~~Complex Numbers~~

(1) b) $\frac{dy}{dx} + f(x)\,y = g(x)$ —— (1)

$$\frac{dz}{dx} = w(x)$$

$$z = \int w(x)\,dx$$

Consider the product rule in differentiation,

$$\frac{d\,u(x)v(x)}{dx} = u(x)\underbrace{\frac{dv(x)}{dx}}_{v'} + \underbrace{\frac{du(x)}{dx}}_{u'}v(x) \quad \text{—— (2)}$$

Express (1) similar to R-H-S of (2) by multiplying with f^n $h(x)$.

That is $\frac{dy}{dx} + f(x)\,y = g(x)$

$u\,v'$ $\quad$ $u'v$

$v = y$

$$\underbrace{h(x)}_{u}\underbrace{\frac{dy}{dx}}_{v'} + \underbrace{h(x)\,f(x)}_{u'}\,\underbrace{y}_{v} = h(x)g(x) \quad \text{—— (3)}$$

$u = h(x)$

$u' = h(x)\,f(x)$

$h'(x) = h(x)\,f(x)$

Coupled ODEs

3) The soln of above ~~first order homogenous~~ diff. eqn is

$$\boxed{h(x) = e^{\int f(x)\,dx}}$$

if $f(x)$ is a constant A

then $$\boxed{h(x) = c\,e^{Ax}}$$

$\because \dfrac{d\beta(x)}{dx} = \alpha(x)\,\beta(x)$

$\Rightarrow \beta(x) = e^{\int \alpha(x)\,dx}$

if $\alpha(x) = A$ (constant)

$\dfrac{d\beta(x)}{dx} = A\,\beta(x)$

$\beta(x) = \underbrace{C}_{\text{constant}} e^{Ax}$

substituting in (3) gives the general soln

$$\boxed{e^{\int f(x)\,dx}\,\frac{dy}{dx} + e^{\int f(x)\,dx} f(x)\,.\,y = e^{\int f(x)\,dx} g(x)}$$

$$\frac{d}{dx}\left(e^{\int f(x)\,dx}\,y\right) = e^{\int f(x)\,dx} g(x)$$

$$e^{\int f(x)\,dx}\,y = \int e^{\int f(x)\,dx} g(x)\,dx + C$$

$$y = \frac{1}{\underbrace{e^{\int f(x)\,dx}}_{IF}}\left[\int \overbrace{e^{\int f(x)\,dx}}^{IF} g(x)\,dx + C\right]$$

$$y = \frac{1}{IF}\left[\int IF\; g(x)\,dx + C\right]$$

1 c) $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ — (1)

The only f^n whose derivation can be expressed as a linear combination is the exponential f^n.

∴ Substituting ~~f~~ $y = e^{mx}$ in (1)

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$(am^2 + bm + c) = 0$$

$$\Rightarrow m_1, m_2$$

sol^n $y = e^{m_1x}$, $y = e^{m_2x}$

General sol^n is $\boxed{y = Ae^{m_1x} + Be^{m_2x}}$

$$a\frac{d^2}{dx^2}(e^{m_1x}) + b\frac{d}{dx}(e^{m_1x}) + ce^{m_1x} = 0 \quad (2)$$

$$a\frac{d^2}{dx^2}(e^{m_2x}) + b\frac{d}{dx}(e^{m_2x}) + ce^{m_2x} = 0 \quad (3)$$

Subs $y = Ae^{m_1x} + Be^{m_2x}$ in LHS of (1) gives

— (4)

Subs (2) & (3) in (4) gives 0

(1) d) $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ — (1).

Separate into two forms of diff. eqn^{ns} using the variable d,

$$\underbrace{\left(a\frac{d^2y}{dx^2} + (b-d)\frac{dy}{dx}\right)}_{\alpha \frac{dz}{dx}} + \underbrace{\left(d\frac{dy}{dx} + cy\right)}_{z} = 0 \quad \text{— (2)}$$

By equating coefficients we can find a general sol^{n} for d in terms of a, b, c

if $z = d\frac{dy}{dx} + cy$

$\frac{dz}{dx} = d\frac{d^2y}{dx^2} + c\frac{dy}{dx}$ — (3).

from (1) $\alpha\frac{dz}{dx} = a\frac{d^2y}{dx^2} + (b-d)\frac{dy}{dx}$ — (4)

multiplying (3) by α & equating to (4) gives,

$\alpha d = a \Rightarrow \alpha = \frac{a}{d}$.

