# Quantum Computation (CO484)

## Quantum States and Evolution

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Autumn 2018

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- The only possible results are eigen-values $\lambda_i$ of **A**.
- The probability of measuring $\lambda_n$ in state $|x\rangle$ is given by:

$$Pr(A = \lambda_n | x) = \langle x | \mathbf{P}_n | x \rangle = \langle x | | \mathbf{P}_n x \rangle$$

with $\mathbf{P}_n = |\lambda_n\rangle\langle\lambda_n|$ the orthogonal projection onto the space generated by eigen-vector $|\lambda_n\rangle = |n\rangle$ of **A**.

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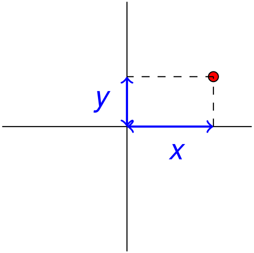
with $i^2 = -1$ or $i = \sqrt{-1}$. The complex conjugate of a complex number $z = x + iy \in \mathbb{C}$ is:

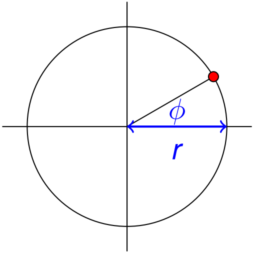
$$z^* = \overline{z} = \overline{x + iy} = x - iy = z^\dagger$$

**Hauptsatz of Algebra**
Complex numbers are algebraically closed: Every polynomial of order $n$ over $\mathbb{C}$ has exactly $n$ roots.

# Polar Coordinates

One can represent numbers $z \in \mathbb{C}$ using the complex plane.




Conversion:

$$x = r \cdot \cos(\phi) \qquad y = r \cdot \sin(\phi)$$

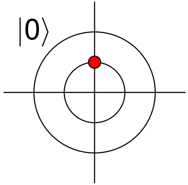
$$r = \sqrt{x^2 + y^2} \qquad \phi = \arctan(\frac{y}{x})$$

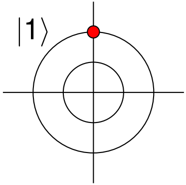
Another representation:

$$(r, \phi) = r \cdot e^{i\phi} \qquad e^{i\phi} = \cos(\phi) + i \sin(\phi),$$

# Computational Quantum States

Consider a simple systems with two **degrees of freedom.**



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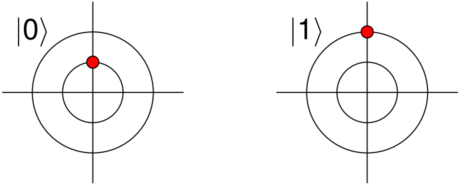


## Definition

A qubit (quantum bit) is a quantum state of the form

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

where $\alpha$ and $\beta$ are complex numbers with $|\alpha|^2 + |\beta|^2 = 1$.

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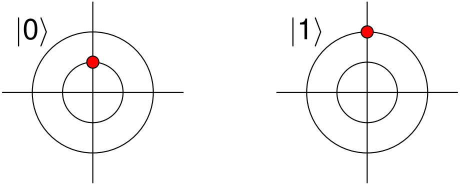


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Qubits live in a two-dimensional complex vector, more precisely, Hilbert space $\mathbb{C}^2$ and are **normalised**, i.e. $\|\,|\psi\rangle\,\| = \langle\psi \mid \psi\rangle = 1$.

# Vector Spaces

A **Vector Space** (over a field $\mathbb{K}$, e.g. $\mathbb{R}$ or $\mathbb{C}$) is a set $\mathcal{V}$ together with two operations:

Scalar Product $\ . \cdot . : \mathbb{K} \times \mathcal{V} \mapsto \mathcal{V}$

Vector Addition $\ . + . : \mathcal{V} \times \mathcal{V} \mapsto \mathcal{V}$

such that $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{K}$:

1. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$
2. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
3. $\exists \mathbf{o} : \mathbf{x} + \mathbf{o} = \mathbf{x}$
4. $\exists -\mathbf{x} : \mathbf{x} + (-\mathbf{x}) = \mathbf{o}$

5. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$
6. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
7. $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$
8. $1\mathbf{x} = \mathbf{x}$ $(1 \in \mathbb{K})$

# Tuple Spaces

## Theorem

*All finite dimensional vector spaces are isomorphic to the (finite) Cartesian product of the underlying field* $\mathbb{K}^n$ *(i.e.* $\mathbb{R}^n$ *or* $\mathbb{C}^n$*).*

$$\vec{x} = (x_1, x_2, x_3, \ldots, x_n) \ \textit{represents} \ \mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{b}_i$$

$$\vec{y} = (y_1, y_2, y_3, \ldots, y_n) \ \textit{represents} \ \mathbf{y} = \sum_{i=1}^{n} y_i \mathbf{b}_i$$

