Quantum Computing CO484

Tutorial*

Sheet 2 - Answers

Exercise 1 *Show that if for a single qubit unitary operation* $\mathbf{U}$ *we have* $\mathbf{U} = \mathbf{V}^2$ *where* $\mathbf{V}$ *is another single qubit unitary operation, then the double controlled* $\mathbf{U}$ *gate, called* $\mathsf{C}^2(\mathbf{U})$ *gate, can be implemented as in Figure 1. Show that for* $\mathbf{V} = (1-i)(\mathbf{I} + i\mathbf{X})/2$ *this implements the Toffoli gate.*

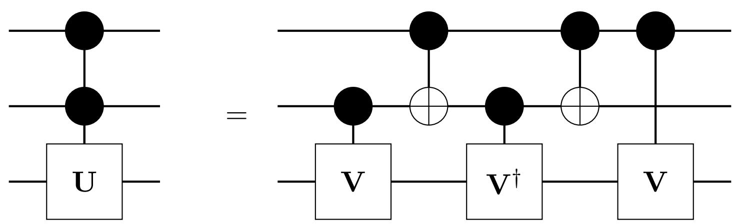

Figure 1: Implementation of $C^2(\mathbf{U})$

Solution We have the following step by step transformation of the input, where x denotes the third qubit:

$$|00x\rangle \mapsto |00x\rangle \mapsto |00x\rangle \mapsto |00x\rangle \mapsto |00x\rangle \mapsto |00x\rangle$$

$$|01x\rangle \mapsto |01\mathbf{V}x\rangle \mapsto \left|01\mathbf{V}^\dagger\mathbf{V}x\right\rangle = |01x\rangle \mapsto |01x\rangle \mapsto |01x\rangle \mapsto |01x\rangle$$

$$|10x\rangle \mapsto |10x\rangle \mapsto |11x\rangle \mapsto \left|11\mathbf{V}^\dagger x\right\rangle \mapsto \left|10\mathbf{V}^\dagger x\right\rangle \mapsto \left|10\mathbf{V}\mathbf{V}^\dagger x\right\rangle = |10x\rangle$$

$$|11x\rangle \mapsto |11\mathbf{V}x\rangle \mapsto |10\mathbf{V}x\rangle \mapsto |10\mathbf{V}x\rangle \mapsto |11\mathbf{V}x\rangle \mapsto |11\mathbf{V}\mathbf{V}x\rangle = |11\mathbf{U}x\rangle .$$

Hence the network implements the $\mathsf{C}^2(\mathbf{U})$ gate. If $\mathbf{V} = (1-i)(\mathbf{I} + i\mathbf{X})/2$, then $\mathbf{V}^2 = (1-i)^2(\mathbf{I} - \mathbf{X}^2 + 2i\mathbf{X})/4 = -2i(2i\mathbf{X})/4 = \mathbf{X}$ and hence the network implements $\mathsf{C}^2(\mathbf{X}) = \mathsf{C}^2(\mathsf{NOT})$, which is the Toffoli gate.

*Based partly on the tutorials by Abbas Edalat.

Exercise 2 *The* rotation operators *around the x, y and z axes are respectively defined as:*

$$\mathbf{R}_x(\theta) = e^{-i\theta\mathbf{X}/2} = \cos(\frac{\theta}{2})\mathbf{I} - i\sin(\frac{\theta}{2})\mathbf{X},$$

$$\mathbf{R}_y(\theta) = e^{-i\theta\mathbf{Y}/2} = \cos(\frac{\theta}{2})\mathbf{I} - i\sin(\frac{\theta}{2})\mathbf{Y},$$

$$\mathbf{R}_z(\theta) = e^{-i\theta\mathbf{Z}/2} = \cos(\frac{\theta}{2})\mathbf{I} - i\sin(\frac{\theta}{2})\mathbf{Z}.$$

The operators $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are the usual Pauli Matrices. Check that $\mathbf{XYX} = -\mathbf{Y}$ and $\mathbf{XZX} = -\mathbf{Z}$; then show that $\mathbf{X}\mathbf{R}_y(\theta)\mathbf{X} = \mathbf{R}_y(-\theta)$ and $\mathbf{X}\mathbf{R}_z(\theta)\mathbf{X} = \mathbf{R}_z(-\theta)$.

Solution It is easy to check the first two equations (using, for example, the Taylor series for exp, sin and cos, and facts like: $\mathbf{X}^0 = \mathbf{I}$, $\mathbf{X}^1 = \mathbf{X}$, and $\mathbf{X}^2 = \mathbf{I}$). We now have

$$\mathbf{R}_y(\theta) = e^{-i\theta\mathbf{Y}/2} = \cos\frac{\theta}{2}\mathbf{I} - i\sin\frac{\theta}{2}\mathbf{Y},$$

Hence $\mathbf{X}\mathbf{R}_y(\theta)\mathbf{X} = \cos\frac{\theta}{2}\mathbf{I} + i\sin\frac{\theta}{2}\mathbf{Y} = \cos\frac{\theta}{2}\mathbf{I} - i\sin\frac{-\theta}{2}\mathbf{Y}$.
Similarly for $\mathbf{X}\mathbf{R}_z(\theta)\mathbf{X}$.

