Information and Coding Theory (CO349)

Coding and Decoding

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Imperial College London

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Information and Codes

What will be covered in this course?

- Data and Information Representation
- Data and Information Transmission

We will look at these problems from a quantitative, statistical point of view as well as form an algebraic point of view.

In German, Dutch, Italian, etc: CS is Informatik – Informatica Information Science rather than Computer Science.

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Practicalities

Two Lecturers

Herbert Wiklicky Slides on `~herbert/teaching.html`
h.wiklicky@imperial.ac.uk
Teaching $3\frac{1}{2}$ weeks until 29 October
Open-book coursework test 24 October (or 29?)

Mahdi Cheraghchi Slides on Piazza
cheraghchi@gmail.com
Teaching $3\frac{1}{2}$ weeks from 31 October
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João Lourenco Ribeiro GTA, Piazza, etc.

Exam: Week 11, 13 December 2018, 2 hours (3 out of 4).

Different classes, different background, different applications.

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Information and Coding (Theory)

Part I: Information Theory

- Simple Codes and Coding
- Revision: Probabilities
- Information Representation
- Information Transmission
- [Information Manifestation]

Part II: Coding Theory

- Revision: Linear Algebra
- Linear Codes
- Algebraic Error-Correcting Codes
- [Information in Cryptography]

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Information and Security

Side-Channel Attacks (Kocher, 1996)
The problem appears in attacks against public encryption algorithms like RSA. In (optimised) versions of de/encoding (using modular exponentation) properties of the secrete key determine the execution time.

How much information about the secret key is revealed?

Differential Privacy (Dwork, 2006)
In large (statistical) databases an attacker can try to reveal information about individuals (de-anonymise), e.g. there are only 3 under-25 with hair loss registered, and there are two people getting hair-loss treatment in SW7. Kim is on the database, 21 and lives at 170 Queen's Gate.

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Information – A Measure of Surprise

The Entropy of a probability distribution **p** is:

$$H(\mathbf{p}) = -\sum \mathbf{p}(x) \log_2(\mathbf{p}(x)).$$

Example (Cover&Thomas)

Consider a Cheltenham Horse Race with 8 horses with winning chances $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64})$, then the entropy is 2.

Label horses $0, 10, 110, 1110, 111100, 111101, 111110, 111111$ then we need only only 2 bits in average to report winner.

For equally strong horses $(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ we have an entropy of 3, and the minimal message length is also 3 bits.

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Surprise vs Prejudice

A father and son are on a fishing trip in the mountains of Wales. On the way back home their car has a serious accident.

The father is immediately killed and declared dead on the site of the accident. However, the son is severely injured and driven by ambulance to the next hospital.

When the son is brought into the operating theatre the surgeon exclaims "I can't do this, he is my son."

Scientific American, 1980s

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Everything You Always Wanted to Know About Logarithms But Were Afraid to Ask

Defined (implicitly) via the equation:

$$x = b^{\log_b(x)}$$

Common logarithms: $b = 2$, $b = 10$, and $b = 2.71828\ldots = e$

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

$$\log_b(x^n) = n\log_b(x), \ldots \qquad \log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

Example

$\log_2(64) = 6$ as $2^6 = 64$; $\log_2(\frac{1}{64}) = -6$ as $\frac{1}{64} = 2^{-6}$ $(2^{-1} = \frac{1}{2})$.

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Messages and Strings

Definition

An alphabet is a finite set S; its members $s \in S$ are referred to as symbols.

Definition

A message m in the alphabet S is a finite sequence of elements in S:

$$m = s_1 s_2 \cdots s_n \quad \text{with } s_i \in S, 1 \leq i \leq n.$$

A message is often also referred to as a string or word in S. The number $n \in \mathbb{N}$ is the length of m.

Notation for length $|m| = n$; prefix $m|_k = s_1 s_2 \cdots s_k$ (with $k \leq n$).

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Notation

Consider also string of length zero, often denoted ε with $|\varepsilon| = 0$.

