Information and Coding Theory CO349

Tutorial - Answers

Sheet 1

Exercise 1 (Entropy) *Consider the Cheltenham horse race (slide 7). Calculate the entropy for winning chances:*

$$\mathbf{p}_1 = (1, 0, 0, 0, 0, 0, 0, 0)$$

$$(\frac{1}{2}, 0, 0, 0, 0, 0, 0, \frac{1}{2})$$

$$(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64})$$

$$(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$$

Compute the corresponding entropies. What does that tell you about the property "entropy" describes?

Solution

$$\begin{aligned}
\mathbf{H}((1,0,0,0,0,0,0,0)) &= -1 \cdot \log_2(1) \\
&= -1(0) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathbf{H}((1,0,0,0,0,0,0,0)) &= -\frac{1}{2} \cdot \log_2(\frac{1}{2}) - \frac{1}{2} \cdot \log_2(\frac{1}{2}) - \frac{1}{2}(-1) - \frac{1}{2}(-1) \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\mathbf{H}((\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64})) &= -\frac{1}{2} \cdot \log_2(\frac{1}{2}) - \frac{1}{4} \cdot \log_2(\frac{1}{4}) - \frac{1}{8} \cdot \log_2(\frac{1}{8}) - \frac{1}{16} \cdot \log_2(\frac{1}{16}) \\
&\quad - \frac{1}{64} \cdot \log_2(\frac{1}{64}) - \frac{1}{64} \cdot \log_2(\frac{1}{64}) - \frac{1}{64} \cdot \log_2(\frac{1}{64}) - \frac{1}{64} \cdot \log_2(\frac{1}{64}) \\
&= -\frac{1}{2} \cdot (-1) - \frac{1}{4} \cdot (-2) - \frac{1}{8} \cdot (-3) - \frac{1}{16} \cdot (-4)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{64}\cdot(-6)-\frac{1}{64}\cdot(-6)-\frac{1}{64}\cdot(-6)-\frac{1}{64}\cdot(-6) \\
= \;& 2
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{H}((\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8})) &= -8\times(-\frac{1}{8}\log_2(\frac{1}{8})) \\
&= -8\times(-\frac{1}{8}(-3)) \\
&= 3
\end{aligned}
$$

The more even (uniform) a distribution, the larger its entropy.

Exercise 2 (Biggs, Example 1.5) *The following messages are coded versions of English sentences. Explain the codes and what is the original messages:*

7 15 15 4 27 12 21 3 11

00111 01111 01111 00100 11011 01100 10101 00011 01011

Solution The message is "GOOD⊙LUCK". The code is $c_1 : \{A, \ldots, Z, \odot\} \to \{1, \ldots, 27\}^*$ and $c_2 : \{A, \ldots, Z, \odot\} \to \mathbb{B}^*$. with

A	$\mapsto$	1	A	$\mapsto$	00001
B	$\mapsto$	2	B	$\mapsto$	00010
C	$\mapsto$	3	C	$\mapsto$	00011
...	...	...	...	...	...
Z	$\mapsto$	26	Z	$\mapsto$	11010
$\odot$	$\mapsto$	27	$\odot$	$\mapsto$	11011

Exercise 3 (Biggs, Example 2.1) *Given a binary code $c : \{s_1, s_2, s_3, s_4\} \to \mathbb{B}^*$ defined by*

$$s_1 \mapsto 00, \quad s_2 \mapsto 010, \quad s_3 \mapsto 100, \quad s_4 \mapsto 111.$$

Is this code UD? Decode

10011101000111010100000.

Solution This one is is PF nad thus also UD (draw tree!). the original message is

$$s_3 s_4 s_2 s_1 s_4 s_2 s_3 s_1.$$

Exercise 4 (Biggs, Example 2.9) *Do PF binary codes with the following parameters*

(i) $n_1 = 0, n_2 = 1, n_3 = 4, n_4 = 3$ *and*

(ii) $n_1 = 1, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 2.$

exist? If so state them. Could they be extended, i.e. more codewords added?

Solution

(i) $n_1 = 0, n_2 = 1, n_3 = 4, n_4 = 3$

$$K = \frac{0}{2} + \frac{1}{4} + \frac{4}{8} + \frac{3}{16} = \frac{15}{16} \leq 1$$

so it exists, e.g.

$$00, 101, 011, 100, 101, 1100, 1101, 1110.$$

Could add codeword(s).