Finite dimensional vectors can be represented as tuples via their coordinates with respect to a base $\{\mathbf{b}_i\}_{i=1}^{n}$.

$$\alpha\vec{x} = (\alpha x_1, \alpha x_2, \alpha x_3, \ldots, \alpha x_n)$$

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots, x_n + y_n)$$

# Hilbert Spaces

A complex vector space $\mathcal{H}$ is called an **Inner Product Space** or **(Pre-)**Hilbert Space if there is a complex valued function $\langle ., . \rangle$ on $\mathcal{H} \times \mathcal{H}$ that satisfies $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{H}$ and $\forall \alpha \in \mathbb{C}$:

1. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$
2. $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \iff \mathbf{x} = \mathbf{o}$
3. $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$
4. $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
5. $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$

The function $\langle ., . \rangle$ is called an inner product on $\mathcal{H}$.

# Caveat: Linear in first or second argument?

**Mathematical Convention:**

$$\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$$

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For physicists it is simply:

$$\langle \mathbf{x} \mid \alpha \mathbf{y} \rangle = \alpha \langle \mathbf{x} \mid \mathbf{y} \rangle$$

# Basis Vectors

A set of vectors $\mathbf{x}_i$ is said to be linearly independent iff

$$\sum \lambda_i \mathbf{x}_i = \mathbf{o} \quad \text{implies that} \quad \forall\, i : \lambda_i = 0$$

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An orthonormal system in a Hilbert space is a set of linearly independent set of vectors with:

$$\left\langle \mathbf{b}_i, \mathbf{b}_j \right\rangle = \delta_{ij} = \begin{cases} 1 & \text{iff } i = j \\ 0 & \text{iff } i \neq j \end{cases}$$

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## Theorem

*For a Hilbert space there exists an orthonormal basis $\{\mathbf{b}\}$. The representation of each vector is unique:*

$$\mathbf{x} = \sum_i x_i \mathbf{b}_i = \sum_i \langle \mathbf{x}, \mathbf{b}_i \rangle \, \mathbf{b}_i$$

# The Finite-Dimensional Hilbert Spaces $\mathbb{C}^n$

We represent vectors and their transpose using coordinates:

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = |x\rangle \,, \quad \vec{y} = (y_1, \ldots, y_n) = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}^T = \langle y|$$

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We can also define a norm (length) $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$.

# Dual and Adjoint States

A linear functional on a vector space $\mathcal{V}$ is a map $f : \mathcal{V} \to \mathbb{K}$ such that (i) $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ and (ii) $f(\alpha\mathbf{x}) = \alpha f(\mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}, \alpha \in \mathbb{K}$.

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## Theorem (Riesz Representation Theorem)

*Every linear functional $f : \mathcal{H} \to \mathbb{C}$ on a Hilbert space $\mathcal{H}$ can be represented by a vector $\mathbf{y}_f$ in $\mathcal{H}$, such that*

$$f(\mathbf{x}) = \langle \mathbf{y}_f, \mathbf{x} \rangle = f_y(\mathbf{x})$$

Dual Hilbert spaces $\mathcal{H}^*$ are isomorphic to the original Hilbert space $\mathcal{H}^*$; in particular we have: $(\mathbb{C}^n)^* = \mathbb{C}^n$.

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We represent vectors or ket-vectors as column vectors; and functionals, dual vector or bra-vectors as row vectors.

# Dirac Notation and Einstein Convention

We will use throughout P.A.M. Dirac's bra-(c)-ket notation:

$$\left\langle \mathbf{x}_i, \mathbf{y}_j \right\rangle = \left\langle \vec{x}_i, \vec{y}_j \right\rangle \text{ denoted as } \langle x_i| \left|y_j\right\rangle = \langle i|\, |j\rangle$$

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We will enumerate the (eigen-)base vectors (of an operator):

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A. Einstein: If in an expression there are matching sub- and super-scripts then this implicitely indicates a summation,

$$\bar{x}_i y^i = \sum_i \bar{x}_i y^i = \langle \vec{x}, \vec{y} \rangle \text{ and } x_i y^{i*} = \sum_i x_i \bar{y}^i = \langle \vec{x} \mid \vec{y} \rangle$$

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We represent the **coordinates** of a qubit (state) or ket-vector as a column vector:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha |0\rangle + \beta |1\rangle$$

with respect to the (orthonormal) **basis** $\{|0\rangle, |1\rangle\}$, i.e. the so-called standard base:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