By S^0 we denote the set of words containing only ε; S^n denotes the set of all strings of length n; and

$$S^* = S^0 \cup S^1 \cup S^2 \cup \cdots$$

Example

With the alphabet $S = \mathbb{B} = \{0, 1\}$ we can produce all binary messages of a certain length n. Concretely, for example:

$$\mathbb{B}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$$

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Coding

Definition

Let S and T be two alphabets. A code c is an *injective* function

$$c : S \to T^*$$

For each symbol $s \in S$ the string $c(s) \in T^*$ is called the codeword for s. The set of all codewords

$$C = \{c(s) \mid s \in S\}$$

is also referred to as the code.

For $|T| = 2$, e.g. $T = \mathbb{B}$, we have a binary, for $|T| = 3$ a ternary, and for $|T| = b$ a b-ary code.

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Example: Morse Code

Example

Development from 1836 by Samuel F. B. Morse (1791–1872).
This is a code $c : \{A, \ldots, Z\} \to \{\bullet, -, \odot\}^*$ (with variations):

A	$\bullet -$	H	$\bullet\bullet\bullet\bullet$	O	$---$	V	$\bullet\bullet\bullet-$
B	$-\bullet\bullet\bullet$	I	$\bullet\bullet$	P	$\bullet--\bullet$	W	$\bullet--$
C	$-\bullet-\bullet$	J	$\bullet---$	Q	$--\bullet-$	X	$-\bullet\bullet-$
D	$-\bullet\bullet$	K	$-\bullet-$	R	$\bullet-\bullet$	Y	$-\bullet--$
E	$\bullet$	L	$\bullet-\bullet\bullet$	S	$\bullet\bullet\bullet$	Z	$--\bullet\bullet$
F	$\bullet\bullet-\bullet$	M	$--$	T	$-$		
G	$--\bullet$	N	$-\bullet$	U	$\bullet\bullet-$		$\odot$

Most famous (after April 1912) perhaps:

$$SOS \mapsto \bullet\bullet\bullet\odot---\odot\bullet\bullet\bullet\odot \equiv \bullet\bullet\bullet---\bullet\bullet\bullet$$

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A code $c : S \to T^*$ is extended to S^* as follows: Given a string $s_1 s_2 \cdots s_n$ in S^* we define

$$c(s_1 s_2 \cdots s_n) = c(s_1)c(s_2) \cdots c(s_n)$$

Note: This way we overload the function c (polymorphism).

Definition

A code $c : S \to T^*$ is uniquely decodeable (short UD) if the extended code function $c : S^* \to T^*$ is *injective*.

This means that every string in T^* corresponds to at most one message in S^*.

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Example

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Let $S = \{x, y, z\}$ and take a code $c : S \to \mathbb{B}^*$ defined as

$$x \mapsto 0 \qquad y \mapsto 10 \qquad z \mapsto 11$$

Take the original stream $yzxxzyxzyy \cdots$ in S^* and encode it.

Is it always possible to decode the encoded stream in $T^* = \mathbb{B}^*$?

Example

Same code as above. Consider encoded stream

$$1000111000 \cdots$$

What is the original stream? Is it uniquely determined?

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$$x \mapsto 0 \qquad y \mapsto 10 \qquad z \mapsto 11$$

Take the original stream $yzxxzyxzyy\cdots$ in S^* and encode it.

Is it always possible to decode the encoded stream in $T^* = \mathbb{B}^*$?

Example

Same code as above. Consider encoded stream

$$1000111000\cdots$$

What is the original stream? Is it uniquely determined?

The Decoding Problem

Given a sequence in T^* can we (sequentially) decode it?

Example

Same code with $C = \{0, 10, 11\}$. Consider $100111000\cdots$
Take prefix of length 1, first 1: it is not in the codebook C.
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Without the 'pause' $\odot$ in the Morse code we could not decode it, e.g. IEEE as $\bullet\,\bullet\,\bullet\,\bullet\,\bullet$ vs $\bullet\,\bullet\odot\bullet\odot\bullet\odot\bullet\odot$.

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Prefix-Free

Definition

A code $c : S \to T^*$ is prefix-free (short PF) if there is no pair of codeworts $q = c(s)$ and $q' = c(s')$ such that

$$q' = qr \quad \text{for some non-empty word } r \in T^*.$$

Example

Let $S = \{w, x, y, z\}$ with $c : S \to \mathbb{B}^*$ defined by

$$w \mapsto 10 \quad x \mapsto 01 \quad y \mapsto 11 \quad z \mapsto 011$$

This code is not PF. It is not UD either, because:

$$wxwy \mapsto 10011011 \text{ as well as } wzz \mapsto 10011011$$

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PF implies UD

Theorem

If a code $c : S \to T^*$ is prefix-free then it is uniquely decodable.

