Quantum Computing CO484

Tutorial*

Sheet 1 - Questions

Exercise 1 *Consider a single qubit system, i.e. a system with $\mathcal{H} = \mathbb{C}^2$.*

(i) Does the following matrix represent an observable *on $\mathbb{C}^2$*

$$\mathbf{A}_1 = \begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix}?$$

What are its eigenbase vectors and eigenvalues?

(ii) Construct an observable $\mathbf{A}_2$ *(i.e. it's matrix representation) on $\mathbb{C}^2$ which has eigenvalues $\{+1, -1\}$ and eigenvectors*

$$\{\frac{1}{\sqrt{2}}(1, 1), \frac{1}{\sqrt{2}}(-1, 1)\}.$$

(iii) What happens if you measure either of the two observables $\mathbf{A}_1$ and $\mathbf{A}_2$ in the state $|x\rangle = \frac{1}{\sqrt{2}}(1, -1)$? Sketch the situation geometrically.

Note: The em outer product $|x\rangle\langle y|$ for vectors $|x\rangle = (x_1, \ldots, x_n)^T$ and $\langle y| = (y_1, \ldots, y_n)$ is an operator/matrix: $(|x\rangle\langle y|)_{ij} = x_i y_j$. It could be treated just as a formal combination of a ket and a bra vector.

Exercise 2 *Consider the complex numbers $\mathbb{C}$.*

(i) Show that for complex numbers x and y we have: $|x + y|^2 = |x|^2 + |y|^2 + 2Re(x^* y)$.

(ii) Compare $x^* x$ with x^2 for a complex number x and explain why the inner product of two complex vectors $v, w \in \mathbb{C}^d$ is defined as $v^\dagger w$ rather than $v^T w$.

*Based partly on the tutorials by Abbas Edalat.

Exercise 3 *Complex numbers (again)*

(i) *Give the cartesian representation of (in polar coordinates) the following complex numbers* $(1, \frac{\pi}{4})$, $(1, \frac{\pi}{2})$ *and* $(1, \pi)$.

(ii) *What is* $1^{\frac{1}{4}}$*?*

(iii) *Proof* **De Moivre's formula**: $(e^{i\theta})^n = \cos(n\theta) + i\sin(n\theta)$

Exercise 4 *Generalise the following notions and properties given for the vector space* $\mathbb{R}^2$ *to* $\mathbb{R}^d$.

(i) *Define linear independence and linear dependence of vectors in* $\mathbb{R}^d$.

(ii) *What is the least integer* n *such that any set of* n *vectors in* $\mathbb{R}^d$ *will be linearly dependent?*

(iii) *What is a basis of* $\mathbb{R}^d$*? How many linearly independent vectors it takes to get a basis for* $\mathbb{R}^d$*?*

(iv) *Define the notion of an* orthonormal *basis for* $\mathbb{R}^d$. *What would be the* standard *basis of* $\mathbb{R}^d$*?*

Exercise 5 *Generalise the following notions and properties given for the vector space* $\mathbb{C}^2$ *to* $\mathbb{C}^d$.

(i) *Define the norm* $\|w\|$ *of a vector and the inner product of two vectors* w_1 *and* w_2 *in* $\mathbb{C}^d$. *What is the dual of a vector* w *in* $\mathbb{C}^d$ *and what can it be identified with?*

(ii) *Define linear independence and linear dependence of vectors in* $\mathbb{C}^d$.

(iii) *What is the least integer* n *such that any set of* n *vectors in* $\mathbb{C}^d$ *will be linearly dependent?*

(iv) *What is a basis of* $\mathbb{C}^d$*? How many linearly independent vectors it takes to get a basis for* $\mathbb{C}^d$*?*

(v) *Define the notion of an* orthonormal *basis for* $\mathbb{C}^d$. *What would be the* standard *basis of* $\mathbb{C}^d$*?*

Exercise 6 *Show that the two vectors* w_1 *and* w_2 *in* $\mathbb{C}^4$ *with* $w_1^T = \frac{1}{2}(1, 1, 1, 1)$ *and* $w_2^T = \frac{1}{2}(1, -1, 1, -1)$ *are orthogonal unit vectors. Find vectors* w_3 *and* w_4 *such that the collection* $\{w_1, w_2, w_3, w_4\}$ *forms an orthonormal basis for* $\mathbb{C}^4$.

Exercise 7 *Linear Maps*

(i) Show that composition of two linear maps (on $\mathbb{C}^2$*) is again linear.*

(ii) Show that if $\mathbf{L}_1, \mathbf{L}_2 : \mathbb{C}^2 \to \mathbb{C}^2$ *are linear, so is* $\alpha_1\mathbf{L}_1 + \alpha_2\mathbf{L}_2 : \mathbb{C}^2 \to \mathbb{C}^2$ *defined by*

$$(\alpha_1\mathbf{L}_1 + \alpha_2\mathbf{L}_2)(v) = \alpha_1\mathbf{L}_1(v) + \alpha_2\mathbf{L}_2(v).$$

(iii) Show that for $|\psi\rangle, |\phi\rangle \in \mathbb{C}^2$*, the map* $|\psi\rangle\langle\phi| : \mathbb{C}^2 \to \mathbb{C}^2$ *defined, using the bracket notation, by* $|\psi\rangle\langle\phi| (|x\rangle) = \langle\phi|x\rangle |\psi\rangle$ *is linear.*

Exercise 8 *Show that the matrix representation of the* NOT *gate – i.e. a linear map on* $\mathbb{C}^2$ *which maps* $|0\rangle \mapsto |1\rangle$ *and* $|1\rangle \mapsto |0\rangle$ *– in the basis* $|0\rangle$, $|1\rangle$ *is given by*

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

What is its matrix representation in the basis $|+\rangle$, $|-\rangle$*?*

Exercise 9 *Unitary Maps and Matrices*

(i) Check that a matrix (in computational basis) is unitary iff its columns (or its rows) form an orthonormal *basis.*

(ii) Check: the matrix for a change in the computational basis is unitary.

(iii) Show that $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$ *and deduce that if a linear map has a unitary matrix representation, then its matrix representation in any computational basis is unitary.*

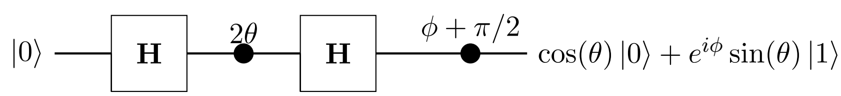
Exercise 10 *Show that* $\langle u|\mathbf{A}v\rangle = \langle\mathbf{A}^\dagger u|v\rangle$*.Deduce that a matrix* $\mathbf{M}$ *is* unitary *iff it preserves all inner products, i.e. iff* $\langle\mathbf{M}u|\mathbf{M}v\rangle = \langle u|v\rangle$ *for all* $u, v \in \mathbb{C}^2$*.*

Exercise 11 *Show that any unitary matrix* $\mathbf{U}$ *can be expressed as*

$$\mathbf{U} = \begin{pmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\gamma/2 & -e^{i(\alpha-\beta/2+\delta/2)}\sin\gamma/2 \\ e^{i(\alpha+\beta/2-\delta/2)}\sin\gamma/2 & e^{i(\alpha+\beta/2+\delta/2)}\cos\gamma/2 \end{pmatrix}$$

where α*,* β*,* δ *and* γ *are real numbers.*

Exercise 12 *Verify the output, up to a global phase, of the following:*


$|0\rangle$
H
2θ
H
$\phi + \pi/2$
$\cos(\theta)\,|0\rangle + e^{i\phi}\sin(\theta)\,|1\rangle$