

Quantum Computation (CO484)

Quantum States and Evolution

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Quantum Postulates

- ▶ The **state** of an (isolated) quantum system is represented by a (normalised) vector in a complex Hilbert space \mathcal{H} .
- ▶ An **observable** is represented by a self-adjoint matrix (operator) \mathbf{A} acting on the Hilbert space \mathcal{H} .
- ▶ The **expected result** (average) when measuring observable \mathbf{A} of a system in state $|x\rangle \in \mathcal{H}$ is given by:

$$\langle A \rangle_x = \langle x | \mathbf{A} | x \rangle = \langle x | \mathbf{A} x \rangle$$

- ▶ The only **possible** results are eigen-values λ_i of \mathbf{A} .
- ▶ The **probability** of measuring λ_n in state $|x\rangle$ is given by:

$$Pr(A = \lambda_n | x) = \langle x | \mathbf{P}_n | x \rangle = \langle x | \mathbf{P}_n x \rangle$$

with $\mathbf{P}_n = |\lambda_n\rangle\langle\lambda_n|$ the orthogonal projection onto the space generated by eigen-vector $|\lambda_n\rangle = |n\rangle$ of \mathbf{A} .

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Complex Numbers

Quantitative information, e.g. measurement results, is usually represented by real numbers \mathbb{R} . For quantum systems we need to consider also complex numbers \mathbb{C} .

A **complex number** $z \in \mathbb{C}$ is a (formal) combinations of two reals $x, y \in \mathbb{R}$:

$$z = x + iy$$

with $i^2 = -1$ or $i = \sqrt{-1}$. The **complex conjugate** of a complex number $z = x + iy \in \mathbb{C}$ is:

$$z^* = \bar{z} = \overline{x + iy} = x - iy = z^\dagger$$

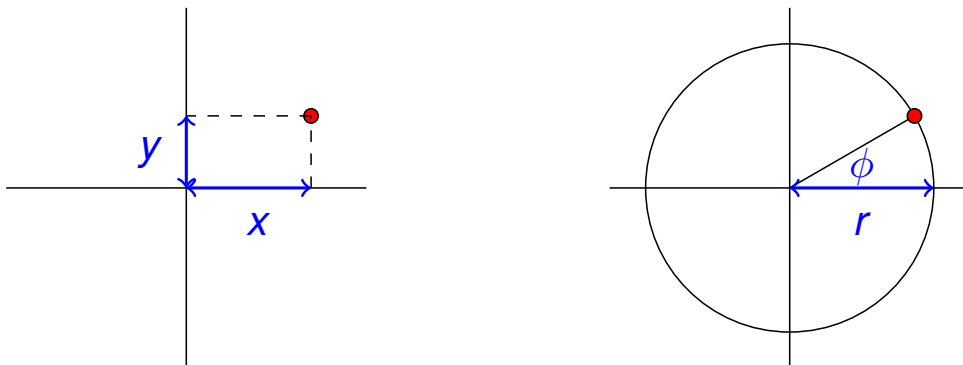
Hauptsatz of Algebra

Complex numbers are algebraically closed: Every polynomial of order n over \mathbb{C} has exactly n roots.

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Polar Coordinates

One can represent numbers $z \in \mathbb{C}$ using the complex plane.



Conversion:

$$x = r \cdot \cos(\phi) \quad y = r \cdot \sin(\phi)$$

$$r = \sqrt{x^2 + y^2} \quad \phi = \arctan\left(\frac{y}{x}\right)$$

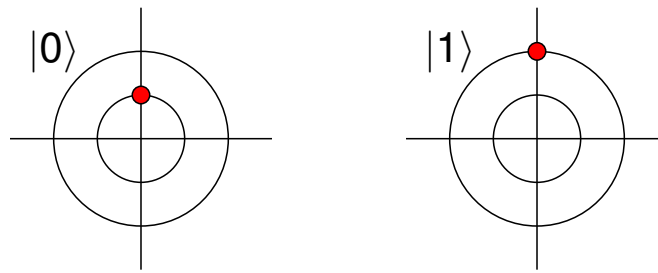
Another representation:

$$(r, \phi) = r \cdot e^{i\phi} \quad e^{i\phi} = \cos(\phi) + i \sin(\phi),$$

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Computational Quantum States

Consider a simple systems with two **degrees of freedom**.