Proof (Indirect).

Take $x = x_1 x_2 \cdots x_m$ and $y = y_1 y_2 \cdots y_n$ such that $c(x) = c(y)$ with $c(x) = c(x_1)c(x_2)\cdots c(x_m)$ and $c(y) = c(y_1)c(y_2)\cdots c(y_n)$.

The prefixes (of length k) must be the same $c(x)|_k = c(y)|_k$ for any k. It might be that $|c(x_1)| \neq |c(y_1)|$.

Assume w.l.o.g. $|c(x_1)| \leq |c(y_1)|$, and take $k = |c(x_1)|$ then $c(x)|_k = c(x_1) = c(y_1)|_k$ and thus $c(y_1) = c(x_1)r$ for some $r \in T^*$. This contradicts PF. Repeat this argument to cover all of $c(x) = c(y)$. Formally: Induction on lengths m and n. □

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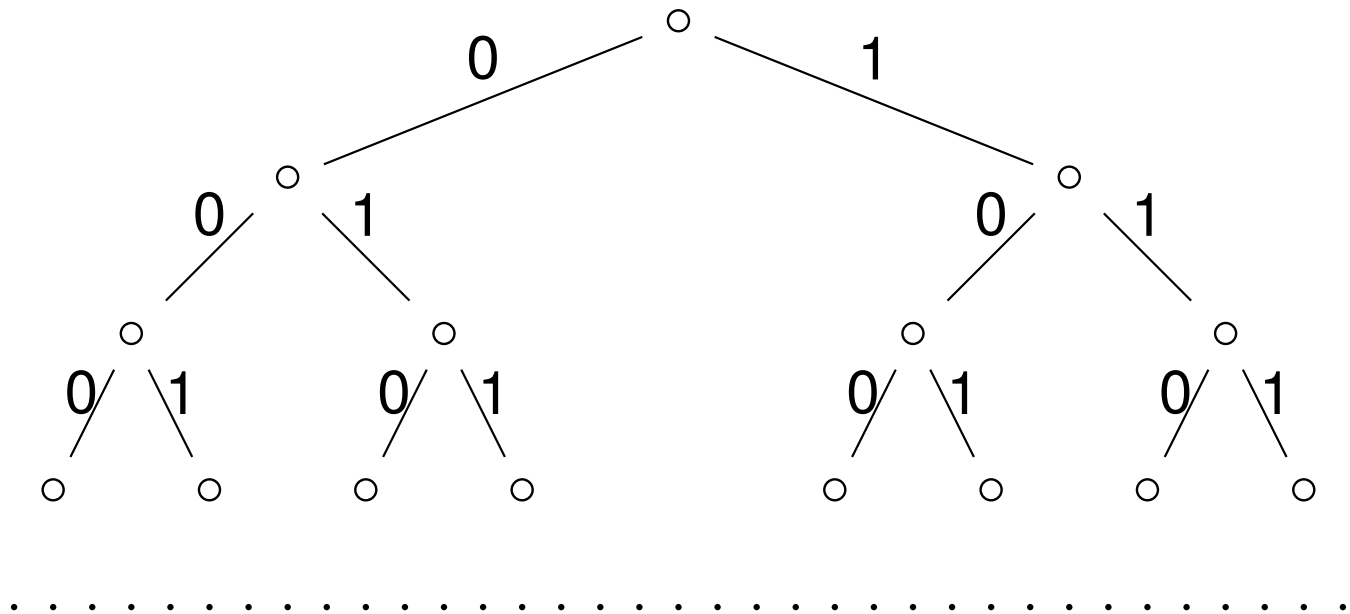
Strings as Paths in Trees

If we concentrate on binary codes with $T = \mathbb{B}$ then we can see any possible codeword in C as a (finite) path in a binary tree:



