The variety generated by completions of representable relation algebras

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Introduction

Last year, Roger Maddux defined a completely new variety of relation algebras!

It is generated by the completions of representable relation algebras.

It has attracted some attention.

In this talk, we show that the representable relation algebras are not finitely axiomatisable within this new variety.

Plan

- Unary relations — boolean algebras
- Binary relations — (representable) relation algebras

- Completions
- Maddux's variety V

- Relation algebras from graphs
- RRA is not finitely axiomatisable over V

- Andréka–Németi on relation algebras not in V (if time)

Unary relations — boolean algebras

Take a set U. A *unary relation* on U is a subset of U.

Consider $(\wp(U), \cup, \setminus, \emptyset, U)$ for any set U.
Algebra. Field of sets.

Cayley: $\mathbf{S}\{$algebra of all permutations on U : U a set$\}$ = $\{$groups$\}$.

(Here & below, $\mathbf{S}\mathcal{K}$ = closure of $\mathcal{K}$ under subalgebras (and isomorphism),
$\mathbf{P}\mathcal{K}$ = closure of $\mathcal{K}$ under products,
$\mathbf{H}\mathcal{K}$ = closure of $\mathcal{K}$ under homomorphic images.)

By analogy, what is $\mathbf{S}\{(\wp(U), \cup, \setminus, \emptyset, U) : U \text{ a set}\}$?
Answer: the class BA of *boolean algebras.*

BA is a nice class:

- elementary, defined by equations — i.e., a *variety*
- finitely axiomatisable by a few explicit equations.

So we feel we understand algebras of unary relations fairly well.

Binary relations — representable relation algebras

A *binary relation* on a set U is a subset of $U \times U$. Let

$$\mathfrak{Re}(U) = (\wp(U \times U),\ \cup,\ \setminus,\ \emptyset,\ U \times U,\ Id_U,\ -^{-1},\ |\ ),$$

where for $a, b \subseteq U \times U$,

- $Id_U = \{(x, x) : x \in U\}$ (identity)
- $a^{-1} = \{(x, y) : (y, x) \in a\}$ (converse)
- $a \mid b = \{(x, y) : \exists z((x, z) \in a \text{ and } (z, y) \in b)\}$ (composition)

Is $\mathbf{S}\{\mathfrak{Re}(U) : U \text{ a set}\}$ a nice class, as with unary relations?

Not very, no. It's not a variety.

But **(Tarski 1955):** $\mathbf{SP}\{\mathfrak{Re}(U) : U \text{ a set}\}$ is a variety — denoted by RRA, the class of 'representable relation algebras'.

So what's a (plain) relation algebra?

Not all algebras here are concrete algebras of relations. So we use 'abstract' symbols $+, -, 0, 1, 1', \smile, ;$ and not $\cup, \setminus, \emptyset, U \times U, \mathrm{Id}_U, ^{-1}, |$.

Tarski (∼1941):
A *relation algebra* is an algebra $\mathcal{A} = (A, +, -, 0, 1, 1', \smile, ;)$ such that

- $(A, +, -, 0, 1)$ is a boolean algebra
- $(A, ;, 1')$ is a monoid
- $a \cdot (b\,;c) = 0$ iff $b \cdot (a\,;c^{\smile}) = 0$, iff $c \cdot (b^{\smile}\,;a) = 0$ $(\forall a, b, c \in A)$

RA denotes the class of all relation algebras. It is a variety.

RRA $\subseteq$ RA. But sadly, RRA $\neq$ RA. In fact (proved later):

Monk (1964): RRA is not finitely axiomatisable in first-order logic.

So the nice simple picture (BA) for unary relations is not replicated.

Completions of relation algebras

Let $\mathcal{B} = (B, +, -, 0, 1)$ be a boolean algebra. For $b, b' \in B$ define $b \leq b'$ if $b + b' = b'$. This is a partial order on B (corresponds to $\subseteq$).