# Representing a Qubit [$*$]

A qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$ can be represented:

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle\,,$$

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with $r_0^2 + r_1^2 = 1$. Take $r_0 = \cos(\rho)$ and $r_1 = \sin(\rho)$ for some $\rho$. Set $\theta/2 = \rho$, then $|\psi\rangle = \cos(\theta/2)e^{i\phi_0}|0\rangle + \sin(\theta/2)e^{i\phi_1}|1\rangle$, with $0 \leq \theta \leq \pi$,

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with $r_0^2 + r_1^2 = 1$. Take $r_0 = \cos(\rho)$ and $r_1 = \sin(\rho)$ for some $\rho$. Set $\theta/2 = \rho$, then $|\psi\rangle = \cos(\theta/2)e^{i\phi_0}|0\rangle + \sin(\theta/2)e^{i\phi_1}|1\rangle$, with $0 \leq \theta \leq \pi$, or equivalently

$$|\psi\rangle = e^{i\gamma}(\cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle),$$

with $\varphi = \phi_1 - \phi_0$ and $\gamma = \phi_0$, with $0 \leq \varphi \leq 2\pi$.

# Representing a Qubit [$*$]

A qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$ can be represented:

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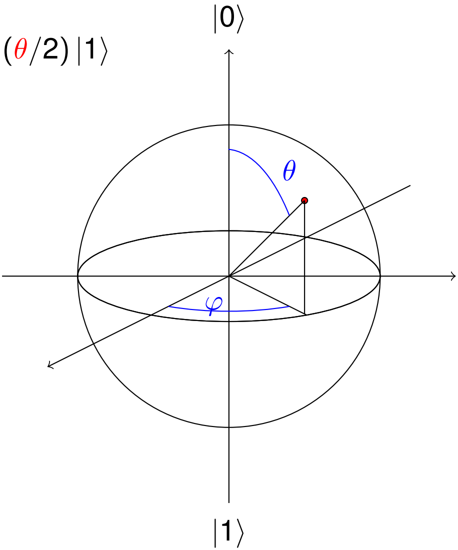
# Bloch Sphere [∗]

$$\cos(\theta/2)\,|0\rangle + e^{i\varphi}\sin(\theta/2)\,|1\rangle$$



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A qubit is therefore represented in the two bases as:

$$\begin{aligned} \alpha\,|0\rangle + \beta\,|1\rangle &= \frac{\alpha}{\sqrt{2}}(|+\rangle + |-\rangle)\frac{\beta}{\sqrt{2}}(|+\rangle - |-\rangle) \\ &= \frac{\alpha+\beta}{\sqrt{2}}\,|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}\,|-\rangle \end{aligned}$$

# Linear Operators

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For $\mathcal{V} = \mathcal{W}$ we talk about a (linear) operator on $\mathcal{V}$.

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then this is enough to know the $T_{ij}$'s to know what **T** is doing to all vectors (as they are representable as linear combinations of the basis vectors).

# Matrices

Using a "mathematical" indexing (starting from 1 rather ten 0), using the first index to indicate a **row** position and second for a **column** position, we can identify **T** with a matrix:

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = (T_{ij})_{i,j=1}^{n} = (T_{ij})$$

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One can also express this, for $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ also as:

$$\mathbf{T}(|\psi\rangle) = \mathbf{T}(\alpha|0\rangle + \beta|1\rangle) = \alpha\mathbf{T}(|0\rangle) + \beta\mathbf{T}(|1\rangle) = \mathbf{T}|\psi\rangle$$

# Matrix Multiplications

The application of a linear opertor **T** (represented by a matrix) to a vector **x** (represented via its coordinates) becomes:

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Finite-dimensional linear operators (matrices) form a vector space and with the multiplication a (linear) **algebra**. Adding the adjoint operation (see below) turns this into a **$C^*$-algebra**.

# Transformations

We can define a linear map **B** which implements the base change $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$:

$$\mathbf{B} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

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Transforming the coordinates $(x_i)$ with respect to $\{|0\rangle, |1\rangle\}$ into coordinates $(y_i)$ using $\{|+\rangle, |-\rangle\}$ can be obtained by:

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Problem: It is not easy to compute inverse $\mathbf{B}^{-1}$, defined on implicitly by $\mathbf{B}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$ the identity (existence?!).