Definition

A **qubit** (quantum bit) is a quantum state of the form

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

where α and β are complex numbers with $|\alpha|^2 + |\beta|^2 = 1$.

Qubits live in a two-dimensional complex vector, more precisely, Hilbert space \mathbb{C}^2 and are **normalised**, i.e.

$$\| |\psi\rangle \| = \langle \psi | \psi \rangle = 1.$$

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Vector Spaces

A **Vector Space** (over a field \mathbb{K} , e.g. \mathbb{R} or \mathbb{C}) is a set \mathcal{V} together with two operations:

Scalar Product $\cdot, \cdot : \mathbb{K} \times \mathcal{V} \mapsto \mathcal{V}$

Vector Addition $+, + : \mathcal{V} \times \mathcal{V} \mapsto \mathcal{V}$

such that $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{K}$:

1. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$
2. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
3. $\exists \mathbf{o} : \mathbf{x} + \mathbf{o} = \mathbf{x}$
4. $\exists -\mathbf{x} : \mathbf{x} + (-\mathbf{x}) = \mathbf{o}$
5. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$
6. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
7. $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$
8. $1\mathbf{x} = \mathbf{x} (1 \in \mathbb{K})$

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Tuple Spaces

Theorem

All finite dimensional vector spaces are isomorphic to the (finite) Cartesian product of the underlying field \mathbb{K}^n (i.e. \mathbb{R}^n or \mathbb{C}^n).

$$\vec{x} = (x_1, x_2, x_3, \dots, x_n) \text{ represents } \mathbf{x} = \sum_{i=1}^n x_i \mathbf{b}_i$$

$$\vec{y} = (y_1, y_2, y_3, \dots, y_n) \text{ represents } \mathbf{y} = \sum_{i=1}^n y_i \mathbf{b}_i$$

Finite dimensional vectors can be represented as tuples via their coordinates with respect to a base $\{\mathbf{b}_i\}_{i=1}^n$.

$$\alpha \vec{x} = (\alpha x_1, \alpha x_2, \alpha x_3, \dots, \alpha x_n)$$

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots, x_n + y_n)$$

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Hilbert Spaces

A complex vector space \mathcal{H} is called an **Inner Product Space** or **(Pre-)Hilbert Space** if there is a complex valued function $\langle \cdot, \cdot \rangle$ on $\mathcal{H} \times \mathcal{H}$ that satisfies $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{H}$ and $\forall \alpha \in \mathbb{C}$:

1. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$
2. $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \iff \mathbf{x} = \mathbf{o}$
3. $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$
4. $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
5. $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$

The function $\langle \cdot, \cdot \rangle$ is called an **inner product** on \mathcal{H} .

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Caveat: Linear in first or second argument?

Mathematical Convention:

$$\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$$

Physical Convention:

$$\langle \mathbf{x} | \alpha \mathbf{y} \rangle = \alpha \langle \mathbf{x} | \mathbf{y} \rangle$$

In mathematics we have:

$$\langle \mathbf{x}, \alpha \mathbf{y} \rangle = \overline{\langle \alpha \mathbf{y}, \mathbf{x} \rangle} = \bar{\alpha} \overline{\langle \mathbf{y}, \mathbf{x} \rangle} = \bar{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle$$

For physicists it is simply:

$$\langle \mathbf{x} | \alpha \mathbf{y} \rangle = \alpha \langle \mathbf{x} | \mathbf{y} \rangle$$