Exercise 3 *Assuming that the single qubit operation $\mathbf{U}$ is given by $\mathbf{U} = e^{i\alpha}\mathbf{AXBXC}$ with $\mathbf{ABC} = \mathbf{I}$,*

$$\mathbf{A} = \mathbf{R}_z(\beta)\mathbf{R}_y(\frac{\gamma}{2})$$

$$\mathbf{B} = \mathbf{R}_y(-\frac{\gamma}{2})\mathbf{R}_z(-\frac{\delta+\beta}{2})$$

$$\mathbf{C} = \mathbf{R}_z(\frac{\delta-\beta}{2})$$

Show that the network in Figure 2 implements the controlled $\mathbf{U}$ gate, $\mathsf{C}(\mathbf{U})$, using the single qubit gate $\mathbf{X}$ the phase gate α and the single qubit gates $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

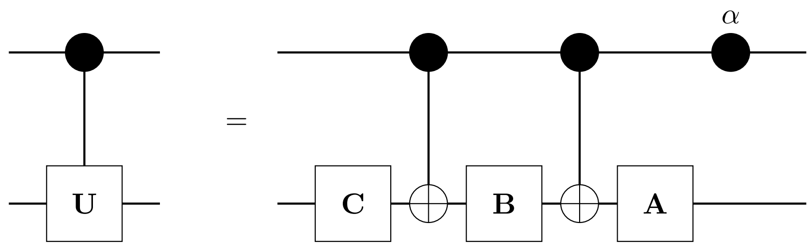

Figure 2: Implementation of $C(\mathbf{U})$

Solution First recall that the phase gate α has matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

and hence it has action:

$$|0\rangle \mapsto |0\rangle\,, \qquad |1\rangle \mapsto e^{i\alpha}\,|1\rangle\,.$$

Therefore, we have the following input-output for the network:

$$|0\rangle \otimes |x\rangle \mapsto |0\rangle \otimes \mathbf{ABC}\,|x\rangle = |0\rangle \otimes |x\rangle\,,$$

$$|1\rangle \otimes |x\rangle \mapsto e^{i\alpha}\,|1\rangle \otimes \mathbf{AXBXC}\,|x\rangle = e^{i\alpha}\,|1\rangle \otimes e^{-i\alpha}\mathbf{U}\,|1\rangle = |1\rangle \otimes \mathbf{U}\,|1\rangle\,.$$

Exercise 4 *Show that the $\mathsf{C^n}(\mathbf{U})$ gate, where $\mathbf{U}$ is single qubit unitary operation, can be implemented using $n-1$ work qubits with input $|0\rangle$ as in the network in Figure 3, depicted for $n=5$. Deduce that any such $\mathsf{C^n}(\mathbf{U})$ gate can be implemented by $O(n)$ single qubit and* CNOT *gates.*

Solution If all of the controlled qubits are $|1\rangle$ then the all the $|0\rangle$ inputs of the work qubits are flipped and hence the control is set for the controlled U gate. Therefore in this case for the target qubit we get: $|\psi\rangle \mapsto U\,|\psi\rangle$; the work qubits are then flipped back to $|0\rangle$ (and the control bits are clearly intact). If on the other hand, any of the controlled qubits is $|0\rangle$, then the input $|0\rangle$ to the connected work qubit and all its underneath work qubits go through intact and hence the control qubit on the controlled U gate is not set. Therefore in this case for the target qubit we get: $|\psi\rangle \mapsto |\psi\rangle$; the flipped work qubits are then flipped back to $|0\rangle$ (and the control bits are again clearly intact).