We say that $\mathcal{B}$ is *complete* if meets and joins (infs and sups) wrt. $\leq$ of *all subsets of B* exist in $\mathcal{B}$.
Example: full power-set algebras $(\wp(U), \cup, \setminus, \emptyset, U)$ are complete.

Monk (1970):
Every relation algebra $\mathcal{A}$ has a *completion:* an algebra $\mathcal{A}^c$ such that:

- $\mathcal{A}$ is a dense subalgebra of $\mathcal{A}^c$ $(\forall x \in \mathcal{A}^c \setminus \{0\} \exists y \in \mathcal{A} \setminus \{0\}(y \leq x))$
- the boolean reduct of $\mathcal{A}^c$ is a complete boolean algebra.

$\mathcal{A}^c$ is unique up to isomorphism over $\mathcal{A}$.

Monk showed that *$\mathcal{A}^c$ is also a relation algebra.*

So what if $\mathcal{A} \in$ RRA? Do we get $\mathcal{A}^c \in$ RRA as well?
Monk asked: *Is* RRA *closed under completions?*

RRA^c **and Maddux's variety** V

Answer (IH, 1997): no. (Sketch proof later.) So

$$\mathsf{RRA}^c \stackrel{\text{def}}{=} \{\mathcal{A}^c : \mathcal{A} \in \mathsf{RRA}\} \not\subseteq \mathsf{RRA}.$$

Maddux (2018):
Let V be the variety generated by RRA^c. That is, $V = \mathbf{HSP}\,\mathsf{RRA}^c$.

Clearly,

$$\mathsf{RRA} \subset V \subseteq \mathsf{RA}.$$

Maddux gave new examples of algebras separating V from RRA.
So the gap between RRA and V is substantial.

Maddux (2018) problem 1.1

1. Is V = RA?
   Andréka–Németi (2018, [3] in the references): no.
   In fact, there are continuum-many varieties between V and RA.
   See later for details (if time).

2. Is V finitely axiomatizable?

3. Is V closed under canonical extensions?
   Yes, by theorem 3.8 of Gehrke–Harding–Venema 2005.

4. Is V closed under completions?

5. Is membership in V decidable for finite algebras?

6. Does V contain any algebras that are not weakly representable?

Maybe V is quite close to RRA after all?

After all,

$$\text{RRA} \subset V \subseteq \text{continuum-many varieties} \subseteq \text{RA}.$$

No: we'll sketch a proof that RRA *is not finitely axiomatisable over V.*

Idea of proof: find a sequence of algebras in $V \setminus \text{RRA}$ having an ultraproduct in RRA.

We do it using relation algebras built from *graphs.*

Relation algebras from graphs

Let $G = (V, E)$ be an (undirected loop-free) graph.
Recall $X \subseteq V$ is *independent (in G)* if $E \cap (X \times X) = \emptyset$.

Write $G \times 3$ for the graph $(V \times 3, E^{\vee})$, where $3 = \{0, 1, 2\}$ and $E^{\vee}((x, i), (y, j))$ iff $E(x, y)$ or $i \neq j$.

Write $A = (V \times 3) \cup \{1'\}$. We say $(x, y, z) \in A^3$ is *consistent* if

- one of x, y, z is 1' and the other two are equal, or
- $\{x, y, z\} \subseteq V \times 3$, and $\{x, y, z\}$ is *not* independent in $G \times 3$.

We define an algebra

$$\mathcal{A}(G) = (\wp(A), \cup, \setminus, \emptyset, A, \{1'\}, {}^{\smile}, ;)$$

where for $a, b \subseteq A$:
$a^{\smile} = a$,
$a \, ; b = \{z \in A : (x, y, z)$ is consistent for some $x \in a, y \in b\}$.

Remark: $a \subseteq V \times 3$ is independent in $G \times 3$ iff $(a \, ; a) \cap a = \emptyset$.

Chromatic number and representability of $\mathcal{A}(G)$

Recall: the *chromatic number* $\chi(G)$ of a graph $G = (V, E)$ is the least n such that V is the union of n independent sets, and ∞ if no such n.