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Basis Vectors

A set of vectors \mathbf{x}_i is said to be **linearly independent** iff

$$\sum \lambda_i \mathbf{x}_i = \mathbf{0} \text{ implies that } \forall i : \lambda_i = 0$$

Two vectors in a Hilbert space are **orthogonal** iff

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

An **orthonormal** system in a Hilbert space is a set of linearly independent set of vectors with:

$$\langle \mathbf{b}_i, \mathbf{b}_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{iff } i = j \\ 0 & \text{iff } i \neq j \end{cases}$$

Theorem

For a Hilbert space there exists an orthonormal basis $\{\mathbf{b}\}$. The representation of each vector is unique:

$$\mathbf{x} = \sum_i x_i \mathbf{b}_i = \sum_i \langle \mathbf{x}, \mathbf{b}_i \rangle \mathbf{b}_i$$

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The Finite-Dimensional Hilbert Spaces \mathbb{C}^n

We represent vectors and their **transpose** using coordinates:

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = |x\rangle, \quad \vec{y} = (y_1, \dots, y_n) = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}^T = \langle y|$$

The **adjoint** of $\vec{x} = (x_1, \dots, x_n)$ is given by

$$\vec{x}^\dagger = (\bar{x}_1, \dots, \bar{x}_n)^T = (x_1^*, \dots, x_n^*)^T$$

The **inner product** is then represented by:

$$\langle \vec{y}, \vec{x} \rangle = \sum_i \bar{y}_i x_i = \sum_i y_i^* x_i$$

We can also define a **norm** (length) $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$.

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Dual and Adjoint States

A **linear functional** on a vector space \mathcal{V} is a map $f : \mathcal{V} \rightarrow \mathbb{K}$ such that (i) $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ and (ii) $f(\alpha\mathbf{x}) = \alpha f(\mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}, \alpha \in \mathbb{K}$.

The space of all linear functionals on \mathcal{V} form the **dual space** \mathcal{V}^* .

Theorem (Riesz Representation Theorem)

Every linear functional $f : \mathcal{H} \rightarrow \mathbb{C}$ on a Hilbert space \mathcal{H} can be represented by a vector \mathbf{y}_f in \mathcal{H} , such that

$$f(\mathbf{x}) = \langle \mathbf{y}_f, \mathbf{x} \rangle = f_y(\mathbf{x})$$

Dual Hilbert spaces \mathcal{H}^* are isomorphic to the original Hilbert space \mathcal{H} ; in particular we have: $(\mathbb{C}^n)^* = \mathbb{C}^n$.

We represent vectors or **ket-vectors** as **column** vectors; and functionals, dual vector or **bra-vectors** as **row** vectors.

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Dirac Notation and Einstein Convention

We will use throughout P.A.M. Dirac's bra-(c)-ket notation:

$$\langle \mathbf{x}_i, \mathbf{y}_j \rangle = \langle \vec{x}_i, \vec{y}_j \rangle \text{ denoted as } \langle x_i | y_j \rangle = \langle i | j \rangle$$

We will enumerate the (eigen-)base vectors (of an operator):

$$\vec{b}_i = \mathbf{b}_i \text{ or } \vec{e}_i = \mathbf{e}_i \text{ are denoted by } |i\rangle$$

but we may need also to specify the coordinates of a vector:

- ▶ Ket-Vectors (column): $|x\rangle = (x_j)_{j=1}^n$ in \mathbb{C}^n .
- ▶ Bra-Vectors (row): $\langle x| = (x^j)_{j=1}^n$ in $(\mathbb{C}^n)^* = \mathbb{C}^n$.