Also $\alpha c = b-d$

$\frac{a}{d}c = b-d$

$ac = bd - d^2$

$d^2 - bd + ac = 0$

$d = \frac{-b \pm \sqrt{b^2-4ac}}{2}$ — (5)

from (2) $\alpha \frac{dz'}{dx} + dz = 0$

$$\frac{dz'}{dx} = -\frac{1}{\alpha} z$$

$\therefore$ solⁿ for z is

$$z = e^{-\frac{1}{\alpha} x}$$

but $\alpha = \frac{a}{d}$

$$z = e^{-\frac{d}{a} x} \quad \text{---} (6)$$

but from (2) $z = d\frac{dy}{dx} + cy$ --- (7)

(6), (7) $d\frac{dy}{dx} + cy = e^{-\frac{d}{a}x}$ $\Rightarrow \frac{dy}{dx} + \frac{c}{d} y = \frac{1}{d} e^{-\frac{d}{a}x}$

Now the solⁿ to the above eqⁿ is

$$y = \frac{1}{e^{\frac{c}{d}x}} \frac{1}{d} \int e^{\frac{c}{d}x} e^{-\frac{d}{a}x} \, dx$$

--- (8)

we know that

Now $am^2 + bm + c = 0$ | obtained by subs. $y = e^{mx}$ in (1)

and if the roots of m are equal

then $b^2 - 4ac = 0$ --- (*)

and $m = -b/2a$

Hence from (5) we can say

$$d = \frac{-b}{2} \quad \text{---} (9)$$

Substituting (9) in (8)

$$y = \frac{1}{e^{\frac{b}{2a}x}} \cdot \frac{-2}{b} \int e^{\frac{2c}{b}x - \frac{b}{2a}x} dx$$

$$= \frac{1}{e^{\frac{b}{2a}x}} \cdot \frac{-2}{b} \int e^{\frac{\overbrace{(4ac-b^2)}^{=0 \text{ from } *}}{2ab}x} dx$$

$$\therefore y = \frac{1}{e^{\frac{b}{2a}x}} \cdot \frac{-2}{b} \underbrace{\int 1 \cdot dx}_{x+c}$$

$$= \left(\underbrace{\frac{-2}{b}}_{A}x - \underbrace{\frac{2}{b}c}_{B}\right) e^{-\frac{b}{2a}x}$$

So the general soln

$$y = (Ax + B)e^{mx} \quad \left[\because m = -\frac{b}{2a}\right]$$

//

Coupled ODEs

- used ~~for~~ to model massive state-space physical & computer systems
- they are of the form,

$$\frac{dy_1}{dx} = ay_1 + by_2 \quad \text{— (1)}$$

$$\frac{dy_2}{dx} = cy_1 + dy_2 \quad \text{— (2)}$$

let $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, write (1), (2) as,

$$\begin{pmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

or

$$\frac{d\vec{y}}{dx} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \vec{y} \quad //$$

For a coupled ODE of type $\frac{d\vec{y}}{dx} = A\vec{y}$ — (1)

substitute $\vec{y} = \vec{v}e^{\lambda x}$ so, $\Rightarrow \frac{d\vec{y}}{dx} = \lambda \vec{v} e^{\lambda x}$ — (2)

also $\frac{d\vec{y}}{dx} = A\vec{y}$ — (2)

so (1), (2) $\Rightarrow$ $\lambda \vec{v} e^{\lambda x} = A\vec{y}$

$\lambda \vec{v} e^{\lambda x} = A\vec{v}e^{\lambda x}$ | $\because \vec{y} = \vec{v}e^{\lambda x}$

Hence $A\vec{v} = \lambda\vec{v}$ — (3)

By eigenvector solution of (3) the sol^n to (1) can be obtained if A is $n \times n$ then for eigen vectors $\vec{u}_1, \vec{u}_2 - - \vec{u}_n$ and eigen values $\lambda_1, \lambda_2 - - \lambda_n$

the general sol^n of ③ will be,

$$\vec{y} = \beta_1 \vec{u}_1 e^{\lambda_1 x} + - - + \beta_n \vec{u}_n e^{\lambda_n x}$$ //

Coupled ODE's example:

$$\frac{dy_1}{dx} = 2y_1 + 8y_2$$

$$\frac{dy_2}{dx} = 5y_1 + 5y_2$$

so
$$\frac{d\vec{y}}{dx} = \underbrace{\begin{pmatrix} 2 & 8 \\ 5 & 5 \end{pmatrix}}_{A} \vec{y}$$