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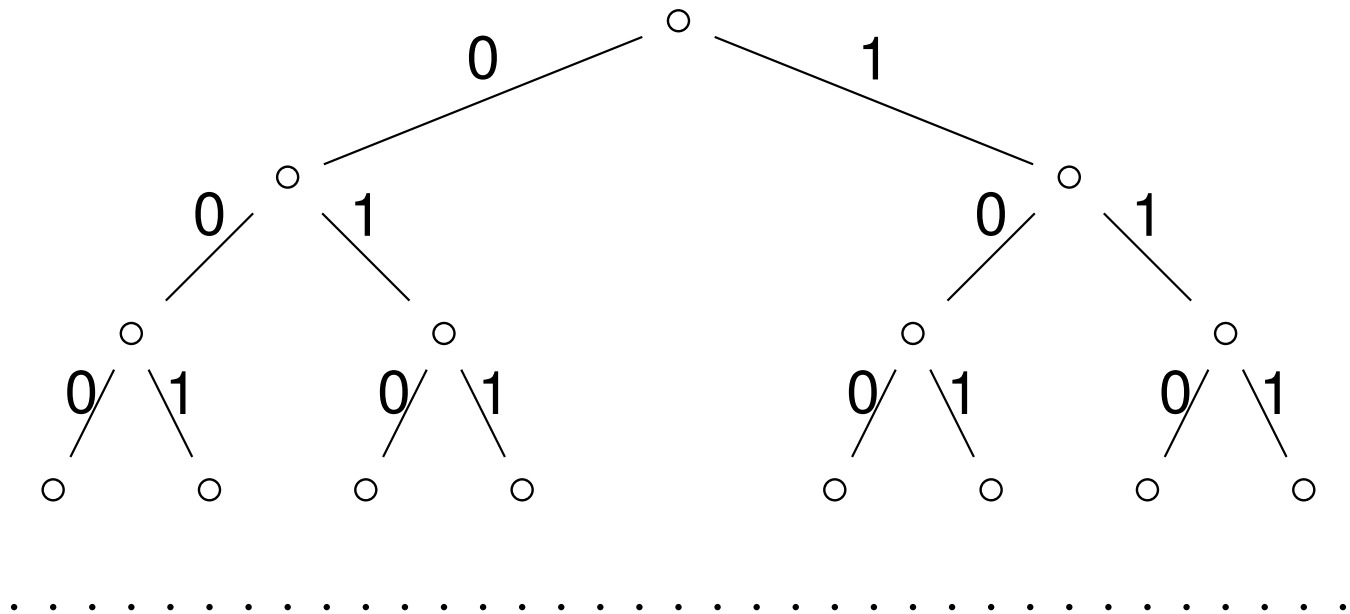

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The codewords in C can be represented by the end-points of the corresponding paths. If a code C is PF then none of its decendents can represent a codeword in C.

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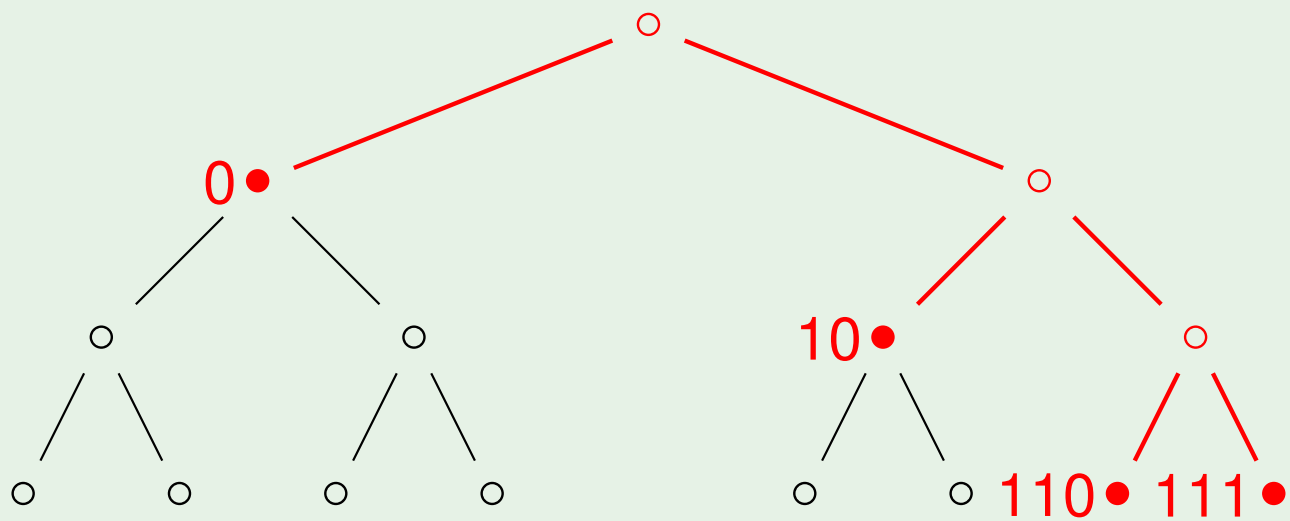