A. Einstein: If in an expression there are matching sub- and super-scripts then this implicitly indicates a summation,

$$\bar{x}_i y^i = \sum_i \bar{x}_i y^i = \langle \vec{x}, \vec{y} \rangle \text{ and } x_i y^{i*} = \sum_i x_i \bar{y}^i = \langle \vec{x} | \vec{y} \rangle$$

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Qubit

The postulates of **Quantum Mechanics** simply require that a computational quantum **state** is represented by a normalised vector in \mathbb{C}^n . A **qubit** is a two-dimensional quantum state in \mathbb{C}^2

We represent the **coordinates** of a qubit (state) or ket-vector as a column vector:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha |0\rangle + \beta |1\rangle$$

with respect to the (orthonormal) **basis** $\{|0\rangle, |1\rangle\}$, i.e. the so-called **standard base**:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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Representing a Qubit [*]

A qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$ can be represented:

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle,$$

where $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. Using polar coordinates we have:

$$|\psi\rangle = r_0 e^{i\phi_0}|0\rangle + r_1 e^{i\phi_1}|1\rangle,$$

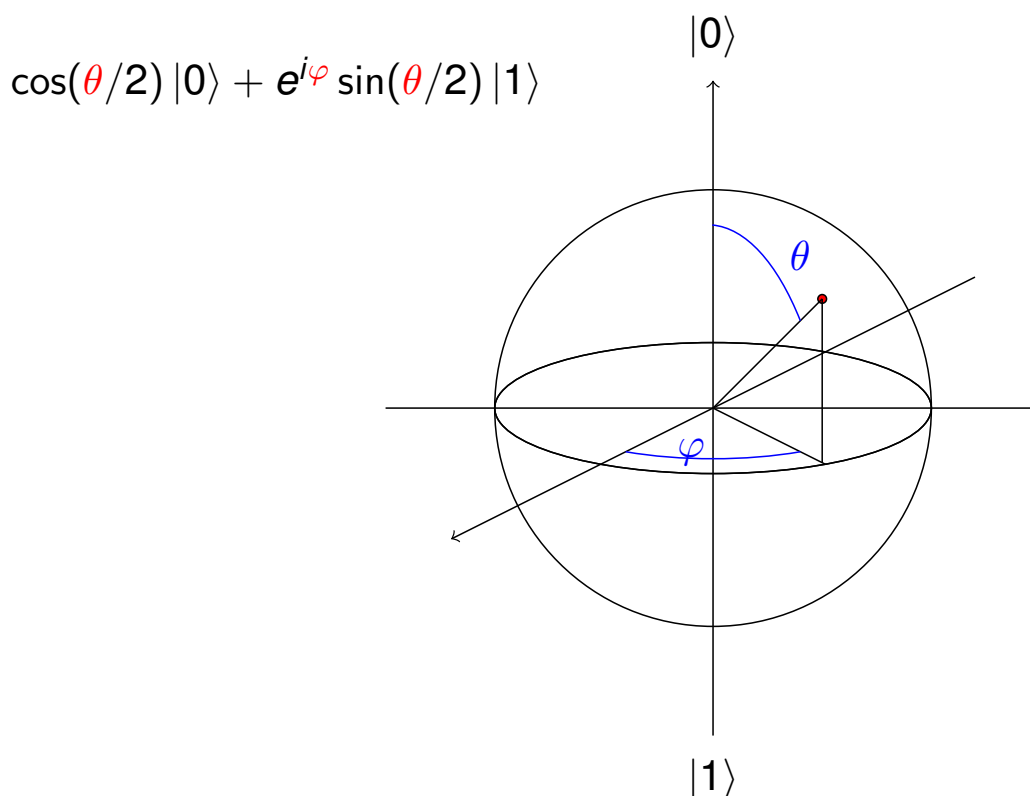
with $r_0^2 + r_1^2 = 1$. Take $r_0 = \cos(\rho)$ and $r_1 = \sin(\rho)$ for some ρ . Set $\theta/2 = \rho$, then $|\psi\rangle = \cos(\theta/2)e^{i\phi_0}|0\rangle + \sin(\theta/2)e^{i\phi_1}|1\rangle$, with $0 \leq \theta \leq \pi$, or equivalently

$$|\psi\rangle = e^{i\gamma}(\cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle),$$

with $\varphi = \phi_1 - \phi_0$ and $\gamma = \phi_0$, with $0 \leq \varphi \leq 2\pi$. The global **phase shift** $e^{i\gamma}$ is physically irrelevant (unobservable).