Well known: $\chi(G) \leq 2$ iff G has no cycles of odd length.

Facts about $\boldsymbol{\mathcal{A}(G)}$

1. $\mathcal{A}(G)$ is always a relation algebra (the ‘3’ in $G \times 3$ is used here).
2. $\mathcal{A}(G)$ is complete (obviously — it's based on $\wp(A)$).
3. (Hirsch–IH 2002) If G is infinite, $\mathcal{A}(G) \in$ RRA iff $\chi(G) = \infty$.

Proof sketch that $\mathcal{A}(G) \in$ RRA $\Rightarrow \chi(G) = \infty$

Suppose $\mathcal{A}(G) \in$ RRA $= \mathbf{SP}\{\mathfrak{Re}(U) : U \text{ a set}\}$.

Then by algebra, we can suppose $\mathcal{A}(G) \subseteq \mathfrak{Re}(U)$ for some set U.
As G is infinite, U must be infinite.

So take pairwise distinct $x_0, x_1, \ldots \in U$.

Suppose for contradiction that $\chi(G) < \infty$.
So $G \times 3$ is the union of independent sets $a_1, \ldots, a_n$ $(n = 3\chi(G))$.

Then in $\mathfrak{Re}(U)$ we have $a_1 \cup \cdots \cup a_n = (U \times U) \setminus Id_U$.
So for each $i < j < \omega$, (x_i, x_j) lies in some a_k $(1 \leq k \leq n)$.

By Ramsey's theorem, we can assume that k is constant.

But then $(x_0, x_1), (x_1, x_2), (x_0, x_2) \in a_k$. So $(x_0, x_2) \in (a_k \,;\, a_k) \cap a_k$.

But a_k is independent, so $(a_k \,;\, a_k) \cap a_k = \emptyset$. Contradiction.

Proof sketch that $\chi(G) = \infty \Rightarrow \mathcal{A}(G) \in$ RRA

Assuming $\chi(G) = \infty$, use games ($\sim$forcing) to construct U and an embedding $\mathcal{A}(G) \hookrightarrow \mathfrak{Re}(U)$.

Key point: since $\chi(G) = \infty$, for each $i < 3$ there is an ultrafilter μ_i of $\mathcal{A}(G)$ such that

- $G \times \{i\} \in \mu_i$
- μ_i contains no independent sets in $G \times 3$.

So essentially, "$\mu_i \,;\, \mu_i = 1$". This makes the game easy.

RRA is not finitely axiomatisable over V — 1/3

Theorem. RRA is not finitely axiomatisable over V.

Proof. Below, k, n are integers with $k \geq 0$ and $n \geq 1$.

- Let K_n be a complete graph with n nodes. So $\chi(K_n) = n$.
- Let E_n be a graph with $\chi(E_n) \geq n$ and no cycles of length $\leq n$. Erdős (1959) constructed finite examples.

Write $+, \sum$ for disjoint union of graphs. Let

$$\begin{aligned} G_n^0 &= \sum_{m \geq n} E_m, \\ G_n^k &= G_n^0 + K_k \quad (k \geq 1). \end{aligned}$$

Remark. These are infinite graphs with chromatic number ∞. So

$$\mathcal{A}(G_n^k) \in \text{RRA} \quad \text{for all } k, n.$$

RRA **is not finitely axiomatisable over** V **— 2/3**

Now for each $k \geq 0$ define non-principal ultraproducts

- $\mathcal{A}^k = \prod_D \{\mathcal{A}(G_n^k) : n \geq 1\}$
- $G^k = \prod_D \{G_n^k : n \geq 1\}$

(for some non-principal ultrafilter D over $\{1, 2, \ldots\}$)

You might think that $\mathcal{A}^k \cong \mathcal{A}(G^k)\ldots$
But $\mathcal{A}^k$ is not necessarily complete, whereas $\mathcal{A}(G^k)$ is complete. In fact,

$$\mathcal{A}(G^k) \cong (\mathcal{A}^k)^c.$$

We saw $\mathcal{A}(G_n^k) \in$ RRA for all k, n.
RRA is elementary, so $\mathcal{A}^k \in$ RRA.
Hence, its completion $\mathcal{A}(G^k)$ is in Maddux's variety $V = \mathbf{HSP}\,\mathrm{RRA}^c$.