Parameters of a Code

Definition

Given a code $c : S \to T^*$, let n_i denote the number of symbols $s \in S$ which are decoded by strings of length i, i.e.

$$n_i = |\{s \in S \mid |c(s)| = i\}| = |\{s \in S \mid c(s) \in T^i\}|$$

We refer to $n_1, n_2, \cdots, n_M$ as parameters of c, where M is the maximal length of a codeword.

The number of all potential codewords of length i is is given by $b^i = |T^i|$, in particular: $b = |T| = b^1$.

Note that $\frac{n_i}{b^i}$ gives the fraction of possible codewords of length i which is actually used by a code ("filling rate").

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Kraft-McMillan Number

Definition

Given a code $c : S \to T^*$ with parameters $n_1, n_2, \cdots, n_M$. The Kraft-McMillan number of c is defined as:

$$K = \sum_{i=1}^{M} \frac{n_i}{b^i} = \frac{n_1}{b} + \frac{n_2}{b^2} + \cdots + \frac{n_M}{b^M}$$

Example

Consider a binary code, i.e. $b = b^1 = 2$. The parameters of the code are $n_1 = 1, n_2 = 1, n_3 = 2$ (like in the last example), then $b^i = 2^i$, i.e. $b^1 = 2, b^2 = 4, b^3 = 8$ and

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Existence of PF Codes

Theorem (L.G. Kraft, 1949)

Given an alphabet T with $|T| = b$ and (code) parameters $n_1, n_2, \cdots, n_M \in \mathbb{N}$. If $K \leq 1$ there exists a PF b-ary code $c : S \to T^*$ with these parameters.

Example

Construct a PF binary code with $n_2 = 2, n_3 = 3, n_4 = 1$.
First check that $K = \frac{8}{16} + \frac{6}{16} + \frac{1}{16} = \frac{15}{16} \leq 1$.

- Codewords of length 2: take any two, e.g. 00 and 01.
- Codewords of length 3: can't take $00\cdots$ or $01\cdots$ but we can take all of form $1s_2s_3$ with $s_2, s_3 \in \mathbb{B}$; in fact we need only three of 100, 101, 110, and (maybe) not 111.
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Proof.

Any partial sum of the one defining K also fulfills $\sum_i \frac{n_i}{b^i} \leq 1$, e.g. $\frac{n_1}{b^1} \leq 1$, $\frac{n_1}{b^1} + \frac{n_2}{b^2} \leq 1$, $\frac{n_1}{b^1} + \frac{n_2}{b^2} + \frac{n_3}{b^3} \leq 1$, etc. This implies:

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or in general:

$$n_i \leq b\left(b^{i-1} - n_1 b^{i-2} - n_2 b^{i-3} - \ldots - n_{i-1}\right).$$

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Consider a code $c : S \to T^*$ with parameters $n_1, n_2, \ldots, n_M$. We aim to encode strings in S of length r. Let $q_r(i)$ be the number of strings of length r in S^* encoded in a string of length i in T^*.

Example

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- For $|c(x_1 x_2)| = 3$: Then $|c(x_1)| = 1$ and $|c(x_2)| = 2$ or vice versa, so we get $q_2(3) = 2 \times 3 + 3 \times 2 = 12$.
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Take $c : S \to T^*$ with parameters $n_1 = 2$, $n_2 = 3$; what is $q_2(i)$? Given $x = x_1 x_2 \in S^2$, then $c(x) \in T$ has lengths 2, 3 or 4.