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Bloch Sphere [*]



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Change of Basis

We can represent (the coordinates of) any vector in \mathbb{C}^n with respect to any basis we like.

For example, we can consider for qubits in \mathbb{C}^2 the (alternative) orthonormal basis:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

and thus, vice versa:

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \quad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

A qubit is therefore represented in the two bases as:

$$\begin{aligned} \alpha|0\rangle + \beta|1\rangle &= \frac{\alpha}{\sqrt{2}}(|+\rangle + |-\rangle) + \frac{\beta}{\sqrt{2}}(|+\rangle - |-\rangle) \\ &= \frac{\alpha + \beta}{\sqrt{2}}|+\rangle + \frac{\alpha - \beta}{\sqrt{2}}|-\rangle \end{aligned}$$

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Linear Operators

Arguably, the best understood and controlled type of functions or maps between two vector spaces \mathcal{V} and \mathcal{W} are those preserving their basic algebraic structure.

Definition

A map $\mathbf{T} : \mathcal{V} \rightarrow \mathcal{W}$ between two vector spaces \mathcal{V} and \mathcal{W} is called a **linear** map if

1. $\mathbf{T}(\mathbf{x} + \mathbf{y}) = \mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y})$ and
2. $\mathbf{T}(\alpha\mathbf{x}) = \alpha\mathbf{T}(\mathbf{x})$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and all $\alpha \in \mathbb{K}$ (e.g. $\mathbb{K} = \mathbb{C}$ or \mathbb{R}).

For $\mathcal{V} = \mathcal{W}$ we talk about a **(linear) operator** on \mathcal{V} .

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Images of the Basis

Like vectors, we can represent a linear operator \mathbf{T} via its “coordinates” as a **matrix**. Again these depend on the **particular basis** we use.

Specifying the image of the base vectors determines – by **linearity** – the operator (or in general a linear map) uniquely.

Suppose we know the images of the basis vectors $|0\rangle$ and $|1\rangle$

$$\mathbf{T}(|0\rangle) = T_{00}|0\rangle + T_{01}|1\rangle$$

$$\mathbf{T}(|1\rangle) = T_{10}|0\rangle + T_{11}|1\rangle$$

then this is enough to know the T_{ij} 's to know what \mathbf{T} is doing to all vectors (as they are representable as linear combinations of the basis vectors).

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Matrices

Using a “mathematical” indexing (starting from 1 rather than 0), using the first index to indicate a **row** position and second for a **column** position, we can identify \mathbf{T} with a matrix:

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = (T_{ij})_{i,j=1}^n = (T_{ij})$$

The **application** of \mathbf{T} to a general vector (qubit) then becomes a simple matrix (pre-)multiplication:

$$\mathbf{T} \left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right) = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} T_{11}\alpha + T_{12}\beta \\ T_{21}\alpha + T_{22}\beta \end{pmatrix}$$

One can also express this, for $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ also as:

$$\mathbf{T}(|\psi\rangle) = \mathbf{T}(\alpha|0\rangle + \beta|1\rangle) = \alpha\mathbf{T}(|0\rangle) + \beta\mathbf{T}(|1\rangle) = \mathbf{T}|\psi\rangle$$

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Matrix Multiplications

The **application** of a linear operator \mathbf{T} (represented by a matrix) to a vector \mathbf{x} (represented via its coordinates) becomes:

$$\mathbf{T}(\mathbf{x}) = \mathbf{T}\mathbf{x} = (T_{ij})(x_j) = \sum_i T_{ij}x_j$$

The standard convention is pre-**multiplication** so as the sequence is the same as with application.