(ii) $n_1 = 1, n_2 = 0, n_3 = 2, n_4 = 3, n_5 = 2.$

$$K = \frac{1}{2} + \frac{0}{4} + \frac{2}{8} + \frac{3}{16} + \frac{2}{32} = \frac{16}{16} \leq 1$$

so it exists, e.g.

$$0, 100, 101, 1100, 1101, 1110, 11110, 11111.$$

Cannot be extended.

Exercise 5 (Biggs, Example 2.13) *If we want to construct a UD code* $c : S \to \mathbb{B}^*$ *with* $|S| = 12$ *with a maximal codeword length* 4. *For which parameters* n_i *does such a code exists?*

Solution The requirments are

$$n_1 + n_2 + n_3 + n_4 = 12$$

and on the Kraft-McMillan number:

$$\frac{n_1}{2} + \frac{n_2}{4} + \frac{n_3}{8} + \frac{n_4}{16} \leq 1$$

Express $n_4 = 12 - n_1 - n_2 - n_3$ in the inequality for K to get:

$$\begin{aligned}
\frac{n_1}{2} + \frac{n_2}{4} + \frac{n_3}{8} + \frac{12 - n_1 - n_2 - n_3}{16} &\leq 1 \\
\frac{8n_1}{2} + \frac{4n_2}{4} + \frac{2n_3}{8} + \frac{12 - n_1 - n_2 - n_3}{16} &\leq 1 \\
8n_1 + 4n_2 + 2n_3 + 12 - n_1 - n_2 - n_3 &\leq 16 \\
7n_1 + 3n_2 + n_3 &\leq 4
\end{aligned}$$

so $n_1 = 0$ and $n_2 \leq 1$. This gives the following seven possible combination of parameters:

n_1	0	0	0	0	0	0	0
n_2	1	1	0	0	0	0	0
n_3	1	0	4	3	2	1	0
n_4	10	11	8	9	10	11	12

Exercise 6 (Product Probability) *Consider* $\Omega_1 = \{0, 1\}$ *and* $\Omega_2 = \{z, o\}$ *and a probability distribution* $Pr(\langle 0, z\rangle) = Pr(\langle 1, o\rangle) = \frac{1}{2}$ *and* $Pr(\langle 0, o\rangle) = Pr(\langle 1, z\rangle) = 0$. *This* **cannot** *be represented as a product* $\mathbf{p}_1 \otimes \mathbf{p}_2$.

In other words: Show that $(\frac{1}{2}, 0, 0, \frac{1}{2})$ *cannot be obtained as a (tensor) product of two two-dimensional vectors. Give an example of a probability distribution which could be represented this way.*

Given a $n \times m$ **matrix A** *and a* $k \times l$ *matrix* **B**:

$$\mathbf{A} = \begin{pmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nm} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} b_{11} & \ldots & b_{1l} \\ \vdots & \ddots & \vdots \\ b_{k1} & \ldots & b_{kl} \end{pmatrix}$$

The **tensor** *or* **Kronecker product** $\mathbf{A} \otimes \mathbf{B}$ *is a* $nk \times ml$ *matrix:*

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & \ldots & a_{1m}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{n1}\mathbf{B} & \ldots & a_{nm}\mathbf{B} \end{pmatrix}$$

Special cases are **square matrices** *($n = m$ and $k = l$) and* **vectors** *(row $n = k = 1$, column $m = l = 1$).*

Solution Assume it could be done, i.e.

$$(\frac{1}{2}, 0, 0, \frac{1}{2}) = (a, b) \otimes (c, d) = (ac, ad, bc, bd).$$

therfore

$$\begin{array}{rcl} \frac{1}{2} & = & ac \\ 0 & = & ad \\ 0 & = & bc \\ \frac{1}{2} & = & bd \end{array}$$

Which is impossible to solve. E.g. from the first equations it follows that a and c are non-zero, from the last equations it follows that b and d must be non-zero, too. But in this case it is impossible that the middle equations hold.

However: $\mathbf{p} = \frac{1}{2}(1, 0) \otimes (1, 0) + \frac{1}{2}(0, 1) \otimes (0, 1)$. Take any distribution on two elements and form its tensor product it, e.g.

$$(\frac{1}{2}, \frac{1}{2}) \otimes (0, 1) = (0, \frac{1}{2}, 0, \frac{1}{2}).$$