- For $|c(x_1 x_2)| = 2$: Then $|c(x_1)| = 1 = |c(x_2)|$, since $n_1 = 2$ we have $q_2(2) = 2 \times 2 = 4$.
- For $|c(x_1 x_2)| = 3$: Then $|c(x_1)| = 1$ and $|c(x_2)| = 2$ or vice versa, so we get $q_2(3) = 2 \times 3 + 3 \times 2 = 12$.
- For $|c(x_1 x_2)| = 4$: Then $|c(x_1)| = 2 = |c(x_2)|$, thus we know that $q_2(4) = 3 \times 3 = 9$.

Encoding Strings

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Generating Functions

Generating functions are an important tool for investigating sequences, e.g. enumerations of combinatorial objects.

Definition

For a sequence of numbers $q(1), q(2), q(3), \ldots$ the generating function $Q(x)$ is given as the polynomial (formal power series) in the unknown variable x:

$$Q(x) = q(1)x + q(2)x^2 + q(3)x^3 + \cdots$$

For a code $c : S \to T^*$ we consider for $1 \leq i \leq rM$:

$$q_r(i) = |\{|c(s)| = i \mid |s| = r\}|$$

and the generating function:

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Counting Principle (CP)

Theorem

Given a UD code $c : S \to T^*$ with $|c(s)| \leq M$ for all $s \in S$ with generating function $Q_r(x)$ then for all $r \geq 1$:

$$Q_r(x) = Q_1(x)^r.$$

Example

Take $c : S \to T^*$ with parameters $n_1 = 2$, $n_2 = 3$ as above.
What is the relationship between Q_1 and Q_2?
Clearly $q_1(1) = 2 = n_1$ and $q_1(2) = 3 = n_2$ and thus

$$Q_1(x) = 2x + 3x^2 \quad \text{and} \quad Q_2(x) = 4x^2 + 12x^3 + 9x^4$$

and thus indeed $(Q_1(x))^2 = Q_1(x)Q_1(x) = Q_2(x)$.

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Proof of the Counting Principle

Proof $*$.

Consider $Q_1(x)^r = (n_1 x + n_2 x^2 + n_3 x^3 + \cdots + n_M x^M)^r$ with $n_j = q_1(j)$. What is the coefficient n to x^k for $1 \leq k \leq rM$? To get any term of the form nx^k multiply r terms from $Q_1(x)$ to get $nx^k = n_{i_1} x^{i_1} \times n_{i_2} x^{i_2} \times \cdots \times n_{i_r} x^{i_r}$ with $i_1 + i_2 + \cdots + i_r = k$ and then $n = n_{i_1} n_{i_2} \cdots n_{i_r}$. The coefficient to x^k is

$$\sum_{i_1+i_2+\cdots+i_r=k} n_{i_1} n_{i_2} \cdots n_{i_r}$$

Consider $Q_r(x)$ or $q_r(k)$, i.e. the number of words of length k encoded from strings of length r: Each such codeword is formed as $c(x_1)c(x_2)\cdots c(x_r)$, we can chose any x_j with $|c(x_j)| = i_j$ as long as $i_1 + i_2 + \cdots + i_r = k$ in which case there $n_{i_1} n_{i_2} \cdots n_{i_r}$ possible (input) stings. In total we have to consider again all partitions of k □

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Kraft-McMillan Number for UD Codes

Theorem (B. McMillan, 1956)

If there exists a uniquely decodable code $c : S \to T^*$ with parameters $n_1, n_2, \cdots, n_M \in \mathbb{N}$ then $K \leq 1$.

$$\exists \text{ UD} \Longrightarrow K \leq 1 \Longrightarrow \exists \text{ PF} \quad \text{and} \quad \text{PF} \Longrightarrow \text{UD}$$

Corollary

The following statements are equivalent for codes $c : S \to T^*$:

- *There exists a UD code with parameters $n_1, n_2, \ldots, n_M$.*
- *It holds that $K \leq 1$ for parameters $n_1, n_2, \ldots, n_M$.*
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The following statements are equivalent for codes $c : S \to T^*$*:*

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Proof of McMillan's Theorem

Proof (Indirect) $*$.