The **composition** of linear operators \mathbf{T} and \mathbf{S} becomes also a matrix/matrix pre-**multiplications**:

$$\mathbf{T} \circ \mathbf{S} = \mathbf{TS} = (T_{ij})(S_{ki}) = \sum_i T_{ij}S_{ki}$$

Some authors use the more “computational” pre-multiplication.

Finite-dimensional linear operators (matrices) form a vector space and with the multiplication a (linear) **algebra**. Adding the adjoint operation (see below) turns this into a **C*-algebra**.

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Transformations

We can define a linear map \mathbf{B} which implements the base change $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$:

$$\mathbf{B} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Transforming the coordinates (x_i) with respect to $\{|0\rangle, |1\rangle\}$ into coordinates (y_i) using $\{|+\rangle, |-\rangle\}$ can be obtained by:

$$\mathbf{B}(x_i)_i = (y_i)_i \text{ and } \mathbf{B}^{-1}(y_i)_i = (x_i)_i$$

The matrix representation \mathbf{T} of an operator using $\{|0\rangle, |1\rangle\}$ can be transformed into the representation \mathbf{S} in $\{|+\rangle, |-\rangle\}$ via:

$$\mathbf{S} = \mathbf{B}\mathbf{T}\mathbf{B}^{-1}$$

Problem: It is not easy to compute **inverse** \mathbf{B}^{-1} , defined on implicitly by $\mathbf{B}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$ the identity (existence?!).

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Adjoint Operator

For a matrix $\mathbf{T} = (T_{ij})$ its **transpose** matrix \mathbf{T}^T is defined as

$$\mathbf{T}^T = (T_{ij}^T) = (T_{ji})$$

the **conjugate** matrix \mathbf{T}^* is defined by

$$\mathbf{T}^* = (T_{ij}^*) = (T_{ij})^* = \overline{(T_{ij})}$$

and the **adjoint** matrix \mathbf{T}^\dagger is given via

$$\mathbf{T}^\dagger = (T_{ij}^\dagger) = (T_{ji}^*) \quad \text{or} \quad \mathbf{T}^\dagger = (\mathbf{T}^*)^T = (\mathbf{T}^T)^*$$

Note that $(\mathbf{TS})^T = \mathbf{S}^T \mathbf{T}^T$ and thus $(\mathbf{TS})^\dagger = \mathbf{S}^\dagger \mathbf{T}^\dagger$.

In **mathematics** the adjoint operator is usually denoted by \mathbf{T}^* (cf. conjugate in physics) and defined implicitly via:

$$\langle \mathbf{T}(\mathbf{x}), \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{T}^*(\mathbf{y}) \rangle \quad \text{or} \quad \langle \mathbf{T}^\dagger \mathbf{x} | \mathbf{y} \rangle = \langle \mathbf{x} | \mathbf{T} \mathbf{y} \rangle$$

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Adjoint Vectors

Bra and ket vectors are also related using the adjoint:

$$|\mathbf{x}\rangle^\dagger = \langle \mathbf{x}|$$

or using their coordinates:

$$(x_i)^\dagger = \left(\begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right)^\dagger = (\bar{x}_1 \quad \cdots \quad \bar{x}_n) = (\bar{x}^i)$$

The adjoint operator specifies the effect on dual vectors:

$$(\mathbf{T} |\mathbf{x}\rangle)^\dagger = |\mathbf{x}\rangle^\dagger \mathbf{T}^\dagger = \langle \mathbf{x}| \mathbf{T}^\dagger$$

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Unitary Operators

A square matrix/operator \mathbf{U} is called **unitary** if

$$\mathbf{U}^\dagger \mathbf{U} = \mathbf{I} = \mathbf{U} \mathbf{U}^\dagger$$

That means \mathbf{U} 's inverse is $\mathbf{U}^\dagger = \mathbf{U}^{-1}$. It also implies that \mathbf{U} is **invertible** and the inverse is easy to compute.

Quantum Mechanics requires that the **dynamics** or **time evolution** of a quantum state, e.g. qubit, is implemented via a unitary operator (as long as there is no measurement).