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In **mathematics** the adjoint operator is usually denoted by $\mathbf{T}^*$ (cf. conjugate in physics) and defined implicitly via:

$$\langle \mathbf{T}(\mathbf{x}), \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{T}^*(\mathbf{y}) \rangle \quad \text{or} \quad \langle \mathbf{T}^\dagger \mathbf{x} \mid \mathbf{y} \rangle = \langle \mathbf{x} \mid \mathbf{T}\mathbf{y} \rangle$$

# Adjoint Vectors

Bra and ket vectors are also related using the adjoint:

$$|x\rangle^{\dagger} = \langle x|$$

or using their coordinates:

$$(x_i)^{\dagger} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^{\dagger} = \begin{pmatrix} \bar{x}_1 & \cdots & \bar{x}_n \end{pmatrix} = (\bar{x}^i)$$

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The adjoint operator specifies the effect on dual vectors:

$$(\mathbf{T}\,|x\rangle)^{\dagger} = |x\rangle^{\dagger}\,\mathbf{T}^{\dagger} = \langle x|\,\mathbf{T}^{\dagger}$$

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The unitary evolution of an (isolated) quantum state/system is a mathematical consequence of being a solution of the Schrödinger equation for some Hamiltonian operator **H**.

# Properties of Unitary Operators

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Any single qubit operation, i.e. unitary $2 \times 2$ matrix $\mathbf{U}$ can be expressed as via 4 (real) parameters:

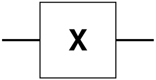
$$\mathbf{U} = \begin{pmatrix} e^{i(\alpha-\beta/2-\delta/2)} \cos\gamma/2 & e^{i(\alpha+\beta/2-\delta/2)} \sin\gamma/2 \\ -e^{i(\alpha-\beta/2+\delta/2)} \sin\gamma/2 & e^{i(\alpha+\beta/2+\delta/2)} \cos\gamma/2 \end{pmatrix}$$

where $\alpha$, $\beta$, $\delta$ and $\gamma$ are real numbers.

# Basic 1-Qubit Operators

Pauli X-Gate

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



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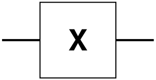


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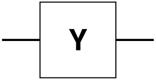

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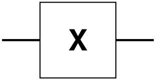


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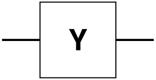


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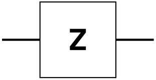
$$\mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



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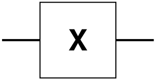


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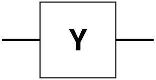


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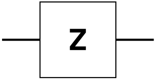
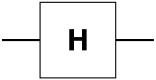



Hadamard Gate

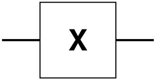
$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



# Basic 1-Qubit Operators

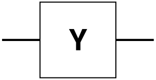
Pauli X-Gate

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



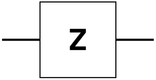

Pauli Y-Gate

$$\mathbf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$




Pauli Z-Gate

$$\mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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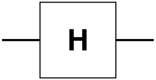
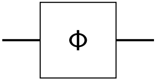
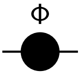



Phase Gate

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$



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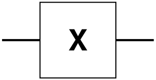


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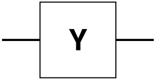


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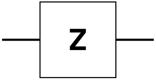


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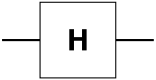


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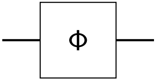


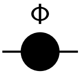


The Pauli-X gate is often referred to as NOT gate. Note that the notation for Hamiltonian and Hadamard gate are both **H**.

# Graphical “Notation”

The product (combination) of unitary operators results in a unitary operator, i.e. with $\mathbf{U}_1, \ldots, \mathbf{U}_n$ unitary, the product $\mathbf{U} = \mathbf{U}_n \ldots \mathbf{U}_1$ is also unitary (Note: $(\mathbf{TS})^\dagger = \mathbf{S}^\dagger \mathbf{T}^\dagger$).

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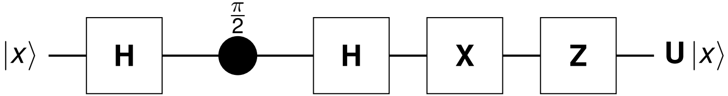

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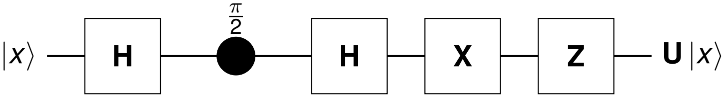
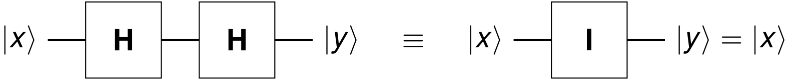



A simple example: $|y\rangle = \mathbf{HH}\,|x\rangle$ or $(|x\rangle\,;\ \mathbf{H};\ \mathbf{H} = |y\rangle)$:




because $\mathbf{H}^2 = \mathbf{I}$, i.e.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$