Fix $r \geq 1$ and take $x = \frac{1}{b}$ in $Q_r(x)$, we get ($M = \max|c(s)|$):

$$Q_r\left(\frac{1}{b}\right) = \frac{q_r(1)}{b} + \frac{q_r(2)}{b^2} + \frac{q_r(3)}{b^3} + \cdots + \frac{q_r(rM)}{b^{rM}}.$$

UD implies $q_r(i) \leq b^i$ and thus $\frac{q_r(j)}{b^j} \leq 1$ for all $1 \leq j \leq rM$, and therefore $Q_r\left(\frac{1}{b}\right) \leq rM$. From $K = Q_1\left(\frac{1}{b}\right)$ and by CP we have:

$$K^r = \left(Q_1\left(\frac{1}{b}\right)\right)^r = Q_r\left(\frac{1}{b}\right) \leq rM \quad \text{thus} \quad \frac{K^r}{r} \leq M \ \ \forall r.$$

Suppose $K > 1$ or $K = 1 + k$ with $k > 0$, then $K^r = (1+k)^r = 1 + rk + \frac{1}{2}r(r-1)k^2 + \cdots$ Furthermore thus $K^r > \frac{1}{2}r(r-1)k^2$ or $\frac{K^r}{r} > \frac{1}{2}(r-1)k^2$ which, for large r, contradicts $\frac{K^r}{r} \leq M$.

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Some Probability Theory

Probability theory is concerned with quantifying or measuring the chances that certain events can happen.

We consider an event space, i.e. a finite set Ω and a set $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ of measurable sets in Ω which form a Boolean algebra (based on union, intersection and complement). For finite sets one can use the power-set $\mathcal{B} = \mathcal{P}(\Omega)$.

Probabilities are then assigned via a measure, i.e. a function $\Pr : \mathcal{B} \to \mathbb{R}$ or $m : \mathcal{B} \to \mathbb{R}$ or $\mu : \mathcal{B} \to \mathbb{R}$.

Note: For (uncountable) infinite Ω one needs to develop a more general measure theory (e.g. with $\mathcal{B}$ a σ-algebra) to overcome problems like $\mu([0,1]) = 1$ but $\mu(x) = 0$ for all $x \in [0,1]$.

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Finite Probability Spaces

Consider a finite measurable spaces $(\Omega, \mathcal{B})$ with $|\Omega| = n$,

Definition

A probability (measure) $\Pr : \mathcal{B}$ on $(\Omega, \mathcal{B})$ has to fulfill

- $\Pr(\Omega) = 1$.
- $0 \leq \Pr(A) \leq 1$ for all $A \in \mathcal{B}$.
- $\Pr(A \cup B) = \Pr(A) + \Pr(B)$ for $A \cap B = \emptyset$.

Some further rules (which follow from the axioms above):

- $\Pr(\emptyset) = 0$,
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Random Distributions

For finite probability spaces $(\Omega, \mathcal{B}, \Pr)$ with $|\Omega| = n$, we can define a probability (measure) via atoms in $\omega \in \Omega$.

Definition

A probability distribution is a function $\mathbf{p} : \Omega \to [0, 1]$, with

$$\sum_{\omega \in \Omega} \mathbf{p}(\omega) = 1.$$

If we enumerate the elements in Ω in some arbitrary way as $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ then we can also represented $\mathbf{p}$ by a (row) vector in $\mathbb{R}^n$.

$$\mathbf{p} = (\mathbf{p}(\omega_1), \mathbf{p}(\omega_2), \ldots, \mathbf{p}(\omega_n))$$

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Random Variables

Probability distributions on a finite Ω define a probability (measure) in the obvious way

$$\Pr(A) = \sum_{\omega \in A} \mathbf{p}(\omega).$$

Definition

A random variable is function $X : \Omega \to \mathbb{R}$.

We can represent random variables as (column) vectors in $\mathbb{R}^n$.

Example

Consider a dice. The event space describes the top face of the dice, i.e. $\Omega = \left\{ \boxed{\cdot}, \boxed{\cdot\,\cdot}, \boxed{\cdot\,\cdot}, \boxed{\cdot\,\cdot}, \boxed{\cdot\,\cdot}, \boxed{\cdot\,\cdot} \right\}$, define X which counts the number of eyes, e.g. $X\left(\boxed{\cdot\,\cdot} \right) = 5$ etc.

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Moments in Probability

Expectation Value. For a random variable X and probability distribution $\mathbf{p}$ we define:

$$\mathbf{E}(X) = \sum_{\omega \in \Omega} \mathbf{p}(\omega) X(\omega) = \sum_i \mathbf{p}_i \mathbf{X}_i = \mu_X$$

One can show: $\mathbf{E}(X+Y) = \mathbf{E}(X) + \mathbf{E}(Y)$ and $\mathbf{E}(\alpha X) = \alpha \mathbf{E}(X)$.

