x}_i)\mathbf{U}(*\mathrm{x}_i \leftarrow 4) \\
\mathbf{U}(*\mathrm{x}_1 \leftarrow 4) &= \sum_{\mathrm{x}_i} \mathbf{P}(\mathrm{x}_1 = \&\mathrm{x}_i)\mathbf{U}(\mathrm{x}_i \leftarrow 4) \\
\mathbf{U}(\mathrm{x}_0 \leftarrow 4) &
\end{aligned}$$

Example

if $[(z_0 \bmod 2 = 0)]^1$ **then**
$[x \leftarrow \&z_1]^2;\ [y \leftarrow \&z_2]^3$
else
$[x \leftarrow \&z_2]^4;\ [y \leftarrow \&z_1]^5$
fi
$[\textbf{stop}]^6$

$$\mathbf{P}(\mathit{even}(z_0)) \otimes \mathbf{E}(1,2) +$$
$$\mathbf{P}(\mathit{odd}(z_0)) \otimes \mathbf{E}(1,4) +$$
$$\mathbf{U}(x \leftarrow \&z_1) \otimes \mathbf{E}(2,3) +$$
$$\mathbf{U}(y \leftarrow \&z_2) \otimes \mathbf{E}(3,6) +$$
$$\mathbf{U}(x \leftarrow \&z_2) \otimes \mathbf{E}(4,5) +$$
$$\mathbf{U}(y \leftarrow \&z_1) \otimes \mathbf{E}(5,6) +$$
$$\mathbf{I} \otimes \mathbf{E}(6,6)$$

Example

$$[\textbf{choose}]^1 \quad \tfrac{1}{2} : ([\mathtt{x} \leftarrow \&\mathtt{z}_1]^2;\ [\mathtt{y} \leftarrow \&\mathtt{z}_2]^3) \quad \textbf{or} \quad \tfrac{1}{2} : ([\mathtt{x} \leftarrow \&\mathtt{z}_2]^4;\ [\mathtt{y} \leftarrow \&\mathtt{z}_1]^5) \quad [\textbf{stop}]^6$$

$$\begin{aligned}
&\tfrac{1}{2} \cdot (\mathbf{I} \otimes \mathbf{E}(1,2)) + \\
&\tfrac{1}{2} \cdot (\mathbf{I} \otimes \mathbf{E}(1,4)) + \\
&\mathbf{U}(\mathtt{x} \leftarrow \&\mathtt{z}_1) \otimes \mathbf{E}(2,3) + \\
&\mathbf{U}(\mathtt{y} \leftarrow \&\mathtt{z}_2) \otimes \mathbf{E}(3,6) + \\
&\mathbf{U}(\mathtt{x} \leftarrow \&\mathtt{z}_2) \otimes \mathbf{E}(4,5) + \\
&\mathbf{U}(\mathtt{y} \leftarrow \&\mathtt{z}_1) \otimes \mathbf{E}(5,6) + \\
&\mathbf{I} \otimes \mathbf{E}(6,6)
\end{aligned}$$

Abstract Branching Probabilities

The abstract tests $\mathbf{P}^\#$ describe the branching probabilities depending on abstract values.

For example, consider $\mathbf{P}(n)$ testing if a variable with values $1, \ldots, n$ is a prime number.

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$$\mathbf{P}(8) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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For example, consider $\mathbf{P}(n)$ testing if a variable with values $1, \ldots, n$ is a prime number.

$$\mathbf{I} - \mathbf{P}(8) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Abstract Branching Probabilities

The abstract tests $\mathbf{P}^{\#}$ describe the branching probabilities depending on abstract values.

For example, consider $\mathbf{P}(n)$ testing if a variable with values $1, \ldots, n$ is a prime number.

Abstraction used could be parity testing for even/odd-ness.

$$\mathbf{A}_p^{\dagger}\mathbf{P}(5)\mathbf{A}_p = \begin{pmatrix} 0.50000 & 0.00000 \\ 0.00000 & 0.66667 \end{pmatrix}$$

$$\mathbf{A}_p^{\dagger}(\mathbf{I} - \mathbf{P}(5))\mathbf{A}_p = \begin{pmatrix} 0.50000 & 0.00000 \\ 0.00000 & 0.33333 \end{pmatrix}$$

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$$\mathbf{A}_p^{\dagger}\mathbf{P}(10)\mathbf{A}_p = \begin{pmatrix} 0.20000 & 0.00000 \\ 0.00000 & 0.60000 \end{pmatrix}$$

$$\mathbf{A}_p^{\dagger}(\mathbf{I} - \mathbf{P}(10))\mathbf{A}_p = \begin{pmatrix} 0.80000 & 0.00000 \\ 0.00000 & 0.40000 \end{pmatrix}$$

Abstract Branching Probabilities

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$$\mathbf{A}_p^{\dagger}\mathbf{P}(100)\mathbf{A}_p = \begin{pmatrix} 0.02000 & 0.00000 \\ 0.00000 & 0.48000 \end{pmatrix}$$

$$\mathbf{A}_p^{\dagger}(\mathbf{I} - \mathbf{P}(100))\mathbf{A}_p = \begin{pmatrix} 0.98000 & 0.00000 \\ 0.00000 & 0.52000 \end{pmatrix}$$

Abstract Branching Probabilities

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For example, consider $\mathbf{P}(n)$ testing if a variable with values $1, \ldots, n$ is a prime number.