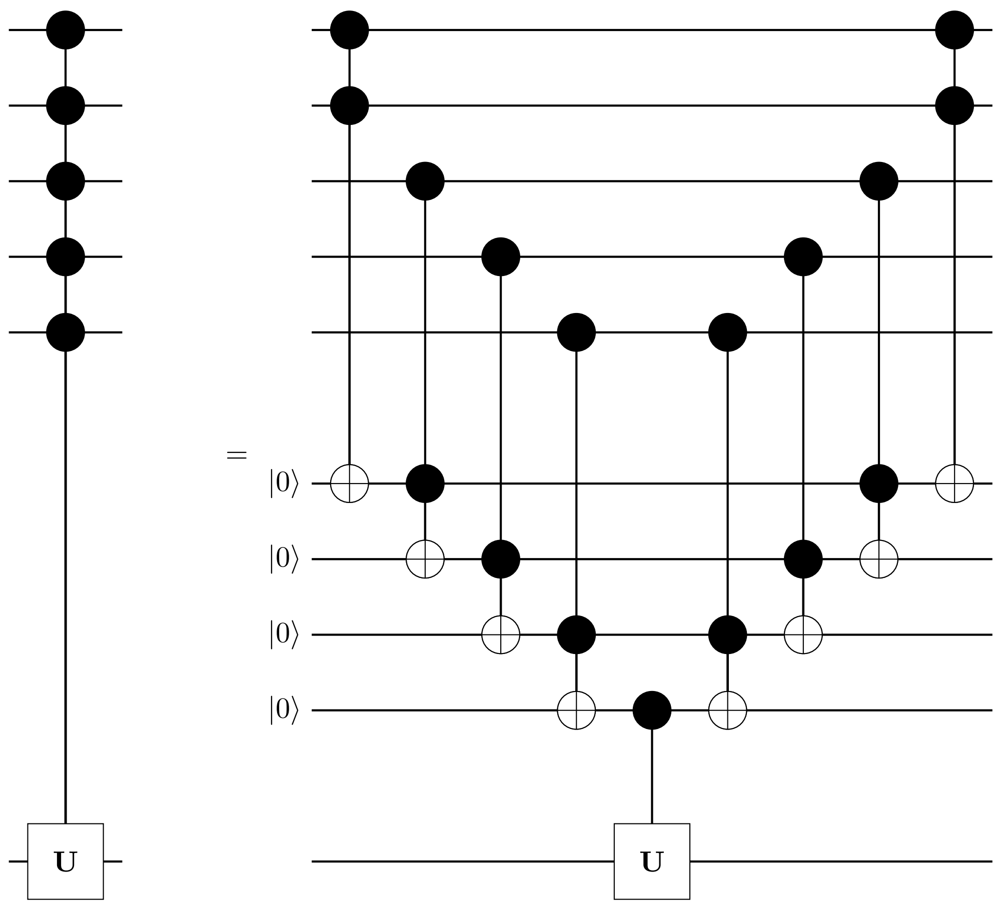

Figure 3: Implementation of $C^n(\mathbf{U})$

Each Toffoli gate can be implemented by $O(1)$ CNOT and single qubit gates. The $\mathsf{C}(\mathbf{U})$ gate also needs $O(1)$ CNOT and single qubit gates. Hence the $2(n-1)$ Toffoli gates in the network and the $\mathsf{C}(\mathbf{U})$ gate can be implemented using $O(n)$ CNOT and single qubit gates.

Exercise 5 *Check that the output of the network in Figure 4 is indeed as in the table on its left up to the normalisation factor* $1/\sqrt{2}$.

in	out
$\|00\rangle$	$\|00\rangle + \|11\rangle$
$\|01\rangle$	$\|01\rangle + \|10\rangle$
$\|10\rangle$	$\|00\rangle - \|11\rangle$
$\|11\rangle$	$\|01\rangle - \|10\rangle$

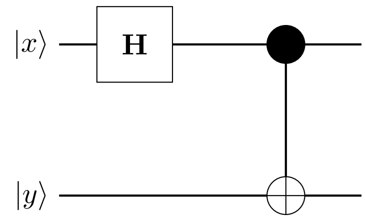

Figure 4: The Bell or EPR states

Solution

$$
\begin{aligned}
|00\rangle &\mapsto \tfrac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \mapsto \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\\
|01\rangle &\mapsto \tfrac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \mapsto \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\\
|10\rangle &\mapsto \tfrac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \mapsto \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\\
|11\rangle &\mapsto \tfrac{1}{\sqrt{2}}(|01\rangle - |11\rangle) \mapsto \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}
$$

Exercise 6 *Let* $f : \{0,1\} \to \{0,1\}$ *be a Boolean function which outputs a bit for each input bit and let* $\mathbf{U}_f : |x, y\rangle \mapsto |x, y \oplus f(x)\rangle$*. Show that* $\mathbf{U}_f$ *is unitary.*

Solution We have shown in general that such a transformation is unitary. Here is a direct proof. The matrix for $\mathbf{U}_f$ is easily seen to be:

$$
\begin{pmatrix}
1 - f(0) & f(0) & 0 & 0\\
f(0) & 1 - f(0) & 0 & 0\\
0 & 0 & 1 - f(1) & f(1)\\
0 & 0 & f(1) & 1 - f(1)
\end{pmatrix}
$$

which is unitary since it is a permutation matrix i.e. there is exactly one non-zero entry namely 1 in each row and these non-zero entries belong to distinct columns. Hence each column is a unit vector and the different column vectors are orthogonal.

Exercise 7 *Show that*

$$
\mathbf{H}^{\otimes n}\,|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle
$$

Solution We have:

$$
\begin{aligned}
\mathbf{H}^{\otimes n}\,|0\rangle^{\otimes n} &= (\mathbf{H}\,|0\rangle)^{\otimes n} = \\
&= \frac{1}{\sqrt{2^n}}(|0\rangle + |1\rangle)^{\otimes n} = \\
&= \frac{1}{\sqrt{2^n}} \sum_{x_1, x_2, \cdots, x_n \in \{0,1\}} |x_1 x_2 \cdots x_n\rangle = \\
&= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle
\end{aligned}
$$