

Quantum Computation (CO484)

Quantum Gates and Circuits

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The idea is to define similar **quantum gates**, taking two (or n) qubits at input and producing some output. Contrary to classical gates we have to use **unitary**, i.e. reversible, gates in quantum circuits.

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$$|x, y\rangle \mapsto |x, y \oplus x\rangle \quad \text{with} \quad y \oplus x = (y + x) \bmod 2$$

i.e. $|00\rangle \mapsto |00\rangle, |01\rangle \mapsto |01\rangle, |10\rangle \mapsto |11\rangle, |11\rangle \mapsto |10\rangle$.

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We represent the CNOT-gate graphically and as a matrix (with respect to the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$) as:

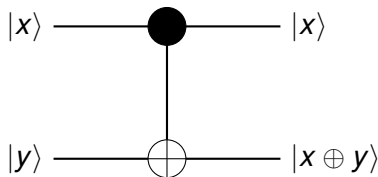
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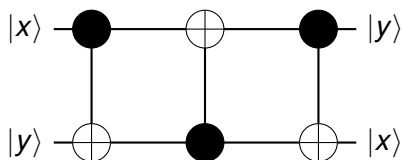
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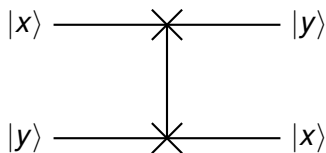
$$\mathbf{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Swapping Gate

We can exploit the CNOT-Gate to **SWAP** two qubits:



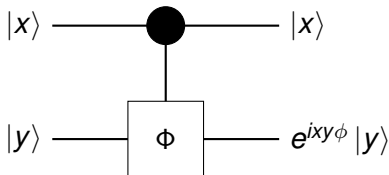
is depicted by (shorthand):



Exercise: Check that this really maps $|x\rangle \otimes |y\rangle$ into $|y\rangle \otimes |x\rangle$ (for all $|x\rangle$ and $|y\rangle$ not just base vectors?).

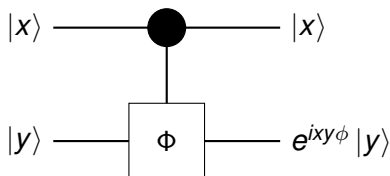
Controlled Phase Gate

The **controlled phase**-gate is depicted as follows (for base vectors $|x\rangle, |y\rangle \in \{|0\rangle, |1\rangle\}$):



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Its matrix/operator representation is given by:

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix}$$

on any two qubits, i.e. vectors in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

General Controlled Gate

In general, we can **control** any single qubit transformation $\mathbf{U} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by another qubit, i.e. such that for all $|y\rangle \in \mathbb{C}^2$:

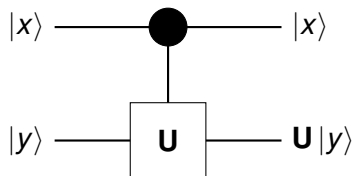
$$\begin{aligned} |0\rangle \otimes |y\rangle &\mapsto |0\rangle \otimes |y\rangle \\ |1\rangle \otimes |y\rangle &\mapsto |1\rangle \otimes \mathbf{U}|y\rangle \end{aligned}$$

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The diagrammatic representation is:



Toffoli Gate

The **Toffoli**-gate is a 3-qubit quantum gate on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^8$ with the following behaviour $\mathbf{T} : |x, y, z\rangle \mapsto |x', y', z'\rangle$ and matrix representation (standard base enumeration):

<i>input</i>			<i>output</i>		
<i>x</i>	<i>y</i>	<i>z</i>	<i>x'</i>	<i>y'</i>	<i>z'</i>
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
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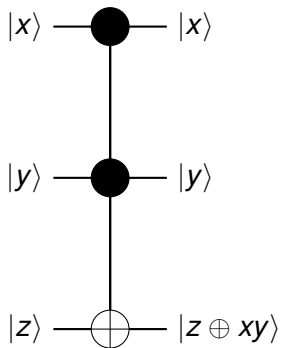
$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Toffoli Gate Usage

The Toffoli gate can be used to implement a reversible version of NAND and a FANOUT gate.