Abstraction used could be parity testing for even/odd-ness.

$$\mathbf{A}_p^{\dagger}\mathbf{P}(1000)\mathbf{A}_p = \begin{pmatrix} 0.00200 & 0.00000 \\ 0.00000 & 0.33400 \end{pmatrix}$$

$$\mathbf{A}_p^{\dagger}(\mathbf{I} - \mathbf{P}(1000))\mathbf{A}_p = \begin{pmatrix} 0.99800 & 0.00000 \\ 0.00000 & 0.66600 \end{pmatrix}$$

Abstract Branching Probabilities

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For example, consider $\mathbf{P}(n)$ testing if a variable with values $1, \ldots, n$ is a prime number.

Abstraction used could be parity testing for even/odd-ness.

$$\mathbf{A}_p^{\dagger}\mathbf{P}(10000)\mathbf{A}_p = \begin{pmatrix} 0.00020 & 0.00000 \\ 0.00000 & 0.24560 \end{pmatrix}$$

$$\mathbf{A}_p^{\dagger}(\mathbf{I} - \mathbf{P}(10000))\mathbf{A}_p = \begin{pmatrix} 0.99980 & 0.00000 \\ 0.00000 & 0.75440 \end{pmatrix}$$

Transforming **if** into **choose**

Based on the abstract branching probabilities we can replace tests, e.g. in **if**'s, by probabilistic choices. In a a first phase, we need to determine the probabilities of abstract values.

If we have the probabilities of z_0 being even or odd we can compute the probabilities of the **then** and **else** branch using $\mathbf{P}^{\#}$. For z_0 being even and odd with the same probability:

$$[\textbf{choose}]^1$$
$$\quad \tfrac{1}{2} : ([x \leftarrow \& z_1]^2;\ [y \leftarrow \& z_2]^3)$$
$$\textbf{or}$$
$$\quad \tfrac{1}{2} : ([x \leftarrow \& z_2]^4;\ [y \leftarrow \& z_1]^5)$$
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If we have the probabilities of $\mathtt{z}_0$ being even or odd we can compute the probabilities of the **then** and **else** branch using $\mathbf{P}^{\#}$. For $\mathtt{z}_0$ being even and odd with the same probability:

if $[(\mathtt{z}_0 \text{ mod } 2 = 0)]^1$ **then**
$[\mathtt{x} \leftarrow \&\mathtt{z}_1]^2$; $[\mathtt{y} \leftarrow \&\mathtt{z}_2]^3$
else
$[\mathtt{x} \leftarrow \&\mathtt{z}_2]^4$; $[\mathtt{y} \leftarrow \&\mathtt{z}_1]^5$
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[**stop**]^6

Probabilistic Pointer Analysis

The typical result of a probabilistic pointer analysis is a so-called points-to matrix: records for every program point the probability that a pointer refers to particular (other) variable.

Consider again our standard example.

if $[(z_0 \bmod 2 = 0)]^1$ **then**
    $[x \leftarrow \&z_1]^2$; $[y \leftarrow \&z_2]^3$
else
    $[x \leftarrow \&z_2]^4$; $[y \leftarrow \&z_1]^5$
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$[$**stop**$]^6$

Where do x and y point to with what probabilities?

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$$
\begin{array}{l}
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\textbf{else} \\
\quad [\mathtt{x} \leftarrow \&\mathtt{z}_2]^4;\ [\mathtt{y} \leftarrow \&\mathtt{z}_1]^5 \\
\textbf{fi} \\
[\textbf{stop}]^6
\end{array}
$$

Where do $\mathtt{x}$ and $\mathtt{y}$ point to with what probabilities?

Points-To Matrix vs Points-To Tensor

if $[(z_0 \bmod 2 = 0)]^1$ **then**
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else
$[x \leftarrow \&z_2]^4;\ [y \leftarrow \&z_1]^5$
fi
$[\textbf{stop}]^6$

Points-To Matrix

	$\&x$	$\&y$	$\&z_0$	$\&z_1$	$\&z_2$
x	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
y	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$

Points-To Matrix vs Points-To Tensor

if $[(\mathtt{z}_0 \text{ mod } 2 = 0)]^1$ **then**
$[\mathtt{x} \leftarrow \&\mathtt{z}_1]^2$; $[\mathtt{y} \leftarrow \&\mathtt{z}_2]^3$
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$[\mathtt{x} \leftarrow \&\mathtt{z}_2]^4$; $[\mathtt{y} \leftarrow \&\mathtt{z}_1]^5$
fi
$[\textbf{stop}]^6$

Points-To Matrix

$$(0,0,0,\frac{1}{2},\frac{1}{2}) \quad — \quad (0,0,0,\frac{1}{2},\frac{1}{2}).$$

Points-To Matrix vs Points-To Tensor

if $[(\mathtt{z}_0 \bmod 2 = 0)]^1$ **then**
$[\mathtt{x} \leftarrow \&\mathtt{z}_1]^2;\ [\mathtt{y} \leftarrow \&\mathtt{z}_2]^3$
else
$[\mathtt{x} \leftarrow \&\mathtt{z}_2]^4;\ [\mathtt{y} \leftarrow \&\mathtt{z}_1]^5$
fi
$[\textbf{stop}]^6$

Points-To Matrix

$$(0, 0, 0, \frac{1}{2}, \frac{1}{2}) \quad — \quad (0, 0, 0, \frac{1}{2}, \frac{1}{2}).$$

Points-To Tensor

$$\frac{1}{2} \cdot (0, 0, 0, 1, 0) \otimes (0, 0, 0, 0, 1) + \frac{1}{2} \cdot (0, 0, 0, 0, 1) \otimes (0, 0, 0, 1, 0)$$