Information and Coding Theory CO349

Tutorial

Sheet 3 - Answers

Exercise 1 (Biggs, Ex 5.15) *Consider the channel* Γ *which accepts as input* $1, 2, \ldots, 2n$ *and outputs* 0 *iff the input is* **even** *and* 1 *for* **odd** *input. Write down the channel matrix.*

Show that for any input distribution $\mathbf{p}$ *and* $\mathbf{q} = \mathbf{p}\Gamma$ *we have* $H(\mathbf{p}) - H(\Gamma; \mathbf{p}) = H(\mathbf{q})$.

Furthermore, show that the channel capacity $\gamma = 1$ *and explain this.*

Solution The channel matrix is a $2n \times 2$ matrix:

$$\Gamma_{id} = \begin{cases} 1 & \text{iff } i \bmod 2 = d - 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{or} \quad \Gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 1 & 0 \end{pmatrix}$$

For $\mathbf{p} = (p_1, p_2, \ldots, p_{2n})$ we have

$$\mathbf{t}_{id} = \begin{cases} p_i & \text{iff } i \bmod 2 = d - 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{or} \quad \mathbf{t} = \begin{pmatrix} 0 & p_1 \\ p_2 & 0 \\ 0 & p_3 \\ \vdots & \vdots \\ p_{2n} & 0 \end{pmatrix}$$

This implies that $H(\mathbf{t}) = H(\mathbf{p})$. With this it easy to see that:

$$H(\mathbf{p}) - H(\Gamma; \mathbf{p}) = H(\mathbf{p}) - H(\mathbf{p} \mid \mathbf{q}) = H(\mathbf{p}) - H(\mathbf{t}) + H(\mathbf{q}) = H(\mathbf{q}).$$

The capacity is the maximum of $H(\mathbf{p}) - H(\Gamma; \mathbf{p})$ or (from above) the maximum of $H(\mathbf{q})$. The later is the maximum entropy for $\mathbf{q} = (q, 1 - q)$ – i.e. $h(\mathbf{q})$ – which we know is 1.

The channel transmits one bit of information, namely whether the input is **even** or **odd**.

Exercise 2 (Exam 2015) *Consider the following channel (matrix)* $\Gamma = (\Gamma_{ij})_{i,j=1}^{8}$ *with*

$$\Gamma_{ij} = \frac{1}{8}$$

for $i, j = 1, \ldots, 8$, *and the distributions* $\mathbf{p} = (1, 0, 0, 0, 0, 0, 0, 0)$ *and* $\mathbf{p}^* = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$. *Is* Γ *indeed a channel matrix?*

Describe briefly what effect this channel has on any message sent through it. Compute $\mathbf{H}(\Gamma; \mathbf{p})$ *and* $\mathbf{H}(\mathbf{p})$ *as well as* $\mathbf{H}(\Gamma; \mathbf{p}^*)$ *and* $\mathbf{H}(\mathbf{p}^*)$, *respectively.*

Solution This channel just produces noise, independently of the input, e.g. $\mathbf{p}\Gamma = \mathbf{p}^* = \mathbf{q}$ as well as $\mathbf{p}^*\Gamma = \mathbf{p}^* = \mathbf{q}^*$. It is a channel (matrix) as it is stochastic (i.e. row normalised).

Entropy for input distributions: $\mathbf{H}(\mathbf{p}) = 0$ and $\mathbf{H}(\mathbf{p}^*) = 3 (= \mathbf{H}(\mathbf{q}) = \mathbf{H}(\mathbf{q}^*))$.

For $\mathbf{p}$ we have

$$\mathbf{t} = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For $\mathbf{p}^*$ we have

$$\mathbf{t}^* = \begin{pmatrix} \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} \\ \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} \\ \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} \\ \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} \\ \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} \\ \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} \\ \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} \\ \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} \end{pmatrix}$$

Note that $\mathbf{t} = \mathbf{p} \otimes \mathbf{p}^*$ and $\mathbf{t}^* = \mathbf{p}^* \otimes \mathbf{p}^*$ so by product theorem we can compute the channel entropy easily. One can also compute it directly.

Channel Entropy: $\mathbf{H}(\mathbf{t}) = \mathbf{H}(\mathbf{p}) + \mathbf{H}(\mathbf{p}^*) = 0 + 3 = 3$ thus $\mathbf{H}(\Gamma; \mathbf{p}) = \mathbf{H}(\mathbf{t}) - \mathbf{H}(\mathbf{q}) = 3 - 3 = 0$; and $\mathbf{H}(\mathbf{t}^*) = \mathbf{H}(\mathbf{p}^*) + \mathbf{H}(\mathbf{p}^*) = 3 + 3 = 6$ thus $\mathbf{H}(\Gamma; \mathbf{p}^*) = \mathbf{H}(\mathbf{t}^*) - \mathbf{H}(q^*) = 6 - 3 = 3$.

Exercise 3 (Biggs, Ex 6.15) *If require a (binary) code $C \subseteq \mathbb{B}^6$ with information rate $\rho \geq 0.35$, what is the smallest number of code-words in C.*

Solution For this we need $\rho = \frac{\log_2(|C|)}{6} \geq 0.35$.

This means that $\log_2(|C|) \geq 0.35 \times 6 = 2.1$, or simply $|C| = 2^{\log_2(|C|)} \geq 2^{2.1} = 4.2871$. That implies that C must contain at least 5 code-words.