Gaussian elimination

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When solving N simultaneous equation the Gaussian elimination stage of the algorithm comprises $N-1$ steps. In the basic algorithm, the i th step eliminates nonzero subdiagonal elements in column i by subtracting the i th row from each row j in the range $[i+1, N]$. Hence, the algorithm sweeps down the matrix from the top left corner to the bottom right corner, leaving zero offdiagonal elements in the lower triangular part.

Example Q.3(a) from Tutorial Sheet 3

In this case $N = 3$.

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 1 & 2 & 3 & | & 14 \\ 1 & 4 & 9 & | & 36 \end{bmatrix} \tag{1}$$

$$\downarrow$$

$i = 1$, $j = 2, 3$
r2 = r2 - r1, r3 = r3 - r1

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 2 & | & 8 \\ 0 & 3 & 8 & | & 30 \end{bmatrix} \tag{2}$$

$$\downarrow$$

$i = 2$, $j = 3$
r3 = r3 - 3(r2)

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 2 & | & 8 \\ 0 & 0 & 2 & | & 6 \end{bmatrix} \tag{3}$$

The third row above can be normalized by dividing by 2. Thus we get a upper triangular matrix. When going through this procedure if a zero appears in a diagonal term the complete row with that term should be swapped with another row before continuing.

In order to set the upper off-diagonal terms to zero we repeat the above steps starting from the last row (before we started from the first row). That is,
$i = 3$, $j = 1, 2$

r2 = r2 - 2(r3), r1 = r1 - r3

$$\begin{bmatrix} 1 & 1 & 0 & | & 3 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \tag{4}$$

$$\downarrow$$

$i = 2$, $j = 1$
r1 = r1 - r2

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix} \tag{5}$$

1 Rank and Nullity

1. Rank is the number of non-all-zero rows in reduced matrix that is obtained from Gaussian elimination. For determining rank Gaussian elimination till you reach a upper triangular matrix (with only the lower triangular offdiagonal elements equal to zero) is sufficient.

Nullity = Dimension of matrix, N - Rank

or

2. In the case of Q4(b), the condition for a valid solution to exist gives an equation of u, v, w. This equation defines a plane of allowed solutions and can be written in general form as follows,

$$(u, v, w) \cdot (a, b, c) = 0 \tag{6}$$

That is of the form,

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{s} \tag{7}$$

The dimension of the range of a plane is 2 (i.e. any $\mathbf{r} = (u, v, w)$ can be defined by two independent variables and one dependent variable), hence rank of the matrix A in Q4(b) is 2.