Example

Consider a (fair) dice and previous random variable X, then

$$\mathbf{E}(X) = 1\frac{1}{6} + 2\frac{1}{6} + 3\frac{1}{6} + 4\frac{1}{6} + 5\frac{1}{6} + 6\frac{1}{6} = \frac{21}{6}$$

Variance. For random variable X and distribution $\mathbf{p}$ we define:

$$\mathrm{Var}(X) = \mathbf{E}((X - \mathbf{E}(X))^2 = \mathbf{E}(X^2) - (\mathbf{E}(X))^2.$$

The standard deviation is $\sigma_X = \sqrt{\mathrm{Var}(X)}$.

Moments in Probability

Expectation Value. For a random variable X and probability distribution $\mathbf{p}$ we define:

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One can show: $\mathbf{E}(X + Y) = \mathbf{E}(X) + \mathbf{E}(Y)$ and $\mathbf{E}(\alpha X) = \alpha \mathbf{E}(X)$.

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Bayes Theorem and Independence

Given two subsets A and B in a probability space $(\Omega, \mathcal{B}, \Pr)$. The conditional probability of A given that B has happened is

$$\Pr_B(A) = \Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

One can show Bayes Theorem which states:

$$\Pr_A(B) = \frac{\Pr_B(A)\Pr(B)}{\Pr(A)} = \frac{\Pr(A \mid B)\Pr(B)}{\Pr(A)} = \Pr(B \mid A).$$

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Products and Probability

Given two probability spaces $(\Omega_1, \Pr_1)$ and $(\Omega_2, \Pr_2)$, to keep things simple use $\mathcal{B}_i = \mathcal{P}(\Omega_i)$. We can define a probability Pr on the cartesian product $\Omega = \Omega_1 \times \Omega_2$ via:

$$\Pr(\langle \omega_1, \omega_2 \rangle) = \Pr_1(\omega_1)\Pr_1(\omega_2)$$

If $\Pr_1$ and $\Pr_2$ correspond to probability distributions $\mathbf{p}_1$ and $\mathbf{p}_2$ and $\mathbf{p}$ to Pr then $\mathbf{p} = \mathbf{p}_1 \otimes \mathbf{p}_2$, i.e. the tensor product.

Caveat: Not all distributions on $\Omega_1 \times \Omega_2$ are a product.

Example

Consider $\Omega_1 = \{0, 1\}$ and $\Omega_2 = \{z, o\}$ and probability $\Pr(\langle 0, z \rangle) = \Pr(\langle 1, o \rangle) = \frac{1}{2}$ and $\Pr(\langle 0, o \rangle) = \Pr(\langle 1, z \rangle) = 0$ cannot be represented as a product $\mathbf{p}_1 \otimes \mathbf{p}_2$.
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Tensor/Kronecker Product

Given a $n \times m$ matrix $\mathbf{A}$ and a $k \times l$ matrix $\mathbf{B}$:

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} b_{11} & \dots & b_{1l} \\ \vdots & \ddots & \vdots \\ b_{k1} & \dots & b_{kl} \end{pmatrix}$$

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Special cases are square matrices ($n = m$ and $k = l$) and vectors (row $n = k = 1$, column $m = l = 1$).

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Correlation

The covariance of two random variables X and Y is:

$$\text{Cov}(X, Y) = \mathbf{E}((X - \mathbf{E}(X))\mathbf{E}((Y - \mathbf{E}(Y)) = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y)$$

The correlation coefficient is $\rho(X, Y) = \text{Cov}(X, Y)/(\sigma_X \sigma_Y)$.

For independent random variables X and Y – i.e. if we have $\Pr((X = x_j) \cap (Y = y_k)) = \Pr(X = x_j)\Pr(Y = y_k)$:

$$\mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y) \text{ and } \text{Cov}(X, Y) = \rho(X, Y) = 0.$$

Note that $\rho(X, Y) = 0$ does **not** imply independence.

Example

$\Pr(X = x) = \frac{1}{3}$ for $x = -1, 0, 1$ and $Y = X^2$, then $\rho(X, Y) = 0$.

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