Solve for eigen values / vectors of matrix A

$$A\vec{u} = \lambda \vec{u}$$

$$\begin{vmatrix} 2-\lambda & 8 \\ 5 & 5-\lambda \end{vmatrix} = 0$$

$$|(2-\lambda)(5-\lambda) - 40| = 0$$

$$|10 + \lambda^2 - 7\lambda - 40| = 0$$

$$\lambda^2 - 7\lambda - 30 = 0$$

$$(\lambda - 10)(\lambda + 3) = 0$$

$$\lambda = -3, 10$$

2×1
15×2
5×6
10×3

$\lambda_1 = -3$ $\begin{pmatrix} 5 & 8 \\ 5 & 8 \end{pmatrix} \begin{pmatrix} \vec{u}_x \\ \vec{u}_y \end{pmatrix} = 0$

$$5\vec{u}_x + 8\vec{u}_y = 0$$

$$\therefore \quad u_x = -\frac{8}{5} u_y$$

$$u_y = -\frac{5}{8} u_x$$

$$\therefore \quad \vec{u}_1 = \begin{pmatrix} 1 \\ -5/8 \end{pmatrix} = \begin{pmatrix} 8 \\ -5 \end{pmatrix}$$

$\lambda_2 = 10$ $\begin{pmatrix} -8 & 8 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = 0$

$$-\cancel{8}u_x + \cancel{8}u_y = 0$$

$$\cancel{5}u_x - \cancel{5}u_y = 0$$

$$u_x = u_y$$

$$\vec{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore $\vec{y} = B_1 \begin{pmatrix} 8 \\ -5 \end{pmatrix} e^{-3x} + B_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{10x}$

Coupled ODE

$$\begin{bmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\vec{y}}$$

$$\frac{d\vec{y}}{dx} = A\vec{y} \quad \text{— (1)}$$

Let $\vec{y} = \underbrace{\vec{u}}_{2\times 1} e^{\lambda x}$ — (2) $\Rightarrow \frac{d\vec{y}}{dx} = \lambda \vec{u} e^{\lambda x}$ — (3)

sub. from (1), (2).

(1) & (2)

$$A\vec{u}\cancel{e^{\lambda x}} = \lambda \vec{u}\, \cancel{e^{\lambda x}} \qquad [\because e^{\lambda x} \neq 0]$$

$$(\underline{A} - \lambda \underline{I})\vec{u} = 0$$

to find λ $\quad |\underline{A} - \lambda \underline{I}| = 0$

$\vdots$

$$\lambda \in \{\lambda_1, \lambda_2\}$$

$$\underbrace{\vec{u}}_{2\times 1} \in \{\vec{u}_1, \vec{u}_2\} \Rightarrow$$

Let $\vec{V} = \underbrace{[\vec{u}_1\ \vec{u}_2]}_{2\times 2}$ and $\boxed{\underbrace{\vec{y}}_{2\times 1} = \underbrace{\vec{V}}_{2\times 2}\underbrace{\vec{z}}_{2\times 1}}$

for some $\vec{z}$ — (A)

$\therefore$ we may re-write (1) as,

$$\vec{V}\frac{d\vec{z}}{dx} = A\vec{V}\vec{z}$$

$$\frac{d\vec{z}}{dx} = \vec{V}^{-1} A \vec{V}\vec{z} \quad \text{— (4)}$$

where $\vec{V}^{-1} A \vec{V} = \underbrace{[\vec{u}_1\ \vec{u}_2]^{-1}}_{2\times 2} \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}}_{2\times 2} \underbrace{[\vec{u}_1\ \vec{u}_2]}_{2\times 2}$

$= D$

$$\vec{V}D = A\vec{V}$$
$$= A\begin{bmatrix}\vec{u_1} & \vec{u_2}\end{bmatrix}$$
$$\begin{bmatrix}\vec{u_1} & \vec{u_2}\end{bmatrix} D = \begin{bmatrix}\lambda_1 \vec{u_1} & \lambda_2 \vec{u_2}\end{bmatrix}$$
$$\therefore D = \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}$$

where $D = \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}$

from ④

$$\frac{d\vec{z}}{dx} = \underbrace{\begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}}_{D}\vec{z}$$

$$\begin{bmatrix}\frac{dz_1}{dx} \\ \frac{dz_2}{dx}\end{bmatrix} = \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix} \cdot \begin{bmatrix}z_1 \\ z_2\end{bmatrix}$$