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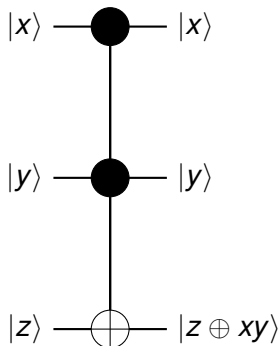
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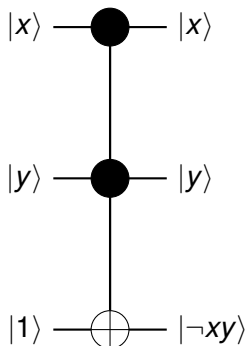
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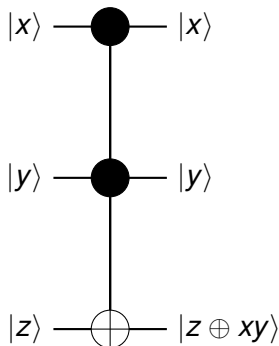
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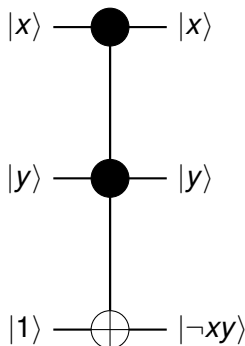
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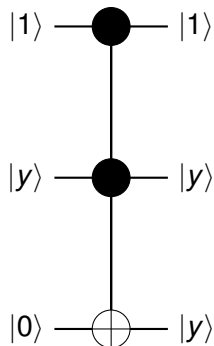
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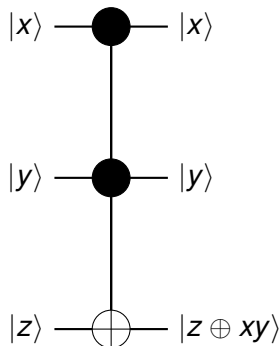
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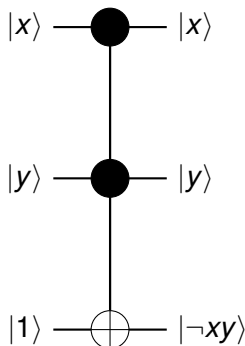
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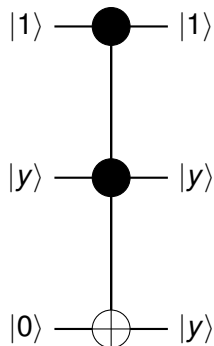
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Toffoli



NAND



FANOUT

This works only with $x, y \in \{0, 1\}$.

Linear Maps from Functions

In general, we can take any (binary) function

$$f : \{0, 1\}^n \rightarrow \{0, 1\}^m$$

and define a corresponding linear map \mathbf{T}_f

$$\mathbf{T}_f : (\mathcal{V}(\{0, 1\}))^{\otimes n} \rightarrow (\mathcal{V}(\{0, 1\}))^{\otimes m} \text{ or } \mathbf{T}_f : (\mathbb{C}^2)^{\otimes n} \rightarrow (\mathbb{C}^2)^{\otimes m}$$

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Once we know or specify the image of all base vectors we know the (matrix representation) of \mathbf{T}_f via

$$\mathbf{T}_f |x\rangle = |f(x)\rangle$$

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Problem: \mathbf{T}_f is, in general, **not unitary**, i.e. reversible.

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Reversibility makes it impossible to have a quantum device \mathbf{U}_f which **just** computes a general function f , i.e. $\mathbf{U}_f : |x\rangle \mapsto |f(x)\rangle$.