But by Łoś's theorem, G^0 has no cycles. So $\chi(G^0) \leq 2$.
And for $k \geq 1$, $G^k \cong G^0 + K_k$. So $\chi(G^k) \leq \max(2, k) < \infty$.

So $\mathcal{A}(G^k) \notin$ RRA. (As promised, this shows $\mathrm{RRA}^c \not\subseteq$ RRA.)

RRA **is not finitely axiomatisable over** V **— 3/3**

We have $\mathcal{A}(G^k) \in V \setminus$ RRA for each $k \geq 0$.

Now one final lot of ultraproducts:

- $\mathcal{B} = \prod_D \{\mathcal{A}(G^k) : k \geq 1\}$,
- $G = \prod_D \{G^k : k \geq 1\}$.

As before, $\mathcal{B}^c \cong \mathcal{A}(G)$. In particular, $\mathcal{B} \subseteq \mathcal{A}(G)$ up to isomorphism.

Recall $G^k \cong G^0 + K_k$ for each $k \geq 1$.
By Łoś's theorem, $K_k \hookrightarrow G$ for every k. So $\chi(G) = \infty$.
So $\mathcal{A}(G) \in$ RRA.
Hence its subalgebra $\mathcal{B} \in$ RRA too.

Thus, an ultraproduct $\mathcal{B}$ of algebras $\mathcal{A}(G^k)$ in $V \setminus$ RRA lies in RRA.
By Łoś's theorem, RRA is not finitely axiomatisable over V. $\square$

Extra: Andréka–Németi, 'Varieties generated by completions' [3]

Showed $V \neq$ RA, and $\exists$ continuum-many varieties between V and RA.

We take a peek into this.

A relation algebra $\mathcal{A}$ is *persistently finite* if all simple relation algebras extending $\mathcal{A}$ are finite.

Frias–Maddux (1997) found simple persistently finite relation algebras "$\mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$", and infinitely many more, not in RRA.
(In fact they have *no* proper simple extensions. They fail 'Equation L', so are all very far from RRA — not even in RA_5. They are integral.)

Andréka–Németi's result

Theorem (Andréka–Németi [3]). Let $\mathcal{A}$ be a simple persistently finite relation algebra. Then $\mathcal{A} \in V$ iff $\mathcal{A} \in$ RRA. (They prove more.)

Amended 22 August 2019 and 5 March 2020:

I have deleted the short 'proof' of this theorem given in the talk, since it had a gap — I claimed without justification that $V = \mathbf{HSPUp}\{\mathcal{S}^c : \mathcal{S} \in \text{RRA simple}\}$. This is in fact true, but the proof is no simpler than [3].

See [3] for a proof of the theorem.

Andréka–Givant–Németi coset relation algebras

So $\mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5, \ldots$ separate V from RA — as Maddux (2018) suggested.
Andréka–Németi gave more examples, using *coset relation algebras* (related to *measurable relation algebras)* of Givant.
These generalise 'group relation algebras'. Many long papers on them.

Andréka–Németi [3]:
Every finite simple coset relation algebra is persistently finite.

Now Andréka–Givant [1, theorem 5.2] had constructed a finite simple coset relation algebra $\mathcal{C} \notin$ RRA.
Hence, by theorem above, $\mathcal{C} \notin V$ as well.

Andréka–Givant–Németi [2] extended: constructed infinitely many $\mathcal{C}$.
Again, they are apparently very far from RRA — not even in RA_5.
They are non-integral.

Andréka–Németi [3] used these $\mathcal{C}$, or Frias–Maddux algebras, to exhibit continuum-many varieties between V and RA.

References

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