$\therefore$ we get a set of <u>de-coupled</u> ODE

$$\frac{dz_1}{dx} = \lambda_1 z_1$$
$$\frac{dz_2}{dx} = \lambda_2 z_2$$

$\therefore$ the general sol^n for $\{z_k\}_{k=1,2}$ will be,

$$z_1 = c_1 e^{\lambda_1 x} \quad \& \quad z_2 = c_2 e^{\lambda_2 x}$$

Substituting back in Ⓐ

$$\vec{y} = \vec{V}\vec{z}$$
$$= \begin{bmatrix}\vec{u_1} & \vec{u_2}\end{bmatrix}\begin{bmatrix}z_1 \\ z_2\end{bmatrix}$$
$$\vec{y} = \vec{u_1} z_1 + \vec{u_2} z_2$$

$\Rightarrow$ the general soln to the original coupled ODE is,

$$\vec{y} = c_1 \vec{u_1} e^{\lambda_1 x} + c_2 \vec{u_2} e^{\lambda_2 x}$$

if $\vec{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$ then express the above general soln in terms of u_x or u_y.

i.e. if $u_x = -u_y$ and $u_x = 2u_y$

then $\vec{u_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\vec{u_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

OR

$\vec{u_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{u_2} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$

$$\therefore \vec{y} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{\lambda_1 x} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{\lambda_2 x}$$

or

$$\vec{y} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\lambda_1 x} + c_2 \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} e^{\lambda_2 x}$$

- Cauchy Theorem (another def of convergence w/o the limit)
need to have

if you pick an arbitrarily small ϵ
then you ~~will~~ can always find a suff. large
N, s.t. $\forall\, n, m > N$

$$\underbrace{|a_n - a_m| < \epsilon}$$

~~the seq converges to a value less than~~
any two elements a_n, a_m are ~~ϵ~~
separated by $< \epsilon$

Tutorial sheet 5

(3)

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$$a_n = \sum_{i=1}^{n} \frac{1}{n}$$

$$a_m = \sum_{i=1}^{m} \frac{1}{m}$$

$$a_n - a_m = \sum_{i=1}^{n} \frac{1}{n} - \sum_{i=1}^{m} \frac{1}{m}$$

for any $n \geq m > N$ for $N \in \mathbb{N}$

$n > m > N$

$\div n$

$\cancel{1 > \frac{m}{n} > \frac{N}{n}}$

$\cancel{1 - \frac{m}{n} > 0}$

$\cancel{\frac{n-m}{n} > 0}$

$$a_n - a_m = 1 + \frac{1}{2} \cdots \frac{1}{n} - \left(1 + \frac{1}{2} + \cdots \frac{1}{m}\right)$$

$$= \sum_{k=m+1}^{n} \frac{1}{k}$$

$$= \frac{1}{m+1} + \frac{1}{m+2} \cdots + \frac{1}{n}$$

$$= \frac{\frac{d}{(m+1)} + \frac{d}{(m+2)} + \cdots + \frac{d}{n}}{\underbrace{(m+1)(m+2) \cdots n}_{=d}}$$

$\frac{d}{n}$ is the smallest value

$$\therefore = \frac{\frac{d}{(m+1)} \cdots \frac{d}{n}}{d} > \frac{d}{n} \frac{(n-m)}{\cancel{d}}$$

$$a_n - a_m > \frac{(n-m)}{n} \quad \text{—(1)}$$

for ~~a cert~~ $\epsilon < \frac{1}{2}$ and some N,

If we pick a $n > 2m$ then

$$|a_n - a_m| > \frac{1}{2} \quad \text{from (1)}$$

which means when $n > 2m$, $|a_n - a_m| \not< \epsilon$.

Thus ~~which~~ whatever value we pick for N, when $n > 2m$ and $\epsilon < \frac{1}{2}$.

$$|a_n - a_m| \not< \epsilon.$$

$\therefore$ Therefore from Cauchy Theorem, a_n does not converge.

$$\therefore \quad a_n \to \infty \quad \text{as} \quad n \to \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

//

Cauchy Theorem states that ~~for~~

~~when $\forall n, m > N$ s.t. $N \in \mathbb{N}$~~

~~$|a_n - a_m|$~~

$\forall \epsilon > 0$, you can find a big $\exists N$,

$$|a_n - a_m| < \epsilon \quad \text{for all } n, m > N$$