The unitary evolution of an (isolated) quantum state/system is a mathematical consequence of being a solution of the Schrödinger equation for some Hamiltonian operator \mathbf{H} .

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Properties of Unitary Operators

Unitary operators generalise in some sense permutations (in fact every permutation of base vectors gives rise to a simple unitary map). They can also be seen as generalised rotations.

Unitary operators also preserve the “geometry” of a Hilbert space, i.e. they preserve the inner product:

$$\langle x | \mathbf{U}^\dagger \mathbf{U} | y \rangle = \langle x | y \rangle.$$

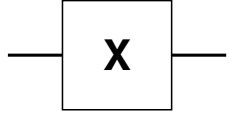
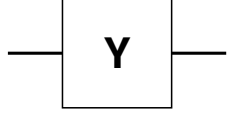
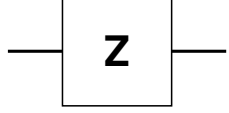
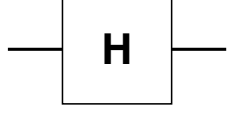
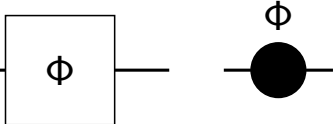
Any single qubit operation, i.e. unitary 2×2 matrix \mathbf{U} can be expressed as via 4 (real) parameters:

$$\mathbf{U} = \begin{pmatrix} e^{i(\alpha-\beta/2-\delta/2)} \cos \gamma/2 & e^{i(\alpha+\beta/2-\delta/2)} \sin \gamma/2 \\ -e^{i(\alpha-\beta/2+\delta/2)} \sin \gamma/2 & e^{i(\alpha+\beta/2+\delta/2)} \cos \gamma/2 \end{pmatrix}$$

where α , β , δ and γ are real numbers.

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Basic 1-Qubit Operators

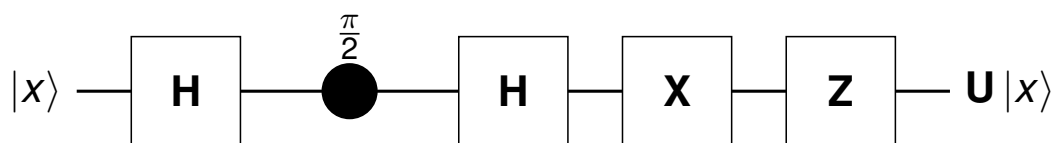
Pauli X-Gate	$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	
Pauli Y-Gate	$\mathbf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	
Pauli Z-Gate	$\mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	
Hadamard Gate	$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	
Phase Gate	$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$	

The Pauli-X gate is often referred to as NOT gate. Note that the notation for Hamiltonian and Hadamard gate are both **H**.

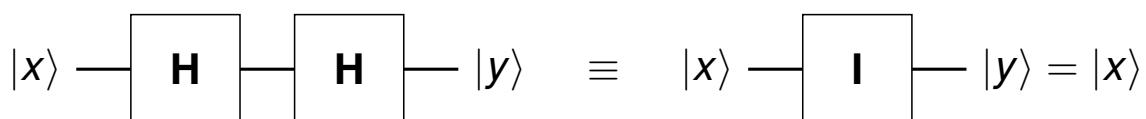
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Graphical “Notation”

The product (combination) of unitary operators results in a unitary operator, i.e. with $\mathbf{U}_1, \dots, \mathbf{U}_n$ unitary, the product $\mathbf{U} = \mathbf{U}_n \dots \mathbf{U}_1$ is also unitary (Note: $(\mathbf{TS})^\dagger = \mathbf{S}^\dagger \mathbf{T}^\dagger$).



A simple example: $|y\rangle = \mathbf{H}\mathbf{H}|x\rangle$ or $(|x\rangle; \mathbf{H}; \mathbf{H} = |y\rangle)$:



because $\mathbf{H}^2 = \mathbf{I}$, i.e.

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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