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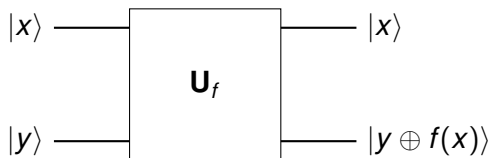
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Graphically represented by the diagram/quantum circuit:



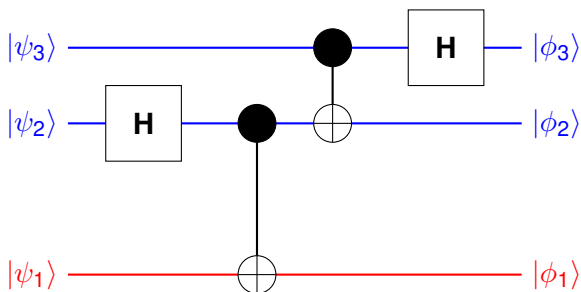
Quantum Circuit Model

We can specify a **quantum algorithm** on qubit registers – i.e. a unitary operator $\mathbf{U} : (\mathbb{C}^2)^{\otimes n} \rightarrow (\mathbb{C}^2)^{\otimes n}$ – using a combination of (standardised) quantum gates – like Hadamard, Pauli, etc. – and maybe “oracles” like \mathbf{U}_f as well as measurements.

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For example, the quantum circuit for **teleportation** (without correction) as an operator on $(\mathbb{C}^2)^{\otimes 3}$ is given as follows:



Calculations for Small Quantum Circuits

Circuits with few qubits can “implemented”, e.g. in `octave`, etc.

$$q_0 = [1, 0]'$$

$$q_1 = [0, 1]'$$

$$H = (1/\sqrt{2}) * [1, 1; 1, -1]$$

$$CX = [1, 0, 0, 0; 0, 1, 0, 0; \\ 0, 0, 0, 1; 0, 0, 1, 0]$$

$$S1 = \text{kron}(\text{eye}(2), H, \text{eye}(2))$$

$$S2 = \text{kron}(\text{eye}(2), CX)$$

$$S3 = \text{kron}(CX, \text{eye}(2))$$

$$S4 = \text{kron}(H, \text{eye}(2), \text{eye}(2))$$

$$T = S1 * S2 * S3 * S4$$

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For general unitary operators **U** on \mathbb{C}^n – in particular on m qubits, i.e. $\mathbb{C}^{2^m} = (\mathbb{C}^2)^{\otimes m}$ – an analogue result guarantees that **2×2 unitary matrices** make up all unitary operators.

See e.g.: A. Yu. Kitaev, A. H. Shen, M. N. Vyalyi: Classical and Quantum Computation, AMS, 2002, p70.

Unitary Operators on \mathbb{C}^n

Theorem

An arbitrary unitary operator \mathbf{U} on the space \mathbb{C}^n can be represented as a product of $\frac{n(n-1)}{2}$ matrices of the form:

$$\begin{pmatrix} 1 & \dots & 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & \ddots & \vdots & & & & & \vdots \\ 0 & \dots & 1 & & & & & \vdots \\ \vdots & & & a & b & & & \vdots \\ \vdots & & & c & d & & & \vdots \\ \vdots & & & & & & 1 & \dots & 0 \\ \vdots & & & & & & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 & \dots & 1 \end{pmatrix}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a 2×2 unitary matrix (on \mathbb{C}^2).

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Definition

A set of gates $\mathcal{G} = \{\mathbf{G}_1, \dots\}$ is said to be **approximaty universal** if any n -qubit operator \mathbf{U} (with $n \geq 1$) can be approximated to arbitrary accuracy, i.e. for all $\varepsilon > 0$ there exists a circuit \mathbf{V} which is constructed of gates in \mathcal{G} and their controlled versions such that we have $e(\mathbf{U}, \mathbf{V}) < \varepsilon$.

(Approximatly) Universal Gates

A possible set of approximatly universal gates (e.g. Kaye, Laflamme, Mosca: Introduction to Quantum Computing, p71):

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \Phi\left(\frac{\pi}{4}\right) = \begin{pmatrix} 1 & 0 \\ 0 & e^{j\frac{\pi}{4}} \end{pmatrix}$$

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