Modal logics of elementary classes of Kripke frames via hybrid logic

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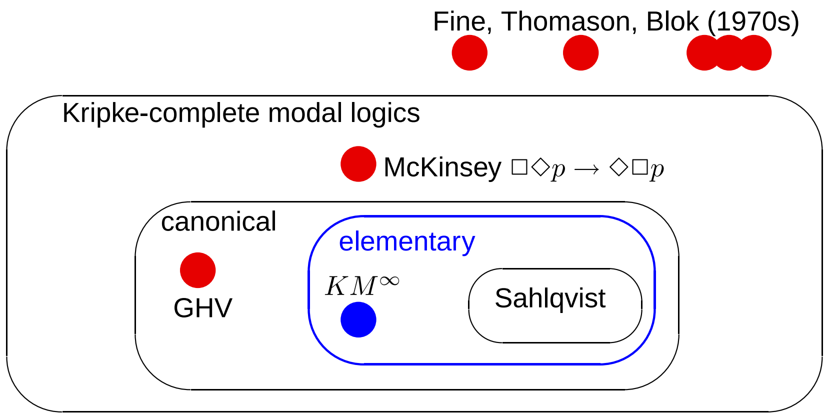
Thanks to Nick for inviting me

introduction

In this talk, *elementary logic* means the modal logic of an *elementary* class of Kripke frames (defined by a first-order theory).

- how to axiomatise?
- partial results — Sahlqvist's theorem. KM^∞
- hybrid logic
- modal approximants of hybrid formulas
- axioms for any elementary modal logic

the world of modal logics



elementary $\Rightarrow$ canonical (Fine–van Benthem 1970s)

elementary modal logics

Many common logics are elementary. Eg, all *Sahlqvist-axiomatisable* ones (Kracht found matching first-order fragment).

But *some elementary logics aren't Sahlqvist.*
E.g., logic KM^∞ from Lemmon notes (1966), extending McKinsey:

$$KM^\infty : \qquad \Diamond \bigwedge_{i \leq k} (\Box p_i \vee \Box\neg p_i) \quad (\text{all } k \geq 1).$$

- class of all frames for KM^∞ is non-elementary
- any axiomatisation has infinitely many non-canonical formulas

Still, KM^∞ *is elementary:* it is logic of class of frames satisfying

$$\forall x \exists y (xRy \wedge \forall zt(yRz \wedge yRt \to z = t))$$

— equivalently, those validating hybrid formula $\Diamond\exists i \Box i$.
Proof: compactness shows canonical frame for KM^∞ satisfies this.

hybrid logic

$$\Phi := i \mid \top \mid \bot \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \Diamond\Phi \mid \Box\Phi \mid \forall i\Phi \mid \exists i\Phi$$

We only consider *pure formulas* — with no propositional atoms.
Sentence — no free nominals.

semantics

Fix Kripke frame $\mathcal{F} = (W, R)$.

Assignments/valuations into $\mathcal{F}$ are maps $h : \{\text{nominals}\} \to W$.

- $\mathcal{F}, h, w \models i$ iff $w = h(i)$
- Boolean and modal operators as usual
- $\mathcal{F}, h, w \models \forall i\varphi$ iff $\mathcal{F}, g, w \models \varphi$ for all assignments g with $g(j) = h(j)$ for all $j \neq i$.

modal approximants of hybrid formulas

Idea: approximate nominals by modally-definable (clopen) sets.

Given any finite set S of modal formulas, can assign i to

$$\bigwedge\{\alpha : \alpha \in X\} \wedge \bigwedge\{\neg\beta : \beta \in S \setminus X\}$$

for any $X \subseteq S$.

Simulate $\forall i\varphi$ by *conjunction* over all $X \subseteq S$.

Simulate $\exists i\varphi$ by *disjunction* over all $X \subseteq S$.

We get a *modal approximant* of a hybrid formula with respect to S.

In canonical model, nominals denote maximal consistent sets.
As $S \to \infty$, clopens $\to$ nominals.
Hope: as $S \to \infty$, approximants converge to the hybrid formula.

example: approximants of $\Diamond\exists i\Box i$ axiomatise KM^∞

Recall $\quad \Diamond \bigwedge_{i \leq k} (\Box p_i \vee \Box\neg p_i) \quad (k \geq 1)$

axiomatises logic KM^∞ of class of frames validating

$$\varphi = \Diamond\exists i\Box i.$$

Approximate φ w.r.t. finite set $S = \{p_1, \ldots, p_k\}$ of atoms:

$$\varphi_S = \overbrace{\Diamond}^{\Diamond} \overbrace{\bigvee_{X \subseteq S}}^{\exists i} \overbrace{\Box}^{\Box} \overbrace{\Big(\bigwedge_{p \in X} p \wedge \bigwedge_{p \in S \setminus X} \neg p\Big)}^{i}.$$

φ_S is equivalent to the kth axiom of KM^∞.
Conclude $\{\varphi_S : S \text{ finite}\}$ axiomatises KM^∞!
In canonical model, all φ_S are valid — forces φ valid on can. frame.

soundness and completeness wish list

1. *if hybrid sentence φ valid in $\mathcal{F}$ then all approximants valid in $\mathcal{F}$.*

   Clopens are coarser than nominals, so should be OK if φ *is positive* (monotonic).

   Problem: $\forall$. $\quad \bigwedge_{X \subseteq S}$ may include inconsistent X.

   Solution: *relativise* $\forall$ to exclude inconsistent X.

   $\forall i\, i$ — bad. $\quad \forall i(\Diamond i \to i)$ — OK.

   General form: $\quad \Diamond(j \wedge \Diamond j') \wedge \forall i(\Box(j \to \Box(j' \to \Diamond i)) \to \varphi)$

   Call $\forall$-relativised-but-otherwise-positive formulas *quasipositive.*
   Equivalently expressive: *positive* sentences of $\mathcal{H}(@, \downarrow)$.

2. *In canonical model (or a descriptive frame), if approximants of hybrid φ are valid, then φ is valid in the underlying Kripke frame.*

   Can prove by extending Sahlqvist's completeness theorem to quasipositive sentences (and even hybrid Sahlqvist formulas!)

theorems

1. If a quasipositive sentence φ is valid in a frame $\mathcal{F}$, then all approximants of φ are valid in $\mathcal{F}$.
2. In canonical model (or a descriptive frame), if all approximants of a quasipositive sentence φ are valid, then φ is valid in the underlying frame.

So *the approximants of φ axiomatise the logic of the class of frames satisfying φ.* This logic is canonical.
φ and its approximants are 'canonical pseudo-correspondents'.

3. The modal logic of a class of frames defined by a first-order theory T is the logic of the class of frames defined by the quasipositive consequences of T.

So *the elementary modal logics are precisely those axiomatised by the approximants of sets of quasipositive sentences.*

remarks

1. applies to multiple polyadic modalities
2. New proof of Fine–van Benthem theorem that elementary modal logics are canonical. 'Explains' canonicity of KM^∞ and other non-Sahlqvist logics by Sahlqvist-like means.
3. Syntactic characterisation of elementary modal logics, by approximants. New way to study them.
4. connection between modal and hybrid logic
5. axioms can be 'natural' — eg KM^∞, Hughes's logic
6. some logics need infinitely many quasipositive sentences
7. *open problem* to find finite axiomatisation where one exists

Reference: I. Hodkinson, Hybrid formulas and elementarily generated modal logics, Notre Dame J. Formal Logic, to appear.